Université Lyon 1

Séminaire Physique mathématique de l'ICJ

Perturbation theory for the Φ^4_3 measure, revisited with Hopf Algebras

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18 November 2022

Joint work with Tom Klose (Berlin)

PROJET FINANCÉ PAR L'ANR - PROJECT FUNDED BY THE ANR

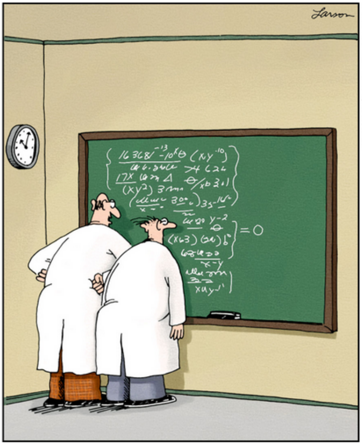
Project PERISTOCH

Disclaimer

I have a degree in physics, but my knowledge in Quantum Field Theory (QFT) is limited.
My main areas of expertise are

- Dynamical systems (ordinary differential equations and iterated maps)
- Stochastic differential equations
- Stochastic partial differential equations (SPDEs)

I have become aware of the deep connections between SPDEs and QFT only quite recently, but do find them very interesting.



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!"

The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2} N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\mu_{N,\varepsilon}(\mathrm{d}y) = \dfrac{1}{Z_{N,\varepsilon}} \mathrm{e}^{-V_{N,\varepsilon}(y)}\,\mathrm{d}y$

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- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4 \Big) \, \mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

$$\mu_\varepsilon(\mathrm{d}\phi) \text{ ``}=\text{''} \frac{1}{Z_\varepsilon} \mathrm{e}^{-V_\varepsilon(\phi)} \text{ ``}\mathrm{d}\phi\text{''}$$

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- Alternative: Spectral Galerkin approx. (Fourier modes with $|k| \leqslant N$)

The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Big(\frac{1}{2} \|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 \Big) \mathrm{d}x = \frac{1}{2}\langle \phi, (-\Delta + 1)\phi \rangle$

  μ_0 is Gaussian free field with covariance $(-\Delta + 1)^{-1}$

  (well-defined since $(-\Delta + 1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1} < \infty$)

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- $\varepsilon > 0$:

$$\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu_0} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x}$$

  where

$$\frac{Z_\varepsilon}{Z_0} = \mathbb{E}^{\mu_0}\Big[\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x}\Big] \text{``="} \frac{1}{Z_0} \int \mathrm{e}^{-V_0(\phi) - \frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x} \,\mathrm{d}\phi$$

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  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Big[\int_\Lambda \phi_{\mathrm{GFF}}(x)^{2n} \, \mathrm{d}x\Big] \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1} \Big)^n < C^n$$

  so that $\frac{Z_\varepsilon}{Z_0} = 1 + \mathcal{O}(\varepsilon)$

The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$

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- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2, |k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF},N}(x)^2]\,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} = \mathrm{Tr}\big[(-\Delta_N + 1)^{-1}\big] \sim \log N$$

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- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

If (X, Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X;C)H_m(Y;C')] = n!\delta_{nm}\mathbb{E}[XY]$

Consequence: $\sup_N \mathbb{E}\Big[\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^{2n}:\mathrm{d}x\Big] < \infty \qquad \forall n$

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- Gibbs measure defined as in 1d case, with

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 + \frac{\varepsilon}{4} :\phi(x)^4:\Big)\, \mathrm{d}x$$

The case $d = 3$

Theorem: Potential needs exactly 4 counterterms:

$$V(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \tfrac{\varepsilon}{4} {:}\phi(x)^4{:}_{C_N^{(1)}} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)} \Big)\,\mathrm{d}x$$

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where

$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 3! \int_\Lambda G_N(x)^3\,\mathrm{d}x = \mathcal{O}(\log N)$$

$$C_N^{(3)} = \frac{4!}{2!4^2} \int_\Lambda G_N(x)^4\,\mathrm{d}x = \mathcal{O}(N)$$

$$C_N^{(4)} = \frac{2^3}{3!4^3}\binom{4}{2}^3 \int_\Lambda \int_\Lambda G_N(x)^2 G_N(y)^2 G_N(x-y)^2\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\log N)$$

and $G_N(x) = \displaystyle\sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ if the Green function of Δ_N

Some literature

- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x) - \varepsilon\phi(t,x)^3 + \xi(t,x)$$

- Parisi & Wu (1981):
  Stochastic quantization
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures

Graphical notations

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{\varepsilon}{4}$, $\beta = \frac{1}{2}\varepsilon^2 C_N^{(2)}$, $\gamma = \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$

  Then $\dfrac{Z_{N,\varepsilon}}{Z_{N,0}} = \mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$

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- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \mathrm{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\, \mathrm{d}x$$

  For instance

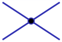
$C_N^{(1)} = \Pi_N$ 

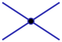
$C_N^{(2)} = 3!\Pi_N$ 

$C_N^{(3)} = \dfrac{4!}{2!4^2}\Pi_N$ 

$C_N^{(4)} = \dfrac{2^3}{3!4^3}\dbinom{4}{2}^3 \Pi_N$ 

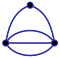
Cumulant expansion

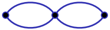
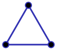
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

  Examples:

  $\mu_2 = \alpha^2 4! \Pi_N$  $+ \beta^2 2! \Pi_N$ 

  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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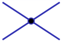
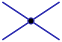
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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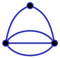
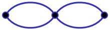
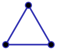
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

  $-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!}$ $\qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$

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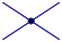
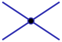
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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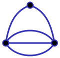
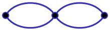
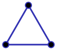
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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- Linked Cluster Theorem: κ_n linear combinations of connected graphs

  Proof: for instance Peccati & Taqqu (2011)

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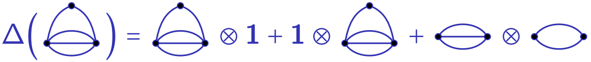
$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \bar\Gamma \otimes (\Gamma / \bar\Gamma) \qquad (\mathbf{1}\text{: empty graph})$$

Example: 

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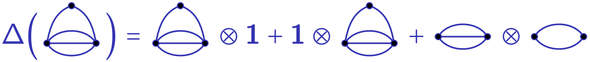
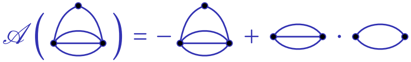
  Example: 
- (Twisted) antipode: $\mathscr{A} : \mathscr{G} \to \mathscr{G}$, $\quad \mathscr{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \mathscr{A}(\bar{\Gamma}) \cdot (\Gamma / \bar{\Gamma})$

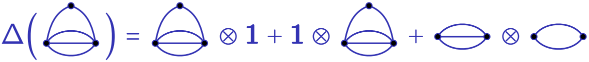
  Example: 

Hopf algebras and renormalisation

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

  $$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \bar\Gamma \otimes (\Gamma/\bar\Gamma) \qquad (\mathbf{1}\text{: empty graph})$$

  Example: 

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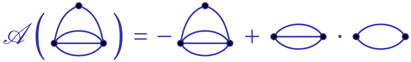
  Example: 

- Character: linear form $g : \mathscr{G} \to \mathbb{R}$ such that $\langle g, \Gamma_1 \cdot \Gamma_2 \rangle = \langle g, \Gamma_1 \rangle \langle g, \Gamma_2 \rangle$

  Renormalisation map: $M^g : \mathscr{G} \to \mathscr{G}$, $M^g(\Gamma) := (g \otimes \mathrm{id})\Delta\Gamma$

  Property: If $\langle f \circ g, \Gamma \rangle = \langle f \otimes g, \Delta\Gamma \rangle$ and $\langle \mathscr{A}^*(f), \Gamma \rangle = \langle f, \mathscr{A}(\Gamma) \rangle$

  then $M^{g \circ h} = M^g M^h$ and $(M^g)^{-1} = M^{\mathscr{A}^*(g)}$ $\Rightarrow$ group structure

BPHZ renormalisation

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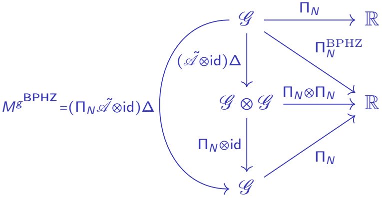

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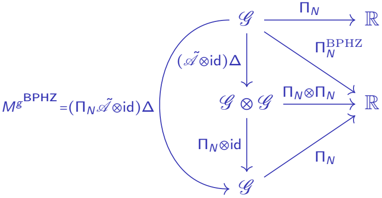

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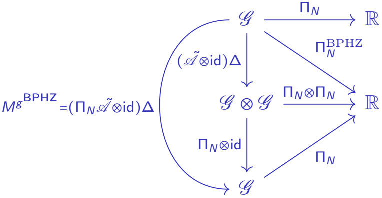

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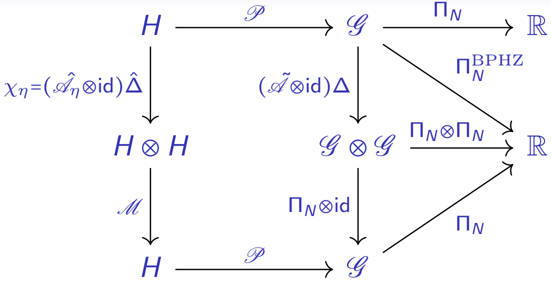
Theorem: Write $\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$

Then $\displaystyle\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!} (-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)})$ $\qquad \deg \Gamma_{pp}^{(k)} = p - 3$

Consequence: all terms in cumulant expansion bounded uniformly in N

Commutative diagram



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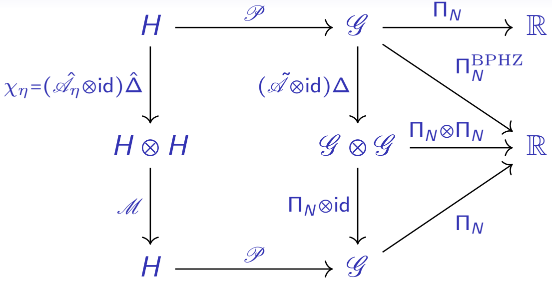

- $H = \mathrm{span}\{\mathbf{X}^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\}$ $\quad \mathbf{X}^{\boldsymbol{n}} := X^{n_1} Y^{n_2}$ (Ebrahimi-Fard et al)

  $$\hat{\Delta}\mathbf{X}^{\boldsymbol{n}} = \sum_{\substack{\boldsymbol{k},\boldsymbol{m} \in \mathbb{N}_0^2 \\ \boldsymbol{k} \cdot \boldsymbol{m} = \boldsymbol{n}}} \binom{\boldsymbol{n}}{\boldsymbol{m}, \boldsymbol{k}} \mathbf{X}^{\boldsymbol{k}} \otimes \mathbf{X}^{\boldsymbol{m}}, \quad \hat{\mathcal{A}}_\eta \mathbf{X}^{\boldsymbol{n}} = (2\ell - 1)!!(-2\eta Y)^\ell \mathbf{1}_{\boldsymbol{n}=(2\ell,0)}$$

- $\chi_\eta(\mathbf{X}^{\boldsymbol{n}}) = (\hat{\mathcal{A}}_\eta \otimes \mathrm{id})\hat{\Delta}\mathbf{X}^{\boldsymbol{n}}$ $\qquad \mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$

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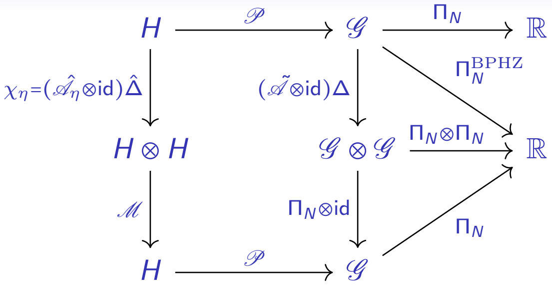

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Lemma: $(\mathscr{M} \circ \chi_\eta)\, \mathrm{e}^{-\alpha X} = \mathrm{e}^{-\alpha X - \beta Y}$

Zimmermann's forest formula

- Zimmermann forest formula: $\mathscr{A}(\Gamma) = -\sum_{\mathscr{F}}(-1)^{|\mathscr{F}|}\mathscr{C}_{\mathscr{F}}\Gamma$

  where sum ranges over all forests $\mathscr{F}$ (set of subgraphs, pairwise vertex-disjoint or included) and $\mathscr{C}_{\mathscr{F}}$ extracts all subgraphs in $\mathscr{F}$

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Lemma: The diagram commutes, that is,

$\mathscr{P} \circ \mathscr{M} \circ \chi_\eta = (\Pi_N \tilde{\mathscr{A}} \otimes \mathrm{id})\Delta \circ \mathscr{P}$

Borel resummation: The Φ^4_0 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

  $$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi$$

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  $$Z_{\mathrm{Borel}}(\varepsilon) = \int_0^{\infty} \mathrm{e}^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-t}\,\mathcal{B}Z(\varepsilon t)\,\mathrm{d}t$$

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Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \operatorname{Re}\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\text{Borel}}(\varepsilon)$ in D_R

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- Control remainder by using
  - Taylor formula with remainder in integral form
  - Cauchy–Schwarz inequality
  - Sharp estimates on $\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}_0(X^{2n})$ (Hairer 2018, B & Bruned 2019)

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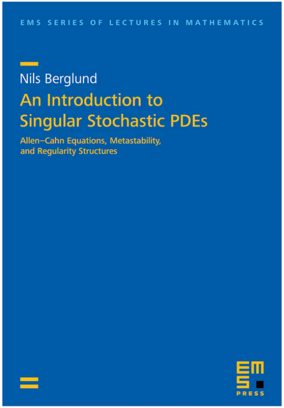

Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/Lyon_nov22.pdf