TU Munich — Critical Earth Workshop

Introduction to stochastic calculus

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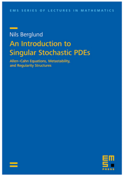
Contents

Some references

Parts 1 and 2:

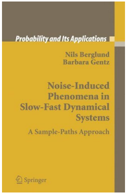
- N. B., *Long-time dynamics of stochastic differential equations*, Lecture notes, `https://arxiv.org/abs/2106.12998` (2021)

Part 2 (potential theory):



- N. B., *An introduction to singular stochastic PDEs: Allen-Cahn equations, metastability and regularity structures*, Lecture notes, `https://arxiv.org/abs/1901.07420` (2019)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)

Part 3:



- N. B. and Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)

PART I

Some basics on stochastic differential equations

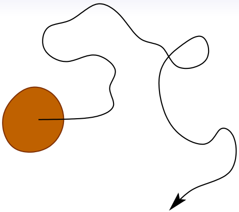
Brownian motion

Paradigm: Brownian motion

R. Brown, 1827

A. Einstein, 1905: $\dfrac{\langle x^2 \rangle}{t} = \dfrac{k_{\mathrm{B}} T}{6\pi\eta r}$

J. Perrin, 1909:
"weighing the hydrogen atom"



Wiener process $\{W_t\}_{t\geqslant 0}$: scaling limit of random walk $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$

Stochastic differential equation

Mathematician's notation: $\qquad \mathrm{d}x_t = \underbrace{f(x_t)\,\mathrm{d}t}_{\text{exterior force}} + \underbrace{g(x_t)\,\mathrm{d}W_t}_{\text{random force}}$

Physicist's notation: $\qquad \dot{x} = f(x) + g(x)\xi, \qquad \langle \xi(s)\xi(t) \rangle = \delta(s-t)$

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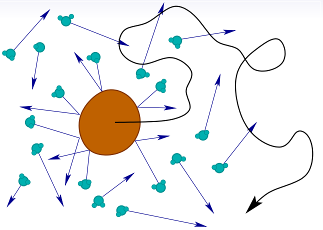

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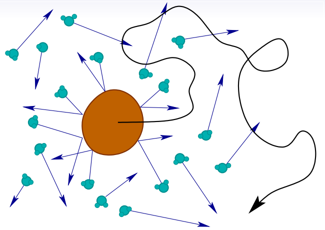

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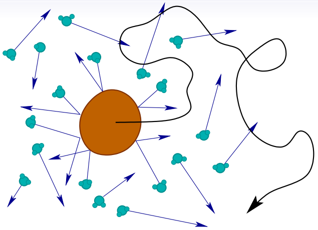

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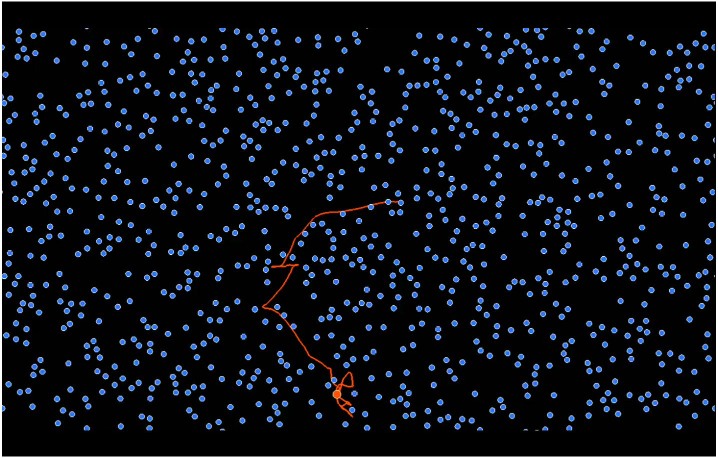

(Online: https://youtu.be/WUzUjcLAjOo)

The symmetric random walk

Definition: SRW

$S_n = \sum_{i=1}^{n} Y_i$

where the Y_i are i.i.d. random variables such that

$\mathbb{P}\{Y_i = 1\} = \mathbb{P}\{Y_i = -1\} = \frac{1}{2}$

Properties:

- **Centred:** $\mathbb{E}[S_n] = 0 \ \forall n$
- **Variance:** $\mathrm{Var}(S_n) = n \ \forall n$
- **Law:** $\mathbb{P}\{S_n = k\} = \frac{1}{2^n}\binom{n}{(n+k)/2}$, $k \in \{-n, -n+2, \dots, n-2, n\}$
- **Independent increments:** $S_n - S_m$ indep of $(S_1, \dots, S_m)$ for $n > m$
- **Stationary increments:** $S_n - S_m$ and S_{n-m} have same law

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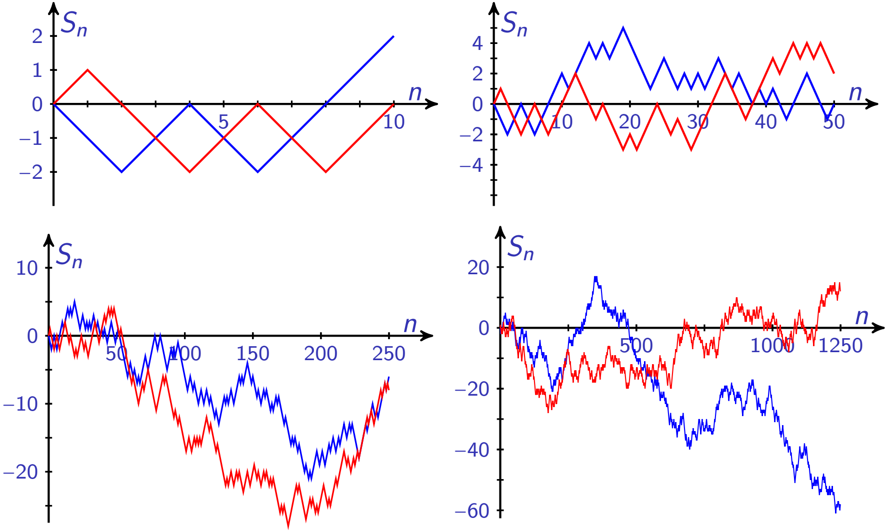
Formally, $W_t = \lim_{n\to\infty} \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$



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Definition: Wiener process

$\{W_t\}_{t\geqslant 0}$ satisfies

1. $W_0 = 0$
2. **Independent increments:**
   For $0 \leqslant s < t$, $W_t - W_s$ is independent of $\{W_u : 0 \leqslant u \leqslant s\}$
3. **Stationary and Gaussian increments:**
   For $0 \leqslant s < t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$

Theorem

There exists a unique process satisfying this definition, and whose sample paths $t \mapsto W_t(\omega)$ are continuous.
One can thus consider the Wiener process as a probability distribution on continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$

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Properties of the Wiener process

- **Markov property:** $\mathbb{P}\{W_{t+s} \in A | W_t = x\} = \int_A \underbrace{\frac{\mathrm{e}^{-(y-x)^2/2s}}{\sqrt{2\pi s}}}_{p_s(y|x)} \mathrm{d}y$
- **Differential property:** $(W_{t+s} - W_t)_{s \geqslant 0}$ is a Wiener process
- **Scaling property:** $(cW_{t/c^2})_{t \geqslant 0}$ is a Wiener process for any $c > 0$
- **Symmetry:** $(-W_t)_{t \geqslant 0}$ is a Wiener process
- **Gaussian process:** centred with covariance $\mathbb{E}[W_s W_t] = \min\{s, t\}$
- **Non-differentiability of paths:** The paths $t \mapsto W_t(\omega)$ are almost surely nowhere Lipschitzian, and thus nowhere differentiable.

The Wiener process and martingales

- $\mathscr{F}_t = \sigma(\{W_s\}_{0 \leqslant s \leqslant t})$: information available at time t
- $\mathbb{E}\{X|\mathscr{F}_t\}$: best estimate of random var X given information at time t
  - $\mathbb{E}\{X|\mathscr{F}_t\} = X$ if X depends only on information in $\mathscr{F}_t$
  - $\mathbb{E}\{X|\mathscr{F}_t\} = \mathbb{E}[X]$ if X independent of $\mathscr{F}_t$

Definition: Martingale

$\{X_t\}_{t \geqslant 0}$ is a martingale if $\mathbb{E}\{X_t|\mathscr{F}_s\} = X_s$ for all $t > s$

- W_t is a martingale
- W_t^2 is not a martingale (but it is a submartingale)
- $W_t^2 - t$ is a martingale
- For any $c \in \mathbb{R}$, $\mathrm{e}^{cW_t - c^2 t/2}$ is a martingale

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Stochastic differential equations driven by a Wiener process

- Aim: define solutions of

$$dx_t = f(x_t)\,dt + dW_t$$

- Transform into integral equation:

$$x_t = x_0 + \int_0^t f(x_s)\,ds + W_t$$

- Use Banach's fixed-point argument in some appropriate space
- Typical result: If f is locally Lipschitz and does not grow too fast (or is confining), then there exists a unique global solution

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Problem: Define the stochastic integral

- Let X_t be piecewise constant on intervals $[t_k, t_{k+1})$. Define Itô integral

$$\int_0^t X_s\,dW_s = \sum_k X_{t_k}\big[W_{t_{k+1}} - W_{t_k}\big]$$

Lemma: Itô isometry

If X_s depends only on information in $\mathscr{F}_s$, then $\mathbb{E}\int_0^t X_s\,dW_s = 0$ and

$$\mathbb{E}\Big[\Big(\int_0^t X_s\,dW_s\Big)^2\Big] = \int_0^t \mathbb{E}[X_s^2]\,ds$$

Allows to define Itô integral for non-piecewise constant integrands by approximating sequence

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An important example

$$\int_0^t W_s \, \mathrm{d}W_s = \frac{1}{2} W_t^2 - \frac{1}{2} t$$

Remark: Stratonovich integral defined by

$$\int_0^t X_s \circ \mathrm{d}W_s = \sum_k \frac{X_{t_k} + X_{t_{k+1}}}{2} \left[W_{t_{k+1}} - W_{t_k} \right]$$

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Itô's formula

Let x_t satisfy $\mathrm{d}x_t = f_t\,\mathrm{d}t + g_t\,\mathrm{d}W_t$

Lemma: Itô formula (chain rule for stochastic integrals)

Let $y_t = u(t, x_t)$. Then

$$\mathrm{d}y_t = \frac{\partial u}{\partial t}(t, x_t)\,\mathrm{d}t + \frac{\partial u}{\partial x}(t, x_t)\big[f_t\,\mathrm{d}t + g_t\,\mathrm{d}W_t\big] + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, x_t) g_t^2\,\mathrm{d}t$$

- Mnemotechnic: $\mathrm{d}W_t^2 = \mathrm{d}t$
- Example: $\mathrm{d}(W_t^2) = \mathrm{d}t + 2W_t\,\mathrm{d}W_t$
- Example: $M_t = \mathrm{e}^{cW_t - c^2 t/2} \Rightarrow \mathrm{d}M_t = cM_t\,\mathrm{d}W_t$
- Multidim case: $u(t, x_t^{(1)}, \dots, x_t^{(n)})$ with $\mathrm{d}x_t^{(i)} = f_t^{(i)}\,\mathrm{d}t + g_t^{(i)}\,\mathrm{d}W_t$

$$\mathrm{d}y_t = \frac{\partial u}{\partial t}\,\mathrm{d}t + \sum_i \frac{\partial u}{\partial x_i}\,\mathrm{d}x_t^{(i)} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}\,\mathrm{d}x_t^{(i)}\,\mathrm{d}x_t^{(j)}$$

where $\mathrm{d}x_t^{(i)}\,\mathrm{d}x_t^{(j)} = g_t^{(i)} g_t^{(j)}\,\mathrm{d}t$

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- Multidim case: $u(t, x_t^{(1)}, \dots, x_t^{(n)})$ with $\mathrm{d}x_t^{(i)} = f_t^{(i)} \, \mathrm{d}t + g_t^{(i)} \, \mathrm{d}W_t$

$$\mathrm{d}y_t = \frac{\partial u}{\partial t} \, \mathrm{d}t + \sum_i \frac{\partial u}{\partial x_i} \, \mathrm{d}x_t^{(i)} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \, \mathrm{d}x_t^{(i)} \, \mathrm{d}x_t^{(j)}$$

  where $\mathrm{d}x_t^{(i)} \, \mathrm{d}x_t^{(j)} = g_t^{(i)} g_t^{(j)} \, \mathrm{d}t$
- Several Wiener processes: $\mathrm{d}W_t^{(i)} \, \mathrm{d}W_t^{(j)} = \delta_{ij} \, \mathrm{d}t$

Examples of solvable SDEs

- Ornstein–Uhlenbeck process

$$\mathrm{d}x_t = -ax_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \quad \Rightarrow \quad x_t = \mathrm{e}^{-at}\,x_0 + \sigma\int_0^t \mathrm{e}^{-a(t-s)}\,\mathrm{d}W_s$$

- Multiplicative noise

$$\mathrm{d}x_t = ax_t\,\mathrm{d}t + \sigma x_t\,\mathrm{d}W_t \quad \Rightarrow \quad x_t = x_0\exp\Big\{\big(a - \tfrac{1}{2}\sigma^2\big)t + \sigma W_t\Big\}$$

- **Remark:** Numerical simulation:
  - Euler–Maruyama scheme:

$$x_{t+\Delta t} \simeq x_t + f(x_t)\Delta t + g(x_t)\sqrt{\Delta t}Z \qquad Z \text{ iid } \sim\mathcal{N}(0,1)$$

  - Milstein scheme:

$$x_{t+\Delta t} \simeq x_t + f(x_t)\Delta t + g(x_t)\sqrt{\Delta t}Z + \frac{1}{2}g(x_t)g'(x_t)\Delta t(Z^2-1)$$

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Semi-groups and generators

Consider $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + g(x_t)\,\mathrm{d}W_t$ in $\mathbb{R}^n$ with global in time solutions

- Transition density: $p_t(x,y) \equiv p(y,t|x,0)$:

$$\mathbb{P}^x\{x_t \in A\} := \mathbb{P}\{x_t \in A | x_0 = x\} = \int_A p_t(x,y)\,\mathrm{d}y$$

- Markov semi-group: Let φ be a test function (bounded)

$$(P_t\varphi)(x) = \mathbb{E}^x[\varphi(x_t)] = \int_{\mathbb{R}^n} p_t(x,y)\varphi(y)\,\mathrm{d}y$$

Markov property $\Rightarrow$ Semi-group property: $P_s \circ P_t = P_{t+s}$

- Infinitesimal generator:

$$(\mathscr{L}\varphi)(x) = \lim_{t\to 0+} \frac{(P_t\varphi)(x) - \varphi(x)}{t} = \left.\frac{\mathrm{d}(P_t\varphi)(x)}{\mathrm{d}t}\right|_{t=0}$$

Thus formally, $P_t = \mathrm{e}^{t\mathscr{L}}$

- Dual semigroup: $Q_t = \mathrm{e}^{t\mathscr{L}^\dagger}$ (where $\langle \mathscr{L}\varphi, \psi\rangle = \langle \varphi, \mathscr{L}^\dagger\psi\rangle$)

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The case of the Wiener process

Let $x_t = W_t$ in $\mathbb{R}^n$ ($f = 0, g = \mathbb{1}$)

- Transition density:

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t^n}} \, \mathrm{e}^{\|x-y\|^2/2t}$$

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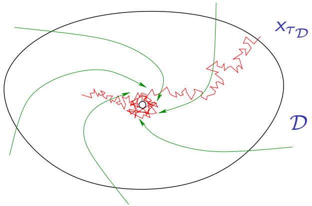
PART II

The stochastic exit problem

Definition of the stochastic exit problem

SDE $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$ in $\mathbb{R}^n$
Given $\mathcal{D} \subset \mathbb{R}^n$, define first-exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$$



First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$ defines harmonic measure $\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$

Questions:

- ▷ Properties of $\tau_{\mathcal{D}}$ – expectation, distribution?
- ▷ Properties of exit location?
- ▷ What if $\partial\mathcal{D}$ is invariant under the deterministic flow?
- ▷ Effective dynamics when there are several attractors?

Methods:

- ▷ PDE methods (boundary value problems)
- ▷ Theory of large deviations (Freidlin and Wentzell), $\sigma \ll 1$
- ▷ Potential-theoretic methods (best understood in gradient case)

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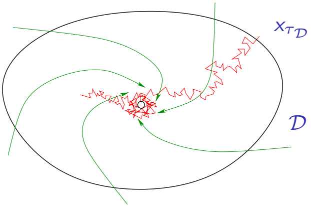

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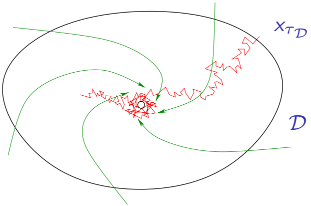

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Arrhenius' law and Eyring–Kramers law

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^n \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$,
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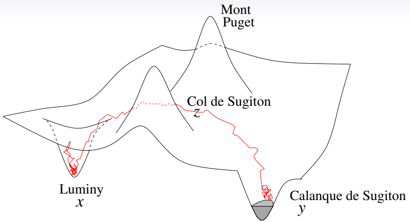

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Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_n$
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$$\mathbb{E}[\tau_y^x] = 2\pi\sqrt{\frac{\lambda_2 \dots \lambda_n}{|\lambda_1|\nu_1 \dots \nu_n}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\bigl[1 + \mathcal{O}_\varepsilon(1)\bigr]$$

Arrhenius' law: proved by Freidlin, Wentzell (1979) using large deviations
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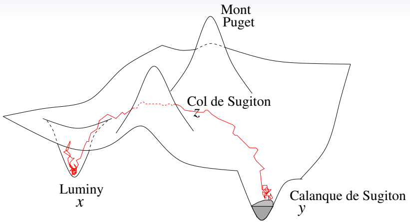

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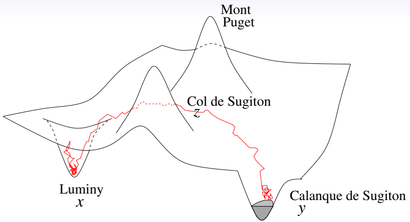

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Dynkin's formula

Theorem: Dynkin's formula (Itô's formula for stopping times)

Let τ be a random stopping time, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Then

$$\mathbb{E}^x\Big[\varphi(x_\tau)\Big] = \varphi(x) + \mathbb{E}^x\Big[\int_0^\tau (\mathscr{L}\varphi)(x_s)\,\mathrm{d}s\Big]$$

Consequence: For $\mathcal{D} \subset \mathbb{R}^n$, $\psi : \partial\mathcal{D} \to \mathbb{R}$, $\theta : \mathcal{D} \to \mathbb{R}$, $\tau = \tau_{\mathcal{D}}$

$$u(x) = \mathbb{E}^x\Big[\psi(x_\tau) - \int_0^\tau \theta(x_s)\,\mathrm{d}s\Big] \quad \Leftrightarrow \quad \begin{cases} (\mathscr{L}u)(x) = \theta(x) & x \in \mathcal{D} \\ u(x) = \psi(x) & x \in \partial\mathcal{D} \end{cases}$$

Applications:

- Mean exit time of Brownian motion from a ball
- Recurrence/transience of Brownian motion
- For a 1-dim diffusion, committor $h(x) = \mathbb{P}^x\{\tau_a < \tau_b\}$
- For $\mathrm{d}x_t = -V'(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$ in $\mathbb{R}$ and $x > a$

$$\mathbb{E}^x[\tau_a] = \frac{1}{\varepsilon}\int_a^x \int_{x_2}^\infty \mathrm{e}^{[V(x_2)-V(x_1)]/\varepsilon}\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \Rightarrow \quad \text{Eyring–Kramers law}$$

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Applications:

- Mean exit time of Brownian motion from a ball
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The Feynman–Kac formula

Theorem: Feynman–Kac formula

For $q : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$

$$v(t,x) = \mathbb{E}^x\Big[\mathrm{e}^{-\int_0^t q(x_s)\,\mathrm{d}s}\varphi(x_t)\Big] \quad \Leftrightarrow \quad \begin{cases} \frac{\partial v}{\partial t} = \mathscr{L}v - q(x)v & t > 0, x \in \mathbb{R}^n \\ v(0,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases}$$

Combining Dynkin and F–K: for $\mathcal{D} \subset \mathbb{R}$, $\theta : \mathcal{D} \to \mathbb{R}$, $\varphi : \partial\mathcal{D} \to \mathbb{R}$, $\tau = \tau_{\mathcal{D}}$

$$v(x) = \mathbb{E}^x\Big[\mathrm{e}^{-\lambda\tau}\varphi(x_\tau) - \int_0^\tau \mathrm{e}^{-\lambda s}\theta(x_s)\,\mathrm{d}s\Big]$$

satisfies

$$\begin{cases} (\mathscr{L}v)(x) = \lambda v(x) + \theta(x) & x \in \mathcal{D} \\ v(x) = \varphi(x) & x \in \partial\mathcal{D} \end{cases}$$

Example: $x_t = x + W_t$, $\mathcal{D} = (-a,a)$, $x \in \mathcal{D}$

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Large-deviation theory

- Consider the SDE

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{\varepsilon}g(x_t)\,\mathrm{d}W_t$$

with an invertible diffusion matrix $D(x) = g(x)g(x)^T$ (ellipticity)

- For a path $\gamma : [0, T] \to \mathbb{R}^n$, define the rate (or action) function

$$\mathscr{I}(\gamma) = \frac{1}{2}\int_0^T \langle \dot\gamma(t) - f(\gamma(t)), D(\gamma(t))^{-1}\dot\gamma(t) - f(\gamma(t))\rangle \,\mathrm{d}t$$

- **Remark 1:** If $D(x) = \mathbb{1}$ then $\mathscr{I}(\gamma) = \frac{1}{2}\int_0^T \|\dot\gamma(t) - f(\gamma(t))\|^2 \,\mathrm{d}t$
- **Remark 2:** $\mathscr{I}(\gamma) \geqslant 0$ and $\mathscr{I}(\gamma) = 0 \Leftrightarrow \dot\gamma(t) = f(\gamma(t))$
- Large-deviation principle (imprecise version):
The proba that x_t stays close to $\gamma(t)$ on $[0, T]$ has order $\mathrm{e}^{-\mathscr{I}(\gamma)/\varepsilon}$
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Let Γ be a set of paths $\gamma : [0, T] \to \mathbb{R}^n$. Then

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Large-deviation theory for scaled Wiener process

Scaled Wiener process: $x_t = \sqrt{\varepsilon} W_t$

- The rate function
$$\mathscr{I}(\gamma) = \frac{1}{2}\int_0^T \|\dot{\gamma}(t))\|^2 \, \mathrm{d}t$$
is the Lagrangian action of a free particle

- To estimate:
$$\mathbb{P}\Big\{\tau_{[-h,h]} < T\Big\} = \mathbb{P}\Big\{\exists t \in [0,T] \colon |x_t| > h\Big\} = \mathbb{P}\bigg\{\sup_{t\in[0,T]} |x_t| > h\bigg\}$$

- Define set of paths
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- **Conclusion:** $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big\{\tau_{[-h,h]} < T\Big\} = -\frac{h^2}{2T}$

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Large-deviation theory for the OU process

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$$\gamma_t(s) = \begin{cases} \frac{h - x_0\,\mathrm{e}^{-t}}{2\sinh(t)}\,\mathrm{e}^{s} + \frac{x_0\,\mathrm{e}^{t} - h}{2\sinh(t)}\,\mathrm{e}^{-s} & 0 \leqslant s \leqslant t \\ h\,\mathrm{e}^{-(s-t)} & t < s \leqslant T \end{cases}$$

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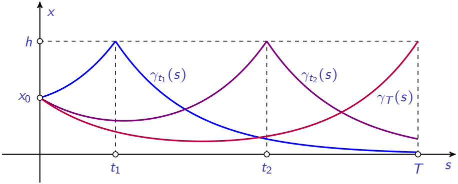

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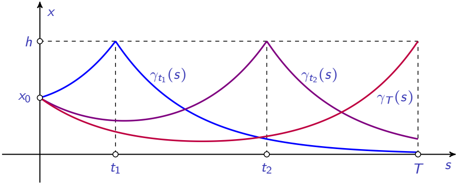
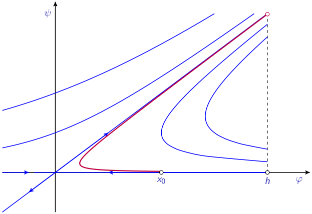


$$\inf_{\gamma\in\Gamma} \mathscr{I}(\gamma) = \mathscr{I}(\gamma_T) = \frac{(h\,\mathrm{e}^{T/2} - x_0\,\mathrm{e}^{-T/2})^2}{2\sinh(T)}$$

$$\lim_{T\to\infty}\inf_{\gamma\in\Gamma}\mathscr{I}(\gamma) = h^2$$



- Hamilton eq: $H(\varphi,\psi) = \frac{1}{2}\psi^2 - \varphi\psi$

Large-deviation theory for gradient systems

Gradient SDE: $\mathrm{d}x_t = -\nabla U(x_t)\,\mathrm{d}t + \sqrt{\varepsilon}W_t$

- Rate function:

$$\begin{aligned}\mathscr{I}(\gamma) &= \frac{1}{2}\int_0^T \|\dot\gamma(s) + \nabla U(\gamma(s))\|^2\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^T \|\dot\gamma(s) - \nabla U(\gamma(s))\|^2\,\mathrm{d}s + 2\int_0^T \langle\dot\gamma(s), \nabla U(\gamma(s))\rangle\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^T \|\dot\gamma(s) - \nabla U(\gamma(s))\|^2\,\mathrm{d}s + 2\big[U(\gamma(T)) - U(\gamma(0))\big]\end{aligned}$$

- If $\gamma(0)$ is a potential minimum reachable from $\gamma(T)$

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Large-deviation theory and exit from a domain

- $\mathcal{D}$ with unique equilibrium x^* and inward-pointing flow on $\partial\mathcal{D}$
- Quasipotential: For $y \in \partial\mathcal{D}$

$$V(x^*, y) = \inf_{t>0} \inf\left\{\mathscr{I}(\gamma) : \gamma(0) = x^*, \gamma(t) = y\right\}$$

Theorem [Freidlin & Wentzell]

Let $\overline{V} = \inf_{y \in \partial\mathcal{D}} V(x^*, y)$

For $x_0 \in \mathcal{D}$, $\lim_{T\to\infty} \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{x_0}\{\tau_{\mathcal{D}} \leqslant T\} = -\overline{V}$

Theorem [Freidlin & Wentzell]

- For any $\eta > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}^{x_0}\{\mathrm{e}^{(\overline{V}-\eta)/\varepsilon} < \tau_{\mathcal{D}} < \mathrm{e}^{(\overline{V}+\eta)/\varepsilon}\} = 1$
- Arrhenius' law: $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}^{x_0}[\tau_{\mathcal{D}}] = \overline{V}$
- For any closed $\mathcal{N} \subset \partial\mathcal{D}$ such that $\inf_{y\in\mathcal{N}} V(x^*, y) > \overline{V}$
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The potential-theoretic approach

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- Generator: $\mathscr{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla \cdot \mathrm{e}^{-V/\varepsilon}\,\nabla$
- Invariant probability: $\pi(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x \quad \Rightarrow \quad \mathscr{L}^\dagger \pi = 0$
- Reversible: $\langle f, \mathscr{L} g\rangle = \langle \mathscr{L} f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} f(x) g(x)\,\mathrm{d}x$
- Dirichlet form: $\mathscr{E}(f) = \langle f, -\mathscr{L} f\rangle = \varepsilon \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} |\nabla f(x)|^2\,\mathrm{d}x$
  $\mathscr{E}(f,g) = \langle f, -\mathscr{L} g\rangle$
- Committor: $A, B \subset \mathbb{R}^n, A \cap B = \varnothing$

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (\mathscr{L} h_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

- Equilibrium measure: $e_{AB}(\mathrm{d}x) = (-\mathscr{L} h_{AB})(\mathrm{d}x)$ measure on $x \in \partial A$

$$\Rightarrow \qquad h_{AB}(x) = \int_A G_B(x,y) e_{AB}(\mathrm{d}y) \qquad G_B = \mathscr{L}_{B^c}^{-1}$$

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Capacity: $\operatorname{cap}(A,B) = \displaystyle\int_{\partial A} e^{-V(x)/\varepsilon}\, e_{AB}(dx) = \mathscr{E}(h_{AB})$

$\Rightarrow \nu_{AB}(dx) = \frac{1}{\operatorname{cap}(A,B)}\, e^{-V(x)/\varepsilon}\, e_{AB}(dx)$ is a probability measure on ∂A

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$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^x[\tau_B]\nu_{AB}(dx) = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} e^{-V(x)/\varepsilon}\, h_{AB}(x)\, dx$$

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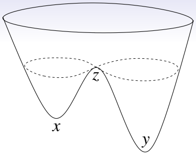
Proving the Eyring–Kramers law



$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

▷ A, B small balls around x, y

▷ Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1\dots\nu_d}}\,\mathrm{e}^{-V(x)/\varepsilon}$

▷ Variational arguments: $\displaystyle\mathrm{cap}(A,B) \simeq \varepsilon\sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}}\sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2\dots\lambda_d}}\,\mathrm{e}^{-V(z)/\varepsilon}$
Test function for Dirichlet principle: $1d$ committor

$$h(x_1) = \int_{x_1}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi \left[\int_{-\delta}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi\right]^{-1} \qquad \delta = \mathcal{O}([\varepsilon\log(\varepsilon^{-1})]^{1/2})$$

▷ Use Harnack inequalities to show that $\mathbb{E}^{\nu_{A,B}}[\tau_B] \simeq \mathbb{E}^x[\tau_B]$
Alternative: coupling argument by [Martinelli, Olivieri & Scoppola]

Eyring–Kramers law:

$$\mathbb{E}^x[\tau_B] = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\nu_1\dots\nu_d}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}(\varepsilon^{1/2}\log(\varepsilon^{-1})^{3/2})\big]$$

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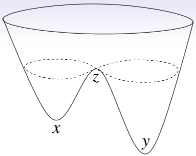

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- Variational arguments: $\operatorname{cap}(A,B) \simeq \varepsilon\sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}}\sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2\dots\lambda_d}}\,\mathrm{e}^{-V(z)/\varepsilon}$
  Test function for Dirichlet principle: $1d$ committor

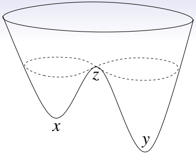
$$h(x_1) = \int_{x_1}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi \left[\int_{-\delta}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi\right]^{-1} \qquad \delta = \mathcal{O}([\varepsilon\log(\varepsilon^{-1})]^{1/2})$$

- Use Harnack inequalities to show that $\mathbb{E}^{\nu_{A,B}}[\tau_B] \simeq \mathbb{E}^x[\tau_B]$
  Alternative: coupling argument by [Martinelli, Olivieri & Scoppola]

Eyring–Kramers law:

$$\mathbb{E}^x[\tau_B] = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\nu_1\dots\nu_d}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\left[1 + \mathcal{O}(\varepsilon^{1/2}\log(\varepsilon^{-1})^{3/2})\right]$$

Proving the Eyring–Kramers law



$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

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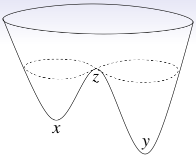

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Potential-theoretic approach: non-reversible case

[Landim, Mariani & Seo 2019]:

- EDS $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t)\,\mathrm{d}W_t$ with generator $\mathscr{L}$
- Invariant measure $\mathrm{d}\pi = \mathrm{e}^{-V(x;\varepsilon)/\varepsilon}\,\mathrm{d}x$, V sat. Hamilton–Jacobi eq.
- Decompose $\mathscr{L} = \mathscr{L}_\mathrm{s} + \mathscr{L}_\mathrm{a}$ where
  - $\mathscr{L}_\mathrm{s} = \mathrm{e}^{V/\varepsilon}\nabla\cdot D\,\mathrm{e}^{-V/\varepsilon}\nabla$, where $D = gg^T$, is self-adjoint wrt π
  - $\mathscr{L}_\mathrm{a} = c\cdot\nabla$ (c explicitly known) is skew-symmetric: $\mathscr{L}_\mathrm{a}^\dagger = -\mathscr{L}_\mathrm{a}$
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Theorem: [LMS 2019] For any $A, B \subset \mathbb{R}^n$, $A \cap B = \varnothing$

$$\int_{\partial A}\mathbb{E}^x\big[\tau_B\big]\,\mathrm{d}\nu_{AB} = \frac{1}{\mathrm{cap}(A,B)}\int_{B^c} h^*_{AB}(x)\,\mathrm{d}\pi$$

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- $\mathrm{cap}(A,B)$: capacity, satisfies Dirichlet and Thomson principles
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Main difficulty: Compute/estimate invariant measure $\mathrm{d}\pi$

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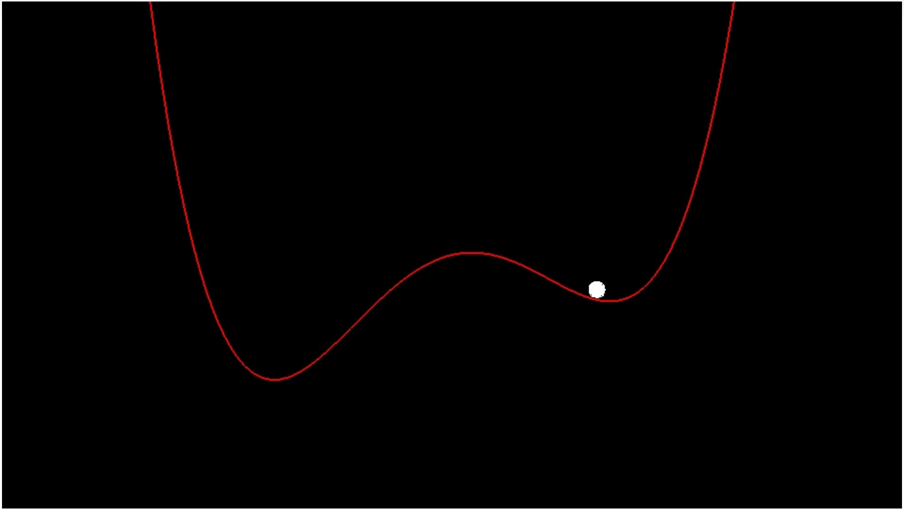
PART III

Slow–fast SDEs and applications

A motivating example: Stochastic resonance



youtu.be/HbJ_I3xbIMg

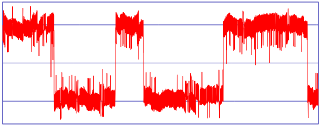
$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Stochastic resonance in an SDE

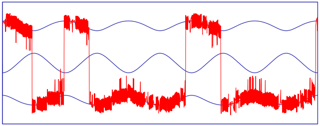
$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

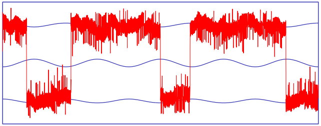
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



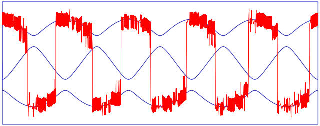
$A = 0,\ \sigma = 0.3$



$A = 0.24,\ \sigma = 0.2$



$A = 0.1,\ \sigma = 0.27$



$A = 0.35,\ \sigma = 0.2$

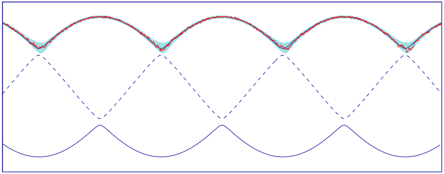
Descriptions of stochastic resonance

- Fokker–Planck equation: [Caroli, Caroli, Roulet & Saint-James '81]
- Two-state Markov chain: [Eckmann & Thomas '82], [Imkeller & Pavljukevich '02], [Herrmann & Imkeller '02]
- Signal-to-noise ratio: [Gammaitoni, Menichella-Saetta & ...'89], [Fox '89], [Jung& Hänggi '89], [McNamara & Wiesenfeld '89]
- Slow forcing: [Jung & Hänggi '91], [Talkner '99], [Talkner & Łuczka '04]
- Large deviations: [Freidlin '00, Freidlin '01]
- Residence-time distributions: [Zhou, Moss & Jung '90], [Choi, Fox & Jung '98], ...
- Overview articles: [Moss, Pierson & O'Gorman '94], [Wiesenfeld & Moss '95], [McNamara & Wiesenfeld '95], [Wiesenfeld & Jaramillo '98], [Gammaitoni, Hänggi, Jung & Marchesoni '98], [Hänggi '02], [Wellens, Shatokhin & Buchleitner '04], ...
- Monograph: [Herrmann, Imkeller, Pavlyukevich & Peithmann '14]

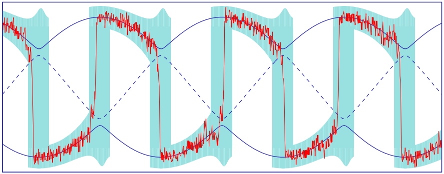
The synchronisation regime

$A_{\mathrm{c}} = \frac{2}{3\sqrt{3}}$, $A = A_{\mathrm{c}} - \delta$, $0 < \delta \ll 1$. Critical noise intensity: $\sigma_{\mathrm{c}} = \max\{\delta, \varepsilon\}^{3/4}$

$\sigma \ll \sigma_{\mathrm{c}}$:
transitions unlikely



$\sigma \gg \sigma_{\mathrm{c}}$:
synchronisation



Theorem [B & Gentz, Annals App. Proba 2002]

- Away from (avoided) bifurcations, sample paths concentrated in σ-neighbourhood of deterministic stable periodic solutions
- $\sigma \ll \sigma_{\mathrm{c}}$: transition probability per period $\leqslant \mathrm{e}^{-\sigma_{\mathrm{c}}^2/\sigma^2}$
- $\sigma \gg \sigma_{\mathrm{c}}$: transition probability per period $\geqslant 1 - \mathrm{e}^{-c\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

One-dimensional time-dependent equations

On the slow time scale εt:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$$

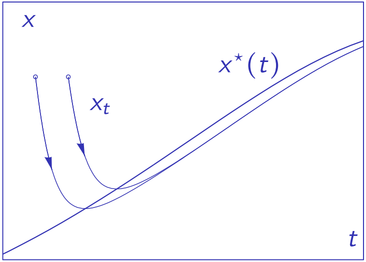
- Equilibrium branch: $\{x = x^\star(t)\}$ where $f(x^\star(t),t) = 0$ for all t
- Stable if $a^\star(t) = \partial_x f(x^\star(t),t) \leqslant -a_0 < 0$ for all t

Then [Tikhonov '52, Fenichel '79]:

- There exists particular solution

$$\bar{x}(t) = x^\star(t) + \mathcal{O}(\varepsilon)$$

- $\bar{x}$ attracts nearby orbits exp. fast
- $\bar{x}$ admits asymptotic series in ε



Slow–fast systems with noise

Stochastic perturbation:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(\xi_t, t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\,\mathrm{d}W_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, b(\xi_s, s)\,\mathrm{d}s}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,\mathrm{d}u$

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Slow–fast systems with noise

Properties of $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \mathrm{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon}\int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon}\,\mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\mathrm{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\,\mathrm{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon\dot v = 2\bar a(t) v + 1$

Lemma [B & Gentz, Proba. Theory Relat. Fields 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\quad \bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t}\frac{|\xi_s^0|}{\sqrt{\bar v(s)}} > h\biggr\} = C_0(t,\varepsilon)\,\mathrm{e}^{-h^2/2\sigma^2}$$

where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}}\frac{1}{\varepsilon}\frac{h}{\sigma}\Bigl|\displaystyle\int_0^t \bar a(s)\,\mathrm{d}s\Bigr|\Bigl[1 + \mathcal{O}(\varepsilon + \tfrac{t}{\varepsilon}\,\mathrm{e}^{-h^2/\sigma^2})\Bigr]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

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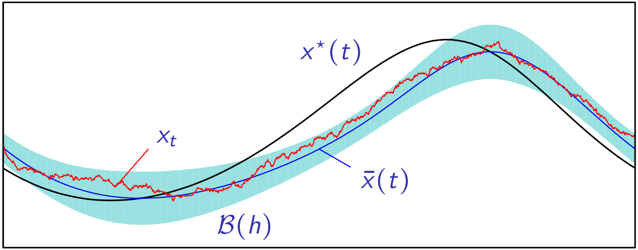
Proof based on Doob's submartingale inequality and partition of $[0,t]$

Slow–fast systems with noise

Nonlinear equation: $\mathrm{d}\xi_t = \dfrac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + b(\xi_t, t)\Big]\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \Big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\} = \Big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\}$



Theorem [B & Gentz, Proba. Theory Relat. Fields 2002]

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\Big[1 + \mathcal{O}(h)\Big]$ (requires $h \leqslant h_0$)

Avoided transcritical bifurcation

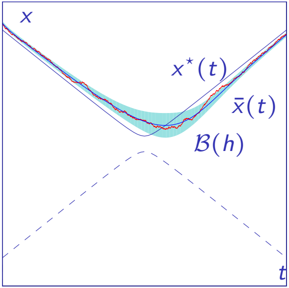
$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2 + \dots\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}\big(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\}\big)$

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



Theorem [B & Gentz, AAP 2002]

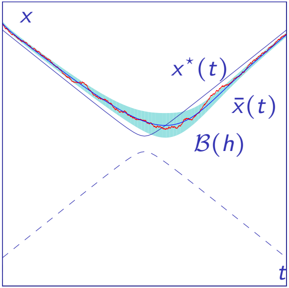
$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ $\quad\Rightarrow$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies $\forall\, t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(\mathrm{e}^{-\kappa\sigma_c^2/\sigma^2})$
- $\sigma > \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Avoided transcritical bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2 + \dots\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$



Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}\big(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\}\big)$

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$

Theorem [B & Gentz, AAP 2002]

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ $\quad\Rightarrow$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

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Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \cdots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

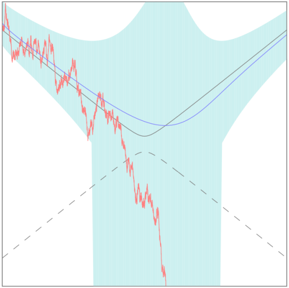
Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant \mathrm{e}^{-n \log q}$

Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, \mathrm{d}s = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

Thm [B & Gentz, AAP 2002]

Transition probability $\geqslant 1 - \mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)}$



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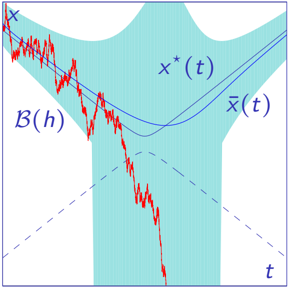

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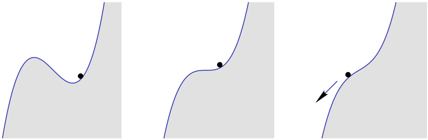
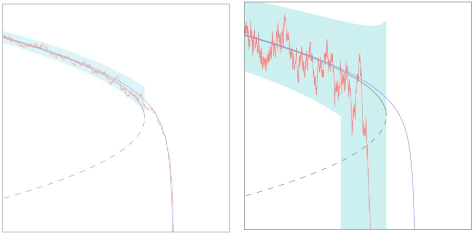
Transition probability $\geqslant 1 - e^{-\kappa\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

Rising variance near tipping points

Saddle–node (fold) bifurcation: $\mathrm{d}x_t = \dfrac{1}{\varepsilon}\Big[-x_t^2 - t\Big]\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$





▷ Fluctuations grow like $\frac{\sigma}{|\bar{a}(t)|^{1/2}} \asymp \frac{\sigma}{\max\{(-t)^{1/4}, \varepsilon^{1/6}\}}$

▷ Early transitions occur if $\sigma \gg \varepsilon^{1/2}$ at time $\asymp -\sigma^{4/3}$

Theorem [B & Gentz, Nonlinearity 2002]

▷ $\sigma < \varepsilon^{1/2}$: transition probability before fold $\leqslant \mathrm{e}^{-c\varepsilon/\sigma^2}$

▷ $\sigma > \varepsilon^{1/2}$: probability of early transition $\geqslant 1 - \mathrm{e}^{-c\sigma^2/(\varepsilon|\log\sigma|)}$

[Carpenter, Brock, *Rising variance: a leading indicator of ecological transition*, Ecology Letters **9**, 311–318 (2006)]

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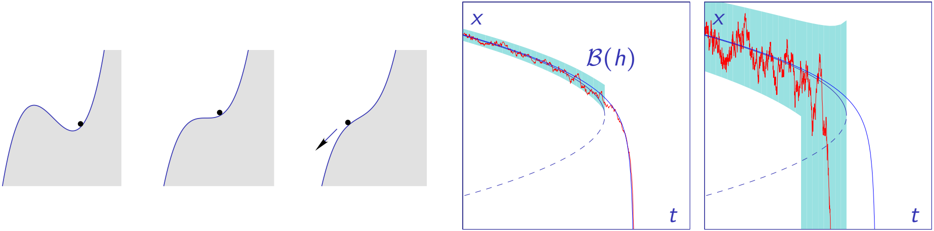

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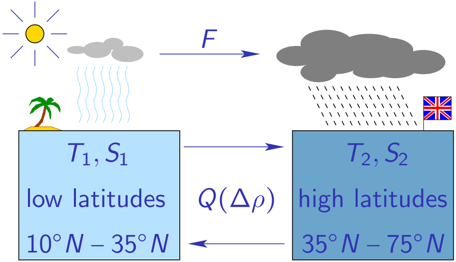
Stommel's box model

- T_i: temperatures
- S_i: salinities
- F: freshwater flux
- $Q(\Delta\rho)$: mass exchange
- $\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$



$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S$$

Model for Q [Cessi]: $Q(\Delta\rho) = \dfrac{1}{\tau_d} + \dfrac{q}{V}\Delta\rho^2$.

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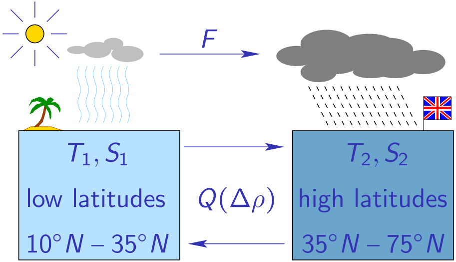

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Stommel's box model

Scaling: $x = \frac{\Delta T}{\theta}$, $y = \frac{\Delta S \alpha_S}{\alpha_T \theta}$, $t \mapsto \tau_d t$

Separation of time scales: $\tau_r \ll \tau_d$, $\varepsilon = \tau_r / \tau_d \ll 1$

$$\varepsilon \dot{x} = -(x-1) - \varepsilon x\big[1 + \eta^2 (x-y)^2\big]$$
$$\dot{y} = \mu - y\big[1 + \eta^2 (x-y)^2\big]$$

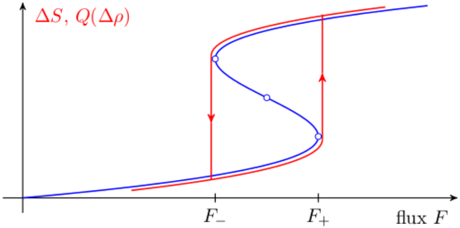
where $\eta^2 = \tau_d (\alpha_T \theta)^2 \frac{q}{V}$, $\mu = \frac{\alpha_S S_0 \tau_d}{\alpha_T \theta H} F$

Slow manifold [Fenichel '79]: $x = 1 + \mathcal{O}(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0$.

Reduced equation on slow manifold:

$$\dot{y} = \mu - y\big[1 + \eta^2 (1-y)^2 + \mathcal{O}(\varepsilon)\big]$$

One or two stable equilibria, depending on μ (and η).



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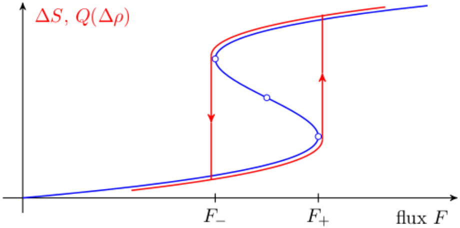
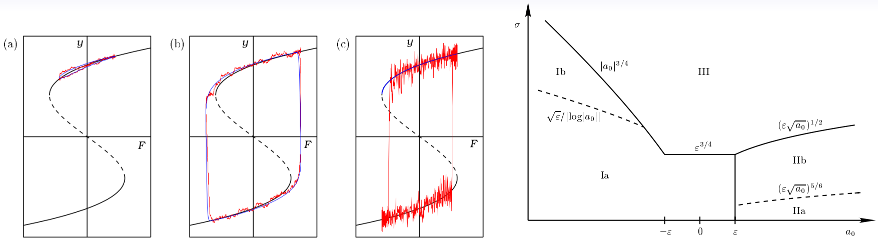
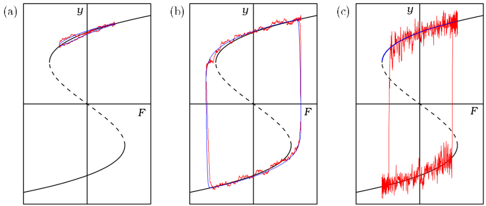


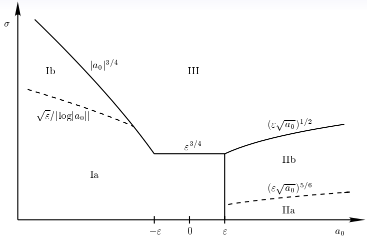
Hysteresis



[B & Gentz, Nonlinearity 2002]

Hysteresis





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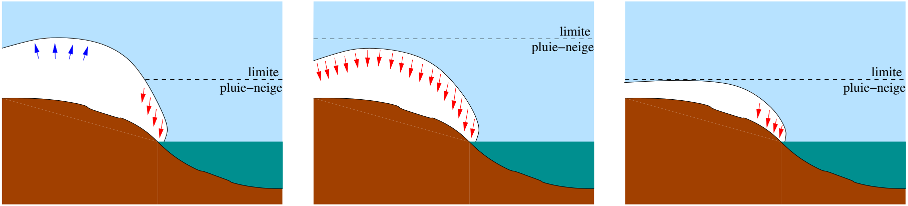


Generalisation to the multidimensional case

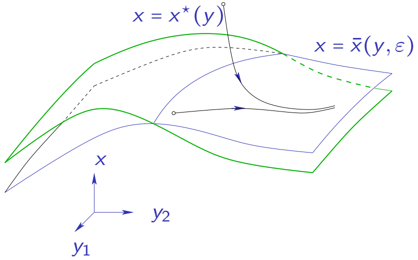
$$\varepsilon\dot{x} = f(x,y) \qquad x \in \mathbb{R}^n \text{ , fast variables}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m \text{ , slow variables}$$

- Critical manifold: $f(x^\star(y),y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $A(y) = \partial_x f(x^\star(y),y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]

$\exists$ slow manifold $x = \bar{x}(y,\varepsilon)$ s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Generalisation to the multidimensional case

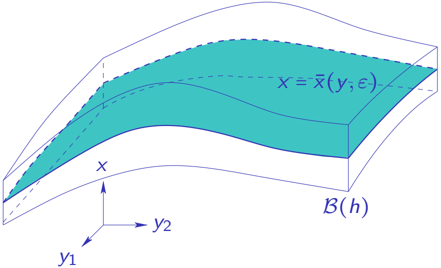
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

$$\mathrm{d}y_t = g(x_t, y_t)\,\mathrm{d}t + \sigma' G(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(slow variables } \in \mathbb{R}^m\text{)}$$

$\mathcal{B}(h)$: confidence set

defined by covariance of linearised equation for $x - \bar{x}(y, \varepsilon)$

$\bar{A}(y) := \partial_x f(\bar{x}(y, \varepsilon), y)$



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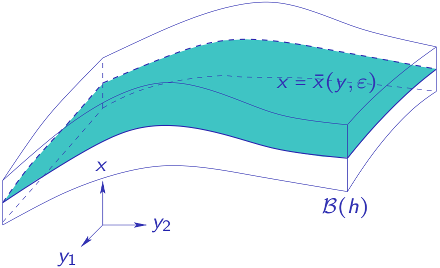

$\bar{X}$: covariance matrix of linearisation, solution of deterministic slow–fast ODE

$$\varepsilon \dot{\bar{X}} = \bar{A}(y)\bar{X} + \bar{X}\bar{A}(y)^{\mathrm{T}} + F(\bar{x}(y, \varepsilon), y) F(\bar{x}(y, \varepsilon), y)^{\mathrm{T}}$$

$$\dot{y} = g(\bar{x}(y, \varepsilon), y)$$

$$\mathcal{B}(h) := \left\{(x, y) : \left\langle \left[x - \bar{x}(y, \varepsilon)\right], \bar{X}(y)^{-1}\left[x - \bar{x}(y, \varepsilon)\right]\right\rangle < h^2\right\}$$

Generalisation to the multidimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

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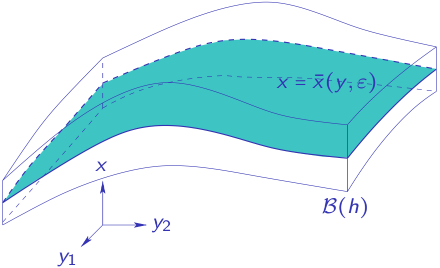


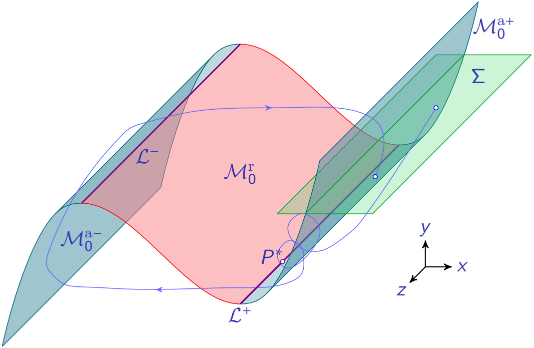
Theorem [B & Gentz, J. Diff. Equ. 2004] Norm. hyperbolic stable case:

$$C_-(t, \varepsilon)\, \mathrm{e}^{-\kappa h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C_+(t, \varepsilon)\, \mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$

Folded-node singularity in dim 3: The Koper model

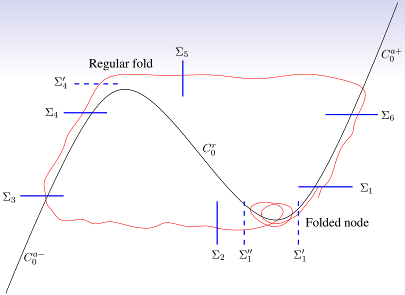
$$
\begin{aligned}
\varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t, y_t, z_t)\,\mathrm{d}W_t
\end{aligned}
$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, ...]
Poincaré map $\Pi : \Sigma \to \Sigma$ is almost $1d$ due to contraction in x-direction

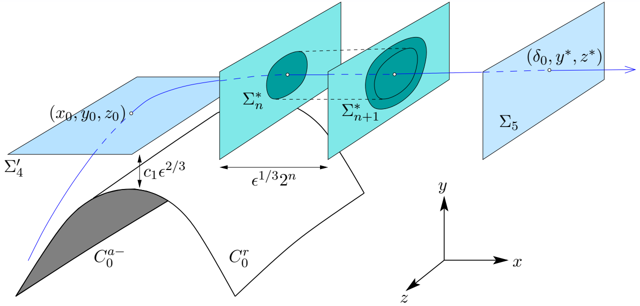
Size of fluctuations

$\mu \ll 1$: eigenvalue ratio
at folded node



Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$\sigma'\varepsilon^{1/4}$

Fold for one fast and two slow variables

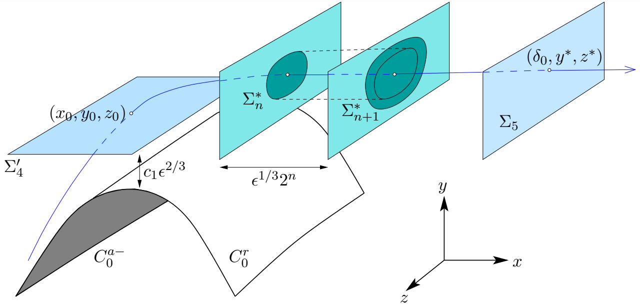


Proposition [B, Gentz & Kuehn, JDDE 2015]:

For $h = \mathcal{O}(\varepsilon^{2/3})$,

$$\mathbb{P}\Big\{\big\|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\big\| > h\Big\}$$
$$\leqslant C|\log\varepsilon|\biggl(\exp\Big\{-\frac{\kappa h^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}}\Big\} + \exp\Big\{-\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon}\Big\}\biggr)$$

Fold for one fast and two slow variables



Proposition [B, Gentz & Kuehn, JDDE 2015]:

For $h = \mathcal{O}(\varepsilon^{2/3})$,

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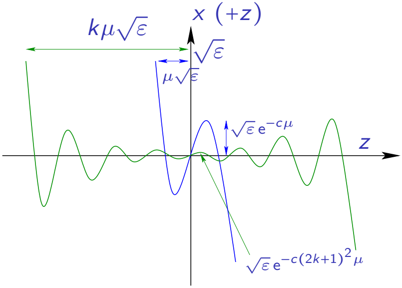
$$\leqslant C|\log\varepsilon|\left(\exp\left\{-\frac{\kappa h^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}}\right\} + \exp\left\{-\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon}\right\}\right)$$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



Theorem 2: size of fluctuations

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$

Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu} \Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma+\sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

- Saturation effect occurs at $k_{\mathrm{c}} \simeq \sqrt{|\log(\sigma+\sigma')|/\mu}$
- For $k > k_{\mathrm{c}}$, behaviour indep. of k and $\Delta z \leqslant \mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma+\sigma')|})$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

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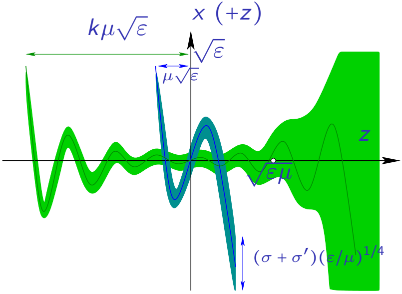

Theorem 3: early escape

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$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

- Saturation effect occurs at $k_c \simeq \sqrt{|\log(\sigma + \sigma')|/\mu}$
- For $k > k_c$, behaviour indep. of k and $\Delta z \leqslant \mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma + \sigma')|}\,)$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

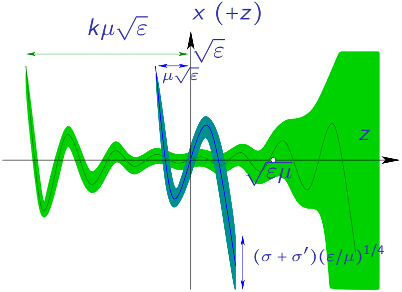
Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard

Theorem 2: size of fluctuations

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



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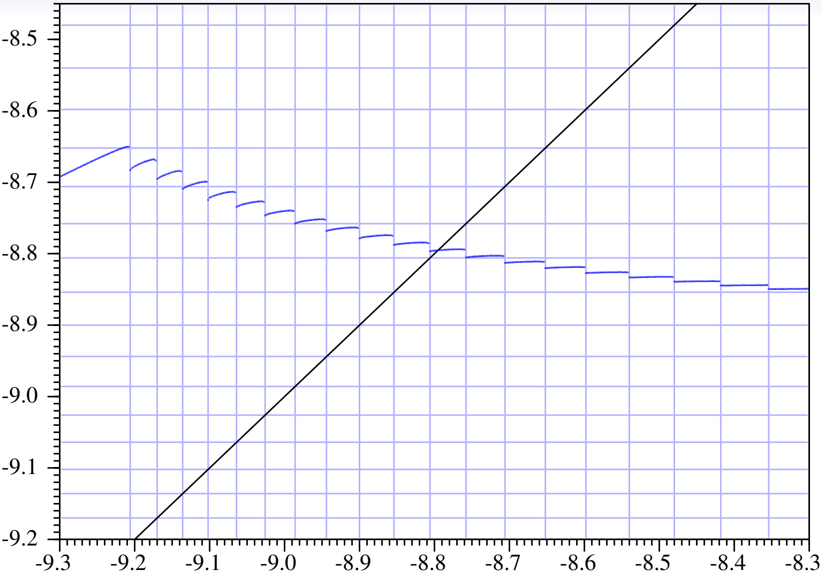
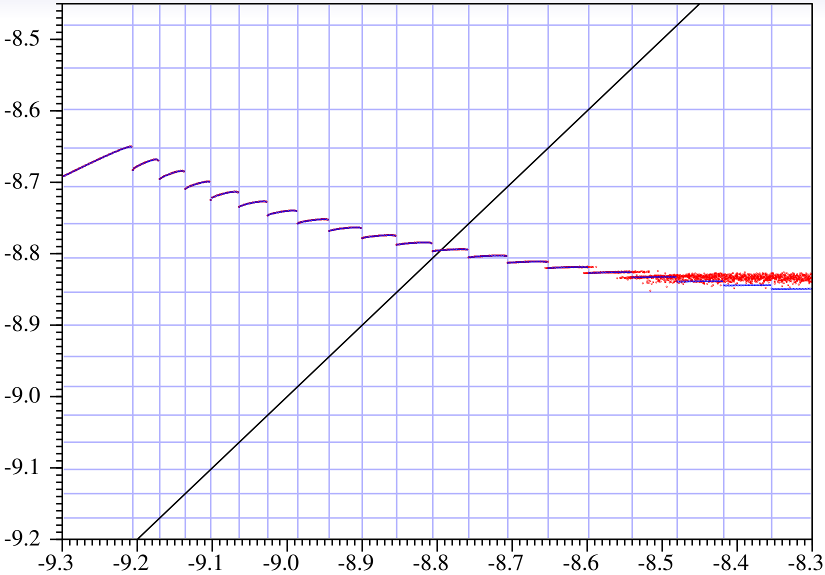
Poincaré map $z_n \mapsto z_{n+1}$



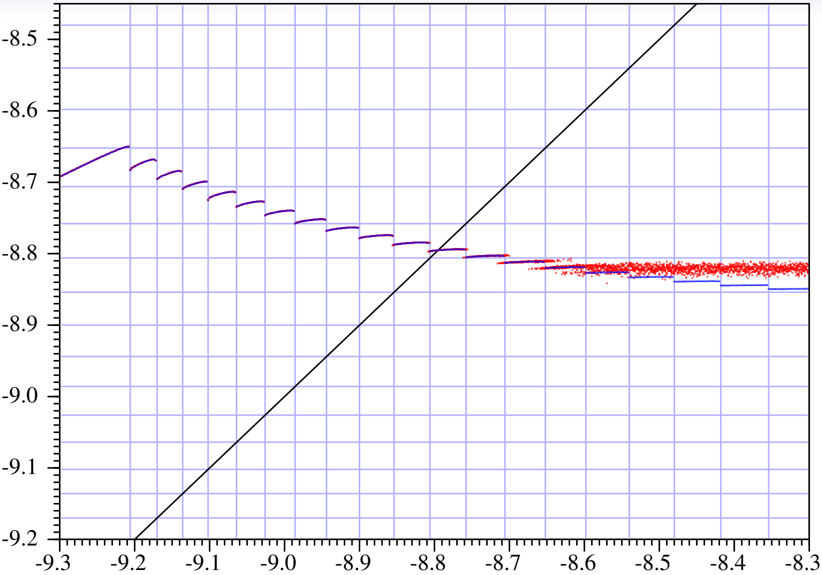
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 0$ – c.f. [Guckenheimer, Chaos, 2008]

Poincaré map $z_n \mapsto z_{n+1}$



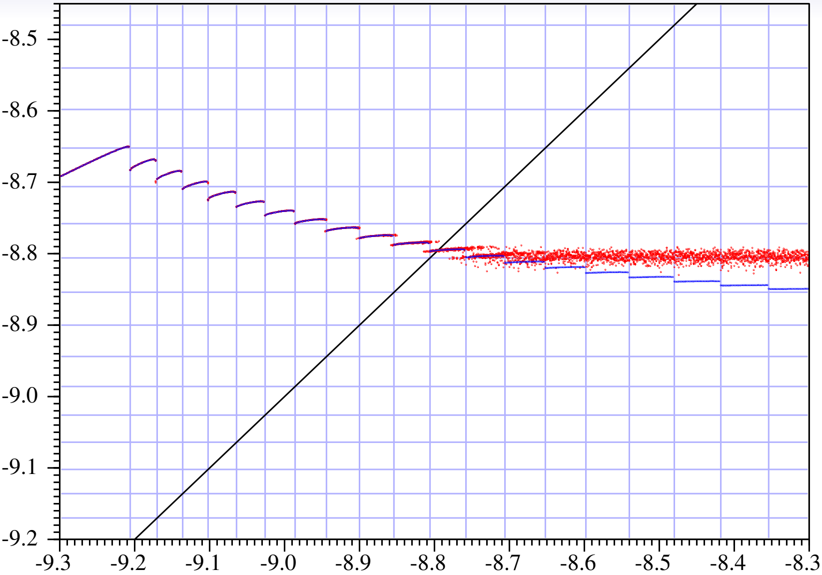
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,\ \sigma = \sigma' = 2 \cdot 10^{-7}$

Poincaré map $z_n \mapsto z_{n+1}$



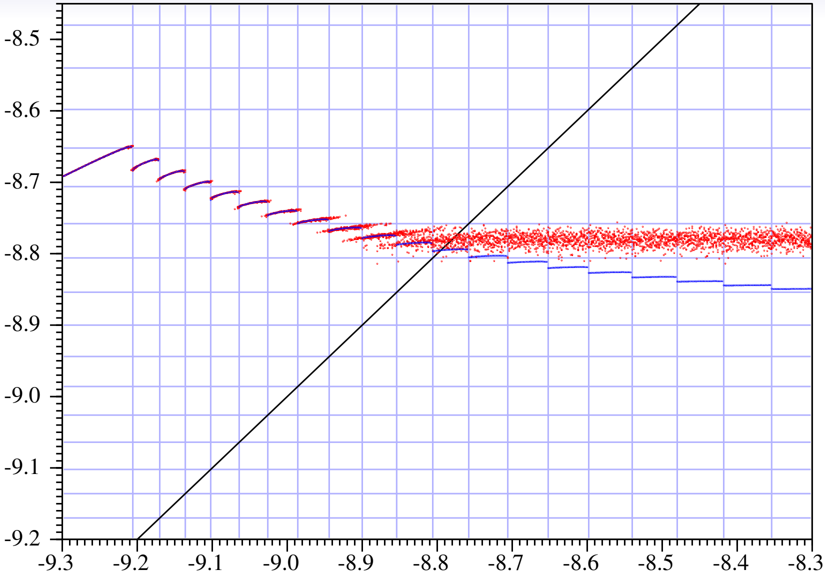
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 2 \cdot 10^{-6}$

Poincaré map $z_n \mapsto z_{n+1}$



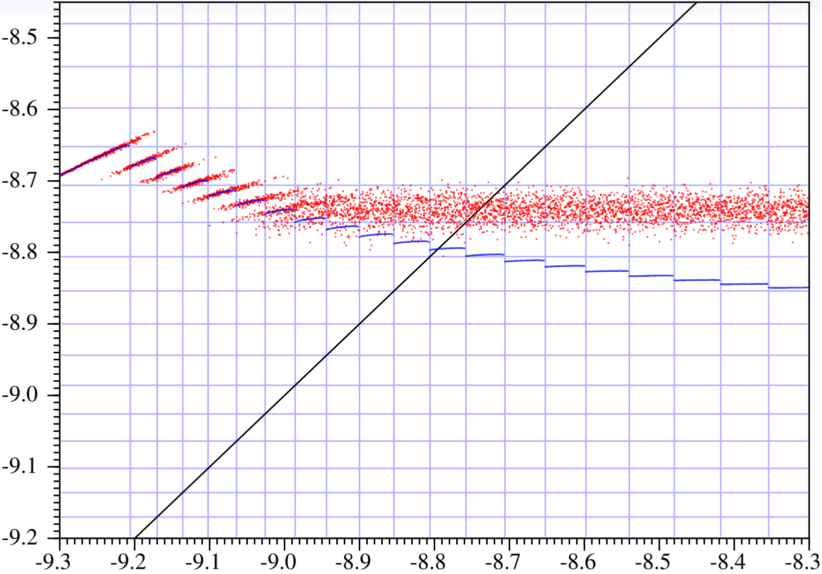
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 2 \cdot 10^{-5}$

Poincaré map $z_n \mapsto z_{n+1}$



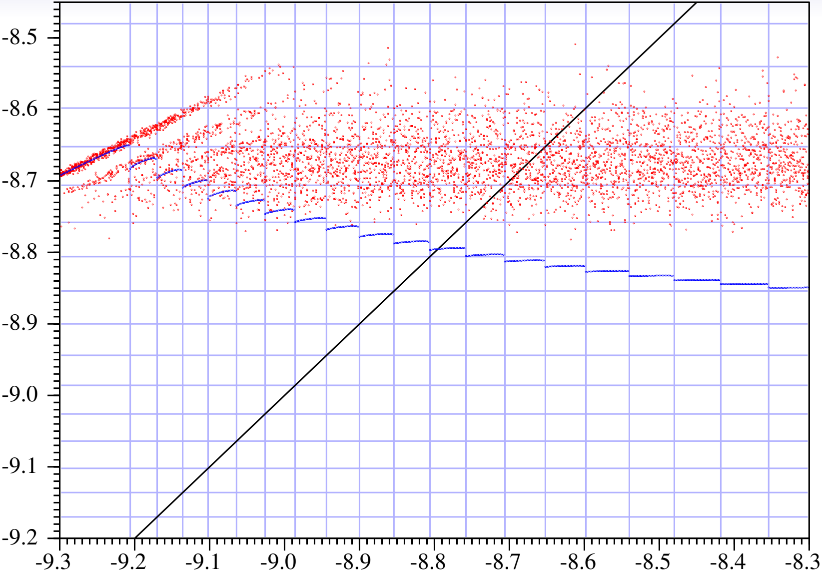
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 2 \cdot 10^{-4}$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 2 \cdot 10^{-3}$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \ \sigma = \sigma' = 10^{-2}$

Some references

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- N. B. and Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model*, Nonlinearity **25**, 2303–2335 (2012)
- N. B., Barbara Gentz and Christian Kuehn, *Hunting French Ducks in a Noisy Environment*, J. Differential Equations **252**, 4786–4841 (2012)
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- N. B., *Kramers' law: Validity, derivations and generalisations*, Markov Processes Relat. Fields **19**, 459–490 (2013)
- Christian Kuehn, N. B., Christian Bick, Maximilian Engel, Tobias Hurth, Annalisa Iuorio, Cinzia Soresina, *A General View on Double Limits in Differential Equations*, Physica D **431**, 133105 (2022)

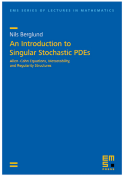
Monographs and lecture notes

Parts 1 and 2:

- N. B., *Long-time dynamics of stochastic differential equations*, Lecture notes, `https://arxiv.org/abs/2106.12998` (2021)

Part 2 (potential theory):

- N. B., *An introduction to singular stochastic PDEs: Allen-Cahn equations, metastability and regularity structures*, Lecture notes, `https://arxiv.org/abs/1901.07420` (2019)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Part 3:

- N. B. and Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)