Nancy, Séminaire du LPCT

An algebraic approach to perturbative renormalisation of the Φ^4_d model

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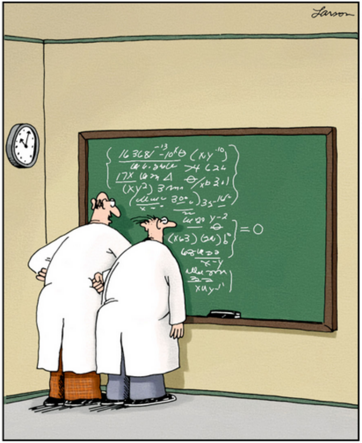
Joint works with Yvain Bruned (Nancy), Tom Klose (Warwick) and Nikolas Tapia (Berlin)

Project PERISTOCH

Renormalisation

- Recipe for turning a classical system into a quantum one: $p \mapsto -\mathrm{i}\,\hbar\nabla$ (De Broglie, Schrödinger, Heisenberg, ...)
- Quantum electrodynamics (QED) (Dirac, ..., Bethe, Tomonaga, Schwinger, Feynman, Dyson)
- Divergences: infrared/ultraviolet, due to self-interaction (can also occur classically) $\longrightarrow$ need for renormalisation (Bogolyubov, Wilson, ...)
- Bare vs observed parameters, analogy with Archimedes force (Connes, Kreimer)



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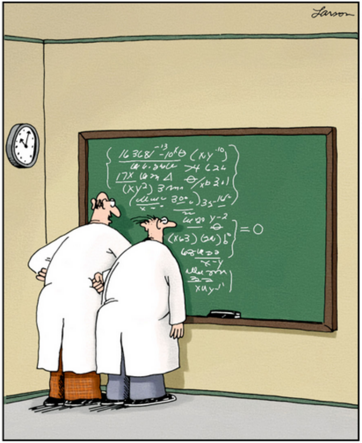

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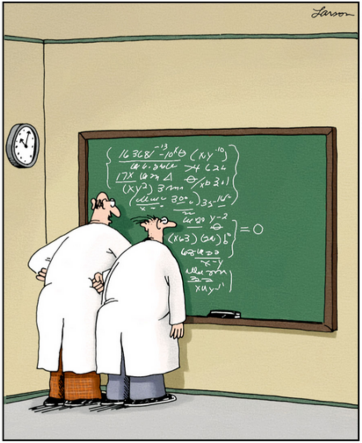

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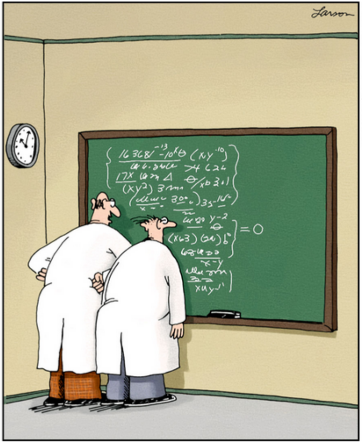

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The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2}N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\nu_{N,\varepsilon}(\mathrm{d}y) = \dfrac{1}{\mathcal{Z}_{N,\varepsilon}} \mathrm{e}^{-V_{N,\varepsilon}(y)}\,\mathrm{d}y$

- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Bigl(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4\Bigr)\,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

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The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Big(\frac{1}{2} \|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 \Big) \,\mathrm{d}x = \frac{1}{2}\langle \phi, (-\Delta + 1)\phi \rangle$

  ν_0 is Gaussian free field with covariance $(-\Delta + 1)^{-1} = G(x-y)$
  (well-defined since $(-\Delta + 1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1} < \infty$)

- $\varepsilon > 0$:

$$\frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\nu_0} = \frac{\mathcal{Z}_0}{\mathcal{Z}_\varepsilon} \mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{\mathcal{Z}_0}{\mathcal{Z}_\varepsilon} \mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x}$$

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  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Big[\Big(\int_\Lambda \phi_{\mathrm{GFF}}(x)^4 \,\mathrm{d}x\Big)^n\Big] \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1}\Big)^{2n} < C^n$$

  so that $\frac{\mathcal{Z}_\varepsilon}{\mathcal{Z}_0} = 1 + \mathcal{O}(\varepsilon)$

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The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:
$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2, |k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x)$$
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  If (X, Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X; C)H_m(Y; C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

  Consequence: $\sup_N \mathbb{E}\Big[\Big(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n:\mathrm{d}x\Big)^2\Big] < \infty \qquad \forall n$
- Gibbs measure defined as in 1d case, with
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The case $d = 3$

Theorem: Potential needs exactly 4 counterterms:

$$V(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla \phi(x)\|^2 + \tfrac{1}{2}\big[1 - \varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \tfrac{\varepsilon}{4} {:}\phi(x)^4{:}_{C_N^{(1)}} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)} \Big)\, \mathrm{d}x$$

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$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 3! \int_\Lambda G_N(x)^3\, \mathrm{d}x = \mathcal{O}(\log N)$$

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Some literature

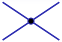
- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Singular stochastic PDEs

$$\partial_t\phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), ...:
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Graphical notations

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{\varepsilon}{4}$, $\beta = \frac{1}{2}\varepsilon^2 C_N^{(2)}$, $\gamma = \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$

  Then $\dfrac{\mathcal{Z}_{N,\varepsilon}}{\mathcal{Z}_{N,0}} = \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \operatorname{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

$C_N^{(1)} = \Pi_N$

$C_N^{(2)} = 3!\,\Pi_N$

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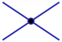
- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
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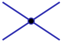
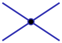
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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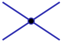
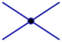
$\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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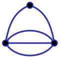
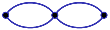
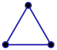
$\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

$- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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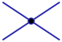
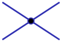
  $\mu_n=(-1)^n\mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big]=(-1)^n\sum_{m=0}^{n}\binom{n}{m}\alpha^m\beta^{n-m}A_{nm}$

  where $A_{nm}=\mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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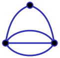
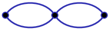
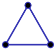
  $\mu_3=-\alpha^3\binom{4}{2}^3 2^3\Pi_N$  $-3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $-3\alpha\beta^2 4!\,\Pi_N$  $-8\beta^3\Pi_N$ 

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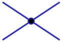
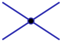
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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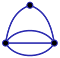
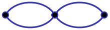
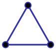
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($ $) = 0$ $\Pi_N($ $) = \mathcal{O}(\log N)$

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- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$
because it contains a subdivergence

Theorem: [Dyson]

If $\deg \bar{\Gamma} > 0$ for all subgraphs $\bar{\Gamma} \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

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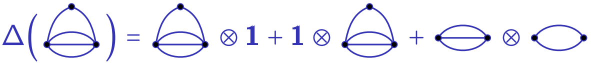
Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

  $$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \bar{\Gamma} \otimes (\Gamma / \bar{\Gamma}) \qquad (\mathbf{1}\text{: empty graph})$$

  Example: 

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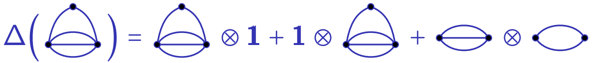
  Example: 

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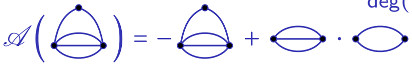
  Example: 

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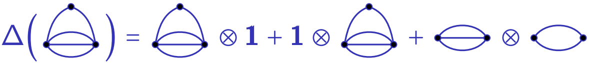
  Example: 

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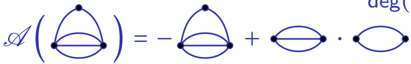
  Example: 

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BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma\rangle = \Pi_N \mathscr{A}(\Gamma)\mathbf{1}_{\deg\Gamma\leqslant 0}$
- Renormalised valuation:

$$
\begin{aligned}
\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) \\
&= (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma \\
&= (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma
\end{aligned}
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$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma)1_{\deg\Gamma\leqslant 0}$$

$$
\begin{tikzcd}
\mathscr{G} \arrow[r, "\Pi_N"] \arrow[dr, "\Pi_N^{\mathrm{BPHZ}}"] \arrow[d, "(\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] \arrow[dd, bend right=60, "M^{g^{\mathrm{BPHZ}}}=(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] & \mathbb{R} \\
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Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg\Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!}(-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg\Gamma_{pp}^{(k)} = p-3$$

Consequence: all terms in cumulant expansion bounded uniformly in N

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\end{aligned}$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma)1_{\deg \Gamma \leqslant 0}$$

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\Pi_N"] \arrow[d, "(\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] \arrow[dr, "\Pi_N^{\mathrm{BPHZ}}"] \arrow[dd, bend right=60, "M^{g^{\mathrm{BPHZ}}}=(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] & \mathbb{R} \\
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Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg \Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!}(-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg \Gamma_{pp}^{(k)} = p-3$$

Consequence: all terms in cumulant expansion bounded uniformly in N

BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma\rangle = \Pi_N \mathscr{A}(\Gamma)\mathbf{1}_{\deg\Gamma\leqslant 0}$
- Renormalised valuation:

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Commutative diagram

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\begin{tikzcd}
e^{-\alpha X} \arrow[r, mapsto, "\mathscr{P}"] \arrow[d, mapsto, "W"'] & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) \arrow[d, mapsto, "M^{g^{\mathrm{BPHZ}}}"'] \arrow[dr, mapsto, "\Pi_N^{\mathrm{BPHZ}}"] & \\
e^{-\alpha X-\beta Y} \arrow[r, mapsto, "\mathscr{P}"] & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) \arrow[r, mapsto, "\Pi_N"] & \log\mathbb{E}[e^{-\alpha X-\beta Y}]
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$$

- $\mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$
- $e^{-\alpha X}, e^{-\alpha X-\beta Y} \in H = \operatorname{span}\{X^n Y^m\}$
- Construction of W inspired by [Ebrahimi-Fard, Patras, Tapia & Zambotti]

Lemma: [B, Klose & Tapia]

- $W(e^{-\alpha X}) = e^{-\alpha X-\beta Y}$
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Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

Commutative diagram

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\begin{aligned}
W(X) &= X\\
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Zimmermann's forest formula

- Zimmermann forest formula: $\mathscr{A}(\Gamma) = -\sum_{\mathscr{F}} (-1)^{|\mathscr{F}|} \mathscr{C}_{\mathscr{F}} \Gamma$

  where sum ranges over all forests $\mathscr{F}$ (set of subgraphs, pairwise vertex-disjoint or included) and $\mathscr{C}_{\mathscr{F}}$ extracts all subgraphs in $\mathscr{F}$

- Our case: $\mathscr{A}(\Gamma) = -\sum_{S \subset \{1,\dots,g\}} (-1)^{|S|}$ $^{|S|} \mathscr{C}_S \Gamma$

  where $\mathscr{C}_S$ extracts all "bubbles" labelled by element of S, g is number of bubbles

  Since $\Pi_N($ $) = \frac{\beta}{3\varepsilon^2} = \frac{\beta}{48\alpha^2}$ we have

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) = -\sum_{S \subset \{1,\dots,g\}} \left(-\frac{\beta}{48\alpha^2}\right)^{|S|} \Pi_N(\mathscr{C}_S \Gamma_{pp}^{(k)})$$

Lemma: [B & Klose] The diagram commutes, that is,

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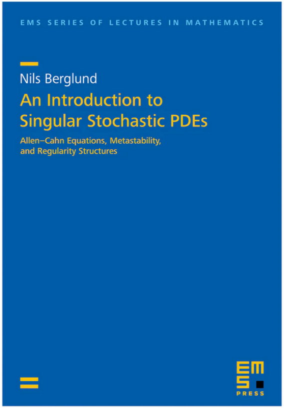
References

- N. B. & Tom Klose, *Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras*, arXiv/2207.08555 (2022)
- N. B. & Yvain Bruned, *BPHZ renormalisation and vanishing subcriticality asymptotics of the fractional Φ^3_d model*, arXiv/1907.13028 (2019), to appear in Stochastics & PDEs: Analysis and Computation
- N. B. & Christian Kuehn, *Model Spaces of Regularity Structures for Space-Fractional SPDEs*, J. Statist. Phys., **168**(1):331–368, 2017
- N. B. & Rita Nader, *Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus*, Electronic J. Probability **29**:1–35 (2024)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/Nancy_2024.pdf

Borel resummation: The Φ^4_0 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

$$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi$$

$$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0}\Big(-\frac{\varepsilon}{4}\Big)^n \frac{(4n-1)!!}{n!} = \sum_{n\geqslant 0} a_n\varepsilon^n, \qquad a_n \sim n!$$

- Borel transform:

$$Z(\varepsilon) \asymp \sum_{n\geqslant 0} a_n\varepsilon^n \frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0}\frac{a_n\varepsilon^n}{n!}\int_0^\infty t^n\,\mathrm{e}^{-t}\,\mathrm{d}t$$

$$Z_{\mathrm{Borel}}(\varepsilon) = \int_0^\infty \mathrm{e}^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\,\mathcal{B}Z(\varepsilon t)\,\mathrm{d}t$$

where $\mathcal{B}Z(t) = \sum_{n\geqslant 0}\frac{a_n}{n!}t^n$

Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \mathrm{Re}\,\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\mathrm{Borel}}(\varepsilon)$ in D_R

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If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k \varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant C r^n n! |\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\mathrm{Borel}}(\varepsilon)$ in D_R

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable
- $e^{-\alpha X} = P(X) + F(X), \qquad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log e^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P})\, e^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$
- $F(X) = S_n + R_n, \quad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \quad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$

  $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \qquad R_n = \frac{(-\alpha)^n}{n!} X^n e^{-\alpha\theta X}$
- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)

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