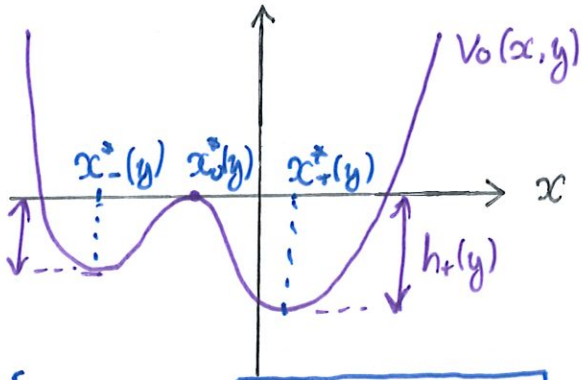
An Eyring-Kramers law for periodically forced bistable systems

$$\begin{cases} dx_t = -\partial_x V_0(x_t, y_t)\,dt + \sigma\, dW_t^x \\ dy_t = \varepsilon\, dt + \sigma\sqrt{\varepsilon}\,\rho\, dW_t^y \end{cases}$$

indep. Wiener processes

$0 < \varepsilon, \sigma \ll 1, \quad \rho > 0$

$x \mapsto V_0(x,y)$ double-well potential
class $\mathcal{C}^4$, confining,
1-periodic in y



$$\begin{cases} \omega_\pm(y) := \sqrt{\partial_{xx} V_0(x_\pm^*(y), y)} \\ \omega_0(y) := \sqrt{|\partial_{xx} V_0(x_0^*(y), y)|} \end{cases}$$

Question: characterise $\tau_+ = \inf\{t > 0 : x_t = x_+^*(y_t)\}$
for initial condition $x_0 = x_-^*(y_0)$ (y_0 fixed)

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1. Static case: $\varepsilon = 0 \Rightarrow dx_t = -\partial_x V_0(x_t)\,dt + \sigma\, dW_t \qquad y \equiv y_0$

[Eyring-Kramers] $\quad \mathbb{E}_{x_-^*}[\tau_+] = \dfrac{2\pi}{\omega_0\,\omega_-}\, e^{2h_-/\sigma^2}\,[1 + \mathcal{O}(\sigma^2)]$

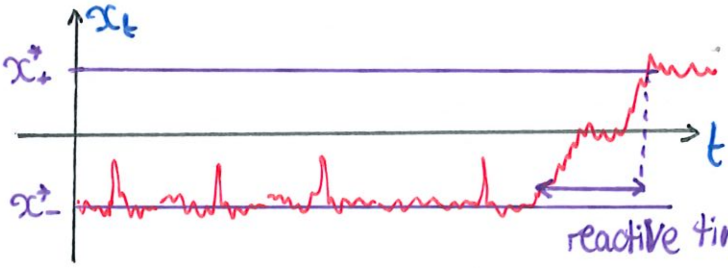
[Day '83] $\quad \lim\limits_{\sigma \to 0} \mathbb{P}_{x_-^*}\{\tau_+ > s\, \mathbb{E}_{x_-^*}[\tau_+]\} = e^{-s}$

[Cérou-Guyader-Lelièvre-Malrieu '13] transition/reactive time

$x_-^* < a < x_0 < x_0^* < b < x_+^*$

$$\lim_{\sigma \to 0} \mathrm{Law}\left(\omega_0^2\, \tau_b - 2\log(\sigma^{-1}) \;\middle|\; \tau_b < \tau_a\right) = \mathrm{Law}\big(Z + \underbrace{T(x_0, b)}_{\text{determ.}}\big)$$

where Z is Gumbel: $\mathbb{P}\{Z \leqslant t\} = e^{-e^{-t}} \quad \forall t \in \mathbb{R}$



reactive time $\sim \dfrac{1}{\omega_0^2}\,[2\log(\sigma^{-1}) + Z + T]$

Similar results for $dx_t = -\nabla V_0(x_t)\,dt + \sigma\, dW_t$
[Bovier et al, Helffer et al, Le Peutrec]
Non-reversible: [Bouchet, Reygner], [LMS], [LP&M]

2. Distribution of τ_0: [B&Gentz '14]

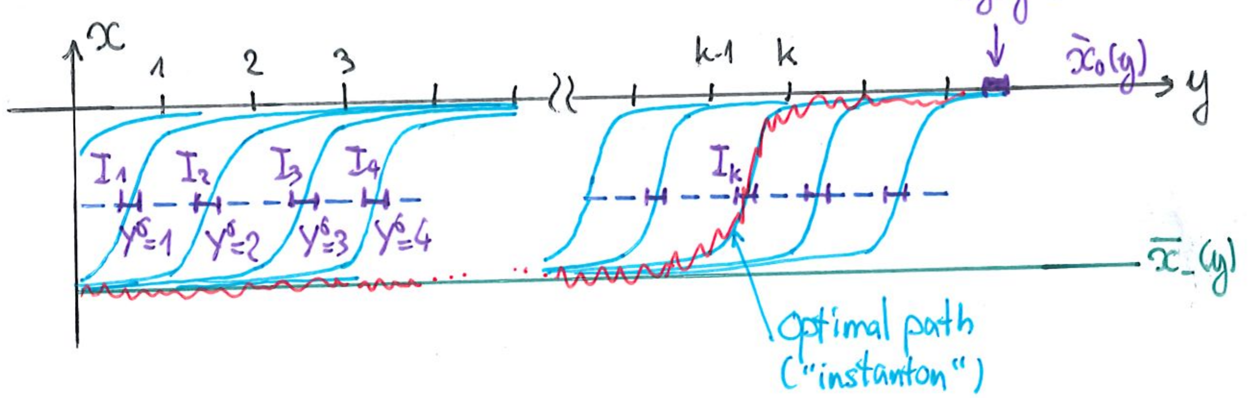
τ_0: hitting time of det. periodic orbit tracking $x_0^*(y)$, denoted $\bar{x}_0(y)$

$$\boxed{\textbf{Thm:}\quad \lim_{\sigma\to 0} \operatorname{Law}\Big(\theta(y_{\tau_0}) - \log(\sigma^{-1}) - \frac{\lambda_+}{\varepsilon} Y^\sigma\Big) = \operatorname{Law}\Big(\frac{Z}{2} - \frac{\log 2}{2}\Big)}$$

- $\theta(y)$ explicit parametrisation of $\bar{x}_0(y)$, $\theta(y+1) = \theta(y) + \frac{\lambda_+}{\varepsilon}$, $\theta' > 0$
- λ_+ Lyapunov exponent of $\bar{x}_0(y)$ $\quad\left(\sim \int_0^1 \omega_0(y)^2\,dy\right)$
- Y^σ asymptotically geometric r.v: $\lim_{n\to\infty} \mathbb{P}\{Y^\sigma = n+1 \mid Y^\sigma = n\} = p(\sigma)$

where $p(\sigma) \asymp e^{-I/\sigma^2}$ $\quad I$ Wentzell-Freidlin quasipotential

idea:



$$\mathbb{P}\{y_{\tau_0} \in [y, y+\Delta]\} \simeq \sum_{k=1}^{\lfloor y \rfloor} \underbrace{\mathbb{P}\{y_{\tau_-} \in I_k\}}_{\simeq\, \mathbb{P}\{Y^\sigma = k\}}\ \underbrace{\mathbb{P}^{I_k}\{y_{\tau_0} \in [y, y+\Delta]\}}_{\simeq\, \mathbb{P}\{\frac{Z}{2} + \text{const} \in [y-k, y-k+\Delta]\}}$$

<u>Rem:</u> $\mathbb{E}[\tau_0] \simeq \dfrac{1}{p(\sigma)} \asymp e^{I/\sigma^2}$ $\qquad$ Sharp asymptotics?

$\mathbb{E}[\tau_+] \simeq \dfrac{2}{p(\sigma)}$

3. Result on $\mathbb{E}[\tau_+]$:

$$\begin{cases} dx_t = -\frac{1}{\varepsilon}\partial_x V_0(x_t,y_t)\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t^x \\ dy_t = \qquad dt + \sigma\varrho\, dW_t^y \end{cases}$$ (time scaled by ε)

$\omega_\pm(y)^2, \omega_0(y)^2$: curvatures of V_0 at $x^*_\pm(y)$, $x_0(y)$

$$r_\pm(y) = \frac{\omega_\pm(y)\,\omega_0(y)}{2\pi}\, e^{-2h_\pm(y)/\sigma^2}$$ (transition rates)

leading eigenvalue of $-\mathcal{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0\,\partial_x$: $\lambda_1(y) = [r_+(y)+r_-(y)][1+O(\sigma^2)]$

Theorem: [B '20, arXiv: 2007.08443]

$$\mathbb{E}^{(x^*_-(y_0),y_0)}[\tau_+] = \frac{2\pi\varepsilon[1+R(\varepsilon,\sigma)]}{\int_0^1 \omega_0(y)\,\omega_-(y)\,e^{-2h_-(y)/\sigma^2}\,dy}$$

with $R(\varepsilon,\sigma) \ll 1$ if $\langle\lambda_1\rangle = \int_0^1 \lambda_1(y)\,dy \ll \varepsilon \ll \langle\lambda_1\rangle^{1/4}$

4. Heuristics:

Two-state jump process: $\ominus \underset{r_+(y)/\varepsilon}{\overset{r_-(y)/\varepsilon}{\rightleftarrows}} \oplus$

$$\frac{d}{dy}\mathbb{P}^{y_0,-}\{\tau_+ > y\} = -\frac{1}{\varepsilon} r_-(y)\,\mathbb{P}^{y_0,-}\{\tau_+ > y\}$$

$$\Rightarrow \mathbb{P}^{y_0,-}\{\tau_+ > y\} = e^{-R_-(y,y_0)/\varepsilon} \qquad R_-(y,y_0) = \int_{y_0}^{y} r_-(\bar y)\,d\bar y$$

$$\mathbb{E}^{y_0,-}[\tau_+] = \int_{y_0}^{\infty} e^{-R_-(y,y_0)/\varepsilon}\,dy$$

$R_-(y_0+n+\bar y, y_0) = nR_-(1,0) + R(y_0+\bar y, y_0)$

$$= \frac{1}{1-e^{-R_-(1,0)/\varepsilon}} \int_0^1 e^{-R_-(y_0+y,y_0)/\varepsilon}\,dy$$

$$\cong \begin{cases} \dfrac{\varepsilon}{R_-(0,1)} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \quad \Leftarrow \varepsilon \gg \langle\lambda_1\rangle \\[2ex] \dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \end{cases}$$

(intermediate ε: stochastic resonance)

5. Sketch of proof: Potential theory

Generator: $\mathcal{L} = \frac{1}{\varepsilon}\mathcal{L}_x + \mathcal{L}_y$
$$\begin{cases} \mathcal{L}_x = \frac{\sigma^2}{2}\partial_{xx} - \partial_x V_0\, \partial_x \\ \mathcal{L}_y = \frac{\rho^2\sigma^2}{2}\partial_{yy} + \partial_y \end{cases}$$

[Landim, Mariani & Seo '19]:

Inv. meas. $d\pi = e^{-2V(x,y)/\sigma^2}\,dx\,dy$, V sat. Hamilton-Jacobi equ.

$$\mathcal{L}f = \underbrace{\frac{\sigma^2}{2\varepsilon} e^{2V/\sigma^2}\nabla\cdot\left[D e^{-2V/\sigma^2}\nabla f\right]}_{=\mathcal{L}_s f} + \underbrace{c\cdot\nabla f}_{=\mathcal{L}_a f} \qquad \begin{cases}\mathcal{L}_s = \mathcal{L}_s^\dagger \\ \mathcal{L}_a = -\mathcal{L}_a^\dagger\end{cases}$$

where
$$\begin{cases} D = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\rho^2\end{pmatrix} \\ c = \frac{1}{\varepsilon}(-\partial_x V_0 + \partial_x V)e_x + (1+\rho^2\partial_y V)e_y \end{cases} \qquad \nabla\cdot\left(e^{-2V/\sigma^2}c\right) = 0$$

Adjoint process: $\mathcal{L}^* = \mathcal{L}_s - \mathcal{L}_a$

Key relation: $A\cap B = \emptyset \qquad A, B \subset \mathbb{R}^2$

$$\boxed{\int_{\partial A}\mathbb{E}_x[\tau_B]\,d\nu_{AB} = \frac{1}{\operatorname{cap}(A,B)}\int_{B^c} h^*_{AB}\,d\pi}$$

capacity

committor $\mathbb{P}^*_{x,y}\{\tau_A < \tau_B\}$ for adjoint process

Var. principles: $\mathcal{D}(\varphi,\psi) = \frac{2\varepsilon}{\sigma^2}\int_{(A\cup B)^c}\varphi\cdot(D^{-1}\psi)\,\frac{dx\,dy}{\pi}$

Vector fields

$\mathcal{D}(\varphi) = \mathcal{D}(\varphi,\varphi) \qquad \Phi_f := \frac{\sigma^2}{2\varepsilon}\pi D\nabla f - \pi f c$

Dirichlet principle:
$$\boxed{\operatorname{cap}(A,B) = \inf_{f:\, f|_A = 1,\, f|_B = 0}\ \inf_{\substack{\varphi:\ \operatorname{div}\varphi = 0 \\ \text{zero flux}}} \mathcal{D}(\Phi_f - \varphi)}$$

Thomson principle:
$$\boxed{\operatorname{cap}(A,B) = \sup_{f:\, f|_A = f|_B = 0}\ \sup_{\substack{\varphi:\ \operatorname{div}\varphi = 0 \\ \text{Flux} = 1}} \frac{1}{\mathcal{D}(\Phi_f - \varphi)}}$$

Good choices of f, φ yield good bounds on $\operatorname{cap}(A,B)$ (given π)

Main difficulty: estimation of π

$$\mathcal{L}^\dagger \pi = 0 \qquad \mathcal{L}^\dagger = \frac{1}{\varepsilon}\mathcal{L}_x^\dagger + \mathcal{L}_y^\dagger \qquad \begin{cases} \mathcal{L}_x^\dagger = \frac{\sigma^2}{2}\partial_{xx} + \partial_x[\partial_x V_0 \,\cdot\,] \\ \mathcal{L}_y^\dagger = \frac{\rho^2\sigma^2}{2}\partial_{yy} - \partial_y \end{cases}$$

$$\pi(x,y) = \underbrace{\pi_0(x|y)}_{=\frac{1}{Z_0(y)} e^{-2V_0(x,y)/\sigma^2}} \sum_{n\geq 0} \alpha_n(y)\phi_n(x|y) \qquad \mathcal{L}_x \phi_n = -\lambda_n(y)\phi_n$$

Case $\rho = 0$:

$$\varepsilon \alpha_n' = -\lambda_n(y)\alpha_n - \frac{\varepsilon}{\sigma^2} f_{n0}(y) - \frac{\varepsilon}{\sigma^2}\sum_{m\geq 1} f_{nm}(y)\alpha_m$$

$$f_{nm}(y) = \sigma^2 \langle \partial_y \pi_m, \phi_n \rangle = -\sigma^2 \langle \pi_m, \partial_y \phi_n \rangle \qquad \pi_m = \pi_0 \phi_m$$

n=0: $\alpha_0(y) = 1$

n=1: $\varepsilon \alpha_1' = -\lambda_1(y)\alpha_1 - \frac{\varepsilon}{\sigma^2} f_{10}(y) - \dots \qquad f_{10}(y) \cong -\frac{\Delta'(y)}{\cosh(\Delta/\sigma^2)}$

$\langle \lambda_1 \rangle \ll \varepsilon$ slow $\qquad e^{2\Delta/\sigma^2} := \frac{\Gamma_-(y)}{\Gamma_+(y)}$

$$\alpha_1(y) = \frac{\tanh(\Delta(y)/\varepsilon) - \delta_1(y)}{\cosh(\Delta(y)/\varepsilon)} \Rightarrow \delta_1(y) \cong \bar{\delta}_1 \cong \frac{\langle \lambda_1 \tanh(\Delta/\varepsilon)\rangle}{\langle \lambda_1 \rangle}$$

n≥2: $\varepsilon \alpha_n' = -\lambda_n \alpha_n - \frac{\varepsilon}{\sigma^2} f_{n0}(y) - \dots$

$\varepsilon \ll \lambda_n$ Fast $\qquad \alpha_n \cong \alpha_n^* = -\frac{\varepsilon}{\sigma^2}\frac{1}{\lambda_n}\left[f_{n0} + \alpha_1 f_{n1}\right]$

Proposition:

$$\pi(x,y) = \pi_0(x|y)\left[1 + \alpha_1(y)\phi_1(x|y) + \Phi_\perp(x,y)\right]$$

$$\alpha_1(y) = \sinh\left(\frac{\Delta(y)}{\sigma^2}\right) - \delta_1(y)\cosh\left(\frac{\Delta(y)}{\sigma^2}\right) \qquad \varepsilon\delta_1' \cong -\lambda_1\left(\delta_1 - \tanh\left(\frac{\Delta}{\sigma^2}\right)\right)$$

$$\langle \pi_0, \Phi_\perp^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2}\cosh\left(\frac{\Delta(y)}{\sigma^2}\right) \qquad \Phi_\perp = \Phi_\perp^* + \Phi_\perp^1$$

source of non-optimal condition $\varepsilon \ll \langle \lambda_1 \rangle^{1/4}$

Case $\rho > 0$: 2nd order ODEs for α_n