One World Dynamics Seminar (Online)

Noise-induced transitions between limit cycles

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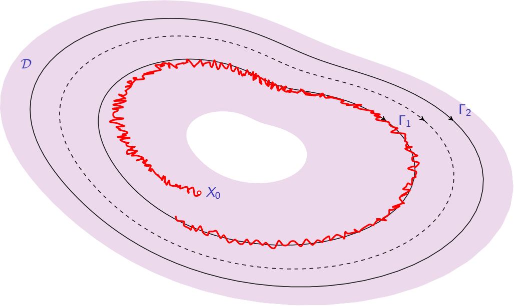
Based on joint works with Barbara Gentz (Bielefeld)

Project PERISTOCH

Do you know that town?

SDE with two limit cycles

$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$



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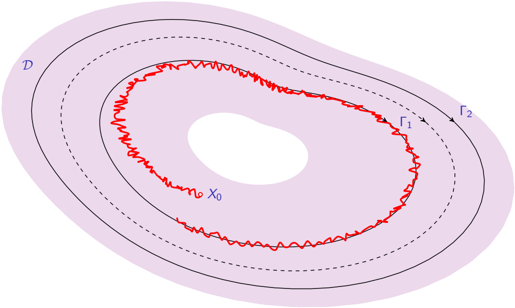

Questions:

- ▷ Distribution of transition times between limit cycles?
- ▷ Distribution of crossing locations of unstable orbit?

Applications:

- ▷ Noise-induced phase slips for synchronisation
- ▷ Stochastic resonance
- ▷ Morris–Lecar model

Synchronization of two coupled oscillators

See e.g. [Pikovsky, Rosenblum, Kurths 2001]

$x_i = (\theta_i, \dot{\theta}_i),\ i = 1, 2$

$$\begin{cases} \dot{x}_1 = f_1(x_1) \\ \dot{x}_2 = f_2(x_2) \end{cases}$$

ϕ_i : good parametrisation of limit cycles

$$\begin{cases} \dot{\phi}_1 = \omega_1 \\ \dot{\phi}_2 = \omega_2 \end{cases}$$

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$x_i = (\theta_i, \dot{\theta}_i), \ i = 1, 2$

$$\begin{cases} \dot{x}_1 = f_1(x_1) + \varepsilon g_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2) + \varepsilon g_2(x_1, x_2) \end{cases}$$

ϕ_i : good parametrisation of limit cycles

$$\begin{cases} \dot{\phi}_1 = \omega_1 + \varepsilon Q_1(\phi_1, \phi_2) \\ \dot{\phi}_2 = \omega_2 + \varepsilon Q_2(\phi_1, \phi_2) \end{cases}$$

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If $\omega_1 \simeq \omega_2$: $\begin{cases} \psi = \phi_1 - \phi_2 \\ \varphi = \frac{\phi_1+\phi_2}{2} \end{cases} \Rightarrow \begin{cases} \dot{\psi} = -\nu + \varepsilon q(\psi, \varphi) \\ \dot{\varphi} = \omega + \mathcal{O}(\varepsilon) \end{cases}$ $\quad \nu = \omega_2 - \omega_1$, $\omega = \frac{\omega_1+\omega_2}{2}$

For small detuning ν: averaging $\Rightarrow$ $\omega \dfrac{\mathrm{d}\psi}{\mathrm{d}\varphi} \simeq -\nu + \varepsilon \bar{q}(\psi)$

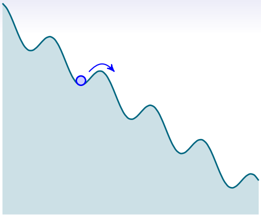
Example: Adler's equation $\bar{q}(\psi) = \sin(\psi)$: Fixed points for $\sin(\psi) = \nu/\varepsilon$

Remark: if $\omega_2/\omega_1 \simeq m/n$ similar behaviour for $\psi = n\phi_1 - m\phi_2$ (Arnold tongues)

Noise-induced phase slips

Averaged equation with noise

$$\omega \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = \underbrace{-\nu + \varepsilon \bar{q}(\psi)}_{-\frac{\partial}{\partial\psi}\left(\nu\psi - \varepsilon \int^{\psi} \bar{q}(x)\,\mathrm{d}x\right)} + \text{noise}$$



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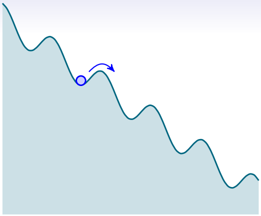
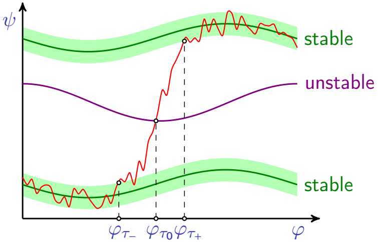


Original equations with noise

$$\begin{cases} \dot\psi = -\nu + \varepsilon q(\psi, \varphi) + \text{noise} \\ \dot\varphi = \omega + \mathcal{O}(\varepsilon) + \text{noise} \end{cases}$$



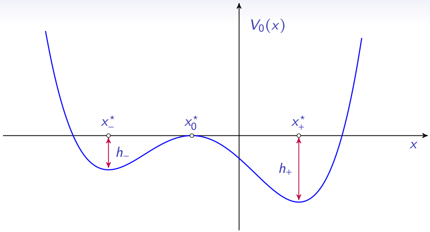
Question: distribution of phases φ_{τ_0} when crossing unstable orbit?
This is a stochastic exit problem.

Static case

$$\mathrm{d}x_t = -V_0'(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

$$\omega_\pm = \sqrt{V_0''(x_\pm^*)} \quad \omega_0 = \sqrt{-V_0''(x_0^*)}$$

τ_x: first-hitting time of x



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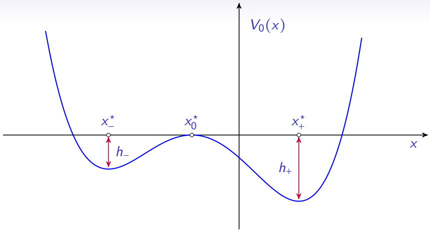

- By Dynkin's equation, $\forall x < x_+^*$,

$$\mathbb{E}^x[\tau_+] = \frac{2}{\sigma^2}\int_x^{x_+^*}\int_{-\infty}^{x_2} e^{2[V_0(x_2)-V_0(x_1)]/\sigma^2}\,dx_1\,dx_2$$

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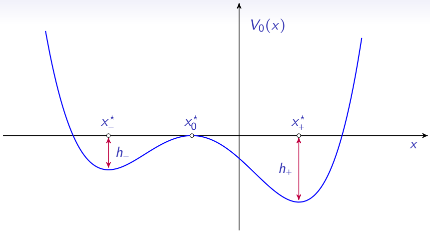

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$\Rightarrow$ Eyring–Kramers law: $\mathbb{E}^{x_-^*}[\tau_+] = \dfrac{2\pi}{\omega_0\omega_-}\,\mathrm{e}^{2h_-/\sigma^2}\,[1+\mathcal{O}(\sigma^2)]$

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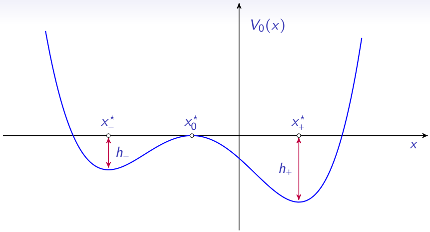

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- [Day 83]: $\lim\limits_{\sigma\to0} \mathrm{Law}\left(\dfrac{\tau_+}{\mathbb{E}^{x_-^*}[\tau_+]}\right) = \mathrm{Law}(\mathscr{E}(1))$ exponential

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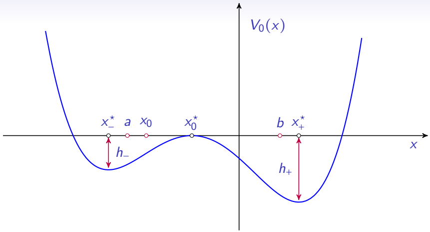

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- [Cérou, Guyader, Lelièvre, Malrieu 13]: Reactive path $x_-^* < a < x_0 < x_0^* < b < x_+^*$

$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\omega_0\tau_b - 2\log(\sigma^{-1}) \,\Big|\, \tau_b < \tau_a\Big) = \mathrm{Law}\Big(\underbrace{\mathcal{G}}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{deterministic}}\Big)$$

Contents

- ▷ **1. Toy model**

  N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit I: Two-level model*, J. Statist. Phys., **114**:1577–1618, 2004

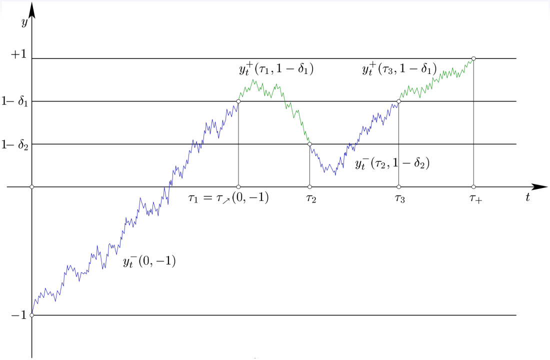
- ▷ **2. General case: distribution of crossing locations**

  N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, SIAM J. Math. Anal., **46**:310–352, 2014

- ▷ **3. General case: sharp asymptotics for exit time**

  N. B., *An Eyring-Kramers law for slowly oscillating bistable diffusions*, Probability and Mathematical Physics, **2–4**:685-743, 2021

1. Toy model



▷ Switch between equations linearized around stable and unstable orbits

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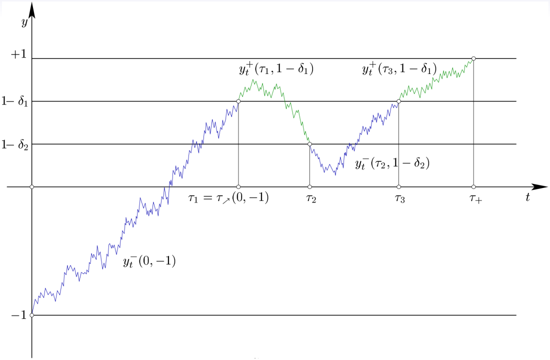


- ▷ Switch between equations linearized around stable and unstable orbits
- ▷ Use André's reflection principle to compute density of hitting time of unstable orbit, starting at $(t, 1-\delta_1)$
- ▷ Transform process around stable orbit to BM by time change and scaling, and use results on first-passage times at curved boundary
- ▷ Use renewal equation to combine both distributions

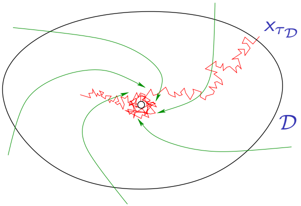
2. General case

Given $\mathcal{D} \subset \mathbb{R}^n$, define first-exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$$

First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$
defines harmonic measure

$$\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$$



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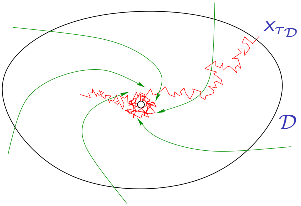

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Facts (following from Dynkin's formula):

- $u(x) = \mathbb{E}^x[\tau_{\mathcal{D}}]$ satisfies $\begin{cases} \mathscr{L}u(x) = -1 & x \in \mathcal{D} \\ u(x) = 0 & x \in \partial\mathcal{D} \end{cases}$
- For $\varphi \in L^\infty(\partial\mathcal{D}, \mathbb{R})$, $h(x) = \mathbb{E}^x[\varphi(x_{\tau_{\mathcal{D}}})]$ satisfies

$$\begin{cases} \mathscr{L}h(x) = 0 & x \in \mathcal{D} \\ h(x) = \varphi(x) & x \in \partial\mathcal{D} \end{cases}$$

where $(\mathscr{L}\varphi)(x) = \sum_i f_i(x) \frac{\partial\varphi}{\partial x_i} + \frac{\sigma^2}{2} \sum_{i,j} (gg^T)_{ij}(x) \frac{\partial^2\varphi}{\partial x_i \partial x_j}$

Freidlin–Wentzell theory

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot\gamma_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot\gamma_t - f(\gamma_t))\,\mathrm{d}t \qquad D = gg^T$$

For a set Γ of paths $\gamma : [0, T] \to \mathbb{R}^n$: $\qquad \mathbb{P}\{(x_t)_{0\leqslant t\leqslant T} \in \Gamma\} \simeq \mathrm{e}^{-\inf_\Gamma I/\sigma^2}$

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Consider domain $\mathcal{D}$ contained in basin of attraction of attractor $\mathcal{A}$

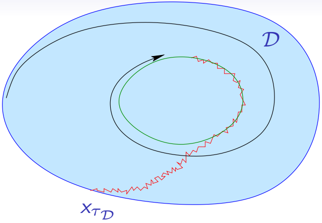
Quasipotential:
$\partial\mathcal{D} \ni y \mapsto V(y) = \inf\{I(\gamma) \colon \gamma : \mathcal{A} \to y \in \partial\mathcal{D}$ in arbitrary time$\}$

- $\lim_{\sigma\to0} \sigma^2 \log \mathbb{E}[\tau_\mathcal{D}] = \overline{V} = \inf_{y\in\partial\mathcal{D}} V(y)$ [Freidlin, Wentzell '69]
- If inf reached at a single point $y^* \in \mathcal{D}$ then
  $\lim_{\sigma\to0} \mathbb{P}\{\|x_{\tau_\mathcal{D}} - y^*\| > \delta\} = 0 \quad \forall\delta > 0$ [Freidlin, Wentzell '69]
- Exponential distr of $\tau_\mathcal{D}$: $\lim_{\sigma\to0} \mathbb{P}\{\tau_\mathcal{D} > s\mathbb{E}[\tau_\mathcal{D}]\} = \mathrm{e}^{-s}$ [Day '83]

Application to exit through unstable orbit

Planar SDE
$dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$

$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit
First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0\colon x_t \notin \mathcal{D}\}$
Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?



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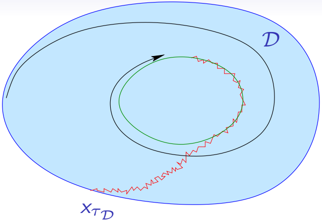

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Quasipotential:

$V(y) = \inf\{I(\gamma)\colon \gamma : \text{stable orbit} \to y \in \partial\mathcal{D} \text{ in arbitrary time}\}$

Theorem [Freidlin, Wentzell '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

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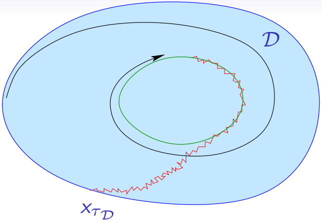

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Problem: V is constant on $\partial\mathcal{D}$!

Most probable exit paths

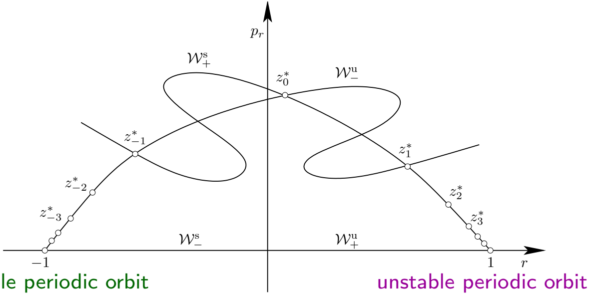
In polar-type coordinates

$$\mathrm{d}\varphi_t = f_\varphi(\varphi_t, r_t)\,\mathrm{d}t + \sigma g_\varphi(\varphi_t, r_t)\,\mathrm{d}W_t \qquad \varphi \in \mathbb{R}/2\pi\mathbb{Z}$$

$$\mathrm{d}r_t = f_r(\varphi_t, r_t)\,\mathrm{d}t + \sigma g_r(\varphi_t, r_t)\,\mathrm{d}W_t \qquad r \in [-1, 1]$$

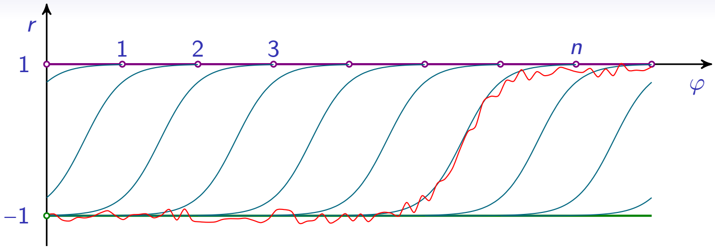
Minimisers of I obey Hamilton equations with Hamiltonian

$H(\gamma, \psi) = \frac{1}{2}\psi^T D(\gamma)\psi + f(\gamma)^T\psi$ where $\psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$



Generically optimal path γ_∞ (for infinite time) is isolated

Random Poincaré maps

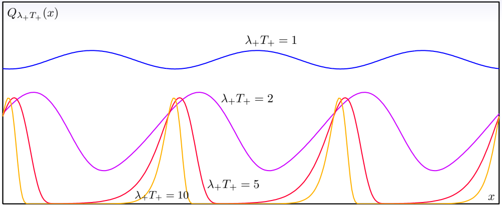


- $R_0, R_1, \dots R_N$ form substochastic Markov chain (killed in $r = 1$)
- Under hypoellipticity cond, transition kernel has smooth density k
  [Ben Arous, Kusuoka, Stroock '84]

$$\mathbb{P}^{R_0}\{R_1 \in B\} = K(R_0, B) := \int_B k(R_0, y)\,\mathrm{d}y$$

- Fredholm theory: spectral decomp $k(x,y) = \sum_{k\geqslant 0} \lambda_k h_k(x) h_k^*(y)$
  $\lambda_0 \in [0,1]$: principal eigenvalue [Perron, Frobenius, Jentzsch, Krein–Rutman]
  $\lim_{n\to\infty} \mathbb{P}\{R_n \in \mathrm{d}x | N > n\} = \dfrac{h_0^*(x)}{\int h_0^*} = \pi_0(x)$ quasistationary distr (QSD)

Main result



Theorem: [B & Gentz, SIAM J Math Analysis 2014]

$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\theta(\varphi_{\tau_0}) - \log(\sigma^{-1}) - \lambda_+ T Y^\sigma\Big) = \mathrm{Law}\Big(\frac{\mathcal{G}}{2} - \frac{\log 2}{2}\Big)$$

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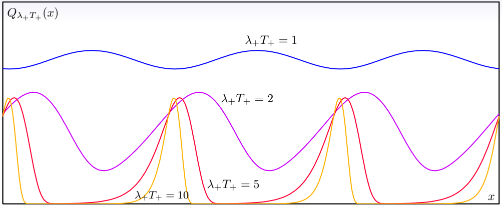

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- $\theta(\varphi)$: explicit parametrisat$^{\text{n}}$ of unstable orbit, $\theta(\varphi + 1) = \theta(\varphi) + \lambda_+ T$
- λ_+: Lyapunov exponent of unstable orbit, T: period
- $Y^\sigma \in \mathbb{N}$: asymptotically geometric $\mathbb{N}$-valued r.v:

$$\lim_{n\to\infty} \mathbb{P}\{Y^\sigma = n+1 | Y^\sigma > n\} = p(\sigma)$$

$p(\sigma) \simeq \mathrm{e}^{-\mathcal{I}/\sigma^2}$, $\mathcal{I}$ Freidlin–Wentzell quasipotential, $\mathbb{E}[\tau_0] \simeq p(\sigma)^{-1}$

- $\mathcal{G}$: Gumbel distribution, $\mathbb{P}\{\mathcal{G} > t\} = \mathrm{e}^{-\mathrm{e}^{-t}}$

Sketch of proof

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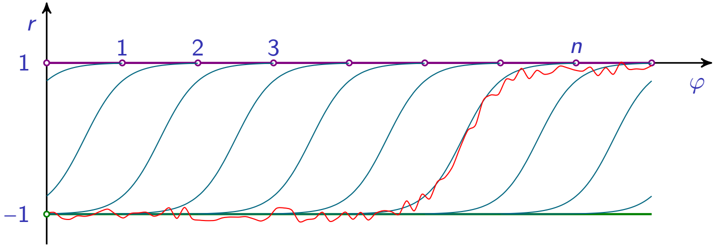

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Theorem: [B & Gentz, SIAM J Math Analysis 2014]

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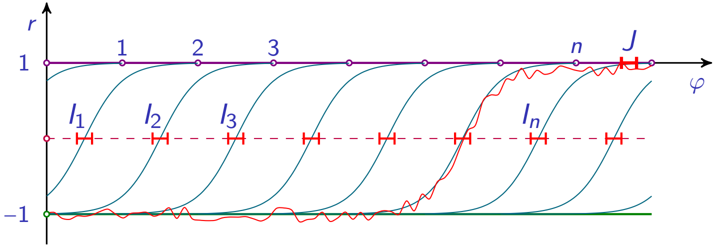

$$\mathbb{P}\{\varphi_{\tau_0} \in J\} \simeq \sum_k \underbrace{\mathbb{P}\{\varphi_{\tau_-} \in I_k\}}_{\simeq \mathbb{P}\{Y^\sigma = k\}} \underbrace{\mathbb{P}^{I_k}\{\varphi_{\tau_0} \in J\}}_{\simeq \mathbb{P}\{\frac{\mathcal{G}}{2} + const \in J-k\}}$$

Why log-periodic oscillations/cycling?

Phase at crossing: $\mathcal{W}_\Delta(t) = \sum_{n=0}^{\infty} \mathbb{P}^{r_0,0}\big\{\theta(\varphi_\tau) \in [n+t, n+t+\Delta]\big\}$

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$\theta(\varphi)$: parametrisation in which effective normal diffusion is constant

$\text{dist}(\gamma_\infty, \text{unst orbit}) \simeq \mathrm{e}^{-\lambda_+ T\theta(\varphi)}$

Escape when

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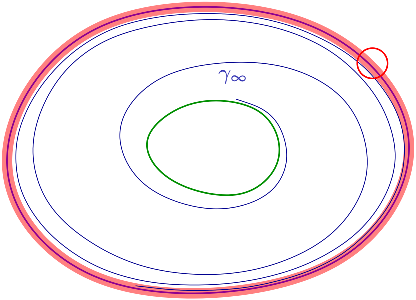

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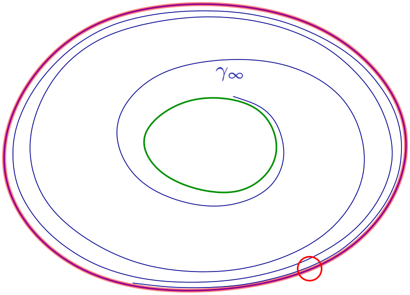

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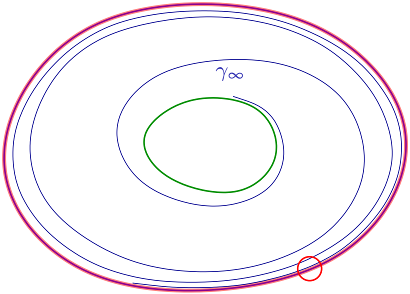


Remark: Distributions of other transition times/phases:

- ▷ Exit from unstable orbit: $\log(\sigma^{-1}) - \log(|\mathcal{N}(0,1)|)$ [Day 95, Bakhtin 08]
- ▷ Between stable orbits: Gumbel [Bakhtin 15]
- ▷ Residence-time distribution: $1/\cosh^2(\theta(\varphi))$ [B & Gentz 05]

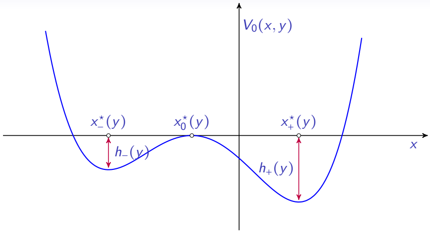
3. Eyring–Kramers-type law for $\mathbb{E}[\tau_+]$

$$\varepsilon = T^{-1}$$

$$\omega_\pm(y) = \sqrt{\partial_{xx} V_0(x^*_\pm(y), y)}$$

$$\omega_0(y) = \sqrt{-\partial_{xx}(x^*_0(y), y)}$$

$$r_\pm(y) = \frac{\omega_\pm(y)\omega_0(y)}{2\pi}\, \mathrm{e}^{-2h_\pm(y)/\sigma^2}$$



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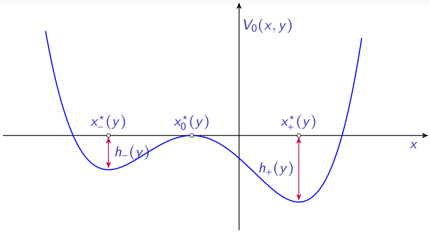

▷ Leading eigenvalue of $-\mathscr{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0 \partial_x$:

$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)]$$

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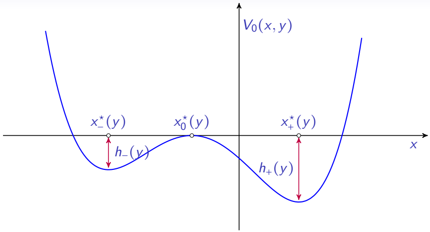

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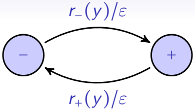
Theorem: [B, PMP 2022]

$$\mathbb{E}^{(x^*_-(y_0),y_0)}[\tau_+] = \frac{2\pi\varepsilon[1 + R(\varepsilon,\sigma)]}{\displaystyle\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y}$$

where $R(\varepsilon,\sigma)$ complicated but small if $\langle\lambda_1\rangle \ll \varepsilon \ll \langle\lambda_1\rangle^{1/4}$

Heuristics: two-state jump process



$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}^{-,y_0}\left\{\tau_+ > y\right\} = -\frac{1}{\varepsilon}r_-(y)\mathbb{P}^{-,y_0}\left\{\tau_+ > y\right\}$$

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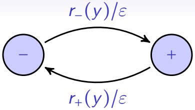

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$$\mathbb{P}^{-,y_0}\big\{\tau_+ > y\big\} = \mathrm{e}^{-R_-(y,y_0)/\varepsilon} \qquad \text{where } R_-(y_1,y_0) = \int_{y_0}^{y_1} r_-(y)\,\mathrm{d}y$$

$$\begin{aligned}
\mathbb{E}^{-,y_0}\big[\tau_+\big] &= \int_{y_0}^{\infty} \mathrm{e}^{-R_-(y,y_0)/\varepsilon}\,\mathrm{d}y \\
&= \frac{1}{1-\mathrm{e}^{-R_-(1,0)/\varepsilon}} \int_0^1 \mathrm{e}^{-R_-(y_0+y,y_0)/\varepsilon}\,\mathrm{d}y \qquad \text{(by periodicity of } r_-\text{)} \\
&\simeq \begin{cases}
\dfrac{\varepsilon}{R_-(1,0)} = \dfrac{2\pi\varepsilon}{\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \\
\dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \\
\text{In between: Stochastic resonance} &
\end{cases}
\end{aligned}$$

Main tool: potential-theoretic approach

- Reversible (gradient) systems: [Bovier, Eckhoff, Gayrard & Klein 2004]
- General case: [Landim, Mariani & Seo, 2019]
- Main relation:

$$\int_{\partial A} \mathbb{E}^{(x,y)}[\tau_B]\,\mathrm{d}\nu_{AB} = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} h^*_{AB}(x,y)\,\mathrm{d}\pi$$

  where

  - ν_{AB}: equilibrium measure on ∂A
  - $\operatorname{cap}(A,B)$: capacity, computable via variation principles
  - h^*_{AB}: committor of adjoint system
  - π: invariant measure

- Main difficulty: estimate invariant measure.
  Done using decomposition in eigenfunctions of static system

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Thanks for your attention!

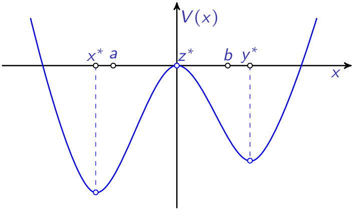
Slides available at https://www.idpoisson.fr/berglund/OWD_22.pdf

Why a Gumbel distribution?

Length of reactive path

[Cérou, Guyader, Lelièvre, Malrieu 2013] :

$dx_t = -V'(x_t)\,dt + \sigma\,dW_t \qquad a < x_0 < 0$



Theorem:

$$\lim_{\sigma\to 0} \mathbb{P}\Big\{\tau_b - \frac{2}{\lambda}|\log\sigma| < t \;\Big|\; \tau_b < \tau_a\Big\} = \frac{1}{\lambda}\Big(\log\frac{2|x_0|b}{\lambda} + I(x_0) + I(b) + \Lambda(t)\Big)$$

where $\lambda = -V''(0)$, $I(x) = \int_x^0 \big(\frac{\lambda}{V'(y)} + \frac{1}{y}\big)\,dy$

and $\Lambda(t) = e^{-e^{-t}}$: distrib. function of standard Gumbel r.v.

Proof uses Doob's h-transform

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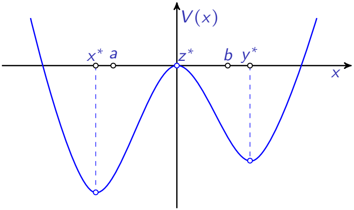

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[Bakhtin 2013] :

Link with extreme-value theory and residual lifetimes for linear case

$\mathrm{d}x_t = \lambda x_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$

Extreme-value theory and residual lifetime

- ▷ $X_1, X_2, \dots$ i.i.d. real r.v. $\qquad M_n = \max\{X_1, \dots, X_n\}$
- ▷ $F(x) = \mathbb{P}\{X_1 \leqslant x\} = 1 - R(x) \quad \Rightarrow \quad \mathbb{P}\{M_n \leqslant x\} = F(x)^n$
- ▷ **Def**: $F \in D(\Phi) \Leftrightarrow \exists (a_n)_n > 0, (b_n)_n$: $\lim_{n\to\infty} F(a_n x + b_n)^n = \Phi(x)$

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  $\Rightarrow \Phi \in$ Gumbel Fréchet Weibull
- [Gnedenko '43]: $F \in D(\Phi) \Leftrightarrow \lim_{n\to\infty} nR(a_n x + b_n) = -\log \Phi(x)$
- [Balkema, de Haan '74]:
  $F \in D(\Lambda) \Leftrightarrow \exists a(\cdot) > 0 \colon \lim_{r\to\infty} \mathbb{P}\{X > r + a(r)x | X > r\} = \mathrm{e}^{-x}$

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$$\mathrm{d}x_t = \lambda x_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t \quad\Rightarrow\quad X_t = \mathrm{e}^{\lambda t}\Big(x_0 + \widetilde{W}_{\sigma^2 \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}}\Big) = \mathrm{e}^{\lambda t}\,\widetilde{X}_t$$

$$\tau = \inf\{t > 0 \colon X_t = 0\}$$

By the reflection principle:

$$\mathbb{P}\Big\{\tau < t + \tfrac{1}{\lambda}|\log\sigma| \;\Big|\; \tau < \infty\Big\} = \mathbb{P}\Big\{\widetilde{X}_{t+\frac{1}{\lambda}|\log\sigma|} > 0 \;\Big|\; \widetilde{X}_\infty > 0\Big\}$$

$$= \mathbb{P}\Big\{N > \frac{x_0}{\sigma}\sqrt{\frac{2\lambda}{1-\sigma^2\,\mathrm{e}^{-2\lambda t}}} \;\Big|\; N > \frac{x_0}{\sigma}\sqrt{2\lambda}\Big\}$$

$$\to \exp\Big\{-x_0^2\lambda\,\mathrm{e}^{-2\lambda t}\Big\} \quad \text{as } \sigma \to 0$$

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$$\Lambda(e^{-x}) = e^{-\Lambda(x)} \qquad \to \exp\Big\{-x_0^2 \lambda e^{-2\lambda t}\Big\} \quad \text{as } \sigma \to 0$$