SLOFADYBIO ANR kickoff meeting

A toolbox to quantify effects of noise on slow–fast dynamical systems

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With Barbara Gentz (Bielefeld), Christian Kuehn (Vienna) and Damien Landon (Dijon)

Slow–fast dynamical systems

Fast variables: $x \in \mathbb{R}^n$ (e.g. membrane potential, prey, atmosphere)
Slow variables: $y \in \mathbb{R}^m$ (e.g. gating variables, predator, ocean)

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon g(x,y) \end{aligned} \quad \overset{t \mapsto \varepsilon t}{\Longleftrightarrow} \quad \begin{aligned} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

$$\Big\downarrow \varepsilon \to 0 \qquad\qquad \Big\downarrow \varepsilon \to 0$$

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= 0 \end{aligned} \quad \not\Longleftrightarrow \quad \begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

Fast system — Slow system

- Averaging [Krylov–Bogoliubov '37, ...], Lyapunov fcts [Tihonov '52, ...]
- Nonstandard analysis, asymptotic/WKB expansions, Gevrey series...
- Geometric singular perturbation theory [Fenichel 1979, ...]
- Normal forms, Newton's polygon, blow-up, topological methods, etc...

Slow–fast dynamical systems perturbed by noise

$$dx_t = \frac{1}{\varepsilon} f(x_t, y_t)\, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\, dW_t \qquad F \in \mathbb{R}^{n\times k}$$

$$W_t \in \mathbb{R}^k$$

$$dy_t = g(x_t, y_t)\, dt + \sigma' G(x_t, y_t)\, dW_t \qquad G \in \mathbb{R}^{m\times k}$$

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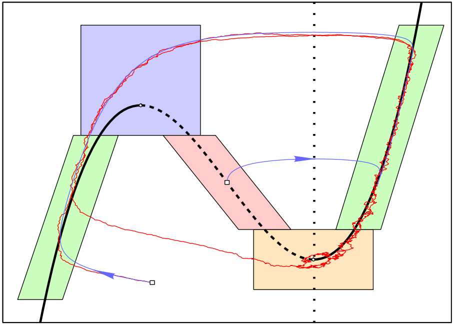


- Near stable normally hyperbolic slow manifold
- Near unstable normally hyperbolic slow manifold
- Near fold points and other bifurcation points

Deterministic Fenichel theory

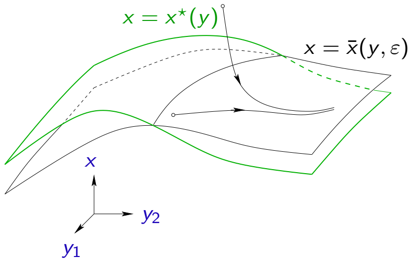
$$\varepsilon\dot{x} = f(x,y) \qquad x \in \mathbb{R}^n \text{, fast variables}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m \text{, slow variables}$$

- Critical manifold: $f(x^\star(y),y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $A(y) = \partial_x f(x^\star(y),y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]

$\exists$ slow manifold $x = \bar{x}(y,\varepsilon)$ s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



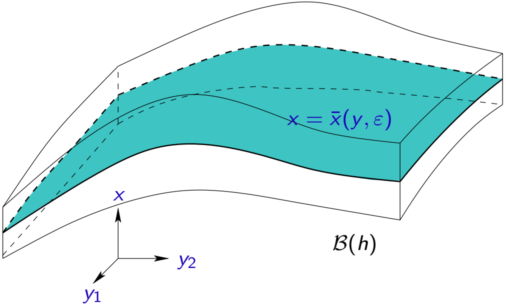
Stochastic Fenichel theory

$$dx_t = \frac{1}{\varepsilon} f(x_t, y_t)\, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\, dW_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

$$dy_t = g(x_t, y_t)\, dt + \sigma' G(x_t, y_t)\, dW_t \qquad \text{(slow variables } \in \mathbb{R}^m\text{)}$$

$\mathcal{B}(h)$: confidence set

defined by covariance of linearised equation for $x - \bar{x}(y, \varepsilon)$



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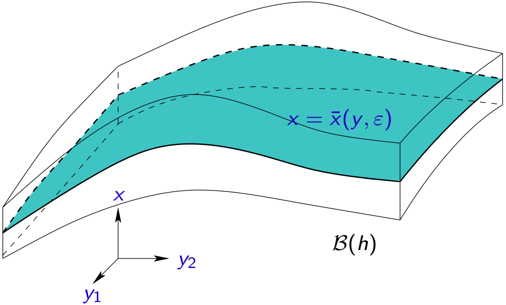
$\bar{A}(y) := \partial_x f(\bar{x}(y,\varepsilon), y)$



$\bar{X}$: covariance matrix of linearisation, solution of deterministic slow–fast ODE

$$\varepsilon \dot{\bar{X}} = \bar{A}(y)\bar{X} + \bar{X}\bar{A}(y)^{\mathrm{T}} + F(\bar{x}(y,\varepsilon), y)F(\bar{x}(y,\varepsilon), y)^{\mathrm{T}}$$
$$\dot{y} = g(\bar{x}(y,\varepsilon), y)$$

$$\mathcal{B}(h) := \left\{(x,y) \colon \left\langle \left[x - \bar{x}(y,\varepsilon)\right], \bar{X}(y)^{-1}\left[x - \bar{x}(y,\varepsilon)\right]\right\rangle < h^2\right\}$$

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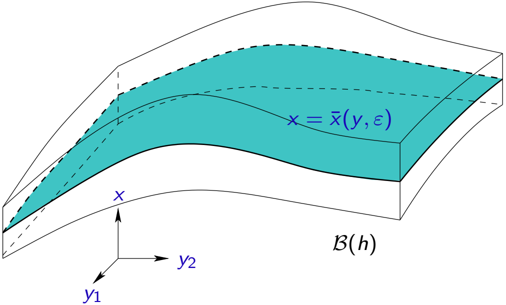


Theorem [B & Gentz, J. Diff. Equ. 2004] Normally hyperbolic stable case:

$$C_-(t, \varepsilon)\, e^{-\kappa h^2/2\sigma^2} \leqslant \mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leqslant C_+(t, \varepsilon)\, e^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$

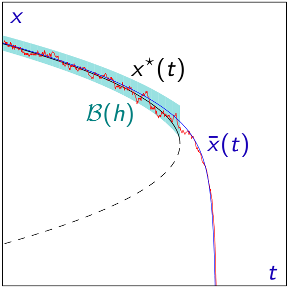
Saddle–node (or fold) bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[-t - x_t^2 + \dots\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{-t}$
Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \varepsilon^{1/3}\})$

$$\bar{a}(t) = \partial_x f(\bar{x}(t), \varepsilon) \asymp \begin{cases} -\sqrt{|t|} & t \leqslant -c\varepsilon^{2/3} \\ -\varepsilon^{1/3} & |t| \leqslant c\varepsilon^{2/3} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



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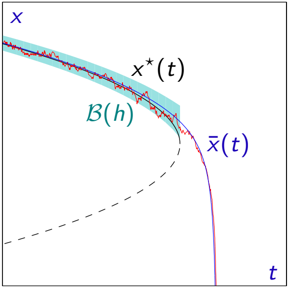

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Theorem [B & Gentz, Nonlinearity 2002]

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leqslant C(t,\varepsilon)\, e^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ $\Rightarrow$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_c = \varepsilon^{1/2}$: result applies $\forall\, t$, $\mathbb{P}\{\text{leaving } \mathcal{B}(h)\} = \mathcal{O}(e^{-\kappa\sigma_c^2/\sigma^2})$
- $\sigma > \sigma_c = \varepsilon^{1/2}$: result applies up to $t \asymp -\sigma^{4/3}$

Saddle–node (or fold) bifurcation

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{4/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

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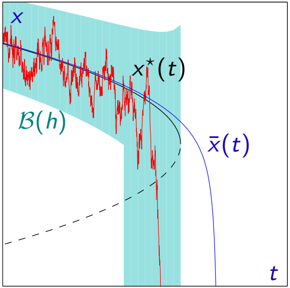
Fold bif.: Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, \mathrm{d}s = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

Proof uses comparison with linearised equations

Thm [B & Gentz, Nonlinearity 2002]

Transition probability $\geqslant 1 - \mathrm{e}^{-\kappa\sigma^2/(\varepsilon|\log\sigma|)}$



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Avoided transcritical bif: $\varepsilon \dot{x} = t^2 + \delta - x^2$

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, \mathrm{d}s = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

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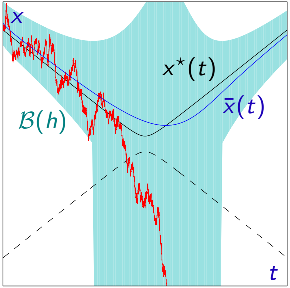
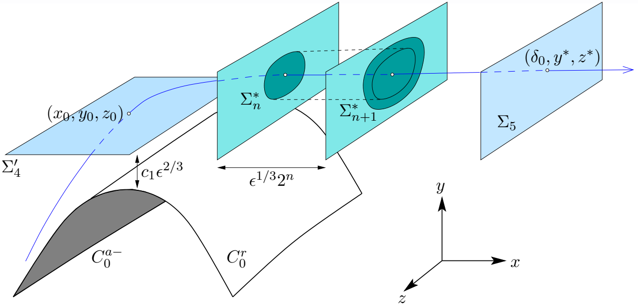


Fold for one fast and two slow variables



Proposition [B, Gentz & Kuehn, JDDE 2015]: For $h = \mathcal{O}(\varepsilon^{2/3})$,

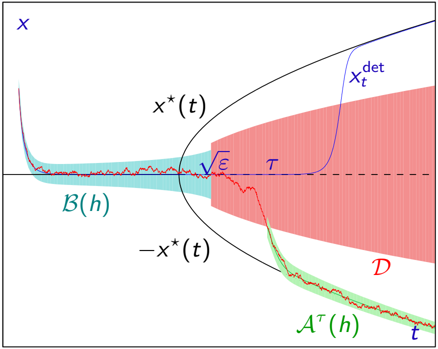
$$\mathbb{P}\Big\{\|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\| > h\Big\}$$

$$\leqslant C|\log\varepsilon|\bigg(\exp\Big\{-\frac{\kappa h^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}}\Big\} + \exp\Big\{-\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon}\Big\}\bigg)$$

Pitchfork bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[t x_t - x_t^3 + \dots\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Deterministic:
bifurcation delay
$x_t \simeq x_0\, \mathrm{e}^{(t^2 - t_0^2)/2\varepsilon}$



Theorem [B & Gentz, PTRF 2002]

- Paths concentrated in $\mathcal{B}(h)$ up to time $\sqrt{\varepsilon}$
  Typical spreading $\sigma\varepsilon^{-1/4}$
- Paths likely to leave $\mathcal{D}$ at time $\sqrt{\varepsilon|\log\sigma|}$
- Paths likely to stay in $\mathcal{A}^\tau(h)$ after leaving $\mathcal{D}$

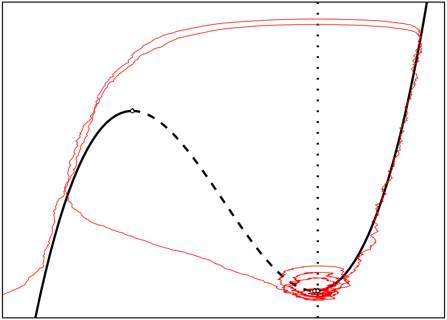
Stochastic FitzHugh–Nagumo equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$

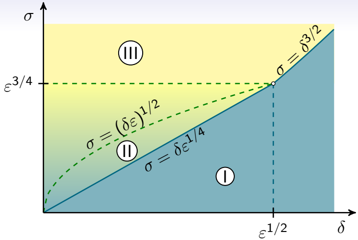
$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$

- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes (white noise)
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

$\varepsilon = 0.1$
$b = 0$
$\delta := \frac{a^2-1}{3} = 0.02$
$\sigma_1 = \sigma_2 = 0.03$



FitzHugh–Nagumo equation: Summary of results



$\sigma_1 = \sigma_2 = \sigma$:
$$\mathbb{P}\{\text{escape}\} \simeq \Phi\biggl(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\biggr)$$

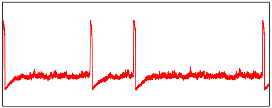
see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes

Theorem [B & Landon, Nonlin. 12]: $\delta \ll \varepsilon^{1/2}$

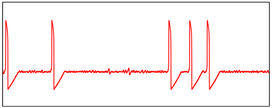
$\mathbb{P}\{\text{escape}\}^{-1} \simeq \mathbb{E}[\#\text{ small oscil}] \simeq e^{\kappa(\varepsilon^{1/4}\delta)^2/\sigma^2}$



Regime II: clusters of spikes

$\#$ small oscillations: asympt geometric

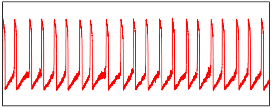
$\sigma = (\delta\varepsilon)^{1/2}$: Geom$(1/2)$



Regime III: repeated spikes

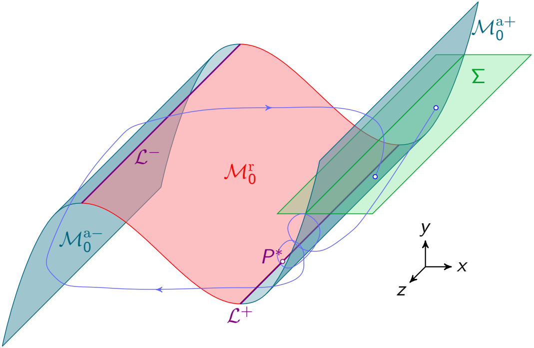
$\mathbb{P}\{N = 1\} \simeq 1$

Interspike interval $\simeq$ constant



Folded-node singularity in dim 3: The Koper model

$$\begin{aligned}
\varepsilon \, dx_t &= [y_t - x_t^3 + 3x_t] \, dt &&+ \sqrt{\varepsilon}\sigma F(x_t, y_t, z_t) \, dW_t \\
dy_t &= [kx_t - 2(y_t + \lambda) + z_t] \, dt &&+ \sigma' G_1(x_t, y_t, z_t) \, dW_t \\
dz_t &= [\rho(\lambda + y_t - z_t)] \, dt &&+ \sigma' G_2(x_t, y_t, z_t) \, dW_t
\end{aligned}$$

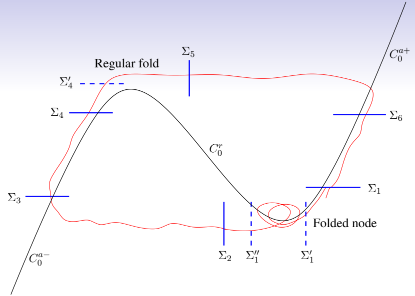


Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, ...]

Poincaré map $\Pi : \Sigma \to \Sigma$ is almost $1d$ due to contraction in x-direction

Size of fluctuations

$\mu \ll 1$: eigenvalue ratio at folded node



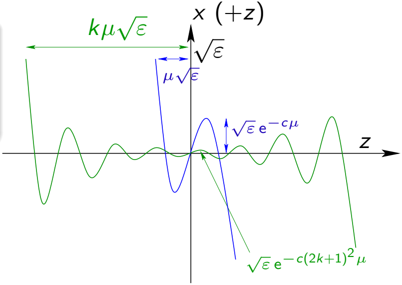
Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$\sigma'\varepsilon^{1/4}$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



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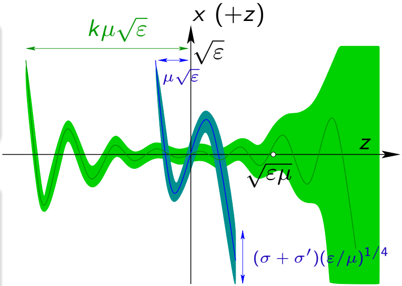
Theorem 2: size of fluctuations

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu} \Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

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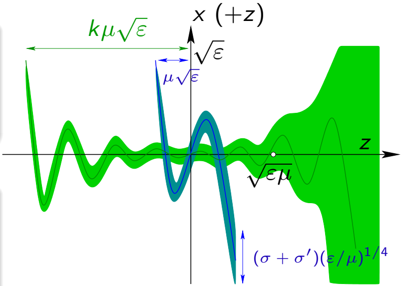

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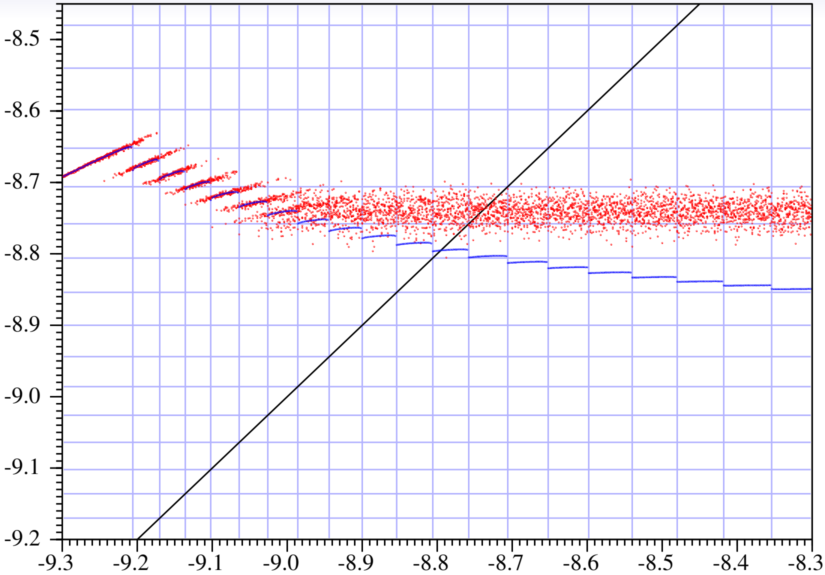
$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

- Saturation effect occurs at $k_c \simeq \sqrt{|\log(\sigma + \sigma')|/\mu}$
- For $k > k_c$, behaviour indep. of k and $\Delta z \leqslant \mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma + \sigma')|}\,)$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-3}$

Summary/Outlook

Noise can cause threshold phenomena

- Below threshold small perturbation of deterministic dynamics
- Above threshold large transitions can occur

Well understood:

- Normally hyperbolic case
- Codimension-1 bifurcations (fold, (avoided) transcritical, pitchfork, Hopf)
- Higher codimension: case studies (folded node, cf. Kuehn)

In progress: theory of random Poincaré maps

Essentially still open:

- Other types of noise (except Ornstein–Uhlenbeck)
- Equations with delay
- Infinite dimensions, in particular with continuous spectrum

Further reading

General slow–fast systems

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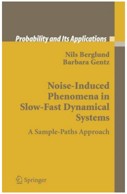

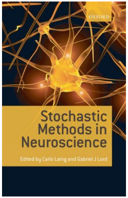

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