

Stochastic equations and particle systems, La Sapienza, Roma

Renormalisation and sharp asymptotics for metastable transition times in stochastic Allen–Cahn equations

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Joint works with Barbara Gentz (Bielefeld), Giacomo Di Gesù (Rome), Hendrik Weber (Münster), Tom Klose (Warwick)





Allen–Cahn and Φ_d^4 equations

- ▷ Allen–Cahn equation [Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + \phi(t, x) - \phi(t, x)^3 + \sqrt{2\varepsilon} \xi$$

- ▷ Dynamic Φ_d^4 model/stochastic quantization eq. [Parisi & Wu 81]

$$\partial_t \phi(t, x) = \Delta \phi(t, x) - \phi(t, x)^3 + \sqrt{2\varepsilon} \xi$$

- ▷ $x \in \mathbb{T}^d$, ξ space-time white noise, $0 < \varepsilon \leq 1$

- ▷ Same solution theory, different dynamics:
Allen–Cahn is metastable, Φ_d^4 is not

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Deterministic Allen–Cahn PDE in $d = 1$

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + f(\phi(t, x))$$

- ▷ $x \in [0, L]$, L : bifurcation parameter
- ▷ $\phi(t, x) \in \mathbb{R}$
- ▷ Either periodic or zero-flux Neumann boundary conditions
- ▷ In this talk: $f(\phi) = \phi - \phi^3$ (results more general)

Energy function:

$$V[\phi] = \int_0^L \left[\frac{1}{2} \phi'(x)^2 - \frac{1}{2} \phi(x)^2 + \frac{1}{4} \phi(x)^4 \right] dx \quad \rightarrow \min$$

Stationary solutions: $\phi_0''(x) = -\phi_0(x) + \phi_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t \psi_t(x) = \psi_t''(x) + [1 - 3\phi_0(x)^2] \psi_t(x)$

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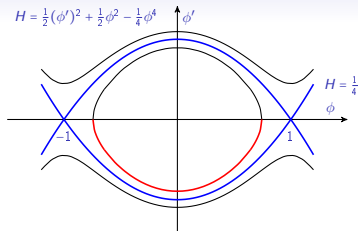
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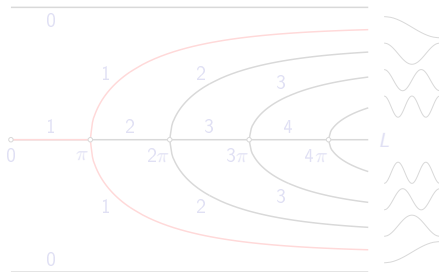
Stationary solutions

$$\phi_0''(x) = -f(\phi_0(x)) = -\phi_0(x) + \phi_0(x)^3$$

- ▷ $\phi_{\pm}(x) \equiv \pm 1$: **stable**
- ▷ $\phi_0(x) \equiv 0$: **unstable**
- ▷ Nonconstant solutions satisfying b.c.
(expressible in terms of Jacobi elliptic fcts)
- ▷ Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$



Number of positive
eigenvalues
(= unstable directions)
Transition state

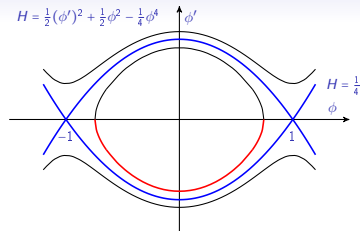


- ▷ Periodic b.c: k families when $L > 2k\pi$

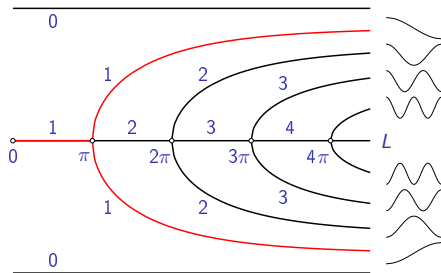
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Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + f(\phi(t, x)) + \sqrt{2\varepsilon} \xi(t, x) \quad (f(\phi) = \phi - \phi^3)$$

Initial condition: ϕ_{in} near $\phi_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k \pi}{L})^2 + 2$

Target: $\phi_+ \equiv 1$, $\tau_+ = \inf\{t > 0: \|\phi(t, \cdot) - \phi_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$\phi_{\text{ts}}(x) = \begin{cases} \phi_0(x) \equiv 0 & \text{if } L \leq \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k \pi}{L})^2 - 1 \\ \phi_1(x) \text{ } \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c. and $L < \pi$

$$\Rightarrow \mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq 2\pi \sqrt{\frac{1}{|\lambda_0| \nu_0}} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$$

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Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- ▷ If $L < \pi - c$ with $c > 0$, then

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- ▷ If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- ▷ Results also for L near π and periodic b.c.

- ▷ Prefactor involves a Fredholm determinant:

Δ_L Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det[(-\Delta_L - 1)(-\Delta_L + 2)^{-1}] = \det[1 - 3(-\Delta_L + 2)^{-1}]$$

converges because $\log \det = \text{Tr} \log$ and $(-\Delta_L + 2)^{-1}$ is trace class

$$(\text{limit} = \frac{\sqrt{2} \sin(L)}{\sinh(\sqrt{2}L)})$$

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Ideas of proof

- ▷ Spectral Galerkin approximation: $\phi(t, x) = \sum_{|k| \leq N} z_k(t) e_k(x)$ (Fourier)
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$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^z[\tau_B] \nu_{AB}(dz) = \frac{1}{\text{cap}(A, B)} \int_{B^c} e^{-V(z)/\varepsilon} h_{AB}(x) dz$$

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(Stochastic) Allen–Cahn equation on $\mathbb{T}^2$

$$d\phi(t, x) = \left[\nu(\varepsilon t) \Delta \phi(t, x) + \phi(t, x) - \phi(t, x)^3 \right] dt + \sigma dW(t, x)$$

(Online: <https://youtu.be/yX0EAxZHNCQ>)

The two-dimensional case

- ▷ Large-deviation principle: [Hairer & Weber, 2015]
- ▷ Naive computation of prefactor fails:

$$\begin{aligned}\log \prod_{k \in (\mathbb{N}^2)^*} \frac{1 - \left(\frac{L}{|k|\pi}\right)^2}{1 + 2\left(\frac{L}{|k|\pi}\right)^2} &\simeq \sum_{k \in (\mathbb{N}^2)^*} \log \left(1 - \frac{3L^2}{|k|^2\pi^2}\right) \\ &\simeq - \sum_{k \in (\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2} \int_1^\infty \frac{r \, dr}{r^2} = -\infty\end{aligned}$$

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- ▷ In fact, the equation needs to be **renormalised**

Theorem: [Da Prato & Debussche 2003]

Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t \phi = \Delta \phi + [1 + 3\varepsilon C(\delta)]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \rightarrow 0$

(Global version: [Mourrat & Weber 2015])

[Mourrat & Weber 2014]: **Renormalised** eq = scaling limit of **Ising–Kac model**

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t e^{(t-s)\Delta} \xi(s, x) ds$ is **distribution**

▷ δ -mollification should be equivalent to Galerkin approx. $|k| \leq N = \delta^{-1}$:

$$w_N(x, t) = \sum_{|k| \leq N} a_k(t) \frac{1}{L} e^{i\Omega k \cdot x} \quad a_k = \int_0^t e^{-\mu_k(t-s)} dW_s^{(k)}$$
$$\mu_k = (\Omega|k|)^2 \quad \Omega = \beta\pi/L$$

▷ $\lim_{t \rightarrow \infty} \int_0^t e^{(\Delta-1)(t-s)} \xi_N(s, x) ds = \phi_N$ is a Gaussian free field, s.t.

$$L^2 C_N := L^2 \mathbb{E} \phi_N^2 = \mathbb{E} \|\phi_N\|_{L^2}^2 = \sum_{|k| \leq N} \frac{1}{2(\mu_k+1)} = \frac{\text{Tr}(P_N[-\Delta+1]^{-1})}{2} \simeq \log(N)$$

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$$:\phi_N^2: = \phi_N^2 - C_N$$

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have zero mean and uniformly bounded variance (when integrated)

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Computation of the prefactor

- ▷ Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- ▷ Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2} \int_{\mathbb{T}^2} [\|\nabla \phi_N(x)\|^2 - \phi_N(x)^2] dx + \frac{1}{4} \int_{\mathbb{T}^2} \phi_N(x)^4 dx$$

- ▷ Using Nelson estimate: $\text{cap}(A, B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0 < |k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- ▷ Symmetry argument:

$$\int_{B^c} h_{A,B}(z) e^{-V_N(z)/\varepsilon} dz = \frac{1}{2} \int e^{-V_N(z)/\varepsilon} dz = \frac{1}{2} \mathcal{Z}_N(\varepsilon)$$

- ▷ $\mathcal{Z}_N(\varepsilon) \simeq 2 \prod_{|k| \leq N} \sqrt{\frac{2\pi\varepsilon}{\nu_k}} e^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N \varepsilon$
- ▷ Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

$$\prod_{0 < |k| \leq N} \frac{\lambda_k}{\lambda_k + 3} e^{3/\lambda_k} \quad \text{converges since} \quad \log \left[\frac{\lambda_k}{\lambda_k + 3} e^{3/\lambda_k} \right] = \mathcal{O}\left(\frac{1}{|k|^4}\right)$$

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Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni \phi_-$, $B \ni \phi_+$, $\exists \mu_N$ probability measures on ∂A :

$$\limsup_{N \rightarrow \infty} \mathbb{E}^{\mu_N}[\tau_B] \leq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} e^{\frac{\nu_k - \lambda_k}{|\lambda_k|}} e^{(V[\phi_{ts}] - V[\phi_-])/\varepsilon} [1 + c_+ \sqrt{\varepsilon}]}$$

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▷ Inverse of prefactor involves Carleman–Fredholm determinant:

$$\det_2(\mathbb{1} + T) = \det(\mathbb{1} + T) e^{-\text{Tr } T}$$

with $T = 3(-\Delta_{\perp} - 1)^{-1}$

$\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leq 3$)

▷ [Tsatsoulis & Weber 2018]: Same result for $\mathbb{E}^{\phi_0}[\tau_B]$

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Allen–Cahn equation in dimension 3

- ▷ Equation needs two counterterms:

$$\partial_t \phi = \Delta \phi + [1 + 3\varepsilon C_1(\delta) - 9\varepsilon^2 C_2(\delta)]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C_1(\delta) \sim \delta^{-1}$, $C_2(\delta) \sim \log(\delta^{-1})$

- ▷ Solution theories:
 - ◊ Regularity structures [Hairer 2014]
 - ◊ Paracontrolled distributions [Gubinelli–Imkeller–Perkowski 2015]
 - ◊ Wilsonian RG [Kupiainen 2016]
- ▷ **Metastability**: one expects equivalent result ($\mathcal{T}$ is still Hilbert–Schmidt)
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Some literature on the static Φ_3^4 model

- ▷ Glimm & Jaffe (1968, 1973), Feldman (1974):
Combinatorics of Feynman diagrams
- ▷ Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
Renormalisation group (integrating out scales)
- ▷ Brydges, Fröhlich & Sokal (1983):
Generating function and skeleton inequalities
- ▷ Brydges, Dimock & Hurd (1995):
Polymer expansions
- ▷ Connes & Kreimer (2000, 2001):
Hopf algebras
- ▷ ...
- ▷ Barashkov & Gubinelli (2020):
Boué–Dupuis formula

Perturbative computation of partition function

▷ Wick powers: $X = \text{diagram of two lines crossing at a point} = \int_{\Lambda} : \phi(x)^4 : dx$, $Y = \text{diagram of a line with a central dot} = \int_{\Lambda} : \phi(x)^2 : dx$

▷ Parameters: $\alpha = \frac{g}{4}$, $\beta = \frac{1}{2}g^2 C_N^{(2)}$, $\gamma = g^2 C_N^{(3)} - g^3 C_N^{(4)}$

Then $\frac{Z_{N,g}}{Z_{N,0}} = \mathbb{E}^{\nu_0} [e^{-\alpha X - \beta Y - \gamma}] = e^{-\gamma} \mathbb{E}^{\nu_0} [e^{-\alpha X - \beta Y}]$

▷ Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a multigraph, $\mathcal{G} = \text{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathcal{V}}} \prod_{e \in \mathcal{E}} G_N(x_{e_+} - x_{e_-}) dx$$

For instance

$$C_N^{(1)} = \Pi_N \text{diagram of a loop}$$

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Cumulant expansion

$$\triangleright \mathbb{E}^{\nu_0} [e^{-\alpha X - \beta Y}] = \sum_{n \geq 0} \frac{1}{n!} \mu_n$$

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$\triangleright$ Linked Cluster Theorem: κ_n projection of μ_n on **connected** graphs

Proof: for instance Peccati & Taqqu (2011)

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Commutative diagram

$$\begin{array}{ccccc}
 e^{-\alpha X} & \xrightarrow{\mathcal{P}} & \sum_p \frac{(-\alpha)^p}{p!} \mathcal{P}(X^p) & & \\
 \downarrow W & & \downarrow M^{\mathcal{G} \text{ BPHZ}} & \searrow \Pi_N^{\text{BPHZ}} & \\
 e^{-\alpha X - \beta Y} & \xrightarrow{\mathcal{P}} & \sum_{n,m} \frac{(-\alpha)^m (-\beta)^{n-m}}{m!(n-m)!} \mathcal{P}(X^m Y^{n-m}) & \xrightarrow{\Pi_N} & \log \mathbb{E}[e^{-\alpha X - \beta Y}]
 \end{array}$$

- ▷ $\mathcal{P} = \Pi_{\text{connected}}(\sum \text{pairings})$
- ▷ $e^{-\alpha X}, e^{-\alpha X - \beta Y} \in H = \text{span}\{X^n Y^m\}$
- ▷ Construction of W inspired by [Ebrahimi-Fard, Patras, Tapia & Zambotti]

Lemma: [B, Klose & Tapia]

- ▷ $W(e^{-\alpha X}) = e^{-\alpha X - \beta Y}$
- ▷ $\mathcal{P} \circ W = M^{\mathcal{G} \text{ BPHZ}} \circ \mathcal{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathcal{A}$

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 e^{-\alpha X} & \xrightarrow{\mathcal{P}} & \sum_p \frac{(-\alpha)^p}{p!} \mathcal{P}(X^p) & & \\
 \downarrow W & & \downarrow M^{\mathcal{G}^{\text{BPHZ}}} & \searrow \Pi_N^{\text{BPHZ}} & \\
 e^{-\alpha X - \beta Y} & \xrightarrow{\mathcal{P}} & \sum_{n,m} \frac{(-\alpha)^m (-\beta)^{n-m}}{m!(n-m)!} \mathcal{P}(X^m Y^{n-m}) & \xrightarrow{\Pi_N} & \log \mathbb{E}[e^{-\alpha X - \beta Y}]
 \end{array}$$

- ▷ $\mathcal{P} = \Pi_{\text{connected}}(\sum \text{pairings})$
- ▷ $e^{-\alpha X}, e^{-\alpha X - \beta Y} \in H = \text{span}\{X^n Y^m\}$
- ▷ Construction of W inspired by [Ebrahimi-Fard, Patras, Tapia & Zambotti]

Lemma: [B, Klose & Tapia]

- ▷ $W(e^{-\alpha X}) = e^{-\alpha X - \beta Y}$
- ▷ $\mathcal{P} \circ W = M^{\mathcal{G}^{\text{BPHZ}}} \circ \mathcal{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathcal{A}$

Commutative diagram

$$\begin{array}{ccccc}
 e^{-\alpha X} & \xrightarrow{\mathcal{P}} & \sum_p \frac{(-\alpha)^p}{p!} \mathcal{P}(X^p) & & \\
 \downarrow W & & \downarrow M^g \text{BPHZ} & \searrow \Pi_N^{\text{BPHZ}} & \\
 e^{-\alpha X - \beta Y} & \xrightarrow{\mathcal{P}} & \sum_{n,m} \frac{(-\alpha)^m (-\beta)^{n-m}}{m!(n-m)!} \mathcal{P}(X^m Y^{n-m}) & \xrightarrow{\Pi_N} & \log \mathbb{E}[e^{-\alpha X - \beta Y}]
 \end{array}$$

- ▷ $\mathcal{P} = \Pi_{\text{connected}}(\sum \text{pairings})$
- ▷ $e^{-\alpha X}, e^{-\alpha X - \beta Y} \in H = \text{span}\{X^n Y^m\}$
- ▷ Construction of W inspired by [Ebrahimi-Fard, Patras, Tapia & Zambotti]

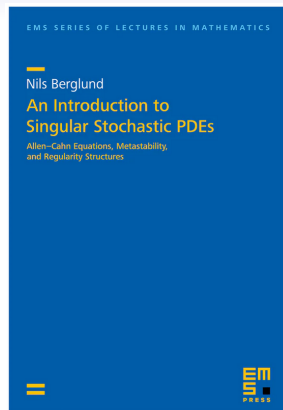
Lemma: [B, Klose & Tapia]

- ▷ $W(e^{-\alpha X}) = e^{-\alpha X - \beta Y}$
- ▷ $\mathcal{P} \circ W = M^g \text{BPHZ} \circ \mathcal{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathcal{A}$

References

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Thanks for your attention!

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