SPA 2017

Invited Session: Regularity structures

A "thermodynamic" characterization of some regularity structures near the subcriticality threshold

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Москва / Moscow, July 24 2017

with Christian Kuehn (TU Munich)

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- $-(-\Delta)^{\rho/2} =: \Delta^{\rho/2}$: Fractional Laplacian
- F polynomial of degree N
- ξ space-time white noise: $\mathbb{E}[\xi(t,x)\xi(y,s)] = \delta(x-y)\delta(t-s)$
  $\langle \xi, \varphi \rangle = W_\varphi \sim \mathcal{N}(0, \|\varphi\|^2_{L^2})$, $\mathbb{E}[W_\varphi W_{\varphi'}] = \langle \varphi, \varphi' \rangle$

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Motivations:

- Better understanding of local subcriticality
- FitzHugh–Nagumo equation [B, Kuehn, EJP 2016]

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Theorem

$d \in \{2,3\}$. $\exists$ choice of renormalisation const $C(\varepsilon)$, $\lim_{\varepsilon \to 0} C(\varepsilon) = \infty$,

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

admits sequence u^ε of local solutions, converging in probability to a limit u as $\varepsilon \to 0$.

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- $d = 2$: [Da Prato & Debussche 2004] $C(\varepsilon) = C_1 \log(\varepsilon^{-1})$
- $d = 3$: [Hairer 2014], also [Catellier & Chouk], [Kupiainen]
  $C(\varepsilon) = C_1\varepsilon^{-1} + C_2 \log(\varepsilon^{-1}) + C_3$

Mild solutions and Hölder spaces

- $(\partial_t - \Delta)u = h \quad \Rightarrow \quad u = G * h$
- $(\partial_t - \Delta)u = F(u) + \xi \quad \Rightarrow \quad u = G * \xi + G * F(u)$
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Hölder spaces $\mathcal{C}^\alpha$ for $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha - 1}$ $\not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^\alpha$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

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Parabolic scaling $\mathcal{C}_\mathfrak{s}^\alpha$: $|x - y| \longrightarrow |t - s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$
Facts:

1. $\alpha \notin \mathbb{Z}$, $f \in \mathcal{C}_\mathfrak{s}^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_\mathfrak{s}^{\alpha+2} \quad$ (Schauder)
2. $\xi \in \mathcal{C}_\mathfrak{s}^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

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Consequence: $G * \xi \in \mathcal{C}^\alpha_{\mathfrak{s}}$ a.s. $\forall \alpha < \frac{2-d}{2} \leqslant 0$ for $d \geqslant 2$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

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\begin{array}{ccc}
(u_0, Z^\varepsilon) & \xrightarrow{\ \mathcal{S}\ } & U \\
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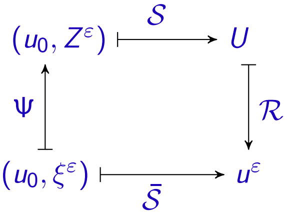

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$1-\frac{d}{2}$ $\quad 5-\frac{3d}{2}$ $\quad 4-d$ $\quad 3-\frac{d}{2}$ $\quad 0$

Locally subcritical: Hölder exponents are bdd below

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

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   Idea: Fixed point equ $U = \mathcal{I}(\Xi + F(U)) + \varphi\mathbf{1} + \dots$
   $|\Xi|_\mathfrak{s} = -\frac{\rho+d}{2} - \kappa =: \alpha_0$
   $|\mathcal{I}(\Xi)^N|_\mathfrak{s} = N(\alpha_0 + \rho) = \frac{N}{2}(\rho - d) - N\kappa$
   $|\mathcal{I}(\Xi)^N|_\mathfrak{s} > |\Xi|_\mathfrak{s} \Leftrightarrow \rho > \rho_\mathrm{c}$
   then induction on fixed point application

Model space

$U = \mathcal{I}(\Xi + F(U)) + \varphi\mathbf{1} + \dots$

- $\mathcal{U}_F$: symbols representing solution U
- $\mathcal{F}_F \supset \mathcal{U}_F$: symbols representing equation (i.e. $\Xi + F(U)$)

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given by induction $\mathcal{W}_0 = \mathcal{U}_0 = \varnothing$ and

$$\mathcal{W}_m = \mathcal{W}_{m-1} \cup \mathcal{U}_{m-1} \cup \dots \cup \mathcal{U}_{m-1}^N \cup \{\Xi\}$$
$$\mathcal{U}_m = \mathcal{I}(\mathcal{W}_m) \cup \{X^k\}$$

with $AB := \{\tau\tau' \colon \tau \in A, \tau' \in B\}$

Then $\mathcal{U}_F = \bigcup_{m \geqslant 0} \mathcal{U}_m, \quad \mathcal{F}_F = \bigcup_{m \geqslant 0} (\mathcal{W}_m \cup \mathcal{U}_m)$

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Questions: Let $\mathcal{A}_F = \{|\tau|_{\mathfrak{s}} \colon \tau \in \mathcal{F}_F\}$

1. Estimate $h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$ (number of negative Hölder exponents)
2. Estimate $c_F = \#\{\tau \in \mathcal{F}_F \colon |\tau|_{\mathfrak{s}} < 0\}$ (number of singular symbols)

Number of negative Hölder exponents

$h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$

Theorem 1

$$\frac{\rho + d}{N+1} \frac{1}{\rho - \rho_c} \leqslant h_F \leqslant 1 + \frac{(\rho + d)dN}{N+1} \frac{1}{\rho - \rho_c}$$

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$p = \#\Xi$, $q = \#\mathcal{I}$, $0 \leqslant |k|_{\mathfrak{s}} =$ polynomial exp.

- $D_0(\mathcal{U}) = \{(p(\tau), q(\tau)) \colon \tau \in \mathcal{U}\} \subset \mathbb{N}_0^2$

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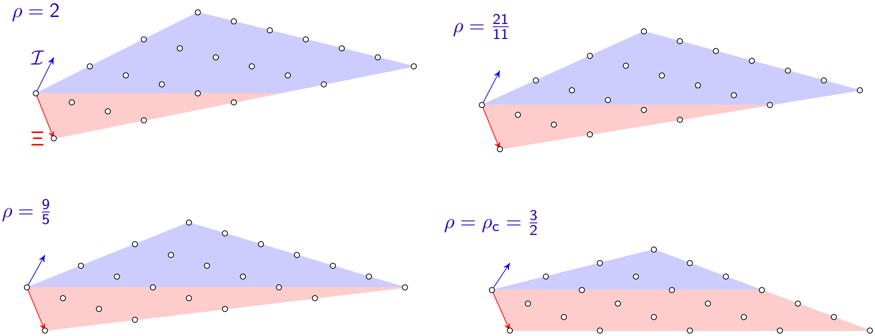
$D_0(\mathcal{W}_3)$ for $N = d = 3$



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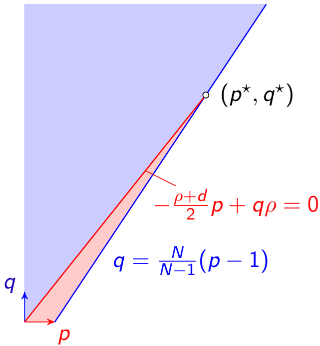
- $D_0(\mathcal{U}) = \{(p(\tau), q(\tau)) \colon \tau \in \mathcal{U}\} \subset \mathbb{N}_0^2$
- $D_0(\mathcal{U}^n) =$ convex env. of $nD_0(\mathcal{U}) \cap \mathbb{N}_0^2$
- $\lim_{m\to\infty} D_0(\mathcal{U}_m) =$ truncated cone
- $|\tau|_{\mathfrak{s}} < 0 \Rightarrow p = 1 + \left\lfloor \frac{N-1}{N} q \right\rfloor$ & $\tau \in$ triangle
- $h_F =$ number of lattice points in triangle
- $q^\star = \frac{(\rho+d)N}{(N+1)(\rho-\rho_c)} + \mathcal{O}(\kappa)$



Number of symbols

$c_F = \#\{\tau \in \mathcal{F}_F \colon |\tau|_{\mathfrak{s}} < 0\}$

Theorem 2

$$C_N^-(\rho - \rho_c)^{3/2}\, e^{\beta_N d/(\rho-\rho_c)} \leqslant c_F \leqslant C_N^+(\rho - \rho_c)^{3/2}\, e^{\beta_N d/(\rho-\rho_c)}$$

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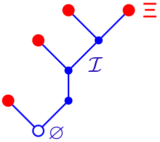
Case $N = 2$: $\tau \to$ trees of degree $\leqslant 3$, p leaves, q edges

$d_i :=$ # vertices of degree i

$$\begin{aligned} d_1 + d_2 + d_3 &= q + 1 \\ d_1 + 2d_2 + 3d_3 &= 2q \\ d_1 &= p + 1_{\{\deg \varnothing = 1\}} \end{aligned}$$



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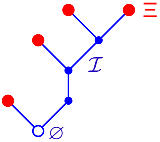
$d_i :=$ # vertices of degree i

$$d_1 + d_2 + d_3 = q + 1$$
$$d_1 + 2d_2 + 3d_3 = 2q$$
$$d_1 = p + 1_{\{\deg \varnothing = 1\}}$$



- $q = 2n \Rightarrow$ binary tree with $q + 1$ vertices
- $q = 2n + 1 \Rightarrow$ binary tree with $q + 2$ vertices minus one edge

Number of symbols

$c_F = \#\{\tau \in \mathcal{F}_F : |\tau|_\mathfrak{s} < 0\}$

Theorem 2

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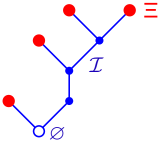

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⚠ One has to count trees up to homeomorphism

Wedderburn–Etherington numbers $W_n \simeq c\,\frac{(1/0.4072\dots)^n}{n^{3/2}}$ [Otter 1948]

Statistical properties

$\Omega = \{\tau \in \mathcal{F}_F \colon |\tau|_{\mathfrak{s}} < 0\}$, $\mathbb{P}$ uniform measure

Properties of random variables $X : \Omega \to \mathbb{R}$ when $\rho \searrow \rho_c$?

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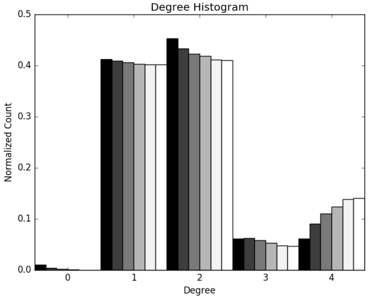
Other interesting random variables:

- Number P of Ξ: function of Q
- Hölder exponent: concentrated in 0^-
- Degree distribution: close to $(\frac{N-1}{N}, 0, \dots, 0, \frac{1}{N})$
- Height and diameter [Broutin & Flajolet]: of order $1/\sqrt{\rho - \rho_c}$



$N = d = 3, \ \rho \in \{1.8, 1.75, 1.7, 1.76, 1.6, 1.59\}$

Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

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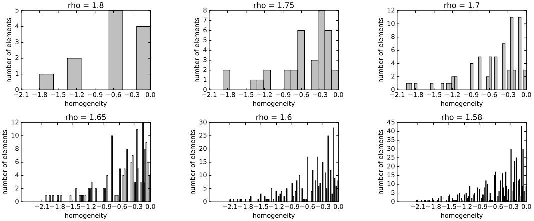

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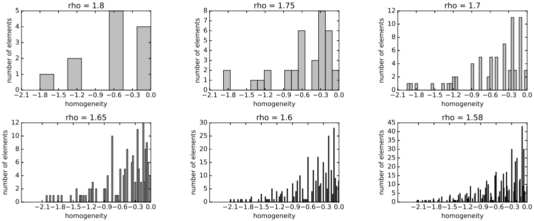

$$\Rightarrow C(\varepsilon) \sim \begin{cases} c_F \log(\varepsilon^{-1}) & \text{if } \varepsilon > \mathrm{e}^{-\gamma_N/(\rho-\rho_c)} \\ c_F \left(\dfrac{\varepsilon}{\mathrm{e}^{-\gamma_N/(\rho-\rho_c)}} \right)^{-(d-\rho)} & \text{if } \varepsilon < \mathrm{e}^{-\gamma_N/(\rho-\rho_c)} \end{cases}$$

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