

SPA 2017

Invited Session: Regularity structures

A “thermodynamic” characterization of some regularity structures near the subcriticality threshold

Nils Berglund

MAPMO, Université d'Orléans, France

Москва / Moscow, July 24 2017

with Christian Kuehn (TU Munich)



Fractional Allen–Cahn-type SPDEs

$$\partial_t u = -(-\Delta)^{\rho/2} u + F(u) + \xi$$

Fractional Allen–Cahn-type SPDEs

$$\partial_t u = -(-\Delta)^{\rho/2} u + F(u) + \xi$$

- ▷ $u = u(t, x) \in \mathbb{R}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $d \geq 1$
- ▷ $-(-\Delta)^{\rho/2} =: \Delta^{\rho/2}$: Fractional Laplacian
- ▷ F polynomial of degree N
- ▷ ξ space-time white noise: $\mathbb{E}[\xi(t, x)\xi(y, s)] = \delta(x - y)\delta(t - s)$
 $\langle \xi, \varphi \rangle = W_\varphi \sim \mathcal{N}(0, \|\varphi\|_{L^2}^2)$, $\mathbb{E}[W_\varphi W_{\varphi'}] = \langle \varphi, \varphi' \rangle$

Fractional Allen–Cahn-type SPDEs

$$\partial_t u = -(-\Delta)^{\rho/2} u + F(u) + \xi$$

- ▷ $u = u(t, x) \in \mathbb{R}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $d \geq 1$
- ▷ $-(-\Delta)^{\rho/2} =: \Delta^{\rho/2}$: Fractional Laplacian
- ▷ F polynomial of degree N
- ▷ ξ space-time white noise: $\mathbb{E}[\xi(t, x)\xi(y, s)] = \delta(x - y)\delta(t - s)$
 $\langle \xi, \varphi \rangle = W_\varphi \sim \mathcal{N}(0, \|\varphi\|_{L^2}^2)$, $\mathbb{E}[W_\varphi W_{\varphi'}] = \langle \varphi, \varphi' \rangle$

Motivations:

- ▷ Better understanding of **local subcriticality**
- ▷ **FitzHugh–Nagumo** equation [B, Kuehn, EJP 2016]

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$

$$\partial_t v = a_1 u + a_2 v$$

Fractional Allen–Cahn-type SPDEs

$$\partial_t u = -(-\Delta)^{\rho/2} u + F(u) + \xi$$

- ▷ $u = u(t, x) \in \mathbb{R}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $d \geq 1$
- ▷ $-(-\Delta)^{\rho/2} =: \Delta^{\rho/2}$: Fractional Laplacian
- ▷ F polynomial of degree N
- ▷ ξ space-time white noise: $\mathbb{E}[\xi(t, x)\xi(y, s)] = \delta(x - y)\delta(t - s)$
 $\langle \xi, \varphi \rangle = W_\varphi \sim \mathcal{N}(0, \|\varphi\|_{L^2}^2)$, $\mathbb{E}[W_\varphi W_{\varphi'}] = \langle \varphi, \varphi' \rangle$

Motivations:

- ▷ Better understanding of **local subcriticality**
- ▷ **FitzHugh–Nagumo** equation [B, Kuehn, EJP 2016]

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$

$$\partial_t v = \delta \Delta v + a_1 u + a_2 v$$

Fractional Allen–Cahn-type SPDEs

$$\partial_t u = -(-\Delta)^{\rho/2} u + F(u) + \xi$$

- ▷ $u = u(t, x) \in \mathbb{R}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $d \geq 1$
- ▷ $-(-\Delta)^{\rho/2} =: \Delta^{\rho/2}$: Fractional Laplacian
- ▷ F polynomial of degree N
- ▷ ξ space-time white noise: $\mathbb{E}[\xi(t, x)\xi(y, s)] = \delta(x - y)\delta(t - s)$
 $\langle \xi, \varphi \rangle = W_\varphi \sim \mathcal{N}(0, \|\varphi\|_{L^2}^2)$, $\mathbb{E}[W_\varphi W_{\varphi'}] = \langle \varphi, \varphi' \rangle$

Motivations:

- ▷ Better understanding of **local subcriticality**
- ▷ **FitzHugh–Nagumo** equation [B, Kuehn, EJP 2016]

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$

$$\partial_t v = \Delta^{\rho/2} v + a_1 u + a_2 v$$

Case $\rho = 2, N = 3$

$$\Phi_d^4 \text{ model: } \partial_t u = \Delta u - u^3 + \xi$$

Case $\rho = 2, N = 3$

$$\Phi_d^4 \text{ model: } \partial_t u = \Delta u - u^3 + \xi$$

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$

with $\varrho_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+2}} \varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ where ϱ compact support, integral 1

Theorem

$d \in \{2, 3\}$. \exists choice of renormalisation const $C(\varepsilon)$, $\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = \infty$,

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

admits sequence u^ε of local solutions, converging in probability to a limit u as $\varepsilon \rightarrow 0$.

Case $\rho = 2, N = 3$

$$\Phi_d^4 \text{ model: } \partial_t u = \Delta u - u^3 + \xi$$

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$

with $\varrho_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+2}} \varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ where ϱ compact support, integral 1

Theorem

$d \in \{2, 3\}$. \exists choice of renormalisation const $C(\varepsilon)$, $\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = \infty$,

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

admits sequence u^ε of local solutions, converging in probability to a limit u as $\varepsilon \rightarrow 0$.

- ▷ $d = 2$: [Da Prato & Debussche 2004] $C(\varepsilon) = C_1 \log(\varepsilon^{-1})$
- ▷ $d = 3$: [Hairer 2014], also [Catellier & Chouk], [Kupiainen]
 $C(\varepsilon) = C_1 \varepsilon^{-1} + C_2 \log(\varepsilon^{-1}) + C_3$

Mild solutions and Hölder spaces

▷ $(\partial_t - \Delta)u = h \quad \Rightarrow \quad u = G * h$

▷ $(\partial_t - \Delta)u = F(u) + \xi \quad \Rightarrow \quad u = G * \xi + G * F(u)$

Which function space?

Mild solutions and Hölder spaces

- ▷ $(\partial_t - \Delta)u = h \Rightarrow u = G * h$
- ▷ $(\partial_t - \Delta)u = F(u) + \xi \Rightarrow u = G * \xi + G * F(u)$

Which function space?

Hölder spaces \mathcal{C}^α for $f : I \rightarrow \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- ▷ $0 < \alpha < 1$: $|f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall x \neq y$
- ▷ $\alpha > 1$: $f \in \mathcal{C}^{[\alpha]}$ and $f' \in \mathcal{C}^{\alpha-1} \not\Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$
- ▷ $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leq C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

Mild solutions and Hölder spaces

$$\triangleright (\partial_t - \Delta)u = h \quad \Rightarrow \quad u = G * h$$

$$\triangleright (\partial_t - \Delta)u = F(u) + \xi \quad \Rightarrow \quad u = G * \xi + G * F(u)$$

Which function space?

Hölder spaces C^α for $f : I \rightarrow \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

$$\triangleright 0 < \alpha < 1: |f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall x \neq y$$

$$\triangleright \alpha > 1: f \in C^{[\alpha]} \text{ and } f' \in C^{\alpha-1} \not\Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$$

$$\triangleright \alpha < 0: f \text{ distribution, } |\langle f, \eta_x^\delta \rangle| \leq C\delta^\alpha \text{ with } \eta_x^\delta(y) = \frac{1}{\delta} \eta\left(\frac{x-y}{\delta}\right)$$

Parabolic scaling C_s^α : $|x - y| \rightarrow |t - s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

$$1. \alpha \notin \mathbb{Z}, f \in C_s^\alpha \quad \Rightarrow \quad G * f \in C_s^{\alpha+2} \quad (\text{Schauder})$$

$$2. \xi \in C_s^\alpha \text{ a.s. } \forall \alpha < -\frac{d+2}{2}$$

Mild solutions and Hölder spaces

$$\triangleright (\partial_t - \Delta)u = h \quad \Rightarrow \quad u = G * h$$

$$\triangleright (\partial_t - \Delta)u = F(u) + \xi \quad \Rightarrow \quad u = G * \xi + G * F(u)$$

Which function space?

Hölder spaces C^α for $f : I \rightarrow \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

$$\triangleright 0 < \alpha < 1: |f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall x \neq y$$

$$\triangleright \alpha > 1: f \in C^{[\alpha]} \text{ and } f' \in C^{\alpha-1} \not\Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$$

$$\triangleright \alpha < 0: f \text{ distribution, } |\langle f, \eta_x^\delta \rangle| \leq C\delta^\alpha \text{ with } \eta_x^\delta(y) = \frac{1}{\delta} \eta\left(\frac{x-y}{\delta}\right)$$

Parabolic scaling C_s^α : $|x - y| \rightarrow |t - s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

$$1. \alpha \notin \mathbb{Z}, f \in C_s^\alpha \quad \Rightarrow \quad G * f \in C_s^{\alpha+2} \quad (\text{Schauder})$$

$$2. \xi \in C_s^\alpha \text{ a.s. } \forall \alpha < -\frac{d+2}{2}$$

Consequence: $G * \xi \in C_s^\alpha$ a.s. $\forall \alpha < \frac{2-d}{2} \leq 0$ for $d \geq 2$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\ \uparrow \Psi & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon \end{array}$$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\ \uparrow \Psi & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon \end{array}$$

$$u = G * (\xi^\varepsilon - u^3) \quad \Rightarrow \quad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \text{polynomial terms}$$

$$U_0 = 0$$

$$U_1 = \mathcal{I}(\Xi) + \varphi \mathbf{1} + \dots \quad U_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi \mathcal{I}(\Xi)^2 + 3\varphi^2 \mathcal{I}(\Xi) + \varphi^3 \mathbf{1} + \dots$$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\ \uparrow \Psi & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon \end{array}$$

$$u = G * (\xi^\varepsilon - u^3) \quad \Rightarrow \quad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \text{polynomial terms}$$

$$U_0 = 0$$

$$U_1 = \mathcal{I}(\Xi) + \varphi \mathbf{1} + \dots \quad U_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi \mathcal{I}(\Xi)^2 + 3\varphi^2 \mathcal{I}(\Xi) + \varphi^3 \mathbf{1} + \dots$$

$$U_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi \mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2 \mathcal{I}(\mathcal{I}(\Xi)) + \varphi \mathbf{1} + \dots$$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

$$\begin{array}{ccc}
 (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\
 \uparrow \Psi & & \downarrow \mathcal{R} \\
 (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon
 \end{array}$$

$$u = G * (\xi^\varepsilon - u^3) \quad \Rightarrow \quad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \text{polynomial terms}$$

$$U_0 = 0$$

$$U_1 = \mathcal{I}(\Xi) + \varphi \mathbf{1} + \dots \quad U_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi \mathcal{I}(\Xi)^2 + 3\varphi^2 \mathcal{I}(\Xi) + \varphi^3 \mathbf{1} + \dots$$

$$U_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi \mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2 \mathcal{I}(\mathcal{I}(\Xi)) + \varphi \mathbf{1} + \dots$$

$$=: \uparrow \quad - \uparrow \uparrow \uparrow \quad - 3\varphi \uparrow \uparrow \quad - 3\varphi^2 \uparrow \quad + \varphi \mathbf{1} + \dots$$

Regularity structures

[Hairer, Inventiones Math. **198**, 269–504, 2014]:

$$\begin{array}{ccc}
 (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\
 \uparrow \Psi & & \downarrow \mathcal{R} \\
 (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon
 \end{array}$$

$$u = G * (\xi^\varepsilon - u^\varepsilon) \quad \Rightarrow \quad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \text{polynomial terms}$$

$$U_0 = 0$$

$$U_1 = \mathcal{I}(\Xi) + \varphi \mathbf{1} + \dots \quad U_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi \mathcal{I}(\Xi)^2 + 3\varphi^2 \mathcal{I}(\Xi) + \varphi^3 \mathbf{1} + \dots$$

$$U_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi \mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2 \mathcal{I}(\mathcal{I}(\Xi)) + \varphi \mathbf{1} + \dots$$

$$\begin{array}{cccccc}
 =: \uparrow & - \updownarrow & - 3\varphi \updownarrow & - 3\varphi^2 \updownarrow & + \varphi \mathbf{1} + \dots \\
 1 - \frac{d}{2} & 5 - \frac{3d}{2} & 4 - d & 3 - \frac{d}{2} & 0
 \end{array}$$

Locally subcritical: Hölder exponents are bdd below

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

$$G_{\rho}(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \xi} e^{-t \|\xi\|^{\rho}} d\xi$$

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

$$G_{\rho}(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \xi} e^{-t \|\xi\|^{\rho}} d\xi$$

Easy to check:

1. $\mathfrak{s} = (\rho, 1, \dots, 1) \Rightarrow G_{\rho}$ regularising of order ρ :

$$f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}, \alpha + \rho \notin \mathbb{Z} \Rightarrow G_{\rho} * f \in \mathcal{C}_{\mathfrak{s}}^{\alpha + \rho}$$

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

$$G_{\rho}(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \xi} e^{-t \|\xi\|^{\rho}} d\xi$$

Easy to check:

1. $\mathfrak{s} = (\rho, 1, \dots, 1) \Rightarrow G_{\rho}$ regularising of order ρ :

$$f \in C_{\mathfrak{s}}^{\alpha}, \alpha + \rho \notin \mathbb{Z} \Rightarrow G_{\rho} * f \in C_{\mathfrak{s}}^{\alpha + \rho}$$

2. $\xi \in C_{\mathfrak{s}}^{\alpha}$ a.s. $\forall \alpha < -\frac{\rho+d}{2}$

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

$$G_\rho(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \xi} e^{-t \|\xi\|^\rho} d\xi$$

Easy to check:

1. $\mathfrak{s} = (\rho, 1, \dots, 1) \Rightarrow G_\rho$ regularising of order ρ :

$$f \in C_{\mathfrak{s}}^\alpha, \alpha + \rho \notin \mathbb{Z} \Rightarrow G_\rho * f \in C_{\mathfrak{s}}^{\alpha+\rho}$$

2. $\xi \in C_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < -\frac{\rho+d}{2}$

3. $\partial_t u = \Delta^{\rho/2} u + \underbrace{F(u)}_{\text{degree } N} + \xi$ loc. subcritical $\Leftrightarrow \rho > \rho_c = d \frac{N-1}{N+1}$

Case of fractional Laplacian

$0 < \rho < 2$, $\Delta^{\rho/2} := -(-\Delta)^{\rho/2}$ generator of a Lévy process with kernel

$$G_\rho(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \xi} e^{-t \|\xi\|^\rho} d\xi$$

Easy to check:

1. $\mathfrak{s} = (\rho, 1, \dots, 1) \Rightarrow G_\rho$ regularising of order ρ :

$$f \in C_{\mathfrak{s}}^\alpha, \alpha + \rho \notin \mathbb{Z} \Rightarrow G_\rho * f \in C_{\mathfrak{s}}^{\alpha+\rho}$$

2. $\xi \in C_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < -\frac{\rho+d}{2}$

3. $\partial_t u = \underbrace{\Delta^{\rho/2} u + F(u)}_{\text{degree } N} + \xi$ loc. subcritical $\Leftrightarrow \rho > \rho_c = d \frac{N-1}{N+1}$

Idea: Fixed point equ $U = \mathcal{I}(\Xi + F(U)) + \varphi \mathbf{1} + \dots$

$$|\Xi|_{\mathfrak{s}} = -\frac{\rho+d}{2} - \kappa =: \alpha_0$$

$$|\mathcal{I}(\Xi)^N|_{\mathfrak{s}} = N(\alpha_0 + \rho) = \frac{N}{2}(\rho - d) - N\kappa$$

$$|\mathcal{I}(\Xi)^N|_{\mathfrak{s}} > |\Xi|_{\mathfrak{s}} \Leftrightarrow \rho > \rho_c$$

then induction on fixed point application

Model space

$$U = \mathcal{I}(\Xi + F(U)) + \varphi \mathbf{1} + \dots$$

- ▷ \mathcal{U}_F : symbols representing solution U
- ▷ $\mathcal{F}_F \supset \mathcal{U}_F$: symbols representing equation (i.e. $\Xi + F(U)$)

Model space

$$U = \mathcal{I}(\Xi + F(U)) + \varphi \mathbf{1} + \dots$$

- ▷ \mathcal{U}_F : symbols representing solution U
- ▷ $\mathcal{F}_F \supset \mathcal{U}_F$: symbols representing equation (i.e. $\Xi + F(U)$)

given by induction $\mathcal{W}_0 = \mathcal{U}_0 = \emptyset$ and

$$\begin{aligned}\mathcal{W}_m &= \mathcal{W}_{m-1} \cup \mathcal{U}_{m-1} \cup \dots \cup \mathcal{U}_{m-1}^N \cup \{\Xi\} \\ \mathcal{U}_m &= \mathcal{I}(\mathcal{W}_m) \cup \{X^k\}\end{aligned}$$

with $AB := \{\tau\tau' : \tau \in A, \tau' \in B\}$

$$\text{Then } \mathcal{U}_F = \bigcup_{m \geq 0} \mathcal{U}_m, \quad \mathcal{F}_F = \bigcup_{m \geq 0} (\mathcal{W}_m \cup \mathcal{U}_m)$$

Model space

$$U = \mathcal{I}(\Xi + F(U)) + \varphi \mathbf{1} + \dots$$

- ▷ \mathcal{U}_F : symbols representing solution U
- ▷ $\mathcal{F}_F \supset \mathcal{U}_F$: symbols representing equation (i.e. $\Xi + F(U)$)

given by induction $\mathcal{W}_0 = \mathcal{U}_0 = \emptyset$ and

$$\begin{aligned}\mathcal{W}_m &= \mathcal{W}_{m-1} \cup \mathcal{U}_{m-1} \cup \dots \cup \mathcal{U}_{m-1}^N \cup \{\Xi\} \\ \mathcal{U}_m &= \mathcal{I}(\mathcal{W}_m) \cup \{X^k\}\end{aligned}$$

with $AB := \{\tau\tau' : \tau \in A, \tau' \in B\}$

$$\text{Then } \mathcal{U}_F = \bigcup_{m \geq 0} \mathcal{U}_m, \quad \mathcal{F}_F = \bigcup_{m \geq 0} (\mathcal{W}_m \cup \mathcal{U}_m)$$

Questions: Let $\mathcal{A}_F = \{|\tau|_s : \tau \in \mathcal{F}_F\}$

1. Estimate $h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$ (number of negative Hölder exponents)
2. Estimate $c_F = \#\{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$ (number of singular symbols)

Number of negative Hölder exponents

$$h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$$

Theorem 1

$$\frac{\rho + d}{N + 1} \frac{1}{\rho - \rho_c} \leq h_F \leq 1 + \frac{(\rho + d)dN}{N + 1} \frac{1}{\rho - \rho_c}$$

Number of negative Hölder exponents

$$h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$$

Theorem 1

$$\frac{\rho + d}{N + 1} \frac{1}{\rho - \rho_c} \leq h_F \leq 1 + \frac{(\rho + d)dN}{N + 1} \frac{1}{\rho - \rho_c}$$

Proof:

$$|\tau|_s = -\frac{\rho+d}{2}p(\tau) + q(\tau)\rho + |k|_s(\tau) - \mathcal{O}(\kappa)$$

$$p = \#\Xi, q = \#\mathcal{I}, 0 \leq |k|_s = \text{polynomial exp.}$$

$$\triangleright D_0(\mathcal{U}) = \{(p(\tau), q(\tau)) : \tau \in \mathcal{U}\} \subset \mathbb{N}_0^2$$

Number of negative Hölder exponents

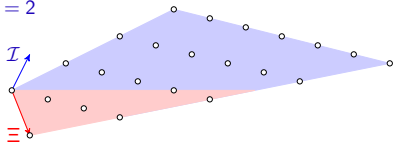
$$h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$$

Theorem 1

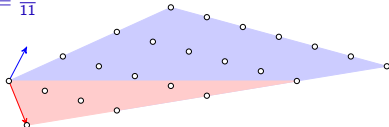
$$\frac{\rho + d}{N + 1} \frac{1}{\rho - \rho_c} \leq h_F \leq 1 + \frac{(\rho + d)dN}{N + 1} \frac{1}{\rho - \rho_c}$$

$D_0(\mathcal{W}_3)$ for $N = d = 3$

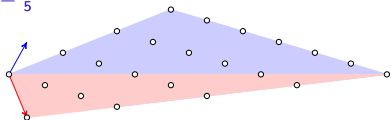
$$\rho = 2$$



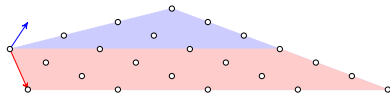
$$\rho = \frac{21}{11}$$



$$\rho = \frac{9}{5}$$



$$\rho = \rho_c = \frac{3}{2}$$



Number of negative Hölder exponents

$$h_F = \#(\mathcal{A}_F \cup \mathbb{R}_-)$$

Theorem 1

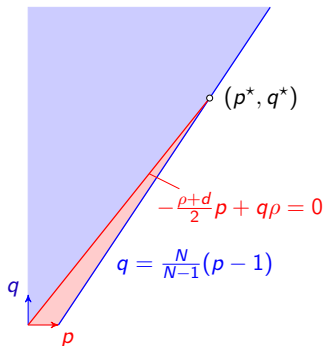
$$\frac{\rho + d}{N + 1} \frac{1}{\rho - \rho_c} \leq h_F \leq 1 + \frac{(\rho + d)dN}{N + 1} \frac{1}{\rho - \rho_c}$$

Proof:

$$|\tau|_s = -\frac{\rho+d}{2}p(\tau) + q(\tau)\rho + |k|_s(\tau) - \mathcal{O}(\kappa)$$

$$p = \#\Xi, q = \#\mathcal{I}, 0 \leq |k|_s = \text{polynomial exp.}$$

- ▷ $D_0(\mathcal{U}) = \{(p(\tau), q(\tau)) : \tau \in \mathcal{U}\} \subset \mathbb{N}_0^2$
- ▷ $D_0(\mathcal{U}^n) = \text{convex env. of } nD_0(\mathcal{U}) \cap \mathbb{N}_0^2$
- ▷ $\lim_{m \rightarrow \infty} D_0(\mathcal{U}_m) = \text{truncated cone}$
- ▷ $|\tau|_s < 0 \Rightarrow p = 1 + \lfloor \frac{N-1}{N}q \rfloor$ & $\tau \in \text{triangle}$
- ▷ $h_F = \text{number of lattice points in triangle}$
- ▷ $q^* = \frac{(\rho+d)N}{(N+1)(\rho-\rho_c)} + \mathcal{O}(\kappa)$



Number of symbols

$$c_F = \#\{\tau \in \mathcal{F}_F : |\tau|_S < 0\}$$

Theorem 2

$$C_N^-(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)} \leq c_F \leq C_N^+(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)}$$

Number of symbols

$$c_F = \#\{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$$

Theorem 2

$$C_N^-(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)} \leq c_F \leq C_N^+(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)}$$

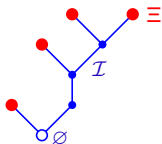
Case $N = 2$: $\tau \rightarrow$ trees of degree ≤ 3 , p leaves, q edges

$d_i := \#$ vertices of degree i

$$d_1 + d_2 + d_3 = q + 1$$

$$d_1 + 2d_2 + 3d_3 = 2q$$

$$d_1 = p + 1_{\{\text{deg } \emptyset = 1\}}$$



Number of symbols

$$C_F = \#\{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$$

Theorem 2

$$C_N^-(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)} \leq C_F \leq C_N^+(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)}$$

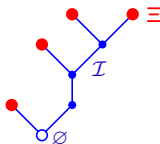
Case $N = 2$: $\tau \rightarrow$ trees of degree ≤ 3 , p leaves, q edges

$d_i := \#$ vertices of degree i

$$d_1 + d_2 + d_3 = q + 1$$

$$d_1 + 2d_2 + 3d_3 = 2q$$

$$d_1 = p + 1_{\{\text{deg } \emptyset = 1\}}$$



- ▷ $q = 2n \Rightarrow$ binary tree with $q + 1$ vertices
- ▷ $q = 2n + 1 \Rightarrow$ binary tree with $q + 2$ vertices minus one edge

Number of symbols

$$c_F = \#\{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$$

Theorem 2

$$C_N^-(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)} \leq c_F \leq C_N^+(\rho - \rho_c)^{3/2} e^{\beta_N d / (\rho - \rho_c)}$$

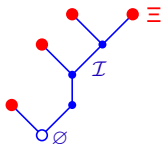
Case $N = 2$: $\tau \rightarrow$ trees of degree ≤ 3 , p leaves, q edges

$d_i := \#$ vertices of degree i

$$d_1 + d_2 + d_3 = q + 1$$

$$d_1 + 2d_2 + 3d_3 = 2q$$

$$d_1 = p + 1_{\{\text{deg } \emptyset = 1\}}$$



- ▷ $q = 2n \Rightarrow$ binary tree with $q + 1$ vertices
- ▷ $q = 2n + 1 \Rightarrow$ binary tree with $q + 2$ vertices minus one edge

⚠ One has to count trees **up to homeomorphism**

Wedderburn–Etherington numbers $W_n \simeq c \frac{(1/0.4072\dots)^n}{n^{3/2}}$ [Otter 1948]

Statistical properties

$\Omega = \{\tau \in \mathcal{F}_F : |\tau|_S < 0\}$, \mathbb{P} uniform measure

Properties of random variables $X : \Omega \rightarrow \mathbb{R}$ when $\rho \searrow \rho_c$?

Statistical properties

$\Omega = \{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$, \mathbb{P} uniform measure

Properties of random variables $X : \Omega \rightarrow \mathbb{R}$ when $\rho \searrow \rho_c$?

Case $X = Q = \#\mathcal{I}$:

- ▷ $\mathbb{E}[Q/q^*] = 1 + \mathcal{O}(\rho - \rho_c)$ et $\text{Var}[Q/q^*] = \mathcal{O}((\rho - \rho_c)^2)$
- ▷ $-\lim_{\rho \searrow \rho_c} (\rho - \rho_c) \log \mathbb{P}\{Q/q^* \leq x\} = \beta_N d(1 - x) \quad \forall x \in [0, 1]$
- ▷ $\mathbb{P}\{Q \notin \mathbb{N}\mathbb{N}\} \leq e^{-\gamma/(\rho - \rho_c)}$

Statistical properties

$\Omega = \{\tau \in \mathcal{F}_F : |\tau|_s < 0\}$, \mathbb{P} uniform measure

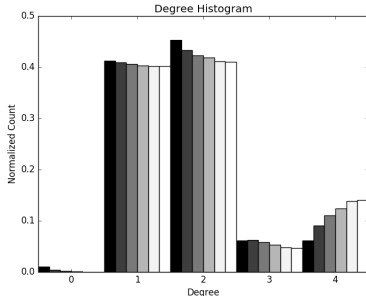
Properties of random variables $X : \Omega \rightarrow \mathbb{R}$ when $\rho \searrow \rho_c$?

Case $X = Q = \#\mathcal{I}$:

- ▷ $\mathbb{E}[Q/q^*] = 1 + \mathcal{O}(\rho - \rho_c)$ et $\text{Var}[Q/q^*] = \mathcal{O}((\rho - \rho_c)^2)$
- ▷ $-\lim_{\rho \searrow \rho_c} (\rho - \rho_c) \log \mathbb{P}\{Q/q^* \leq x\} = \beta_N d(1 - x) \quad \forall x \in [0, 1]$
- ▷ $\mathbb{P}\{Q \notin \mathbb{N}\mathbb{N}\} \leq e^{-\gamma/(\rho - \rho_c)}$

Other interesting random variables:

- ▷ Number P of Ξ : function of Q
- ▷ Hölder exponent: concentrated in 0^-
- ▷ Degree distribution: close to $(\frac{N-1}{N}, 0, \dots, 0, \frac{1}{N})$
- ▷ Height and diameter [Broutin & Flajolet]: of order $1/\sqrt{\rho - \rho_c}$



$N = d = 3$, $\rho \in \{1.8, 1.75, 1.7, 1.76, 1.6, 1.59\}$

Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

BPHZ theory [Bruned, Hairer & Zambotti 2016, Chandra & Hairer 2016]:

We expect

$$C(\varepsilon) \simeq \sum_{\tau \in \mathcal{F}_F: |\tau|_s < 0} \varepsilon^{|\tau|_s} \quad (\varepsilon^{0-} = \log(\varepsilon^{-1}))$$

Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

BPHZ theory [Bruned, Hairer & Zambotti 2016, Chandra & Hairer 2016]:

We expect

$$\begin{aligned} C(\varepsilon) &\simeq \sum_{\tau \in \mathcal{F}_F: |\tau|_s < 0} \varepsilon^{|\tau|_s} & (\varepsilon^{0-} = \log(\varepsilon^{-1})) \\ &= c_F \mathbb{E}[\varepsilon^{|\tau|_s}] \end{aligned}$$

Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

BPHZ theory [Bruned, Hairer & Zambotti 2016, Chandra & Hairer 2016]:

We expect

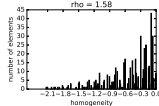
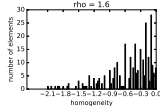
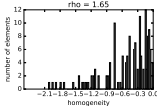
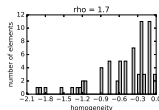
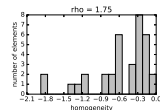
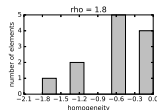
$$C(\varepsilon) \simeq \sum_{\tau \in \mathcal{F}_F: |\tau|_s < 0} \varepsilon^{|\tau|_s} \quad (\varepsilon^{0-} = \log(\varepsilon^{-1}))$$

$$= c_F \mathbb{E}[\varepsilon^{|\tau|_s}]$$

$$\mathbb{P}\{|\tau|_s = -h\} \sim e^{-\gamma_N h / (\rho - \rho_c)}$$

$$\gamma_N = \frac{N+1}{N} \beta_N$$

$$-\frac{\rho+d}{2} < -h < 0$$



Renormalisation

$$\partial_t u^\varepsilon = \Delta^{\rho/2} u^\varepsilon + C(\varepsilon) u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

BPHZ theory [Bruned, Hairer & Zambotti 2016, Chandra & Hairer 2016]:

We expect

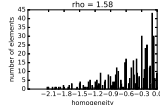
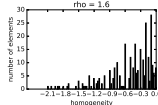
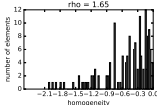
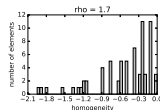
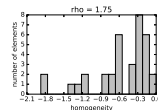
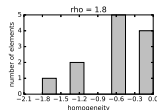
$$C(\varepsilon) \simeq \sum_{\tau \in \mathcal{F}_F: |\tau|_s < 0} \varepsilon^{|\tau|_s} \quad (\varepsilon^{0-} = \log(\varepsilon^{-1}))$$

$$= c_F \mathbb{E}[\varepsilon^{|\tau|_s}]$$

$$\mathbb{P}\{|\tau|_s = -h\} \sim e^{-\gamma_N h / (\rho - \rho_c)}$$

$$\gamma_N = \frac{N+1}{N} \beta_N$$

$$-\frac{\rho+d}{2} < -h < 0$$



$$\Rightarrow C(\varepsilon) \sim \begin{cases} c_F \log(\varepsilon^{-1}) & \text{if } \varepsilon > e^{-\gamma_N / (\rho - \rho_c)} \\ c_F \left(\frac{\varepsilon}{e^{-\gamma_N / (\rho - \rho_c)}} \right)^{-(d-\rho)} & \text{if } \varepsilon < e^{-\gamma_N / (\rho - \rho_c)} \end{cases}$$

References

- ▷ M. Hairer, *A theory of regularity structures*, Inv. Math. **198**, 269–504 (2014)
 - ▷ M. Hairer, *Introduction to Regularity Structures*, lecture notes (2013)
 - ▷ A. Chandra, H. Weber, *Stochastic PDEs, regularity structures, and interacting particle systems*, Annales Mathématiques de la Faculté des Sciences de Toulouse (in press), [arXiv/1508.03616](https://arxiv.org/abs/1508.03616)
-
- ▷ N. B., C. Kuehn, *Regularity structures and renormalisation of FitzHugh–Nagumo SPDEs in three space dimensions*, Elec J Prob 21 (18):1–48 (2016)
 - ▶ N. B., C. Kuehn, *Model spaces of regularity structures in space-fractional SPDEs*, J Statist Phys (168):331–368 (2017)
-
- ▷ Y. Bruned, M. Hairer, L. Zambotti, *Algebraic renormalisation of regularity structures*, [arXiv/1610.08468](https://arxiv.org/abs/1610.08468)
 - ▷ A. Chandra, M. Hairer, *An analytic BPHZ theorem for regularity structures*, [arXiv/1612.08138](https://arxiv.org/abs/1612.08138)
 - ▷ M. Hairer, *An analyst's take on the BPHZ theorem*, [arXiv/1704.08634](https://arxiv.org/abs/1704.08634)

Advertisement

At CIRM, Marseille–Luminy, France:

CONFERENCE

Stochastic Partial Differential Equations

Equations aux dérivées partielles stochastiques

14 - 18 May, 2018

Scientific Committee Comité scientifique

[Sandra Cerrai](#) (University of Maryland)
[Peter Friz](#) (TU Berlin & WIAS)
[Etienne Pardoux](#) (Aix-Marseille Université)

Organizing Committee Comité d'organisation

[Nils Berglund](#) (Université d'Orléans)
[Arnaud Debussche](#) (ENS Rennes)
[François Delarue](#) (Université Nice-Sophia Antipolis)
[Christian Kuehn](#) (TU Munich)

Speakers

[Dirk Blömker](#) (Augsburg)
[Zdzislaw Brzezniak](#) (York)
[Robert Dalang](#) (EPF Lausanne)
[Anne De Bouard](#) (Ecole Polytechnique)
[Franco Flandoli](#) (Pisa)
[Benjamin Gess](#) (MPI Leipzig)
[Massimiliano Gubinelli](#) (Bonn)
[Istvan Gyöngy](#) (Edinburgh)
[Martin Hairer](#) (Warwick)
[Martina Hofmanova](#) (TU Berlin)

[Antti Kupiainen](#) (Helsinki)
[Jonathan Mattingly](#) (Duke)
[Jean-Christophe Mourrat](#) (ENS Lyon)
[Felix Otto](#) (MPI Leipzig)
[Michael Röckner](#) (Bielefeld)
[Marta Sanz-Solé](#) (Barcelona)
[Wilhelm Stannat](#) (TU Berlin)
[Josef Teichmann](#) (ETH Zürich)
[Hendrik Weber](#) (Warwick)
[Maria Westdickenberg](#) (RWTH Aachen)
[Lorenzo Zambotti](#) (UPMC)