SPA 2017

Contributed Session: Interacting particle systems

Metastability for interacting particles in double-well potentials and Allen–Cahn SPDEs

Nils Berglund

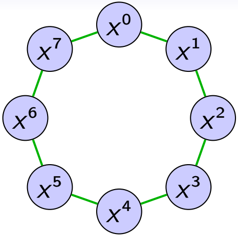
MAPMO, Université d'Orléans, France

Москва / Moscow, July 28 2017

Joint works with Giacomo Di Gesù (Vienna), Bastien Fernandez (Paris), Barbara Gentz (Bielefeld) and Hendrik Weber (Warwick)

Examples of particle systems

Example 1 [B, Fernandez, Gentz, Nonlinearity 2007]



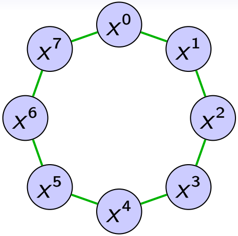
- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site

$$\mathrm{d}x_t^i = \left[x_t^i - (x_t^i)^3\right]\mathrm{d}t + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right]\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^i$$

Examples of particle systems

Example 1 [B, Fernandez, Gentz, Nonlinearity 2007]

- ▷ N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- ▷ Bistable local dynamics
- ▷ Ferromagnetic nearest neighbour coupling
- ▷ Independent noise on each site



$$dx_t^i = \left[x_t^i - (x_t^i)^3\right] dt + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right] dt + \sqrt{2\varepsilon}\, dW_t^i$$

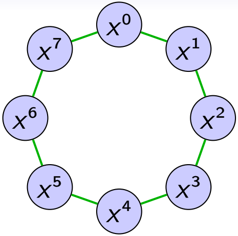
Gradient system $\qquad dx_t = -\nabla V(x_t)\, dt + \sqrt{2\varepsilon}\, dW_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4} \sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

Examples of particle systems

Example 1 [B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site



$$dx_t^i = \left[x_t^i - (x_t^i)^3\right] dt + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right] dt + \sqrt{2\varepsilon}\, dW_t^i$$

Gradient system $\qquad dx_t = -\nabla V(x_t)\, dt + \sqrt{2\varepsilon}\, dW_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

Example 2 Replace circle by 2D torus $(\mathbb{Z}/N\mathbb{Z})^2$

Example 3 [B, Dutercq, J Stat Phys 2016]: Same V + constraint $\sum_i x^i = 0$

Coarsening dynamics with noise

(Link to simulation)

General gradient systems with noise

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$V : \mathbb{R}^N \to \mathbb{R}$: confining potential, class $\mathcal{C}^2$

General gradient systems with noise

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^N \to \mathbb{R}$: confining potential, class $\mathcal{C}^2$

- ▷ Stationary points: $\mathcal{X} = \{x \colon \nabla V(x) = 0\}$
- ▷ Local minima: $\mathcal{X}_0 = \{x \in \mathcal{X} \colon \text{all ev of Hessian } \nabla^2 V(x) \text{ are } > 0\}$
- ▷ Saddles of Morse index 1: $\mathcal{X}_1 = \{x \in \mathcal{X} \colon \nabla^2 V(x) \text{ has 1 negative ev }\}$

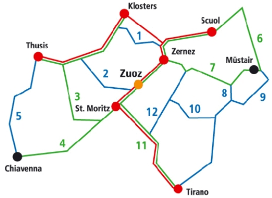
General gradient systems with noise

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

$V : \mathbb{R}^N \to \mathbb{R}$: confining potential, class $\mathcal{C}^2$

- Stationary points: $\mathcal{X} = \{x : \nabla V(x) = 0\}$
- Local minima: $\mathcal{X}_0 = \{x \in \mathcal{X} : \text{all ev of Hessian } \nabla^2 V(x) \text{ are } > 0\}$
- Saddles of Morse index 1: $\mathcal{X}_1 = \{x \in \mathcal{X} : \nabla^2 V(x) \text{ has 1 negative ev }\}$

Dynamics ~ markovian jump process on $\mathcal{G} = (\mathcal{X}_0, \mathcal{E})$, $\mathcal{E} \subset \mathcal{X}_1$
[Фрейдлин & Вентцел' / Freidlin & Wentzell 1970], [Bovier et al 2004]



Rot Rhätische Bahn
Grün ganzjährig offen
Blau Wintersperre

Nr.	Pass	Land	Passhöhe (m.ü.M.)
1	Flüela	CH	2383
2	Albula	CH	2312
3	Julier	CH	2284
4	Maloja	CH	1815
5	Splügen	I - CH	2115
6	Reschen	A - I	1507
7	Ofen	CH	2149
8	Umbrail	CH - I	2502
9	Stilfserjoch	I	2757
10	Foscagno	I	2291
11	Bernina	CH - I	2323
12	Fla. di Livigno	I	2315

Eyring–Kramers law

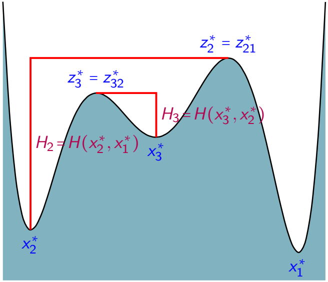
Definition: Communication height

$$H(x_i^*, x_j^*) = \inf_{\gamma: x_i^* \to x_j^*} \sup_t V(\gamma_t) - V(x_i^*)$$
$$= V(z_{ij}^*) - V(x_i^*)$$

Definition: Metastable hierarchy

$x_1^* \prec x_2^* \prec \cdots \prec x_n^* \Leftrightarrow \exists \theta > 0 \colon \ \forall k$

$H_k := H(x_k^*, \{x_1^*, \dots, x_{k-1}^*\})$
$\leqslant \min_{i<k} H(x_i^*, \{x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_k^*\}) - \theta$



Eyring–Kramers law

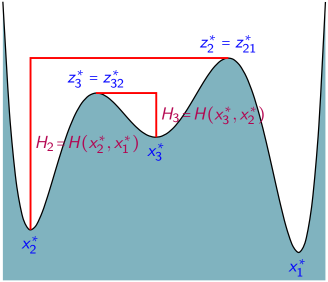
Definition: Communication height

$$H(x_i^*, x_j^*) = \inf_{\gamma: x_i^* \to x_j^*} \sup_t V(\gamma_t) - V(x_i^*)$$
$$= V(z_{ij}^*) - V(x_i^*)$$

Definition: Metastable hierarchy

$x_1^* \prec x_2^* \prec \dots \prec x_n^* \Leftrightarrow \exists \theta > 0\colon\ \forall k$

$$H_k := H(x_k^*, \{x_1^*, \dots, x_{k-1}^*\})$$
$$\leqslant \min_{i<k} H(x_i^*, \{x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_k^*\}) - \theta$$



Theorem: Eyring–Kramers law [Bovier,Eckhoff,Gayrard,Klein 2004]

τ_k = first-hitting time of nbh of $\{x_1^*, \dots, x_k^*\}$ $\qquad \lambda_k = k^{\text{th}}$ ev of generator

$$\mathbb{E}^{x_k^*}[\tau_{k-1}] = \frac{2\pi}{|\lambda_-(z_k^*)|} \sqrt{\frac{|\det \nabla^2 V(z_k^*)|}{\det \nabla^2 V(x_k^*)}}\, e^{H_k/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big] \simeq |\lambda_k|^{-1}$$

Potential landscape for Example 1

$$V(x) = \sum_{i \in \mathbb{Z}/N\mathbb{Z}} U(x^i) + \frac{\gamma}{4} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$$

$\gamma = 0$: $\mathcal{X} = \{-1, 0, 1\}^N$, $\mathcal{X}_0 = \{-1, 1\}^N$, $\mathcal{X}_1 = \{x \in \mathcal{X} :$ one $x^i = 0\}$
$\mathcal{G} =$ hypercube

Potential landscape for Example 1

$$V(x) = \sum_{i\in\mathbb{Z}/N\mathbb{Z}} U(x^i) + \frac{\gamma}{4}\sum_{i\in\mathbb{Z}/N\mathbb{Z}} (x^{i+1}-x^i)^2 \qquad U(\xi) = \tfrac14\xi^4 - \tfrac12\xi^2$$

$\gamma = 0$: $\mathcal{X} = \{-1,0,1\}^N$, $\mathcal{X}_0 = \{-1,1\}^N$, $\mathcal{X}_1 = \{x\in\mathcal{X}\colon \text{one } x^i = 0\}$
$\mathcal{G}$ = hypercube

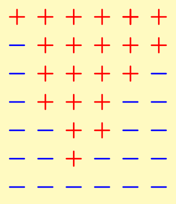
Theorem [BFG, Nonlinearity 2007]

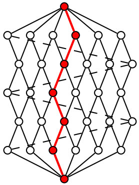
No bifurcation for $0 \leqslant \gamma \leqslant \gamma^*(N)$

where $\gamma^*(N) > \frac14 \quad \forall N \geqslant 2$

$V_\gamma(z^*_\gamma) = V_0(z^*_0) + \gamma(\#\text{ interfaces}) + \cdots$

Ising-like dynamics





Potential landscape for Example 1

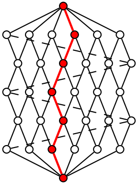
$$V(x) = \sum_{i\in\mathbb{Z}/N\mathbb{Z}} U(x^i) + \frac{\gamma}{4}\sum_{i\in\mathbb{Z}/N\mathbb{Z}} (x^{i+1} - x^i)^2 \qquad U(\xi) = \tfrac{1}{4}\xi^4 - \tfrac{1}{2}\xi^2$$

$\gamma = 0$: $\mathcal{X} = \{-1, 0, 1\}^N$, $\mathcal{X}_0 = \{-1, 1\}^N$, $\mathcal{X}_1 = \{x \in \mathcal{X} \colon \text{one } x^i = 0\}$
$\mathcal{G}$ = hypercube

Theorem [BFG, Nonlinearity 2007]

No bifurcation for $0 \leqslant \gamma \leqslant \gamma^*(N)$
where $\gamma^*(N) > \frac{1}{4} \quad \forall N \geqslant 2$

+	+	+	+	+	+
−	+	+	+	+	+
−	+	+	+	+	−
−	+	+	+	−	−
−	−	+	+	−	−
−	−	+	−	−	−
−	−	−	−	−	−



$V_\gamma(z^*_\gamma) = V_0(z^*_0) + \gamma(\#\text{ interfaces}) + \cdots$
Ising-like dynamics

Theorem [BFG, Nonlinearity 2007]

$\gamma > \dfrac{1}{2\sin^2(\pi/N)} \asymp N^2 \Leftrightarrow \mathcal{X}_0 = \{\pm(1, \dots, 1)\}$, $\mathcal{X}_1 = \{0\} \Leftrightarrow$ Synchronization

Limitations of the standard Eyring–Kramers law

▷ **Question 1:**
What happens when V is invariant under a group of symmetries?
Representation theory of finite groups → clustering of eigenvalues
[B, Dutercq, JoTP 2015]

Limitations of the standard Eyring–Kramers law

- **Question 1:**
  What happens when V is invariant under a group of symmetries?
  Representation theory of finite groups → clustering of eigenvalues
  [B, Dutercq, JoTP 2015]

- **Question 2:**
  What happens when V has saddles with zero eigenvalues?
  ($\det \nabla^2 V(z^*) = 0$ at bifurcations)
  Eyring–Kramers law with ε-dependent prefactor
  [B, Gentz, MPRF 2010]

Limitations of the standard Eyring–Kramers law

- **Question 1:**
  What happens when V is invariant under a group of symmetries?
  Representation theory of finite groups → clustering of eigenvalues
  [B, Dutercq, JoTP 2015]

- **Question 2:**
  What happens when V has saddles with zero eigenvalues?
  ($\det \nabla^2 V(z^*) = 0$ at bifurcations)
  Eyring–Kramers law with ε-dependent prefactor
  [B, Gentz, MPRF 2010]

- **Question 3:**
  What happens when $\gamma \sim N^2$ and $N \to \infty$ in Example 1?
  One expects convergence to Allen–Cahn SPDE

$$\partial_t u(t,x) = \frac{\gamma}{2N^2} \Delta u(t,x) + u(t,x) - u(t,x)^3 + \sqrt{2\varepsilon}\xi(t,x)$$

  where ξ is space-time white noise
  Is there an Eyring–Kramers law for such SPDEs?

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + f(u(x,t))$$

- $x \in [0, L]$, L: bifurcation parameter
- $u(x,t) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + f(u(x,t))$$

- $x \in [0, L]$, L: bifurcation parameter
- $u(x,t) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

Energy function:

$$V[u] = \int_0^L \Big[\frac{1}{2}u'(x)^2 - \frac{1}{2}u(x)^2 + \frac{1}{4}u(x)^4\Big]\,\mathrm{d}x \qquad \to \quad \text{min}$$

Scaling limit of particle system of Example 1 with $\gamma = 2\frac{N^2}{L^2}$

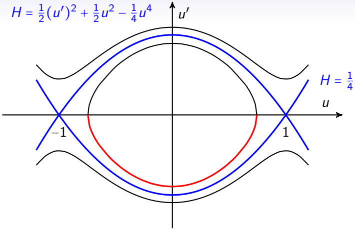
Stationary solutions: $u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t v_t(x) = v_t''(x) + [1 - 3u_0(x)^2]v_t(x)$

Stationary solutions

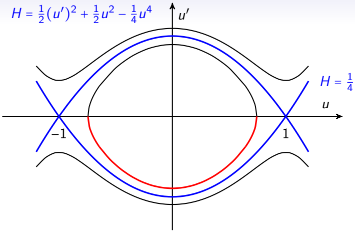
$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$

- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c. (expressible in terms of Jacobi elliptic fcts)



Stationary solutions

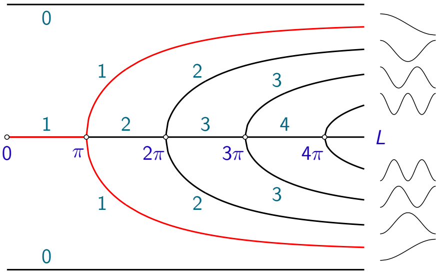
$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$



- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c. (expressible in terms of Jacobi elliptic fcts)
- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues (= unstable directions)

Transition state



- Periodic b.c: k families when $L > 2k\pi$

Eyring–Kramers law for 1D SPDEs: heuristics

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

Initial condition: $u_{\rm in}$ near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k \pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\rm ts}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k \pi}{L})^2 - 1 \\ u_1(x) \ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

Eyring–Kramers law for 1D SPDEs: heuristics

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

Initial condition: u_{in} near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\text{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ u_1(x) \ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq \mathrm{e}^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$\Rightarrow \mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}$

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, Elec J Proba 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}]-V[u_-])/\varepsilon} \Big[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, Elec J Proba 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}} \, \mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon} \Big[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ_k' instead of λ_k

- Results also for L near π and periodic b.c.

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, Elec J Proba 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\Big[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- Results also for L near π and periodic b.c.
- Prefactor involves a Fredholm determinant:

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\Big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\Big] = \det\Big[1 - 3(-\Delta_\perp + 2)^{-1}\Big]$$

converges because $(-\Delta_\perp + 2)^{-1}$ is trace class (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, Elec J Proba 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\rm ts}]-V[u_-])/\varepsilon} \Big[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- Results also for L near π and periodic b.c.
- Prefactor involves a Fredholm determinant:

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[1 - 3(-\Delta_\perp + 2)^{-1}\big]$$

converges because $(-\Delta_\perp + 2)^{-1}$ is trace class (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

- Proof uses spectral Galerkin approx. and potential-theoretic formula

$$\mathbb{E}^{\nu_{A,B}}[\tau_B] = \frac{1}{\mathrm{cap}_A(B)} \int_{B^c} \mathbb{P}^y\{\tau_A < \tau_B\}\, \mathrm{e}^{-V(y)/\varepsilon}\, \mathrm{d}y$$

The two-dimensional case

(Link to simulation)

The two-dimensional case

- ▷ Large-deviation principle: [Hairer & Weber, Ann. Fac. Sc. Toulouse, 2015]
- ▷ Naive computation of prefactor fails ($(-\Delta_\perp + 2)^{-1}$ is not trace class):

$$\log\det\left[1 - 3(-\Delta_\perp + 2)^{-1}\right] \simeq \sum_{k\in(\mathbb{N}_0^2)^*} \log\left(1 - \frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}_0^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

The two-dimensional case

- ▷ Large-deviation principle: [Hairer & Weber, Ann. Fac. Sc. Toulouse, 2015]
- ▷ Naive computation of prefactor fails ($(-\Delta_\perp + 2)^{-1}$ is not trace class):

$$\log\det\left[1 - 3(-\Delta_\perp + 2)^{-1}\right] \simeq \sum_{k \in (\mathbb{N}_0^2)^*} \log\left(1 - \frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq - \sum_{k \in (\mathbb{N}_0^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2} \int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

- ▷ In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]

Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t u = \Delta u + \left[1 + 3\varepsilon C(\delta)\right] u - u^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) = \mathrm{Tr}((-P_{\delta^{-1}}\Delta_\perp)^{-1}) \simeq \log(\delta^{-1})$ admits local solution cv as $\delta \to 0$

(Global version: [Mourrat & Weber 2015])

[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, Elec J Proba 2016]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}} \, e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}} \, e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, Elec J Proba 2016]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

- Prefactor involves Carleman–Fredholm determinant:

  $$\det_2(\mathrm{Id} + T) = \det(\mathrm{Id} + T)\mathrm{e}^{-\operatorname{Tr} T}$$

  Defined whenever T is only Hilbert–Schmidt

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, Elec J Proba 2016]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

- Prefactor involves Carleman–Fredholm determinant:

  $$\det{}_2(\mathrm{Id} + T) = \det(\mathrm{Id} + T)\mathrm{e}^{-\operatorname{Tr} T}$$

  Defined whenever T is only Hilbert–Schmidt
- Proof uses absolute continuity of $\mathrm{e}^{-V/\varepsilon}$ wrt Gaussian free field and Nelson argument for Wick powers: if $X \in n^{\text{th}}$ Wiener chaos then $\mathbb{E}[X^{2p}]^{1/2p} \leqslant C_n(2p-1)^{n/2}\mathbb{E}[X^2]^{1/2}$

Outlook

▷ Dim $d = 3$ (in progress): more difficult because 2 renormalisation constants needed, no Nelson argument ($e^{-V/\varepsilon}$ singular wrt GFF)

Outlook

- ▷ Dim $d = 3$ (in progress): more difficult because 2 renormalisation constants needed, no Nelson argument ($e^{-V/\varepsilon}$ singular wrt GFF)

References: Particle system: [1]; nonquadratic: [2,3]; SPDEs: [4,5]; symmetries: [6,7];

1. N. B., Bastien Fernandez & Barbara Gentz, *Metastability in interacting nonlinear stochastic differential equations I: From weak coupling to synchronisation* & *II: Large-N behaviour*, Nonlinearity **20**, 2551–2581; 2583–2614 (2007)
2. N. B. & Barbara Gentz, *Anomalous behavior of the Kramers rate at bifurcations in classical field theories*, J. Phys. A: Math. Theor. **42**, 052001 (2009)
3. ———, *The Eyring–Kramers law for potentials with nonquadratic saddles*, Markov Processes Relat. Fields **16**, 549–598 (2010)
4. ———, *Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond*, Electronic J. Probability **18**, (24):1–58 (2013)
5. N. B., Giacomo Di Gesù & Hendrik Weber, *An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two*, Electronic J. Probability **22**, (41):1–27 (2017)
6. N. B. & Sébastien Dutercq, *The Eyring–Kramers law for Markovian jump processes with symmetries*, J. Theoretical Probability, **29**, (4):1240–1279 (2016)
7. ———, *Interface dynamics of a metastable mass-conserving spatially extended diffusion*, J. Statist. Phys. **162**, 334–370 (2016)