

Strasbourg, Séminaire Calcul Stochastique

Structures de régularité et renormalisation d'EDPS de FitzHugh–Nagumo

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avec Christian Kuehn (TU Vienne)

Plan

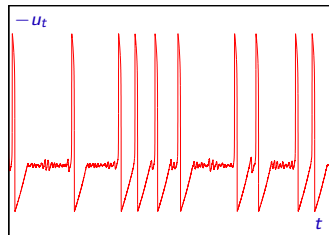
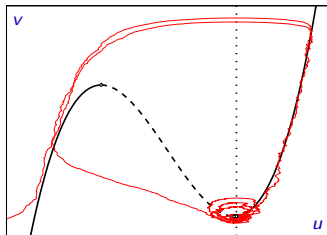
1. Motivation and main result
2. Introduction to regularity structures
3. Extension to FitzHugh–Nagumo

FitzHugh–Nagumo SDE

$$\begin{aligned} du_t &= [u_t - u_t^3 + v_t] dt + \sigma dW_t \\ dv_t &= \varepsilon[a - u_t - bv_t] dt \end{aligned}$$

- ▷ u_t : membrane potential of neuron
- ▷ v_t : gating variable (proportion of open ion channels)

$$\begin{aligned} \varepsilon &= 0.1 \\ b &= 0 \\ a &= \frac{1}{\sqrt{3}} + 0.02 \\ \sigma &= 0.03 \end{aligned}$$



FitzHugh–Nagumo SPDE

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$

$$\partial_t v = a_1 u + a_2 v$$

- ▷ $u = u(t, x) \in \mathbb{R}$, $v = v(t, x) \in \mathbb{R}$ (or \mathbb{R}^n), $(t, x) \in D = \mathbb{R}_+ \times \mathbb{T}^d$, $d = 2, 3$
- ▷ $\xi(t, x)$ Gaussian space-time white noise: $\mathbb{E}[\xi(t, x)\xi(s, y)] = \delta(t-s)\delta(x-y)$
 ξ : distribution defined by $\langle \xi, \varphi \rangle = W_\varphi$, $\{W_h\}_{h \in L^2(D)}$, $\mathbb{E}[W_h W_{h'}] = \langle h, h' \rangle$

([Link to simulation](#))

Main result

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$

where $\varrho_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+2}} \varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ with ϱ compactly supported, integral 1

Theorem [NB & C. Kuehn, preprint 2015, arXiv/1504.02953]

There exists a choice of renormalisation constant $C(\varepsilon)$, $\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = \infty$, such that

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + [1 + C(\varepsilon)]u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon + \xi^\varepsilon$$

$$\partial_t v^\varepsilon = a_1 u^\varepsilon + a_2 v^\varepsilon$$

admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \rightarrow 0$.

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admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \rightarrow 0$.

- ▶ Local solution means up to a random possible explosion time
- ▶ Initial conditions should be in appropriate Hölder spaces
- ▶ $C(\varepsilon) \asymp \log(\varepsilon^{-1})$ for $d = 2$ and $C(\varepsilon) \asymp \varepsilon^{-1}$ for $d = 3$
- ▶ Similar results for more general cubic nonlinearity and $v \in \mathbb{R}^n$

Mild solutions of SPDE

$$\begin{aligned}\partial_t u &= \Delta u + F(u) + \xi \\ u(0, x) &= u_0(x)\end{aligned}$$

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$$\triangleright \partial_t u = \Delta u \quad \Rightarrow \quad u(t, x) = \int G(t, x - y) u_0(y) dy =: (e^{\Delta t} u_0)(x)$$

where $G(t, x)$: heat kernel (compatible with bc)

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$$\triangleright \partial_t u = \Delta u + f \quad \Rightarrow \quad u(t, x) = (e^{\Delta t} u_0)(x) + \int_0^t e^{\Delta(t-s)} f(s, \cdot)(x) ds$$

Notation: $u = Gu_0 + G * f$

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$$\triangleright \partial_t u = \Delta u + \xi + F(u) \quad \Rightarrow \quad u = Gu_0 + G * [\xi + F(u)]$$

Aim: use **Banach's fixed-point theorem** — but which function space?

Hölder spaces

Definition of \mathcal{C}^α for $f : I \rightarrow \mathbb{R}$, with $I \subset \mathbb{R}$ a compact interval:

▷ $0 < \alpha < 1$: $|f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall x \neq y$

▷ $\alpha > 1$: $f \in \mathcal{C}^{[\alpha]}$ and $f' \in \mathcal{C}^{\alpha-1}$

▷ $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leq C\delta^\alpha$

where $\eta_x^\delta(y) = \frac{1}{\delta}\eta\left(\frac{x-y}{\delta}\right)$ for all test functions $\eta \in \mathcal{C}^{-[\alpha]}$

Property: $f \in \mathcal{C}^\alpha$, $0 < \alpha < 1 \Rightarrow f' \in \mathcal{C}^{\alpha-1}$ where $\langle f', \eta \rangle = -\langle f, \eta' \rangle$

Remark: $f \in \mathcal{C}^{1+\alpha} \not\Rightarrow |f(x) - f(y)| \leq C|x - y|^{1+\alpha}$. See e.g $f(x) = x + |x|^{3/2}$

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Case of the heat kernel: $(\partial_t - \Delta)u = f \Rightarrow u = G * f$

Parabolic scaling \mathcal{C}_5^α : $|x - y| \longrightarrow |t - s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

$$\frac{1}{\delta} \eta\left(\frac{x-y}{\delta}\right) \longrightarrow \frac{1}{\delta^{d+2}} \eta\left(\frac{t-s}{\delta^2}, \frac{x-y}{\delta}\right)$$

Schauder estimates and fixed-point equation

Schauder estimate

$$\alpha \notin \mathbb{Z}, f \in \mathcal{C}_s^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_s^{\alpha+2}$$

Fact: in dimension d , space-time white noise $\xi \in \mathcal{C}_s^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

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Fixed-point equation: $u = Gu_0 + G * [\xi + F(u)]$

- ▷ $d = 1$: $\xi \in C_s^{-3/2^-} \Rightarrow G * \xi \in C_s^{1/2^-} \Rightarrow F(u)$ defined
- ▷ $d = 3$: $\xi \in C_s^{-5/2^-} \Rightarrow G * \xi \in C_s^{-1/2^-} \Rightarrow F(u)$ not defined
- ▷ $d = 2$: $\xi \in C_s^{-2^-} \Rightarrow G * \xi \in C_s^{0^-} \Rightarrow F(u)$ not defined

Boundary case, can be treated with Besov spaces
[Da Prato & Debussche 2003]

Why not use mollified noise? Limit $\varepsilon \rightarrow 0$ does not exist

Regularity structures

Basic idea of Martin Hairer [Inventiones Math. **198**, 269–504, 2014]:

Lift mollified fixed-point equation

$$u = Gu_0 + G * [\xi^\varepsilon + F(u)]$$

to a larger space called a **Regularity structure**

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xrightarrow{\mathcal{S}} & U \\ \uparrow \Psi & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xrightarrow{\bar{\mathcal{S}}} & u^\varepsilon \end{array}$$

- ▷ $u^\varepsilon = \bar{\mathcal{S}}(u_0, \xi^\varepsilon)$: classical solution of mollified equation
- ▷ $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- ▷ \mathcal{S} and \mathcal{R} are continuous (in suitable topology)

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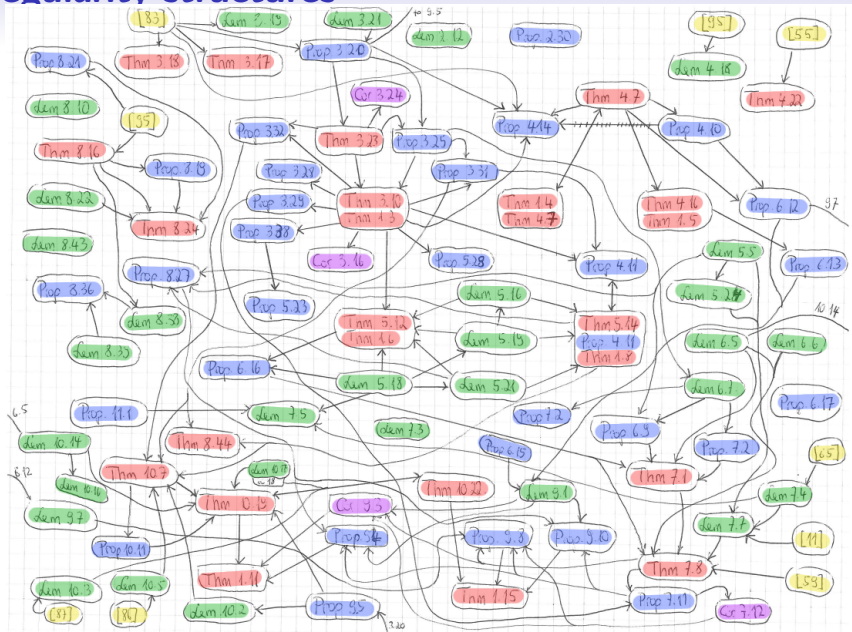
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- ▷ $U = S(u_0, Z^\varepsilon)$: solution map in regularity structure
- ▷ S and \mathcal{R} are continuous (in suitable topology)
- ▷ Renormalisation: modification of the lift Ψ

Alternative approaches for $d = 3$: [Catellier & Chouk '13], [Kupiainen '15]

Regularity structures



Structure of Hairer, Invent. Math. 198:269-504 (2014)

Drawing by Christian Kuehn

Basic idea: Generalised Taylor series

$f : I \rightarrow \mathbb{R}$, $0 < \alpha < 1$

$f \in \mathcal{C}^{2+\alpha} \iff f \in \mathcal{C}^2$ and $f'' \in \mathcal{C}^\alpha$

Associate with f the triple (f, f', f'')

When does a triple (f_0, f_1, f_2) represent a function $f \in \mathcal{C}^{2+\alpha}$?

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$$|f_1(y) - f_1(x) - (y - x)f_2(x)| \leq C|x - y|^{1+\alpha}$$

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Notation: $f = f_0 \mathbf{1} + f_1 X + f_2 X^2$

Regularity structure: Generalised Taylor basis whose basis elements can also be singular distributions

Definition of a regularity structure

Definition [M. Hairer, Inventiones Math 2014]

A **Regularity structure** is a triple (A, T, \mathcal{G}) where

1. **Index set:** $A \subset \mathbb{R}$, bdd below, locally finite, $0 \in A$
2. **Model space:** $T = \bigoplus_{\alpha \in A} T_{\alpha}$, each T_{α} Banach space, $T_0 = \text{span}(\mathbf{1}) \simeq \mathbb{R}$
3. **Structure group:** \mathcal{G} group of linear maps $\Gamma : T \rightarrow T$ such that

$$\Gamma \tau - \tau \in \bigoplus_{\beta < \alpha} T_{\beta} \quad \forall \tau \in T_{\alpha}$$

and $\Gamma \mathbf{1} = \mathbf{1} \quad \forall \Gamma \in \mathcal{G}$.

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Polynomial regularity structure on \mathbb{R} :

- ▷ $A = \mathbb{N}_0$
- ▷ $T_k \simeq \mathbb{R}$, $T_k = \text{span}(X^k)$
- ▷ $\Gamma_h(X^k) = (X - h)^k \forall h \in \mathbb{R}$

Polynomial reg. structure on \mathbb{R}^d : $X^k = X_1^{k_1} \dots X_d^{k_d} \in T_{|k|}$, $|k| = \sum_{i=1}^d k_i$

Regularity structure for $\partial_t u = \Delta u - u^3 + \xi$

New symbols: Ξ , representing ξ , Hölder exponent $|\Xi|_s = \alpha_0 = -\frac{d+2}{2} - \kappa$
 $\mathcal{I}(\tau)$, representing $G * f$, Hölder exponent $|\mathcal{I}(\tau)|_s = |\tau|_s + 2$
 $\tau\sigma$, Hölder exponent $|\tau\sigma|_s = |\tau|_s + |\sigma|_s$

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τ	Symbol	$ \tau _s$	$d = 3$	$d = 2$
Ξ		α_0	$-\frac{5}{2} - \kappa$	$-2 - \kappa$
$\mathcal{I}(\Xi)^3$		$3\alpha_0 + 6$	$-\frac{3}{2} - 3\kappa$	$0 - 3\kappa$
$\mathcal{I}(\Xi)^2$		$2\alpha_0 + 4$	$-1 - 2\kappa$	$0 - 2\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2$		$5\alpha_0 + 12$	$-\frac{1}{2} - 5\kappa$	$2 - 5\kappa$
$\mathcal{I}(\Xi)$		$\alpha_0 + 2$	$-\frac{1}{2} - \kappa$	$0 - \kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\Xi)^2 X_i$		$2\alpha_0 + 5$	$0 - 2\kappa$	$1 - 2\kappa$
1	1	0	0	0
$\mathcal{I}(\mathcal{I}(\Xi)^3)$		$3\alpha_0 + 8$	$\frac{1}{2} - 3\kappa$	$2 - 3\kappa$
...

Fixed-point equation for $\partial_t u = \Delta u - u^3 + \xi$

$$u = G * [\xi^\varepsilon - u^3] + Gu_0 \quad \Rightarrow \quad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \dots$$

$$U_0 = 0$$

$$U_1 = \mathfrak{I} + \varphi \mathbf{1}$$

$$U_2 = \mathfrak{I} + \varphi \mathbf{1} - \mathfrak{Y} - 3\varphi \mathfrak{Y} + \dots$$

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To prove convergence, we need

- ▶ A **model** (Π, Γ) : $\forall z \in \mathbb{R}^{d+1}$, $\Pi_z \tau$ is distribution describing τ near z
 $\Gamma_{z\bar{z}} \in \mathcal{G}$ describes translations: $\Pi_{\bar{z}} = \Pi_z \Gamma_{z\bar{z}}$

- ▶ Spaces of **modelled distributions**

$$\mathcal{D}^\gamma = \left\{ f : \mathbb{R}^{d+1} \rightarrow \bigoplus_{\beta < \gamma} T_\beta : \|f(z) - \Gamma_{z\bar{z}} f(\bar{z})\|_\beta \lesssim \|z - \bar{z}\|_s^{\gamma - \beta} \right\}$$

equipped with a seminorm

- ▶ The **Reconstruction theorem**: provides a unique map $\mathcal{R} : \mathcal{D}^\gamma \rightarrow \mathcal{C}_s^{\alpha_*}$
 $(\alpha_* = \inf A)$ s.t. $|\langle \mathcal{R}f - \Pi_z f(z), \eta_{s,z}^\delta \rangle| \lesssim \delta^\gamma$
 (constructed using **wavelets**)

Canonical model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$

Defined inductively by

$$(\Pi_z^\varepsilon \Xi)(\bar{z}) = \xi^\varepsilon(\bar{z})$$

$$(\Pi_z^\varepsilon X^k)(\bar{z}) = (\bar{z} - z)^k$$

$$(\Pi_z^\varepsilon \tau \sigma)(\bar{z}) = (\Pi_z^\varepsilon \tau)(\bar{z})(\Pi_z^\varepsilon \sigma)(\bar{z})$$

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$$(\Pi_z^\varepsilon \mathcal{I}(\tau))(\bar{z}) = \int G(\bar{z} - z') (\Pi_z^\varepsilon \tau)(z') dz'$$

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Then $\exists \mathcal{K}$ s.t. $\mathcal{R}\mathcal{K}f = G * \mathcal{R}f$ and the following diagrams commute:

$$\begin{array}{ccc} \mathcal{D}^\gamma & \xrightarrow{\mathcal{K}} & \mathcal{D}^{\gamma+2} \\ \mathcal{R} \downarrow & & \downarrow \mathcal{R} \\ \mathcal{C}_5^{\alpha_*} & \xrightarrow{G_*} & \mathcal{C}_5^{\alpha_*+2} \end{array}$$

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \xrightarrow{S} & U \\ \mathcal{R} \downarrow & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \xrightarrow{\bar{S}} & u^\varepsilon \end{array}$$

where $\alpha_* = \inf A$ and $\mathcal{K}f = \mathcal{I}f + \text{polynomial term} + \text{nonlocal term}$

Why do we need to renormalise?

Let $G_\varepsilon = G * \varrho_\varepsilon$ where ϱ_ε is the mollifier

$$(\Pi_{\frac{\varepsilon}{2}}^\varepsilon \uparrow)(z) = (G * \xi^\varepsilon)(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1) \xi(z_1) dz_1$$

belongs to first Wiener chaos, limit $\varepsilon \rightarrow 0$ well-defined

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$$(\Pi_{\frac{\varepsilon}{2}}^\varepsilon \downarrow \downarrow)(z) = (G * \xi^\varepsilon)(z)^2 = \iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2) \xi(z_1) \xi(z_2) dz_1 dz_2$$

diverges as $\varepsilon \rightarrow 0$

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diverges as $\varepsilon \rightarrow 0$

Wick product: $\xi(z_1) \diamond \xi(z_2) = \xi(z_1) \xi(z_2) - \delta(z_1 - z_2)$

$$(\Pi_{\frac{\varepsilon}{2}}^\varepsilon \heartsuit)(z) = \underbrace{\iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2) \xi(z_1) \diamond \xi(z_2) dz_1 dz_2}_{\text{in 2nd Wiener chaos, bdd}} + \underbrace{\int G_\varepsilon(z - z_1)^2 dz_1}_{C_1(\varepsilon) \rightarrow \infty}$$

Renormalised model: $(\widehat{\Pi}_{\frac{\varepsilon}{2}}^\varepsilon \heartsuit)(z) = (\Pi_{\frac{\varepsilon}{2}}^\varepsilon \heartsuit)(z) - C_1(\varepsilon)$

The case of the FitzHugh–Nagumo equations

Fixed-point equation

$$u(t, x) = G * [\xi^\varepsilon + u - u^3 + v](t, x) + Gu_0(t, x)$$
$$v(t, x) = \int_0^t u(s, x) e^{(t-s)a_2} a_1 ds + e^{ta_2} v_0$$

Lifted version

$$U = \mathcal{I}[\Xi + U - U^3 + V] + Gu_0$$
$$V = \mathcal{E}U + Qv_0$$

where \mathcal{E} is an integration map which is not regularising in space

New symbols $\mathcal{E}(\mathcal{I}(\Xi)) = \mathfrak{I}$, etc. . .

We expect U , and thus also V to be α -Hölder for $\alpha < -\frac{1}{2}$

Thus $\mathcal{I}(U - U^3 + V)$ should be well-defined

The standard theory has to be extended, because \mathcal{E} does not correspond to a smooth kernel

Details on implementing \mathcal{E}

Problems:

- ▷ Fixed-point equation requires **diagonal identity** $(\Pi_{t,x}\tau)(t, x) = 0$
- ▷ Usual definition of \mathcal{K} would contain Taylor series

$$\mathcal{J}(z)\tau = \sum_{|k|_s < \alpha} \frac{X^k}{k!} \int D^k G(z - \bar{z})(\Pi_z \tau)(d\bar{z})$$

$$\mathcal{N}f(z) = \sum_{|k|_s < \gamma} \frac{X^k}{k!} \int D^k G(z - \bar{z})(\mathcal{R}f - \Pi_z f(z))(d\bar{z})$$

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Solution:

- ▷ Define $\Pi\mathcal{E}\tau$ only if $-2 < |\tau|_s < 0$ (otherwise $\mathcal{E}\tau = 0$) $\Rightarrow \mathcal{J}(z)\tau = 0$
- ▷ Define \mathcal{K} only for $f = \sum_{|\tau|_s < 0} c_\tau \tau + \sum_{|\tau|_s \geq 0} c_\tau(t, x)\tau =: f_- + f_+$
 \Rightarrow can take $\mathcal{R}f = \Pi_{t,x} f(t, x)$ and thus $\mathcal{N}f = 0$ for these f
- ▷ Time-convolution with Q lifted to

$$(\mathcal{K}^Q f)(t, x) = \sum_{|\tau|_s < 0} c_\tau \mathcal{E}\tau + \sum_{|\tau|_s \geq 0} \int Q(t-s) c_\tau(s, x) ds \tau =: (\mathcal{E}f_- + Qf_+)(t, x)$$

Fixed-point equation

Consider $\partial_t u = \Delta_u + F(u, v) + \xi$ with F a polynomial of degree 3

If (U, V) satisfies fixed-point equation

$$U = \mathcal{I}[\Xi + F(U, V)] + Gu_0 + \text{polynomial term}$$

$$V = \mathcal{E}U_- + \mathcal{Q}U_+ + Qv_0$$

then $(\mathcal{R}U, \mathcal{R}V)$ is solution, provided $\mathcal{R}F(U, V) = F(\mathcal{R}U, \mathcal{R}V)$

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Fixed point is of the form

$$U = \mathfrak{I} + \varphi \mathbf{1} + [a_1 \mathfrak{Y} + a_2 \mathfrak{Y} + a_3 \mathfrak{Y} + a_4 \mathfrak{Y}] + [b_1 \mathfrak{Y} + b_2 \mathfrak{Y} + b_3 \mathfrak{Y}] + \dots$$

$$V = \mathfrak{I} + \psi \mathbf{1} + [\hat{a}_1 \mathfrak{Y} + \hat{a}_2 \mathfrak{Y} + \hat{a}_3 \mathfrak{Y} + \hat{a}_4 \mathfrak{Y}] + [\hat{b}_1 \mathfrak{Y} + \hat{b}_2 \mathfrak{Y} + \hat{b}_3 \mathfrak{Y}] + \dots$$

► Conclusion

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- ▶ Prove existence of fixed point in (modification of) \mathcal{D}^γ with $\gamma = 1 + \bar{\kappa}$
- ▶ Extend from small interval $[0, T]$ up to first exit from large ball
- ▶ Deal with renormalisation procedure

▶ Conclusion

Renormalisation

▷ **Renormalisation group**: group of linear maps $M : T \rightarrow T$

Associated model: Π_z^M s.t. $\Pi_z^M \tau = \Pi M \tau$ where $\Pi_z = \Pi \Gamma_{f_z}$

Allen–Cahn eq.: $M = e^{-C_1 L_1 - C_2 L_2}$ with $L_1 : \vee \rightarrow \mathbf{1}$, $L_2 : \text{trivalent tree} \rightarrow \mathbf{1}$

FHN eq.: the same group suffices because Q is smoothing

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- ▷ Look for r.v. $\hat{\Pi}_z \tau$ s.t. if $\hat{\Pi}_z^{(\varepsilon)} = (\Pi_z^{(\varepsilon)})^{M_\varepsilon}$ then $\exists \kappa, \theta > 0$ s.t.

$$\mathbb{E} |\langle \hat{\Pi}_z \tau, \eta_z^\lambda \rangle|^2 \lesssim \lambda^{2|\tau|_s + \kappa} \quad \mathbb{E} |\langle \hat{\Pi}_z \tau - \hat{\Pi}_z^{(\varepsilon)} \tau, \eta_z^\lambda \rangle|^2 \lesssim \varepsilon^{2\theta} \lambda^{2|\tau|_s + \kappa}$$

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Then $(\hat{\Pi}_z^{(\varepsilon)}, \hat{\Gamma}_z^{(\varepsilon)})$ converges to limiting model, with explicit L^p bounds

- ▷ Renormalised equations have nonlinearity \hat{F} s.t.

$\hat{F}(MU, MV) = MF(U, V) + \text{terms of Hölder exponent } > 0$

FHN eq. with cubic nonlinearity

$$F = \alpha_1 u + \alpha_2 v + \beta_1 u^2 + \beta_2 uv + \beta_3 v^2 + \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uv^2 + \gamma_4 v^3$$

$$\hat{F}(u, v) = F(u, v) - c_0(\varepsilon) - c_1(\varepsilon)u - c_2(\varepsilon)v$$

with the $c_i(\varepsilon)$ depending on C_1, C_2 , **provided** either $d = 2$ or $\gamma_2 = 0$

Concluding remarks

- ▷ Models with $\partial_t u$ of order $u^4 + v^4$ and $\partial_t v$ of order $u^2 + v$ should be renormalisable
Current approach does not work when singular part (t, x) -dependent
- ▷ Global existence: recent progress by J.-C. Mourrat and H. Weber on 2D Allen–Cahn
- ▷ More quantitative results?

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