Symposium of the Collaborative Research Center 910

Theory and applications of random Poincaré maps

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Joint works with Manon Baudel (Ecole des Ponts, Paris), Barbara Gentz (Bielefeld), Christian Kuehn (TU Munich) and Damien Landon

   

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   [Baudel & B, SIAM J. Math. Anal. 2017]

1. Deterministic Poincaré maps

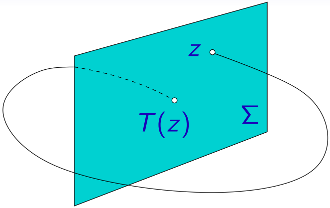
ODE $\dot{z} = f(z) \quad z \in \mathbb{R}^n$

Flow: $z_t = \varphi_t(z_0)$

$\Sigma \subset \mathbb{R}^n$: $(n-1)$-dimensional manifold

Poincaré map (or first-return map):

$T : \Sigma \to \Sigma$



$T(z) = \varphi_\tau(z)$ where $\tau = \inf\{t > 0 \colon \varphi_t(z) \in \Sigma\}$

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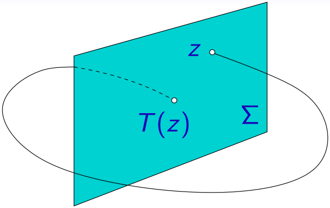

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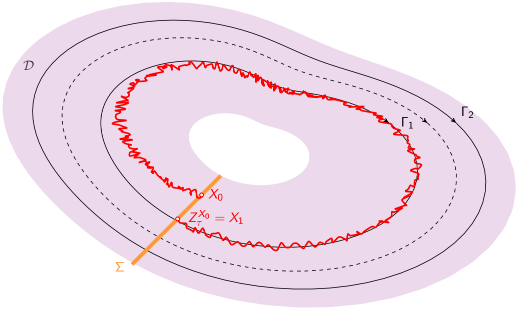
Benefits:

1. Dimension reduction: T is an $(n-1)$-dimensional map
2. Stability of periodic orbits: no neutral direction
3. Bifurcations of periodic orbits easier to study (period doubling, Hopf, ...)

Question: how about SDEs $\quad \mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$?

Random Poincaré maps

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \quad \Rightarrow \quad$ Sample path $(Z_t^{z_0}(\omega))_{t\geqslant 0}$



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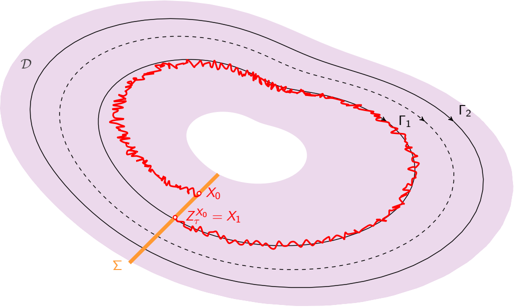

⚠ $z_0 = X_0 \in \Sigma \quad \Rightarrow \quad \inf\{t > 0 \colon Z_t^{X_0} \in \Sigma\} = 0$

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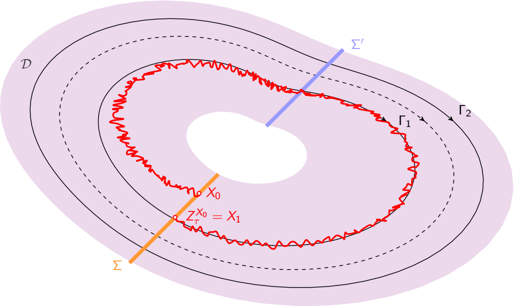


⚠ $z_0 = X_0 \in \Sigma \quad \Rightarrow \quad \inf\{t > 0 \colon Z_t^{X_0} \in \Sigma\} = 0$

Solution: $\tau_0 = 0$, $\tau'_{n+1} = \inf\{t > \tau_n \colon Z_t^{X_0} \in \Sigma'\}$

$\tau_{n+1} = \inf\{t > \tau'_{n+1} \colon Z_t^{X_0} \in \Sigma\}$

$X_n = Z_{\tau_n}^{X_0} \in \Sigma \quad \Rightarrow \quad (X_n)_{n\geqslant 0}$ is a Markov chain $\quad K(x,A) := \mathbb{P}^x\{X_1 \in A\}$

$(X_n,\omega) \mapsto X_{n+1}$: random Poincaré map

[J. Weiss, E. Knobloch, 1990], [P. Hitczenko, G. Medvedev, 2009]

Fredholm theory

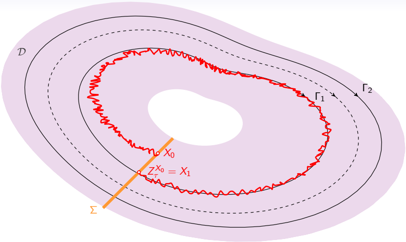
$X_n = Z_{\tau_n}^{X_0} \in \Sigma$

$(X_n)_{n \geqslant 0}$ Markov chain, of kernel

$K(x, A) = \mathbb{P}\{X_{n+1} \in A | X_n = x\}$

Under appropriate conditions

$$K(x, A) = \int_A k(x, y) \, \mathrm{d}y$$



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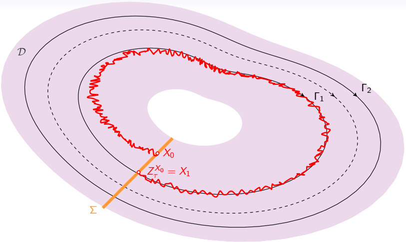

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Markov semigroups:

$$(K\varphi)(x) = \int_\Sigma k(x, y)\varphi(y) \, \mathrm{d}y = \mathbb{E}^x[\varphi(X_1)] \qquad \varphi \in L^\infty$$

$$(\mu K)(y) = \int_\Sigma \mu(x) k(x, y) \, \mathrm{d}x = \mathbb{P}^\mu\{X_1 \in \mathrm{d}y\} \qquad \mu \in L^1$$

K is compact operator $\Rightarrow$ Fredholm theory

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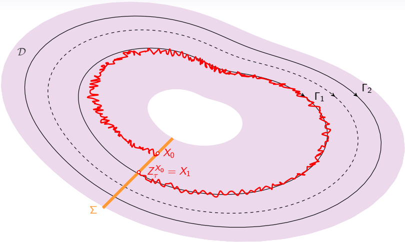

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K is compact operator $\Rightarrow$ Fredholm theory

Spectral decomposition: $\quad K\phi_i = \lambda_i \phi_i \qquad \pi_i K = \lambda_i \pi_i \qquad \langle \pi_i, \phi_j \rangle = \delta_{ij}$

$$k(x, y) = \lambda_0 \phi_0(x) \pi_0(y) + \lambda_1 \phi_1(x) \pi_1(y) + \dots$$

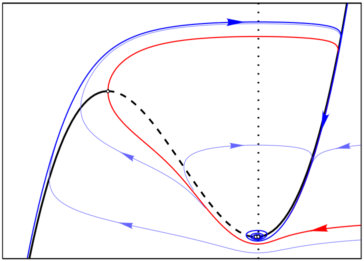
2. Example 1: Stochastic FitzHugh–Nagumo eq.

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t$$ neuron membrane potential

$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t$$ open ion channels

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

$\varepsilon = 0.1$
$\delta = 0.02$



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$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$ neuron membrane potential

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- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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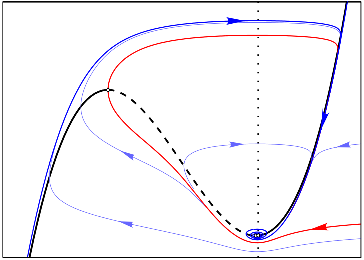

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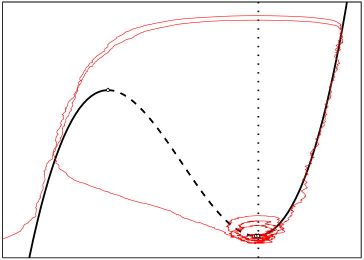
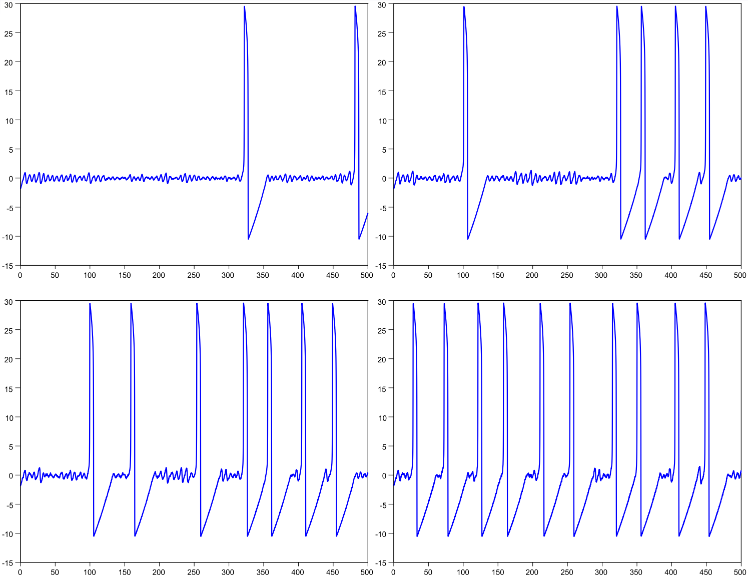
$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

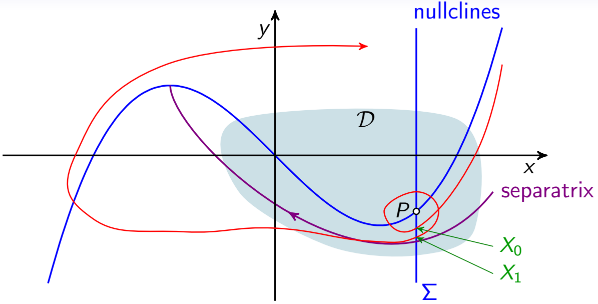


Mixed-mode oscillations (MMOs)



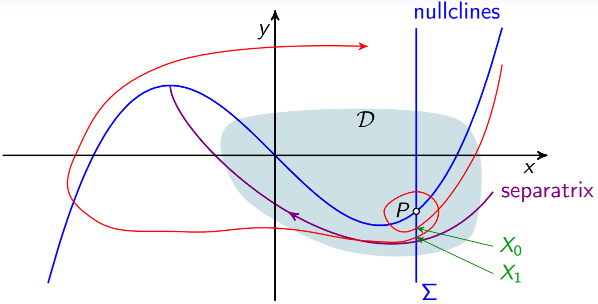
Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$X_0, X_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations: $N = \inf\{n \geqslant 1 \colon X_n \notin \Sigma\}$

Random Poincaré map



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Theorem 1 [B & Landon, Nonlinearity **25**:2303–2335, 2012]

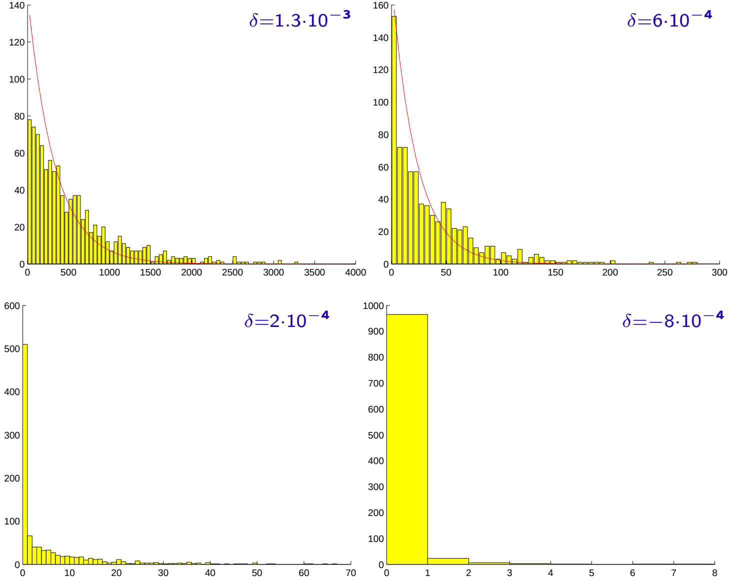
N is asymptotically geometric: $\lim\limits_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the kernel K, $\lambda_0 < 1$ if $\sigma > 0$

Proof: follows from existence of spectral gap

Details

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, Nonlinearity **25**:2303–2335, 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

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Proof: Let $A \subset \Sigma$ have positive Lebesgue measure

$$\lambda_0 \pi_0(A) = \int_\Sigma \pi_0(\mathrm{d}x) K(x,A) \geqslant \int_A \pi_0(\mathrm{d}x) K(x,A) \quad \Rightarrow \quad \lambda_0 \geqslant \inf_{x\in A} K(x,A)$$

$\Rightarrow$ construct A such that $K(x,A)$ exponentially close to 1 for all $x \in A$

Dynamics near the separatrix

Change of variables:

- Translate to Hopf bif. point
- Scale space and time
- Straighten nullcline $\dot{x} = 0$

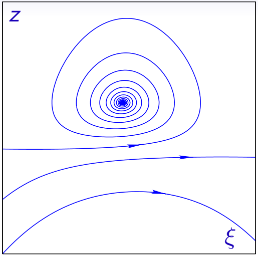
$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t$$

$$\mathrm{d}z_t = \left(\mu + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t$$

where

$$\mu = \frac{\delta}{\sqrt{\varepsilon}}$$



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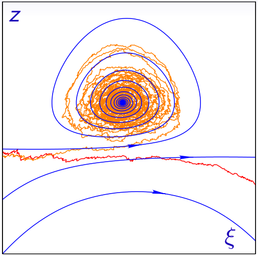

$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t + \tilde{\sigma}_1\,\mathrm{d}W_t^{(1)}$$

$$\mathrm{d}z_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t - 2\tilde{\sigma}_1\xi_t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z\,\mathrm{e}^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Take $A = \{z > \tilde{\mu}^{1-\gamma}\}$ with $0 < \gamma < \frac{1}{4}$ □

From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right) \mathrm{d}t - \tilde{\sigma}_1 t \,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2 \,\mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\Big(-\pi^{1/4} \tfrac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}}\Big) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \,\mathrm{d}y$$

From below to above threshold

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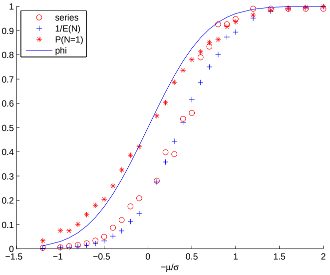


$*$: $\mathbb{P}\{\text{no small osc}\}$

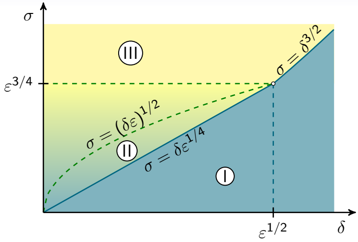
$+$: $1/\mathbb{E}[N]$

$\circ$: $1-\lambda_0$

curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Summary: Parameter regimes



$\sigma_1 = \sigma_2$:

$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\Big)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy$$

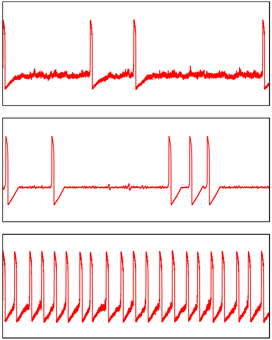
see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential

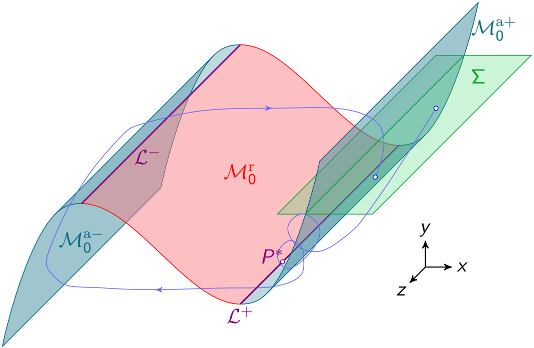
Regime II: clusters of spikes
interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)

Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



3. Example 2: The Koper model

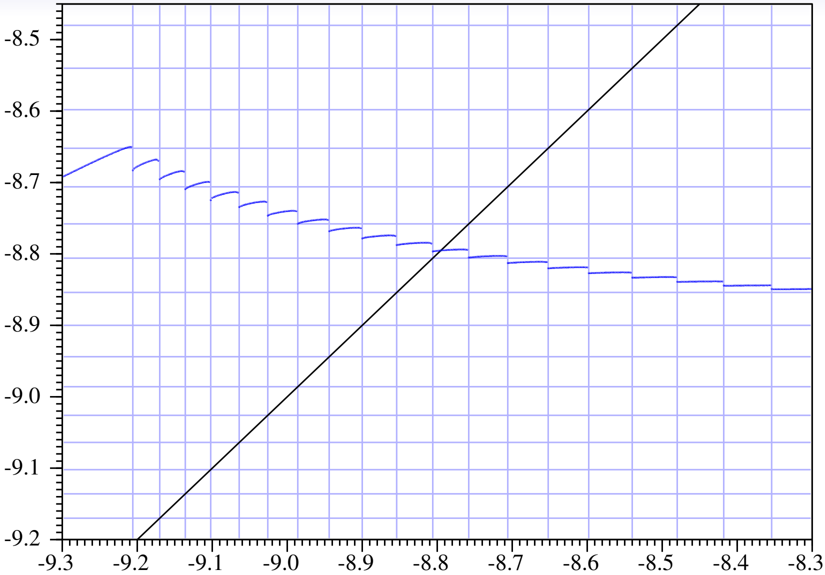
$$\begin{aligned}
\varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t, y_t, z_t)\,\mathrm{d}W_t
\end{aligned}$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, ...]

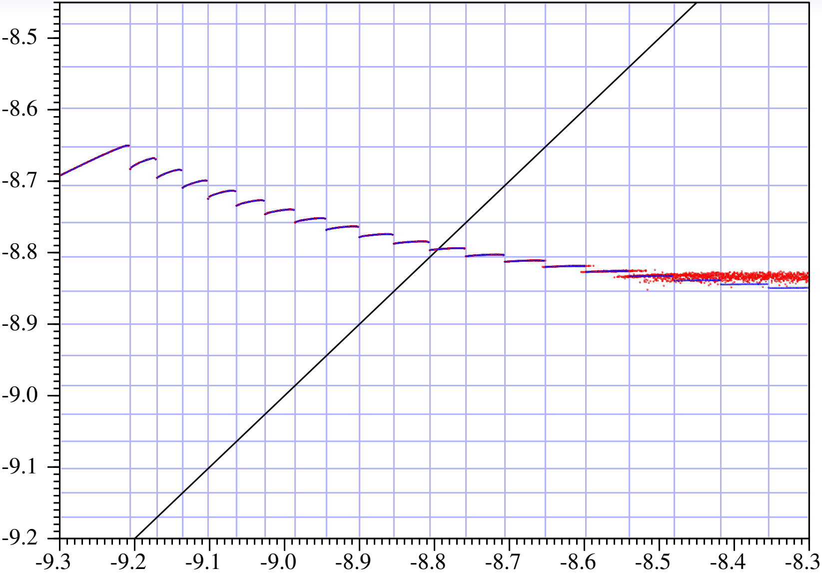
Poincaré map $\Pi : \Sigma \to \Sigma$ is almost $1d$ due to contraction in x-direction

Poincaré map $z_n \mapsto z_{n+1}$



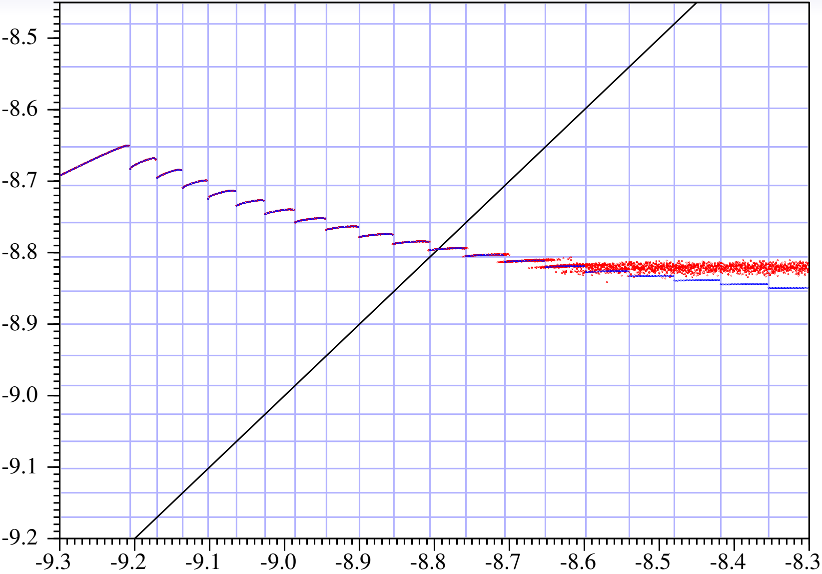
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01$, $\sigma = \sigma' = 0$ – c.f. [Guckenheimer, Chaos, 2008]

Poincaré map $z_n \mapsto z_{n+1}$



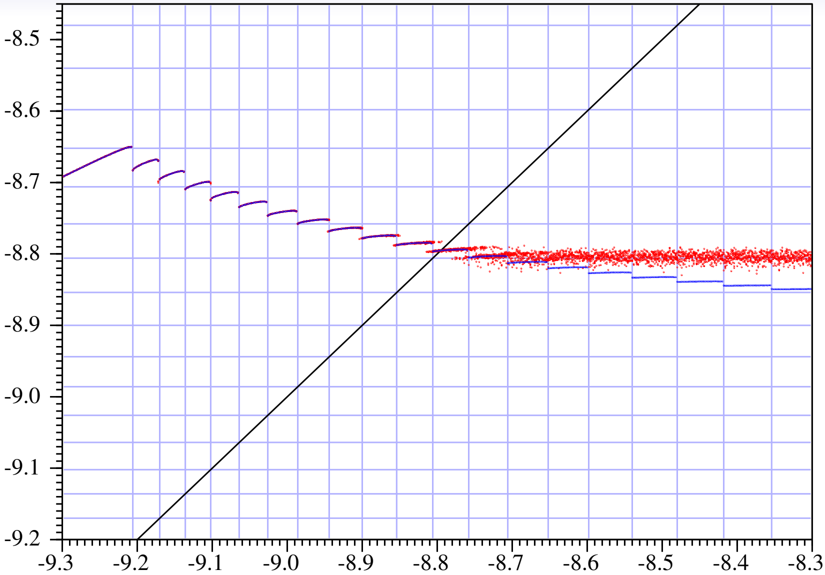
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-7}$

Poincaré map $z_n \mapsto z_{n+1}$



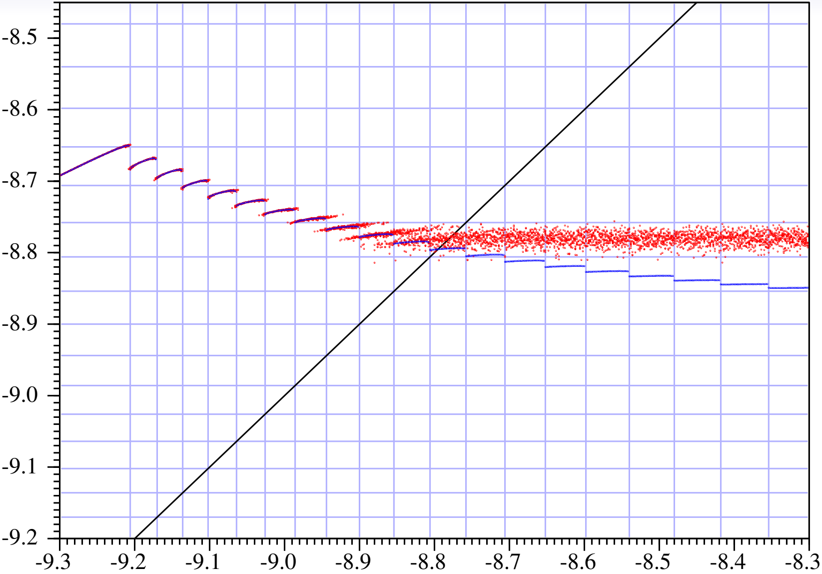
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \sigma = \sigma' = 2 \cdot 10^{-6}$

Poincaré map $z_n \mapsto z_{n+1}$



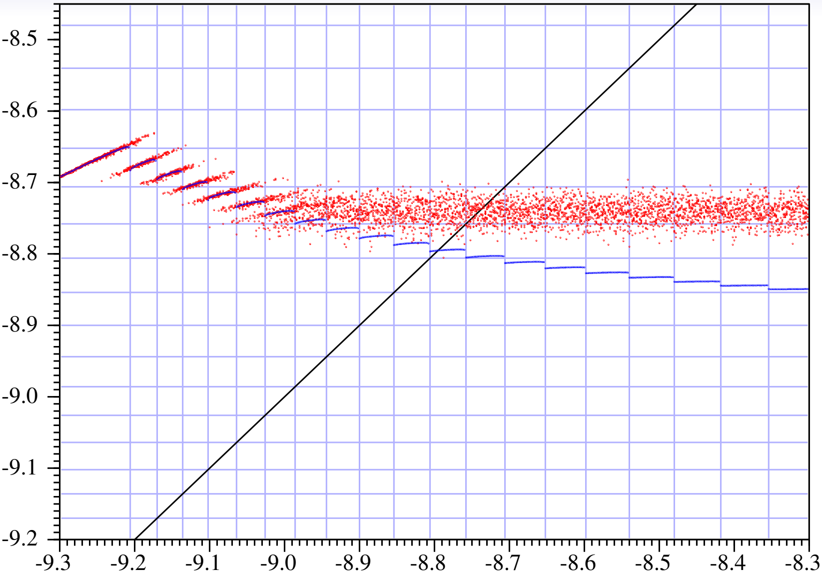
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-5}$

Poincaré map $z_n \mapsto z_{n+1}$



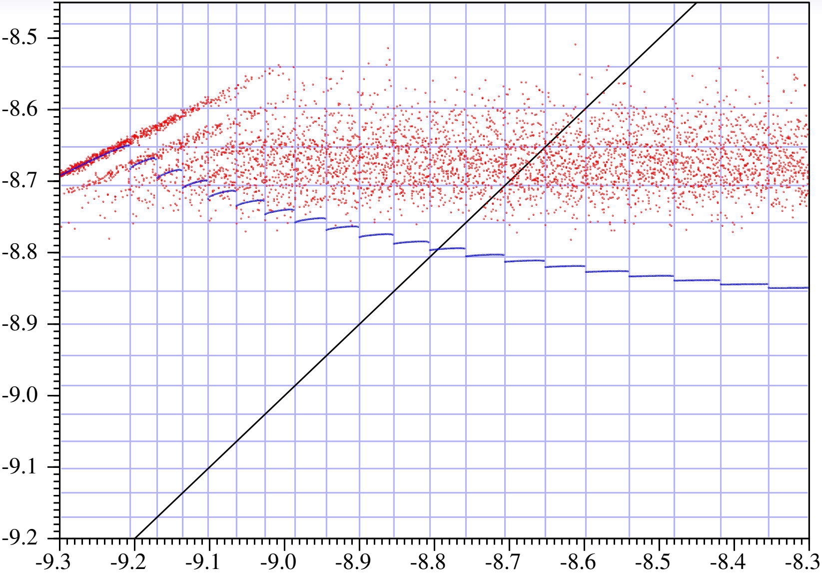
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-4}$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-3}$

Poincaré map $z_n \mapsto z_{n+1}$



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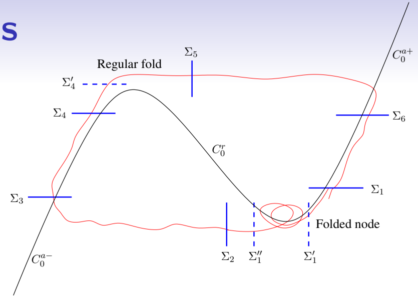
Size of fluctuations



$\mu \ll 1$: eigenvalue ratio at folded node

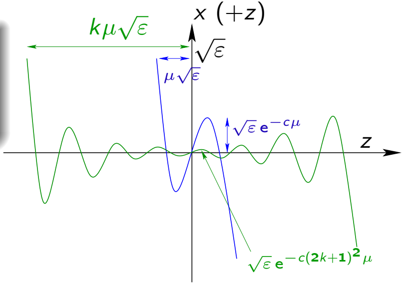
Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$\sigma'\varepsilon^{1/4}$

Main results

[B, Gentz, Kuehn, JDE 2012 & J Dynam Diff Eq 2015]

Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



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Theorem 2: size of fluctuations ▸ More

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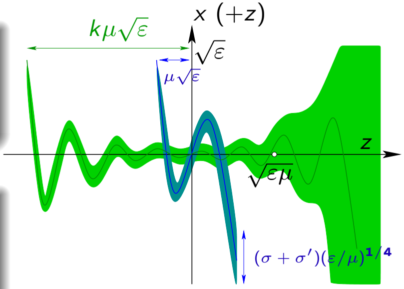

Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu}$ $\Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

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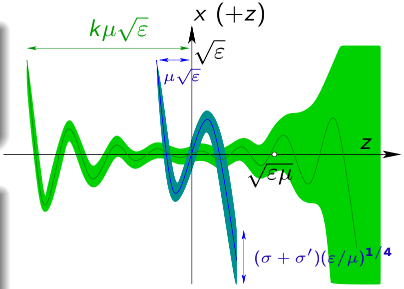

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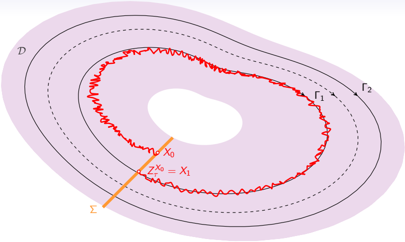
- Saturation effect occurs at $k_{\mathrm{c}} \simeq \sqrt{|\log(\sigma+\sigma')|/\mu}$
- For $k > k_{\mathrm{c}}$, behaviour indep. of k and $\Delta z \leqslant \mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma+\sigma')|}\,)$

4. Spectral theory of random Poincaré maps



$X_n = Z_{\tau_n}^{X_0} \in \Sigma$

$(X_n)_{n\geqslant 0}$ Markov chain, of kernel

$K(x,A) = \mathbb{P}\{X_{n+1} \in A | X_n = x\}$

Under appropriate conditions

$K(x,A) = \int_A k(x,y)\,\mathrm{d}y$

Markov semigroups:

$$(K\varphi)(x) = \int_\Sigma k(x,y)\varphi(y)\,\mathrm{d}y = \mathbb{E}^x[\varphi(X_1)] \qquad \varphi \in L^\infty$$

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Spectral decomposition: $K\phi_i = \lambda_i\phi_i \qquad \pi_i K = \lambda_i\pi_i \qquad \langle\pi_i, \phi_j\rangle = \delta_{ij}$

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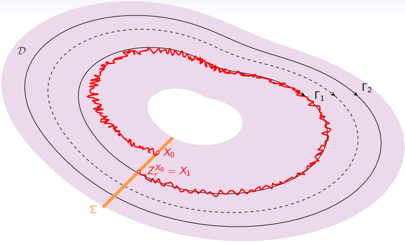

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Assumptions

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \qquad z_t \in \mathcal{D}_0 \subset \mathbb{R}^{d+1}$

Assumption 1: Domain

$f \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}^{d+1})$, $\mathcal{D} \subset \mathcal{D}_0$ positively invariant under deterministic flow

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Other limit sets are unstable points or orbits, no heteroclinic connections

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$g \in \mathcal{C}^1(\mathcal{D}_0, \mathbb{R}^{(d+1)\times k})$
$0 < c_- \|\xi\|^2 \leqslant \langle \xi, g(z)g(z)^{\mathsf{T}}\xi \rangle \leqslant c_+ \|\xi\|^2 \qquad \forall z \in \mathcal{D}\ \forall \xi \in \mathbb{R}^{d+1} \setminus \{0\}$

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Assumption 4: Confinement

$\exists$ Lyapunov function $V \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}_+)$, $\|V(z)\| \to \infty$ as $z \to \partial\mathcal{D}_0$
$(\mathcal{L}V)(z) \leqslant -cV + d\mathbb{1}_{z\in\mathcal{D}}, \quad c > 0,\ d \geqslant 0 \qquad \mathcal{L}$: generator of diffusion

Metastable hierarchy

Freidlin–Wentzell theory:

Rate function: $I_{[0,T]}(\gamma) = \dfrac{1}{2}\displaystyle\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathsf{T}} [gg^{\mathsf{T}}(\gamma_s)]^{-1} (\dot\gamma_s - f(\gamma_s))\,\mathrm{d}s$

Large-deviation principle: $\mathbb{P}\bigl\{(z_t)_{0\leqslant t\leqslant T} \in \Lambda\bigr\} \simeq \mathrm{e}^{-\inf_{\gamma\in\Lambda} I_{[0,T]}(\gamma)/\sigma^2}$

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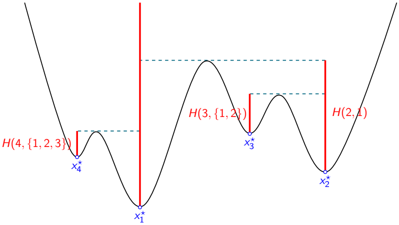
Assumption 5: Metastable hierarchy

$\exists\theta > 0$ s.t. $\forall\, 2 \leqslant k \leqslant N$

$$\min_{\ell<k} H(k,\ell) \leqslant \min_{\substack{i<k\\ j\neq i}} H(i,j) - \theta$$



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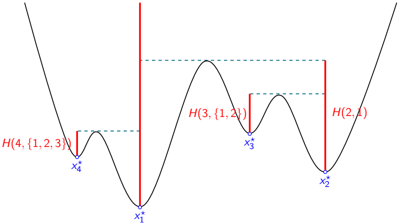

Remark: Using Doob's h-transform, one may replace Assumption 4 by

Assumption 4': Confinement

$\exists\theta' > 0$ such that $\displaystyle\min_i H(i,\partial\mathcal{D}) \geqslant \max_{i\neq j} H(i,j) + \theta'$

Main results [Baudel & B, SIAM J. Math. Anal. **49**:4319–4375, 2017]

$B_k \subset \Sigma$: nbh of $\Gamma_k \cap \Sigma$, $\mathcal{M}_k = \bigcup_{j \leqslant k} B_k$, τ_A, τ_A^+ 1st-passage/return time to A

Theorem 1: Eigenvalues

The N largest eigenvalues of K are real and positive. $\exists\, \theta_k, c > 0$ s.t.

$$\lambda_0 = 1$$

$$\lambda_k = 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\}[1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})] \qquad 1 \leqslant k \leqslant N-1$$

$$|\lambda_k| < 1 - \frac{c}{\log(\sigma^{-1})} =: \rho \qquad k \geqslant N$$

$\mathring{\pi}_0^{k+1}$: QSD on B_{k+1} and $\mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\} \simeq \mathrm{e}^{-H(k+1,\{1,\dots,k\})/\sigma^2}$

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Theorem 2: Eigenfunctions

$$\phi_0(x) = 1 \qquad \phi_k(x) = \mathbb{P}^x\{\tau_{B_{k+1}} < \tau_{\mathcal{M}_k}\}[1 + \mathcal{O}(e^{-\bar\theta/\sigma^2})] + \mathcal{O}(e^{-\bar\theta_k/\sigma^2})$$

$$\pi_k(B_{k+1}) = 1 - \mathcal{O}(e^{-\kappa/\sigma^2}) \qquad \pi_k(B_j) = \mathcal{O}(e^{-\bar\theta/\sigma^2}) \qquad 0 \leqslant k < j < N$$

$$k^n(x,y) = \pi_0(y) + \sum_{i=1}^{N-1} \lambda_i^n \phi_i(x) \pi_i(y) + \mathcal{O}(\rho^n) \qquad n \gg \log(\rho^{-1})$$

Proofs

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Theorem 3: Expected hitting times

Proofs

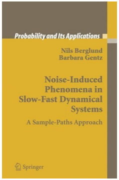
$$\mathbb{E}^x[\tau_{\mathcal{M}_k}] = [1 - \lambda_k]^{-1}[1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})] \qquad \forall x \in B_{k+1},\ 1 \leqslant k \leqslant N-1$$

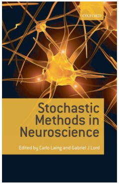
Outlook

- ▷ In progress: cases without metastable hierarchy (eigenvalue crossings)
- ▷ Open: asymptotics beyond large deviations

References:

1. N. B. & Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)
2. ______, *Stochastic dynamic bifurcations and excitability*, in C. Laing and G. Lord, (Eds.), *Stochastic methods in Neuroscience*, p. 65-93, Oxford University Press (2009)





3. N. B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**, 2303-2335 (2012)
4. N. B., Barbara Gentz & Christian Kuehn, *Hunting French Ducks in a Noisy Environment*, J. Differential Equations **252**, 4786–4841 (2012)
5. ______, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, J. Dynam. Diff. Eq. **27**, 83–136 (2015)
6. Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375 (2017)

Proof of asymptotically geometric distribution

Theorem 1 [B & Landon, Nonlinearity 25:2303–2335, 2012]

N is asymptotically geometric: $\lim\limits_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

where $\lambda_0 \in (0,1)$ if $\sigma > 0$ is principal eigenvalue of the kernel K

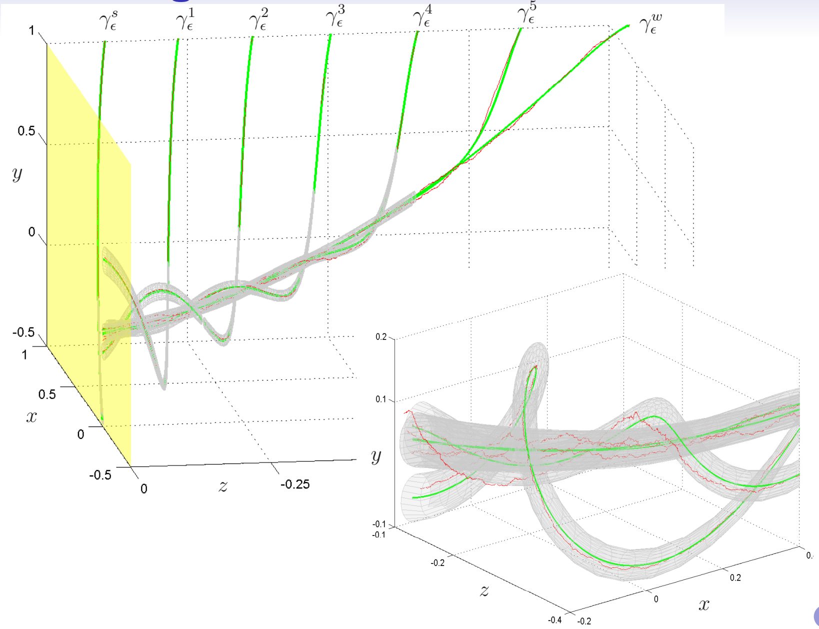
Proof:

Markov chain on Σ, kernel K with density k [Ben Arous, Kusuoka, Stroock '84]

- $\lambda_0 \leqslant \sup\limits_{x\in\Sigma} K(x,\Sigma) < 1$ by ellipticity (k bounded below)
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N > n\} = \mathbb{P}^{\mu_0}\{X_n \in \Sigma\} &= \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\Sigma) \\ &= \int_\Sigma \mu_0(\mathrm{d}x) \lambda_0^n \phi_0(x)[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)] \\ &= \lambda_0^n \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N = n+1\} &= \int_\Sigma \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\mathrm{d}y)[1 - K(y,\Sigma)] \\ &= \lambda_0^n (1-\lambda_0) \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- Existence of spectral gap follows from positivity condition [Birkhoff '57]

Back

Estimating noise-induced fluctuations



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Estimating noise-induced fluctuations

$\zeta_t = (x_t, y_t, z_t) - (x_t^{\text{det}}, y_t^{\text{det}}, z_t^{\text{det}})$

$$\mathrm{d}\zeta_t = \frac{1}{\varepsilon} A(t)\zeta_t \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathcal{F}(\zeta_t, t)\,\mathrm{d}W_t + \frac{1}{\varepsilon} \underbrace{b(\zeta_t, t)}_{=\mathcal{O}(\|\zeta_t\|^2)} \,\mathrm{d}t$$

$$\zeta_t = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t U(t,s)\mathcal{F}(\zeta_s, s)\,\mathrm{d}W_s + \frac{1}{\varepsilon} \int_0^t U(t,s) b(\zeta_s, s)\,\mathrm{d}s$$

where $U(t,s)$ principal solution of $\varepsilon\dot{\zeta} = A(t)\zeta$.

Lemma (Bernstein-type estimate):

$$\mathbb{P}\biggl\{ \sup_{0 \leqslant s \leqslant t} \biggl\| \int_0^s \mathcal{G}(\zeta_u, u)\,\mathrm{d}W_u \biggr\| > h \biggr\} \leqslant 2n \exp\biggl\{ -\frac{h^2}{2V(t)} \biggr\}$$

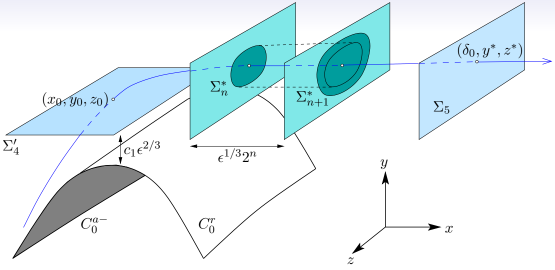
where $\int_0^s \mathcal{G}(\zeta_u, u)\mathcal{G}(\zeta_u, u)^T \,\mathrm{d}u \leqslant V(s)$ a.s. and $n = 3$ space dimension

Remark: more precise results using ODE for covariance matrix of

$$\zeta_t^0 = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t U(t,s)\mathcal{F}(0,s)\,\mathrm{d}W_s$$

Back

Example: analysis near the regular fold



Proposition: For $h_1 = \mathcal{O}(\varepsilon^{2/3})$

$$\mathbb{P}\Big\{\|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\| > h_1\Big\}$$

$$\leqslant C|\log\varepsilon|\bigg(\exp\Big\{-\frac{\kappa h_1^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}}\Big\} + \exp\Big\{-\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon}\Big\}\bigg)$$

Useful if $\sigma, \sigma' \ll \sqrt{\varepsilon}$

Back

Proof of spectral decomposition

Feynman–Kac-type relation: For $|\mathrm{e}^{-u}| > \sup_{x \in A^c} \mathbb{P}^x\{X_1 \in A^c\}$

$$\begin{cases} (K\psi)(x) = \mathrm{e}^{-u}\,\psi(x) & x \in A^c \\ \psi(x) = \phi(x) & x \in A \end{cases} \qquad \Leftrightarrow \qquad \psi(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_A}\,\phi(X_{\tau_A})]$$

To estimate (λ_k, ϕ_k) choose $A = \mathcal{M}_{k+1} = \bigcup_{j=1}^{k+1} B_j$

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Lemma: Though K not reversible $\forall$ disjoint $A_1, A_2 \subset \Sigma$ ⏩ Back

$$\int_{A_1} \pi_0(x)\mathbb{P}^x\{\tau_{A_2}^+ < \tau_{A_1}^+\}\,\mathrm{d}x = \int_{A_2} \pi_0(x)\mathbb{P}^x\{\tau_{A_1}^+ < \tau_{A_2}^+\}\,\mathrm{d}x$$