École Polytechnique
Journée thématique autour de la renormalisation

Renormalisation of static & dynamic Φ^4_d models

Nils Berglund

Institut Denis Poisson, University of Orléans, France

INSTITUT ORLÉANS · TOURS Mathématiques & Physique Théorique DENIS POISSON

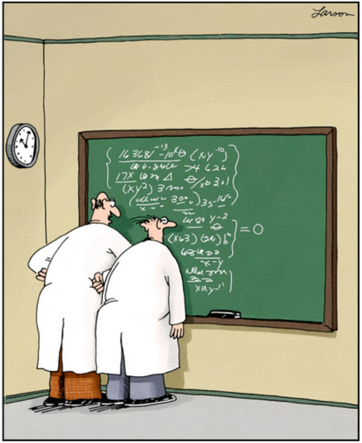
12 June 2023

Partly based on joint works with Tom Klose (Berlin), Barbara Gentz (Bielefeld)
Giacomo Di Gesù (Rome) and Hendrik Weber (Münster)




Renormalisation

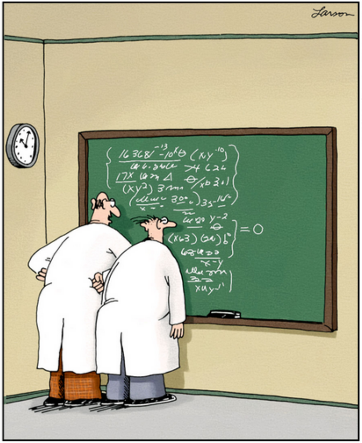
- Recipe for turning a classical system into a quantum one: $p \mapsto -\mathrm{i}\hbar\nabla$ (De Broglie, Schrödinger, Heisenberg, ...)
- Quantum electrodynamics (QED) (Dirac, ..., Bethe, Tomonaga, Schwinger, Feynman, Dyson)
- Divergences: infrared/ultraviolet, due to self-interaction (can also occur classically) $\longrightarrow$ need for renormalisation (Bogolyubov, Wilson, ...)
- Bare vs observed parameters, analogy with Archimedes force (Connes, Kreimer)



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!"

Renormalisation

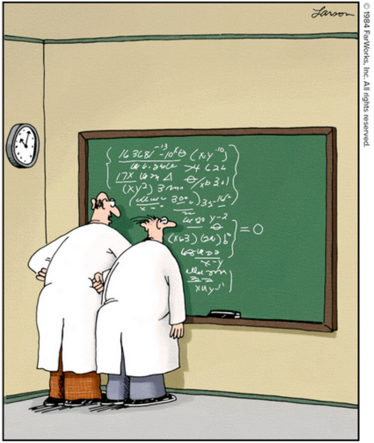
- Recipe for turning a classical system into a quantum one: $p \mapsto -\mathrm{i}\,\hbar\nabla$ (De Broglie, Schrödinger, Heisenberg, . . .)
- Quantum electrodynamics (QED) (Dirac, . . . , Bethe, Tomonaga, Schwinger, Feynman, Dyson)
- Divergences: infrared/ultraviolet, due to self-interaction (can also occur classically) $\longrightarrow$ need for renormalisation (Bogolyubov, Wilson, . . .)
- Bare vs observed parameters, analogy with Archimedes force (Connes, Kreimer)



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!"

Renormalisation

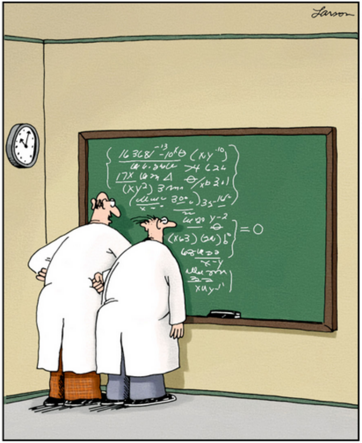
- Recipe for turning a classical system into a quantum one: $p \mapsto -i\hbar\nabla$ (De Broglie, Schrödinger, Heisenberg, . . .)
- Quantum electrodynamics (QED) (Dirac, . . . , Bethe, Tomonaga, Schwinger, Feynman, Dyson)
- Divergences: infrared/ultraviolet, due to self-interaction (can also occur classically) $\longrightarrow$ need for renormalisation (Bogolyubov, Wilson, . . .)
- Bare vs observed parameters, analogy with Archimedes force (Connes, Kreimer)



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!"

Renormalisation

- Recipe for turning a classical system into a quantum one: $p \mapsto -i\hbar\nabla$ (De Broglie, Schrödinger, Heisenberg, ...)
- Quantum electrodynamics (QED) (Dirac, ..., Bethe, Tomonaga, Schwinger, Feynman, Dyson)
- Divergences: infrared/ultraviolet, due to self-interaction (can also occur classically) $\longrightarrow$ need for renormalisation (Bogolyubov, Wilson, ...)
- Bare vs observed parameters, analogy with Archimedes force (Connes, Kreimer)



"No doubt about it, Ellington—we've mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!"

The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2}N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\mu_{N,\varepsilon}(\mathrm{d}y) = \dfrac{1}{Z_{N,\varepsilon}} \mathrm{e}^{-V_{N,\varepsilon}(y)}\,\mathrm{d}y$

- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4\Big)\,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

$$\mu_\varepsilon(\mathrm{d}\phi) \text{ “=” } \frac{1}{Z_\varepsilon}\mathrm{e}^{-V_\varepsilon(\phi)} \text{ “}\mathrm{d}\phi\text{”}$$

- Alternative: Spectral Galerkin approx. (Fourier modes with $|k| \leqslant N$)

The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2}N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\mu_{N,\varepsilon}(\mathrm{d}y) = \dfrac{1}{Z_{N,\varepsilon}}\,\mathrm{e}^{-V_{N,\varepsilon}(y)}\,\mathrm{d}y$

- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Bigl(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4\Bigr)\,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

$$\mu_\varepsilon(\mathrm{d}\phi) \text{ “=” } \frac{1}{Z_\varepsilon}\,\mathrm{e}^{-V_\varepsilon(\phi)} \text{ “}\mathrm{d}\phi\text{”}$$

- Alternative: Spectral Galerkin approx. (Fourier modes with $|k| \leqslant N$)

The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2}N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\mu_{N,\varepsilon}(\mathrm{d}y) = \dfrac{1}{Z_{N,\varepsilon}} \mathrm{e}^{-V_{N,\varepsilon}(y)} \,\mathrm{d}y$

- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4 \Big) \,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

$$\mu_\varepsilon(\mathrm{d}\phi) \text{ “=” } \frac{1}{Z_\varepsilon} \mathrm{e}^{-V_\varepsilon(\phi)} \text{ “}\mathrm{d}\phi\text{”}$$

- Alternative: Spectral Galerkin approx. (Fourier modes with $|k| \leqslant N$)

The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Big(\frac{1}{2} \|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 \Big) \, \mathrm{d}x = \frac{1}{2}\langle \phi, (-\Delta + 1)\phi \rangle$

  μ_0 is Gaussian free field with covariance $(-\Delta+1)^{-1}$

  (well-defined since $(-\Delta+1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1} < \infty$)

- $\varepsilon > 0$:

$$\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu_0} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \, \mathrm{d}x}$$

  where

$$\frac{Z_\varepsilon}{Z_0} = \mathbb{E}^{\mu_0}\Big[\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \, \mathrm{d}x}\Big] \text{“}=\text{”} \frac{1}{Z_0}\int \mathrm{e}^{-V_0(\phi) - \frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \, \mathrm{d}x} \, \mathrm{d}\phi$$

  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Big[\int_\Lambda \phi_{\mathrm{GFF}}(x)^{2n} \, \mathrm{d}x\Big] \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1}\Big)^n < C^n$$

  so that $\frac{Z_\varepsilon}{Z_0} = 1 + \mathcal{O}(\varepsilon)$

The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Big(\frac12 \|\nabla\phi(x)\|^2 + \frac12 \phi(x)^2 \Big)\,\mathrm{d}x = \frac12 \langle \phi, (-\Delta + 1)\phi \rangle$

  μ_0 is Gaussian free field with covariance $(-\Delta+1)^{-1}$
  (well-defined since $(-\Delta+1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1} < \infty$)

- $\varepsilon > 0$:

$$\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu_0} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}$$

  where

$$\frac{Z_\varepsilon}{Z_0} = \mathbb{E}^{\mu_0}\Big[\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}\Big] \text{ “=” } \frac{1}{Z_0}\int \mathrm{e}^{-V_0(\phi)-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}\,\mathrm{d}\phi$$

  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Big[\int_\Lambda \phi_{\mathrm{GFF}}(x)^{2n}\,\mathrm{d}x\Big] \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1}\Big)^n < C^n$$

  so that $\frac{Z_\varepsilon}{Z_0} = 1 + \mathcal{O}(\varepsilon)$

The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Bigl(\frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2\Bigr)\,\mathrm{d}x = \frac{1}{2}\langle\phi, (-\Delta+1)\phi\rangle$

  μ_0 is Gaussian free field with covariance $(-\Delta+1)^{-1}$

  (well-defined since $(-\Delta+1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1} < \infty$)

- $\varepsilon > 0$:

$$\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu_0} = \frac{Z_0}{Z_\varepsilon}\,\mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{Z_0}{Z_\varepsilon}\,\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}$$

  where

$$\frac{Z_\varepsilon}{Z_0} = \mathbb{E}^{\mu_0}\Bigl[\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}\Bigr] \text{ ``=''} \frac{1}{Z_0}\int \mathrm{e}^{-V_0(\phi)-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x}\,\mathrm{d}\phi$$

  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Bigl[\int_\Lambda \phi_{\mathrm{GFF}}(x)^{2n}\,\mathrm{d}x\Bigr] \lesssim \Bigl(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k+1}\Bigr)^n < C^n$$

  so that $\frac{Z_\varepsilon}{Z_0} = 1 + \mathcal{O}(\varepsilon)$

The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2,|k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF},N}(x)^2]\,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k+1} = \mathrm{Tr}\big[(-\Delta_N+1)^{-1}\big] \sim \log N$$

- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

  If (X,Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X;C)H_m(Y;C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

  Consequence: $\sup_N \mathbb{E}\bigg[\bigg(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n:\mathrm{d}x\bigg)^2\bigg] < \infty \qquad \forall n$

- Gibbs measure defined as in 1d case, with

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}:\phi(x)^4:\Big)\,\mathrm{d}x$$

The case $d = 2$

- $(-\Delta+1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2, |k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}\big[\phi_{\mathrm{GFF},N}(x)^2\big]\,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k+1} = \mathrm{Tr}\Big[(-\Delta_N+1)^{-1}\Big] \sim \log N$$

- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

If (X,Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X;C)H_m(Y;C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

Consequence: $\sup_N \mathbb{E}\bigg[\Big(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n:\mathrm{d}x\Big)^2\bigg] < \infty \qquad \forall n$

- Gibbs measure defined as in 1d case, with

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}:\phi(x)^4:\Big)\,\mathrm{d}x$$

The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2, |k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF},N}(x)^2] \,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} = \mathrm{Tr}\big[(-\Delta_N + 1)^{-1}\big] \sim \log N$$

- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

  If (X, Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X; C) H_m(Y; C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

  Consequence: $\sup_N \mathbb{E}\Bigg[\bigg(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n: \mathrm{d}x\bigg)^2\Bigg] < \infty \qquad \forall n$

- Gibbs measure defined as in 1d case, with

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}:\phi(x)^4:\Big) \,\mathrm{d}x$$

The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2,|k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF},N}(x)^2]\,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} = \mathrm{Tr}\big[(-\Delta_N + 1)^{-1}\big] \sim \log N$$

- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

  If (X, Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X; C)H_m(Y; C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

  Consequence: $\sup_N \mathbb{E}\Bigg[\bigg(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n:\,\mathrm{d}x\bigg)^2\Bigg] < \infty \qquad \forall n$

- Gibbs measure defined as in 1d case, with

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac12\|\nabla\phi(x)\|^2 + \tfrac12\phi(x)^2 + \tfrac{\varepsilon}{4}:\phi(x)^4:\Big)\,\mathrm{d}x$$

The case $d = 3$

Theorem: Potential needs exactly 4 counterterms:

$$V(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \tfrac{\varepsilon}{4} {:}\phi(x)^4{:}_{C_N^{(1)}} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)} \Big)\,\mathrm{d}x$$

where

$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 3! \int_\Lambda G_N(x)^3\,\mathrm{d}x = \mathcal{O}(\log N)$$

$$C_N^{(3)} = \frac{4!}{2!4^2} \int_\Lambda G_N(x)^4\,\mathrm{d}x = \mathcal{O}(N)$$

$$C_N^{(4)} = \frac{2^3}{3!4^3} \binom{4}{2}^3 \int_\Lambda \int_\Lambda G_N(x)^2 G_N(y)^2 G_N(x-y)^2\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\log N)$$

and $G_N(x) = \sum_{|k| \leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ if the Green function of Δ_N

The case $d = 3$

Theorem: Potential needs exactly 4 counterterms:

$$V(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \tfrac{\varepsilon}{4} {:}\phi(x)^4{:}_{C_N^{(1)}} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)} \Big)\,\mathrm{d}x$$

where

$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 3! \int_\Lambda G_N(x)^3\,\mathrm{d}x = \mathcal{O}(\log N)$$

$$C_N^{(3)} = \frac{4!}{2!4^2} \int_\Lambda G_N(x)^4\,\mathrm{d}x = \mathcal{O}(N)$$

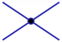
$$C_N^{(4)} = \frac{2^3}{3!4^3}\binom{4}{2}^3 \int_\Lambda \int_\Lambda G_N(x)^2 G_N(y)^2 G_N(x-y)^2\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\log N)$$

and $G_N(x) = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ if the Green function of Δ_N

Some literature

- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Graphical notations

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{\varepsilon}{4}$, $\beta = \frac{1}{2}\varepsilon^2 C_N^{(2)}$, $\gamma = \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$

  Then $\dfrac{Z_{N,\varepsilon}}{Z_{N,0}} = \mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \operatorname{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

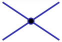
$C_N^{(1)} = \Pi_N$

$C_N^{(2)} = 3!\Pi_N$

$C_N^{(3)} = \dfrac{4!}{2!4^2}\Pi_N$

$C_N^{(4)} = \dfrac{2^3}{3!4^3}\binom{4}{2}^3\Pi_N$

Graphical notations

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{\varepsilon}{4}$, $\beta = \frac{1}{2}\varepsilon^2 C_N^{(2)}$, $\gamma = \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$

  Then $\dfrac{Z_{N,\varepsilon}}{Z_{N,0}} = \mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \mathrm{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

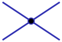
$C_N^{(1)} = \Pi_N$ 

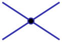
$C_N^{(2)} = 3!\Pi_N$ 

$C_N^{(3)} = \dfrac{4!}{2!4^2}\Pi_N$ 

$C_N^{(4)} = \dfrac{2^3}{3!4^3}\dbinom{4}{2}^3 \Pi_N$ 

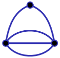
Cumulant expansion

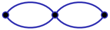
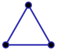
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

  Examples:

  $\mu_2 = \alpha^2 4! \Pi_N$  $+ \beta^2 2! \Pi_N$ 

  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

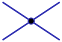
- Cumulant expansion: (Leonov & Shiraev)

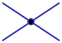
  $$-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

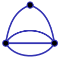
Cumulant expansion

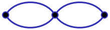
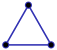
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

  Examples:

  $\mu_2 = \alpha^2 4! \Pi_N$  $+ \beta^2 2! \Pi_N$ 

  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4! \, \Pi_N$  $- 8\beta^3 \Pi_N$ 

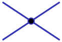
- Cumulant expansion: (Leonov & Shiraev)

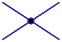
  $$-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

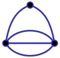
Cumulant expansion

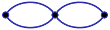
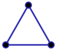
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+\,\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

  Examples:

  $\mu_2 = \alpha^2 4!\Pi_N$  $+\,\beta^2 2!\Pi_N$ 

  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $-\,3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $-\,3\alpha\beta^2 4!\,\Pi_N$  $-\,8\beta^3\Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

  $-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

Divergences and subdivergences

- ▷ Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- ▷ Examples:

deg( ) = 0 Π_N( ) = $\mathcal{O}(\log N)$

deg( ) = −1 Π_N( ) = $\mathcal{O}(N)$

deg( ) = 0 Π_N( ) = $\mathcal{O}(\log N)$

- ▷ It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

However, deg( ) = 1, while Π_N( ) = $\mathcal{O}(\log N)$ ‘
because it contains a subdivergence

Theorem: [Dyson]

If $\deg \bar\Gamma > 0$ for all subgraphs $\bar\Gamma \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($  $) = 0$ $\Pi_N($  $) = \mathcal{O}(\log N)$

$\deg($  $) = -1$ $\Pi_N($  $) = \mathcal{O}(N)$

$\deg($  $) = 0$ $\Pi_N($  $) = \mathcal{O}(\log N)$

- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$ ‘
because it contains a subdivergence

Theorem: [Dyson]

If $\deg \bar\Gamma > 0$ for all subgraphs $\bar\Gamma \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

Divergences and subdivergences

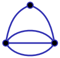
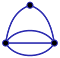
- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($  $) = 0$ $\quad\quad$ $\Pi_N($  $) = \mathcal{O}(\log N)$

$\deg($  $) = -1$ $\quad\quad$ $\Pi_N($  $) = \mathcal{O}(N)$

$\deg($  $) = 0$ $\quad\quad$ $\Pi_N($  $) = \mathcal{O}(\log N)$

- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

  However, $\deg($  $) = 1$, while $\Pi_N($  $) = \mathcal{O}(\log N)$ ‘
  because it contains a subdivergence 

Theorem: [Dyson]

If $\deg \bar\Gamma > 0$ for all subgraphs $\bar\Gamma \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

Divergences and subdivergences

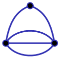
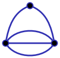
- **Degree** of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ **divergent** if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($  $) = 0$ $\qquad \Pi_N($  $) = \mathcal{O}(\log N)$

$\deg($  $) = -1$ $\qquad \Pi_N($  $) = \mathcal{O}(N)$

$\deg($  $) = 0$ $\qquad \Pi_N($  $) = \mathcal{O}(\log N)$

- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

  However, $\deg($  $) = 1$, while $\Pi_N($  $) = \mathcal{O}(\log N)$ ‘
  because it contains a **subdivergence** 

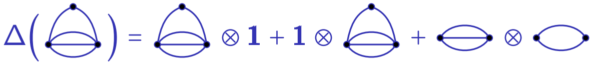
Theorem: [Dyson]

If $\deg \bar{\Gamma} > 0$ for all subgraphs $\bar{\Gamma} \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \bar\Gamma \otimes (\Gamma / \bar\Gamma) \qquad (\mathbf{1}\text{: empty graph})$$

Example:


- (Twisted) antipode: $\mathscr{A} : \mathscr{G} \to \mathscr{G}$, $\quad \mathscr{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \mathscr{A}(\bar\Gamma) \cdot (\Gamma / \bar\Gamma)$

Example:

- Character: linear form $g : \mathscr{G} \to \mathbb{R}$ such that $\langle g, \Gamma_1 \cdot \Gamma_2 \rangle = \langle g, \Gamma_1 \rangle \langle g, \Gamma_2 \rangle$

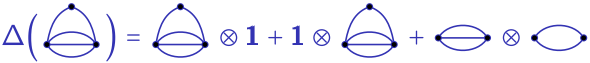
Renormalisation map: $M^g : \mathscr{G} \to \mathscr{G}$, $M^g(\Gamma) := (g \otimes \mathrm{id})\Delta\Gamma$

Property: If $\langle f \circ g, \Gamma \rangle = \langle f \otimes g, \Delta\Gamma \rangle$ and $\langle \mathscr{A}^*(f), \Gamma \rangle = \langle f, \mathscr{A}(\Gamma) \rangle$ then $M^{g \circ h} = M^g M^h$ and $(M^g)^{-1} = M^{\mathscr{A}^*(g)}$ $\Rightarrow$ group structure

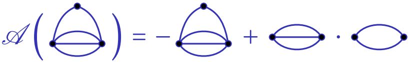
Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \bar{\Gamma} \otimes (\Gamma / \bar{\Gamma}) \qquad (\mathbf{1}\text{: empty graph})$$

Example: 

- (Twisted) antipode: $\mathscr{A} : \mathscr{G} \to \mathscr{G}$, $\quad \mathscr{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \mathscr{A}(\bar{\Gamma}) \cdot (\Gamma / \bar{\Gamma})$

Example: 

- Character: linear form $g : \mathscr{G} \to \mathbb{R}$ such that $\langle g, \Gamma_1 \cdot \Gamma_2 \rangle = \langle g, \Gamma_1 \rangle \langle g, \Gamma_2 \rangle$

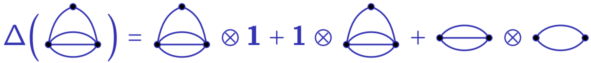
Renormalisation map: $M^g : \mathscr{G} \to \mathscr{G}$, $M^g(\Gamma) := (g \otimes \mathrm{id})\Delta\Gamma$

Property: If $\langle f \circ g, \Gamma \rangle = \langle f \otimes g, \Delta\Gamma \rangle$ and $\langle \mathscr{A}^*(f), \Gamma \rangle = \langle f, \mathscr{A}(\Gamma) \rangle$ then $M^{g \circ h} = M^g M^h$ and $(M^g)^{-1} = M^{\mathscr{A}^*(g)}$ $\Rightarrow$ group structure

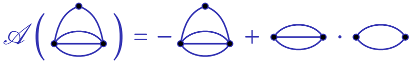
Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

  $$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \bar{\Gamma} \otimes (\Gamma / \bar{\Gamma}) \qquad (\mathbf{1}\text{: empty graph})$$

  Example: 

- (Twisted) antipode: $\mathscr{A} : \mathscr{G} \to \mathscr{G}$, $\quad \mathscr{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1} \neq \bar{\Gamma} \subsetneq \Gamma \\ \deg(\bar{\Gamma}) \leqslant 0}} \mathscr{A}(\bar{\Gamma}) \cdot (\Gamma / \bar{\Gamma})$

  Example: 

- Character: linear form $g : \mathscr{G} \to \mathbb{R}$ such that $\langle g, \Gamma_1 \cdot \Gamma_2 \rangle = \langle g, \Gamma_1 \rangle \langle g, \Gamma_2 \rangle$

  Renormalisation map: $M^g : \mathscr{G} \to \mathscr{G}$, $M^g(\Gamma) := (g \otimes \mathrm{id}) \Delta \Gamma$

  Property: If $\langle f \circ g, \Gamma \rangle = \langle f \otimes g, \Delta \Gamma \rangle$ and $\langle \mathscr{A}^*(f), \Gamma \rangle = \langle f, \mathscr{A}(\Gamma) \rangle$ then $M^{g \circ h} = M^g M^h$ and $(M^g)^{-1} = M^{\mathscr{A}^*(g)}$ $\Rightarrow$ group structure

BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
- Renormalised valuation:

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) = (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma = (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma) 1_{\deg \Gamma \leqslant 0}$$

Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg \Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^n \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!} (-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg \Gamma_{pp}^{(k)} = p - 3$$

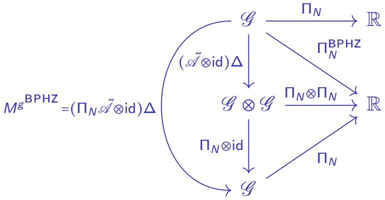
Consequence: all terms in cumulant expansion bounded uniformly in N

BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma\rangle = \Pi_N \mathscr{A}(\Gamma)\mathbf{1}_{\deg\Gamma\leqslant 0}$
- Renormalised valuation:

$$\begin{aligned}\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma)\\ &= (g^{\mathrm{BPHZ}}\otimes\Pi_N)\Delta\Gamma\\ &= (\Pi_N\tilde{\mathscr{A}}\otimes\Pi_N)\Delta\Gamma\end{aligned}$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma)1_{\deg\Gamma\leqslant 0}$$



Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg\Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^n \binom{n}{m}\alpha^m\beta^{n-m}\sum_k b_{nm}^{(k)}\Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^{\infty}\frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty}\frac{1}{p!}(-\alpha)^p\sum_k b_{pp}^{(k)}\Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg\Gamma_{pp}^{(k)} = p-3$$

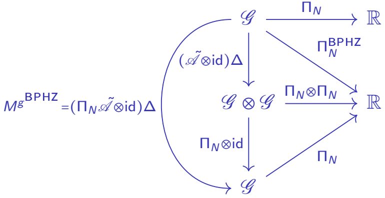
Consequence: all terms in cumulant expansion bounded uniformly in N

BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
- Renormalised valuation:

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) = (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma = (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma) 1_{\deg \Gamma \leqslant 0}$$



Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg \Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!}(-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg \Gamma_{pp}^{(k)} = p - 3$$

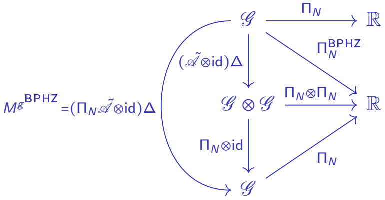
Consequence: all terms in cumulant expansion bounded uniformly in N

BPHZ renormalisation

- BPHZ character: $\langle g^{\text{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
- Renormalised valuation:

$$\begin{aligned} \Pi_N^{\text{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\text{BPHZ}}}(\Gamma) \\ &= (g^{\text{BPHZ}} \otimes \Pi_N)\Delta\Gamma \\ &= (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma \end{aligned}$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma) 1_{\deg \Gamma \leqslant 0}$$



Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg \Gamma > 0$ then $\Pi_N^{\text{BPHZ}}(\Gamma)$ bdd uniformly in N

Theorem: [B & Klose]

Write $\kappa_n = (-1)^n \sum_{m=0}^n \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

$$\sum_{n=2}^\infty \frac{\kappa_n}{n!} = -\sum_{p=2}^\infty \frac{1}{p!} (-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\text{BPHZ}}(\Gamma_{pp}^{(k)}) \qquad \deg \Gamma_{pp}^{(k)} = p - 3$$

Consequence: all terms in cumulant expansion bounded uniformly in N

Commutative diagram

$$
\begin{array}{ccccc}
e^{-\alpha X} & \xmapsto{\mathscr{P}} & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) & & \\
\Big\downarrow \chi & & \Big\downarrow M^{g^{\mathrm{BPHZ}}} & \searrow \Pi_N^{\mathrm{BPHZ}} & \\
e^{-\alpha X-\beta Y} & \xmapsto{\mathscr{P}} & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) & \xmapsto{\Pi_N} & \log \mathbb{E}[e^{-\alpha X-\beta Y}]
\end{array}
$$

- $\mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$
- $e^{-\alpha X}, e^{-\alpha X-\beta Y} \in H = \operatorname{span}\{X^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\} \qquad X^{\boldsymbol{n}} := X^{n_1} Y^{n_2}$
- Construction of χ inspired by Ebrahimi-Fard et al

Lemma: [B & Klose]

- $\chi(e^{-\alpha X}) = e^{-\alpha X-\beta Y}$
- $\mathscr{P} \circ \chi = M^{g^{\mathrm{BPHZ}}} \circ \mathscr{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

Commutative diagram

$$
\begin{array}{ccccc}
e^{-\alpha X} & \xmapsto{\ \mathscr{P}\ } & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) & & \\
\Big\downarrow \chi & & \Big\downarrow M^{g^{\text{BPHZ}}} & \overset{\Pi_N^{\text{BPHZ}}}{\searrow} & \\
e^{-\alpha X-\beta Y} & \xmapsto{\ \mathscr{P}\ } & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) & \xmapsto{\ \Pi_N\ } & \log \mathbb{E}[e^{-\alpha X-\beta Y}]
\end{array}
$$

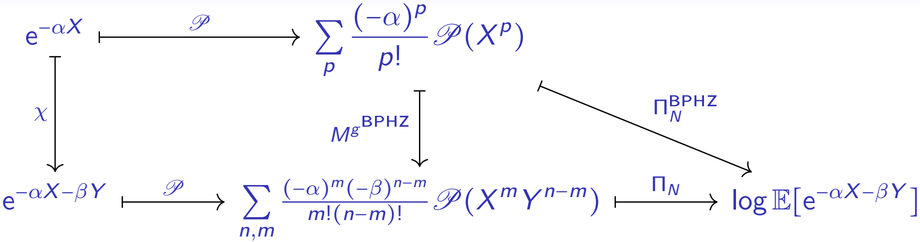
- ▷ $\mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$
- ▷ $e^{-\alpha X}, e^{-\alpha X-\beta Y} \in H = \text{span}\left\{X^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\right\}$ $\quad X^{\boldsymbol{n}} := X^{n_1} Y^{n_2}$
- ▷ Construction of χ inspired by Ebrahimi-Fard et al

Lemma: [B & Klose]

- ▷ $\chi(e^{-\alpha X}) = e^{-\alpha X-\beta Y}$
- ▷ $\mathscr{P} \circ \chi = M^{g^{\text{BPHZ}}} \circ \mathscr{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

Commutative diagram



- $\mathscr{P} = \Pi_{\mathrm{connected}}\left(\sum_{\mathrm{pairings}}\right)$
- $e^{-\alpha X}, e^{-\alpha X - \beta Y} \in H = \mathrm{span}\{\mathrm{X}^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\} \qquad \mathrm{X}^{\boldsymbol{n}} := X^{n_1} Y^{n_2}$
- Construction of χ inspired by Ebrahimi-Fard et al

Lemma: [B & Klose]

- $\chi(e^{-\alpha X}) = e^{-\alpha X - \beta Y}$
- $\mathscr{P} \circ \chi = M^{g^{\mathrm{BPHZ}}} \circ \mathscr{P}$

Proof of commutativity based on Zimmermann's forest formula for $\mathscr{A}$

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), ...:
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), . . . :
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), . . . :
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), . . . :
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x)^3 + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- Parisi & Wu (1981):
  Stochastic quantization
- Faris & Jona-Lasinio (1982), …:
  1d case: Well-posed, large-deviation principle
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures, Banach spaces of modeled distributions
  Ad-hoc renormalisation for Φ^4_3 and PAM (parabolic Anderson model)
- Bruned, Chandra, Chevyrev, Hairer, Zambotti (2016+):
  Solution theory and renormalisation for general locally subcritical (superrenormalisable) parabolic SPDEs, using BPHZ renormalisation

Mild solutions and Hölder–Besov spaces

- $(\partial_t - \Delta)\phi = h \quad \Rightarrow \quad \phi = G * h$ where G heat kernel
- $(\partial_t - \Delta)\phi = F(\phi) + \xi \quad \Rightarrow \quad \phi = G * \xi + G * F(\phi)$

  Use Banach's fixed point theorem, but on which function space?

Definition: Hölder–Besov spaces $\mathcal{C}^\alpha$

For $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha - 1} \qquad (\not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^\alpha)$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

Parabolic scaling $\mathcal{C}_\mathfrak{s}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

1. $\alpha \notin \mathbb{Z}$, $f \in \mathcal{C}_\mathfrak{s}^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_\mathfrak{s}^{\alpha+2} \qquad$ (Schauder)
2. $\xi \in \mathcal{C}_\mathfrak{s}^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

Consequence: $G * \xi \in \mathcal{C}_\mathfrak{s}^\alpha$ a.s. $\forall \alpha < \frac{2-d}{2} \leqslant 0$ for $d \geqslant 2$

Mild solutions and Hölder–Besov spaces

- $(\partial_t - \Delta)\phi = h \quad \Rightarrow \quad \phi = G * h$ where G heat kernel
- $(\partial_t - \Delta)\phi = F(\phi) + \xi \quad \Rightarrow \quad \phi = G * \xi + G * F(\phi)$

  Use Banach's fixed point theorem, but on which function space?

Definition: Hölder–Besov spaces $\mathcal{C}^\alpha$

For $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha - 1} \qquad (\not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^\alpha)$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

Parabolic scaling $\mathcal{C}_\mathfrak{s}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

1. $\alpha \notin \mathbb{Z}$, $f \in \mathcal{C}_\mathfrak{s}^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_\mathfrak{s}^{\alpha+2} \qquad$ (Schauder)
2. $\xi \in \mathcal{C}_\mathfrak{s}^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

Consequence: $G * \xi \in \mathcal{C}_\mathfrak{s}^\alpha$ a.s. $\forall \alpha < \frac{2-d}{2} \leqslant 0$ for $d \geqslant 2$

Mild solutions and Hölder–Besov spaces

- $(\partial_t - \Delta)\phi = h \quad \Rightarrow \quad \phi = G * h$ where G heat kernel
- $(\partial_t - \Delta)\phi = F(\phi) + \xi \quad \Rightarrow \quad \phi = G * \xi + G * F(\phi)$

  Use Banach's fixed point theorem, but on which function space?

Definition: Hölder–Besov spaces $\mathcal{C}^\alpha$

For $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha-1} \qquad (\not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^\alpha)$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

Parabolic scaling $\mathcal{C}_{\mathfrak{s}}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

1. $\alpha \notin \mathbb{Z}$, $f \in \mathcal{C}_{\mathfrak{s}}^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_{\mathfrak{s}}^{\alpha+2} \qquad$ (Schauder)
2. $\xi \in \mathcal{C}_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

Consequence: $G * \xi \in \mathcal{C}_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < \frac{2-d}{2} \leqslant 0$ for $d \geqslant 2$

Mild solutions and Hölder–Besov spaces

- $(\partial_t - \Delta)\phi = h \quad \Rightarrow \quad \phi = G * h$ where G heat kernel
- $(\partial_t - \Delta)\phi = F(\phi) + \xi \quad \Rightarrow \quad \phi = G * \xi + G * F(\phi)$

  Use Banach's fixed point theorem, but on which function space?

Definition: Hölder–Besov spaces $\mathcal{C}^\alpha$

For $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha - 1} \qquad (\not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^\alpha)$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$ with $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$

Parabolic scaling $\mathcal{C}_{\mathfrak{s}}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

Facts:

1. $\alpha \notin \mathbb{Z}$, $f \in \mathcal{C}_{\mathfrak{s}}^\alpha \quad \Rightarrow \quad G * f \in \mathcal{C}_{\mathfrak{s}}^{\alpha+2} \qquad$ (Schauder)
2. $\xi \in \mathcal{C}_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < -\frac{d+2}{2}$

Consequence: $G * \xi \in \mathcal{C}_{\mathfrak{s}}^\alpha$ a.s. $\forall \alpha < \frac{2-d}{2} \leqslant 0$ for $d \geqslant 2$

Regularity structures

$\partial_t \phi = \Delta\phi - \phi^3 + \xi^\delta$ $\qquad$ $\xi^\delta = \varrho^\delta * \xi$ mollified noise, $\varrho^\delta(t,x) = \frac{1}{\delta^5}\varrho\left(\frac{t}{\delta^2}, \frac{x}{\delta}\right)$

$$
\begin{array}{ccc}
(\phi_0, Z^\delta) & \xrightarrow{\ \mathcal{S}\ } & \Phi \\
\Psi \big\uparrow & & \big\downarrow \mathscr{R} \\
(\phi_0, \xi^\delta) & \xrightarrow[\ \bar{\mathcal{S}}\ ]{} & \phi^\delta
\end{array}
$$

$\phi = G * (\xi^\delta - \phi^3) \quad \Leftrightarrow \quad \Phi = \mathcal{I}(\Xi - \Phi^3) + \varphi\mathbf{1}$ + polynomial terms

$\Phi_0 = 0$

$\Phi_1 = \mathcal{I}(\Xi) + \varphi\mathbf{1} + \dots \qquad \Phi_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi\mathcal{I}(\Xi)^2 + 3\varphi^2\mathcal{I}(\Xi) + \varphi^3\mathbf{1} + \dots$

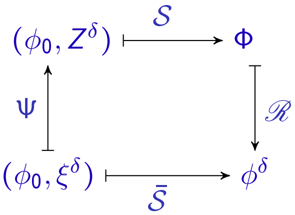
$\Phi_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2\mathcal{I}(\mathcal{I}(\Xi)) + \varphi\mathbf{1} + \dots$

degree $\quad 1-\frac{d}{2} \quad 5-\frac{3d}{2} \quad 4-d \quad 3-\frac{d}{2} \quad 0$

Locally subcritical: Hölder exponents are bdd below

Regularity structures

$\partial_t \phi = \Delta\phi - \phi^3 + \xi^\delta$ $\qquad$ $\xi^\delta = \varrho^\delta * \xi$ mollified noise, $\varrho^\delta(t,x) = \frac{1}{\delta^5}\varrho(\frac{t}{\delta^2}, \frac{x}{\delta})$



$\phi = G * (\xi^\delta - \phi^3) \quad \Leftrightarrow \quad \Phi = \mathcal{I}(\Xi - \Phi^3) + \varphi\mathbf{1} + \text{polynomial terms}$

$\Phi_0 = 0$

$\Phi_1 = \mathcal{I}(\Xi) + \varphi\mathbf{1} + \dots \qquad \Phi_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi\mathcal{I}(\Xi)^2 + 3\varphi^2\mathcal{I}(\Xi) + \varphi^3\mathbf{1} + \dots$

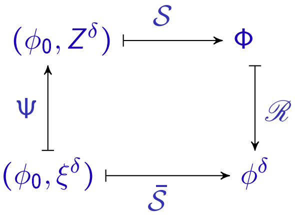
$\Phi_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2\mathcal{I}(\mathcal{I}(\Xi)) + \varphi\mathbf{1} + \dots$

degree $\quad 1-\frac{d}{2} \quad 5-\frac{3d}{2} \quad 4-d \quad 3-\frac{d}{2} \quad 0$

Locally subcritical: Hölder exponents are bdd below

Regularity structures

$\partial_t \phi = \Delta\phi - \phi^3 + \xi^\delta \qquad \xi^\delta = \varrho^\delta * \xi$ mollified noise, $\varrho^\delta(t,x) = \frac{1}{\delta^5}\varrho\left(\frac{t}{\delta^2}, \frac{x}{\delta}\right)$



$\phi = G * (\xi^\delta - \phi^3) \quad \Leftrightarrow \quad \Phi = \mathcal{I}(\Xi - \Phi^3) + \varphi\mathbf{1}$ + polynomial terms

$$\Phi_0 = 0$$

$$\Phi_1 = \mathcal{I}(\Xi) + \varphi\mathbf{1} + \dots \qquad \Phi_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi\mathcal{I}(\Xi)^2 + 3\varphi^2\mathcal{I}(\Xi) + \varphi^3\mathbf{1} + \dots$$

$$\Phi_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2\mathcal{I}(\mathcal{I}(\Xi)) + \varphi\mathbf{1} + \dots$$

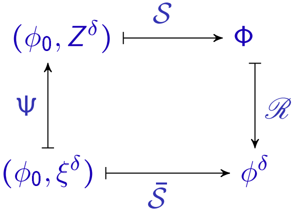
$$=: \quad - \quad - 3\varphi \quad - 3\varphi^2 \quad + \varphi\mathbf{1} + \dots$$

degree	$1-\frac{d}{2}$	$5-\frac{3d}{2}$	$4-d$	$3-\frac{d}{2}$	0

Locally subcritical: Hölder exponents are bdd below

Regularity structures

$\partial_t \phi = \Delta\phi - \phi^3 + \xi^\delta$ $\qquad$ $\xi^\delta = \varrho^\delta * \xi$ mollified noise, $\varrho^\delta(t,x) = \frac{1}{\delta^5}\varrho(\frac{t}{\delta^2}, \frac{x}{\delta})$



$\phi = G * (\xi^\delta - \phi^3) \quad \Leftrightarrow \quad \Phi = \mathcal{I}(\Xi - \Phi^3) + \varphi\mathbf{1}$ + polynomial terms

$\Phi_0 = 0$

$\Phi_1 = \mathcal{I}(\Xi) + \varphi\mathbf{1} + \dots \qquad \Phi_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi\mathcal{I}(\Xi)^2 + 3\varphi^2\mathcal{I}(\Xi) + \varphi^3\mathbf{1} + \dots$

$\Phi_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2\mathcal{I}(\mathcal{I}(\Xi)) + \varphi\mathbf{1} + \dots$

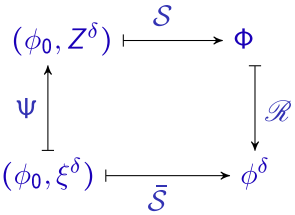
$=:$  $-$  $- 3\varphi$  $- 3\varphi^2$  $+ \varphi\mathbf{1} + \dots$

degree $\quad 1-\frac{d}{2} \quad 5-\frac{3d}{2} \quad 4-d \quad 3-\frac{d}{2} \quad 0$

Locally subcritical: Hölder exponents are bdd below

Regularity structures

$\partial_t \phi = \Delta\phi - \phi^3 + \xi^\delta$ $\qquad$ $\xi^\delta = \varrho^\delta * \xi$ mollified noise, $\varrho^\delta(t,x) = \frac{1}{\delta^5}\varrho(\frac{t}{\delta^2}, \frac{x}{\delta})$



$\phi = G * (\xi^\delta - \phi^3) \quad \Leftrightarrow \quad \Phi = \mathcal{I}(\Xi - \Phi^3) + \varphi\mathbf{1} +$ polynomial terms

$$\Phi_0 = 0$$

$$\Phi_1 = \mathcal{I}(\Xi) + \varphi\mathbf{1} + \dots \qquad \Phi_1^3 = \mathcal{I}(\Xi)^3 + 3\varphi\mathcal{I}(\Xi)^2 + 3\varphi^2\mathcal{I}(\Xi) + \varphi^3\mathbf{1} + \dots$$

$$\Phi_2 = \mathcal{I}(\Xi) - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\mathcal{I}(\mathcal{I}(\Xi)^2) - 3\varphi^2\mathcal{I}(\mathcal{I}(\Xi)) + \varphi\mathbf{1} + \dots$$

$=:$  $-$  -3φ  $-3\varphi^2$  $+\varphi\mathbf{1} + \dots$

degree $\quad 1-\frac{d}{2} \quad 5-\frac{3d}{2} \quad 4-d \quad 3-\frac{d}{2} \quad 0$

Locally subcritical: Hölder exponents are bdd below

Renormalisation

$$
\begin{array}{ccc}
(\phi_0, M^g Z^\delta) & \xrightarrow{\ \mathcal{S}\ } & \Phi_M \\
\Big\uparrow{\scriptstyle M^g \circ \Psi} & & \Big\downarrow{\scriptstyle \mathscr{R}^g} \\
(\phi_0, \xi^\delta) & \xrightarrow[\ \bar{\mathcal{S}}_\Upsilon\ ]{} & \phi^\delta
\end{array}
$$

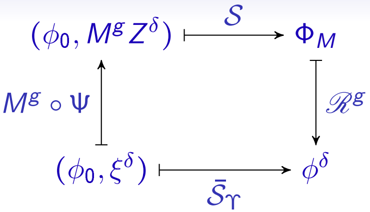
- $M^g Z^\delta$: renormalised model, compatible with $M^g \tau = (g \otimes \mathrm{id})\hat{\Delta}\tau$
  g BPHZ character
  $\langle g, \vee \rangle = -C_\delta^{(1)} = \mathcal{O}(\delta^{-1})$, $\langle g, \text{tree} \rangle = -C_\delta^{(2)} = \mathcal{O}(\log(\delta^{-1}))$
- $\bar{\mathcal{S}}_\Upsilon$: solution map of renormalised SPDE

$$\partial_t \phi^\delta = \Delta\phi^\delta - (\phi^\delta)^3 + \left[3C_\delta^{(1)} - 9C_\delta^{(2)}\right]\phi^\delta + \xi^\delta$$

Theorem [Hairer 2014]:

For initial conditions in $\mathcal{C}^\eta$ with $\eta > -\frac{2}{3}$, the sequence of (local in time) ϕ^δ converges in probability to a limit as $\delta \to 0$

Renormalisation



- $M^g Z^\delta$: renormalised model, compatible with $M^g \tau = (g \otimes \mathrm{id})\hat{\Delta}\tau$
  g BPHZ character
  $\langle g, \vee \rangle = -C_\delta^{(1)} = \mathcal{O}(\delta^{-1})$, $\langle g, \vee \rangle = -C_\delta^{(2)} = \mathcal{O}(\log(\delta^{-1}))$
- $\bar{\mathcal{S}}_\Upsilon$: solution map of renormalised SPDE

$$\partial_t \phi^\delta = \Delta\phi^\delta - (\phi^\delta)^3 + \left[3C_\delta^{(1)} - 9C_\delta^{(2)}\right]\phi^\delta + \xi^\delta$$

Theorem [Hairer 2014]:

For initial conditions in $\mathcal{C}^\eta$ with $\eta > -\frac{2}{3}$, the sequence of (local in time) ϕ^δ converges in probability to a limit as $\delta \to 0$

Renormalisation

$$
\begin{array}{ccc}
(\phi_0, M^g Z^\delta) & \xrightarrow{\quad \mathcal{S} \quad} & \Phi_M \\
\uparrow {\scriptstyle M^g \circ \Psi} & & \downarrow {\scriptstyle \mathscr{R}^g} \\
(\phi_0, \xi^\delta) & \xrightarrow[\quad \bar{\mathcal{S}}_\Upsilon \quad]{} & \phi^\delta
\end{array}
$$

- $M^g Z^\delta$: renormalised model, compatible with $M^g \tau = (g \otimes \mathrm{id})\hat{\Delta}\tau$
  g BPHZ character
  $\langle g,$  $\rangle = -C_\delta^{(1)} = \mathcal{O}(\delta^{-1})$, $\langle g,$  $\rangle = -C_\delta^{(2)} = \mathcal{O}(\log(\delta^{-1}))$
- $\bar{\mathcal{S}}_\Upsilon$: solution map of renormalised SPDE

$$\partial_t \phi^\delta = \Delta\phi^\delta - (\phi^\delta)^3 + \left[3C_\delta^{(1)} - 9C_\delta^{(2)}\right]\phi^\delta + \xi^\delta$$

Theorem [Hairer 2014]:

For initial conditions in $\mathcal{C}^\eta$ with $\eta > -\frac{2}{3}$, the sequence of (local in time) ϕ^δ converges in probability to a limit as $\delta \to 0$

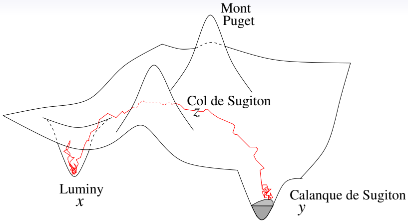
(Stochastic) Allen–Cahn equation on $\mathbb{T}^2$

$\partial_t\phi(t,x) = \nu(\varepsilon t)\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3 + \sigma\xi(t,x)$

(Online: https://youtu.be/yX0EAxZHNCQ)

Metastability in gradient SDEs

$$dx_t = -\nabla V(x_t)\, dt + \sqrt{2\varepsilon}\, dW_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$
when starting in x

Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

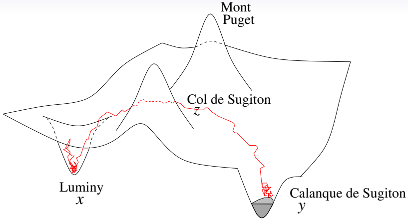
$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon} \left[1 + \mathcal{O}_\varepsilon(1)\right]$$

Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, ...

Metastability in gradient SDEs

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 : x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$
when starting in x

Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

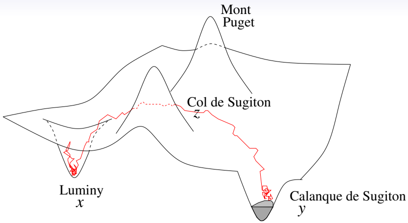
$$\mathbb{E}[\tau_y^x] = 2\pi\sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1|\nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, ...

Metastability in gradient SDEs

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$
when starting in x

Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon} \bigl[1 + \mathcal{O}_\varepsilon(1)\bigr]$$

Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, . . .

Eyring–Kramers law for 1D SPDEs

$\partial_t \phi = \Delta\phi + \phi - \phi^3 + \sqrt{2\varepsilon}\,\xi, \qquad x \in [0, L], \; \tau_+ = \inf\{t > 0\colon \|\phi_t - 1\|_\infty < \delta\}$

Ev at saddle $\phi = 0$: $\lambda_k = (\frac{k\pi}{L})^2 - 1$, ev at minima $\phi = \pm 1$: $\nu_k = (\frac{k\pi}{L})^2 + 2$

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{-1}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\; \mathrm{e}^{(V[0]-V[-1])/\varepsilon} \Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k. Results also for L near π and periodic b.c.

- Prefactor involves a Fredholm determinant: $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[1 - 3(-\Delta_\perp + 2)^{-1}\big]$$

converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class

Eyring–Kramers law for 1D SPDEs

$\partial_t \phi = \Delta\phi + \phi - \phi^3 + \sqrt{2\varepsilon}\,\xi, \qquad x \in [0,L],\ \tau_+ = \inf\{t > 0 \colon \|\phi_t - 1\|_\infty < \delta\}$

Ev at saddle $\phi = 0$: $\lambda_k = \left(\frac{k\pi}{L}\right)^2 - 1$, ev at minima $\phi = \pm 1$: $\nu_k = \left(\frac{k\pi}{L}\right)^2 + 2$

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{-1}[\tau_+] = 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k}}\ \mathrm{e}^{(V[0]-V[-1])/\varepsilon}\Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k. Results also for L near π and periodic b.c.

- Prefactor involves a Fredholm determinant:
  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k} = \det\left[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\right] = \det\left[\mathbf{1} - 3(-\Delta_\perp + 2)^{-1}\right]$$

  converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class

Metastability in the 2D Allen–Cahn equation

$\partial_t \phi = \Delta\phi + \left[1 + 3\varepsilon C_N\right]\phi - P_N(\phi^3) + \sqrt{2\varepsilon}\xi_N, \qquad C_N = \mathrm{Tr}\left[(-\Delta_N + 1)^{-1}\right]$

- Large-deviation principle: [Hairer & Weber, 2015]
- Without renormalisation, naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right) \simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} = -\infty$$

- With renormalisation, $V(0) - V(-1) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
  Inverse of prefactor involves Carleman–Fredholm determinant:
  $\det_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\mathrm{e}^{-\mathrm{Tr}\,T}$ with $T = 3(-\Delta_\perp - 1)^{-1}$
  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

Theorem: [B, Di Gesù, Weber, 2017, Tsatsoulis & Weber 2018]
For $L < \pi$, Eyring–Kramers law holds with finite prefactor given by Carleman–Fredholm determinant

Remark: If $C_N = \mathrm{Tr}\left[(-\Delta_N + \theta)^{-1}\right]$, prefactor depends on θ

Metastability in the 2D Allen–Cahn equation

$$\partial_t \phi = \Delta\phi + \big[1 + 3\varepsilon C_N\big]\phi - P_N(\phi^3) + \sqrt{2\varepsilon}\,\xi_N, \qquad C_N = \operatorname{Tr}\big[(-\Delta_N + 1)^{-1}\big]$$

- Large-deviation principle: [Hairer & Weber, 2015]
- Without renormalisation, naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1 - \big(\frac{L}{|k|\pi}\big)^2}{1 + 2\big(\frac{L}{|k|\pi}\big)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\bigg(1 - \frac{3L^2}{|k|^2\pi^2}\bigg) \simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} = -\infty$$

- With renormalisation, $V(0) - V(-1) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N \varepsilon$
  Inverse of prefactor involves Carleman–Fredholm determinant:
  $\det_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\mathrm{e}^{-\operatorname{Tr} T}$ with $T = 3(-\Delta_\perp - 1)^{-1}$
  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

Theorem: [B, Di Gesù, Weber, 2017, Tsatsoulis & Weber 2018]

For $L < \pi$, Eyring–Kramers law holds with finite prefactor given by Carleman–Fredholm determinant

Remark: If $C_N = \operatorname{Tr}[(-\Delta_N + \theta)^{-1}]$, prefactor depends on θ

Metastability in the 2D Allen–Cahn equation

$$\partial_t \phi = \Delta\phi + \big[1 + 3\varepsilon C_N\big]\phi - P_N(\phi^3) + \sqrt{2\varepsilon}\xi_N, \qquad C_N = \operatorname{Tr}\big[(-\Delta_N + 1)^{-1}\big]$$

- Large-deviation principle: [Hairer & Weber, 2015]
- Without renormalisation, naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1 - \big(\frac{L}{|k|\pi}\big)^2}{1 + 2\big(\frac{L}{|k|\pi}\big)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1 - \frac{3L^2}{|k|^2\pi^2}\right) \simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} = -\infty$$

- With renormalisation, $V(0) - V(-1) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
  Inverse of prefactor involves Carleman–Fredholm determinant:
  $\det_2(\mathbf{1} + T) = \det(\mathbf{1} + T)e^{-\operatorname{Tr} T}$ with $T = 3(-\Delta_\perp - 1)^{-1}$
  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

Theorem: [B, Di Gesù, Weber, 2017, Tsatsoulis & Weber 2018]

For $L < \pi$, Eyring–Kramers law holds with finite prefactor given by Carleman–Fredholm determinant

Remark: If $C_N = \operatorname{Tr}\big[(-\Delta_N + \theta)^{-1}\big]$, prefactor depends on θ

Metastability in the 2D Allen–Cahn equation

$\partial_t \phi = \Delta\phi + \big[1 + 3\varepsilon C_N\big]\phi - P_N(\phi^3) + \sqrt{2\varepsilon}\xi_N, \qquad C_N = \mathrm{Tr}[(-\Delta_N + 1)^{-1}]$

- ▷ Large-deviation principle: [Hairer & Weber, 2015]
- ▷ Without renormalisation, naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\big(\frac{L}{|k|\pi}\big)^2}{1+2\big(\frac{L}{|k|\pi}\big)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\bigg(1 - \frac{3L^2}{|k|^2\pi^2}\bigg) \simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} = -\infty$$

- ▷ With renormalisation, $V(0) - V(-1) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
  Inverse of prefactor involves Carleman–Fredholm determinant:
  $\det_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\mathrm{e}^{-\mathrm{Tr}\,T}$ with $T = 3(-\Delta_\perp - 1)^{-1}$
  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

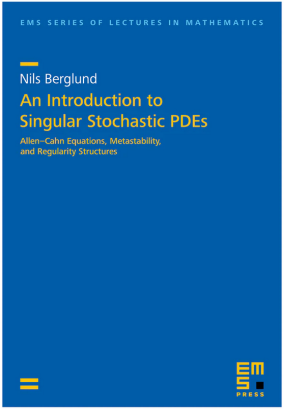
Theorem: [B, Di Gesù, Weber, 2017, Tsatsoulis & Weber 2018]

For $L < \pi$, Eyring–Kramers law holds with finite prefactor given by Carleman–Fredholm determinant

Remark: If $C_N = \mathrm{Tr}[(-\Delta_N + \theta)^{-1}]$, prefactor depends on θ

References

- N. B. & Tom Klose, *Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras*, arXiv/2207.08555 (2022)
- N. B. & Barbara Gentz, *Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond*, Electronic J. Probability **18**, (24):1–58 (2013)
- N. B., Giacomo Di Gesù & Hendrik Weber, *An Eyring–Kramers law for the stoch. Allen–Cahn equation in dimension two*, Electronic J. Probability **22**, 1–27 (2017)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/X_2023.pdf

Borel resummation: The Φ^4_0 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

  $$Z(\varepsilon) = \int_{-\infty}^{\infty} e^{-V(\phi)}\,d\phi = \int_{-\infty}^{\infty} e^{-\phi^2/2}\,e^{-\varepsilon\phi^4/4}\,d\phi$$

  $$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0}\Big(-\frac{\varepsilon}{4}\Big)^n\frac{(4n-1)!!}{n!} = \sum_{n\geqslant 0} a_n\varepsilon^n, \qquad a_n \sim n!$$

- Borel transform:

  $$Z(\varepsilon) \asymp \sum_{n\geqslant 0} a_n\varepsilon^n\frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0}\frac{a_n\varepsilon^n}{n!}\int_0^\infty t^n e^{-t}\,dt$$

  $$Z_{\text{Borel}}(\varepsilon) = \int_0^\infty e^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,dt = \int_0^\infty e^{-t}\,\mathcal{B}Z(\varepsilon t)\,dt$$

  where $\mathcal{B}Z(t) = \sum_{n\geqslant 0}\frac{a_n}{n!}t^n$

Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \operatorname{Re}\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\text{Borel}}(\varepsilon)$ in D_R

Borel resummation: The Φ^4_0 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

  $$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi$$

  $$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0}\Bigl(-\frac{\varepsilon}{4}\Bigr)^n \frac{(4n-1)!!}{n!} = \sum_{n\geqslant 0} a_n\varepsilon^n, \qquad a_n \sim n!$$

- Borel transform:

  $$Z(\varepsilon) \asymp \sum_{n\geqslant 0} a_n\varepsilon^n \frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0} \frac{a_n\varepsilon^n}{n!}\int_0^\infty t^n\,\mathrm{e}^{-t}\,\mathrm{d}t$$

  $$Z_{\text{Borel}}(\varepsilon) = \int_0^\infty \mathrm{e}^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\,\mathcal{B}Z(\varepsilon t)\,\mathrm{d}t$$

  where $\mathcal{B}Z(t) = \sum_{n\geqslant 0}\frac{a_n}{n!}t^n$

Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \operatorname{Re}\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\text{Borel}}(\varepsilon)$ in D_R

Borel resummation: The Φ^4_0 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

$$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi$$

$$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0}\Big(-\frac{\varepsilon}{4}\Big)^n \frac{(4n-1)!!}{n!} = \sum_{n\geqslant 0} a_n\varepsilon^n, \qquad a_n \sim n!$$

- Borel transform:

$$Z(\varepsilon) \asymp \sum_{n\geqslant 0} a_n\varepsilon^n \frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0}\frac{a_n\varepsilon^n}{n!}\int_0^{\infty} t^n\,\mathrm{e}^{-t}\,\mathrm{d}t$$

$$Z_{\mathrm{Borel}}(\varepsilon) = \int_0^{\infty}\mathrm{e}^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,\mathrm{d}t = \int_0^{\infty}\mathrm{e}^{-t}\,\mathcal{B}Z(\varepsilon t)\,\mathrm{d}t$$

where $\mathcal{B}Z(t) = \sum_{n\geqslant 0}\frac{a_n}{n!}t^n$

Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \operatorname{Re}\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^{n} a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\mathrm{Borel}}(\varepsilon)$ in D_R

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable
- $\mathrm{e}^{-\alpha X} = P(X) + F(X), \quad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log \mathrm{e}^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P})\, \mathrm{e}^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$
- $F(X) = S_n + R_n, \quad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \quad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$

  $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \qquad R_n = \frac{(-\alpha)^n}{n!} X^n \,\mathrm{e}^{-\alpha\theta X}$
- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable

- $\mathrm{e}^{-\alpha X} = P(X) + F(X), \quad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log \mathrm{e}^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \, \mathrm{e}^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$

- $F(X) = S_n + R_n, \quad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \quad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$

  $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \qquad R_n = \frac{(-\alpha)^n}{n!} X^n \, \mathrm{e}^{-\alpha\theta X}$

- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable
- $$e^{-\alpha X} = P(X) + F(X), \qquad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$$
  $$\log e^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P})\, e^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$$
- $$F(X) = S_n + R_n, \quad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \quad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$$
  $$(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \qquad R_n = \frac{(-\alpha)^n}{n!} X^n e^{-\alpha\theta X}$$
- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ in D_R: doable
- $e^{-\alpha X} = P(X) + F(X), \quad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log e^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\text{BPHZ}} \circ \mathscr{P})\, e^{-\alpha X} = -(\Pi_N^{\text{BPHZ}} \circ \mathscr{P}) F(X)$
- $F(X) = S_n + R_n, \quad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \quad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$

  $(\Pi_N^{\text{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \quad R_n = \frac{(-\alpha)^n}{n!} X^n e^{-\alpha\theta X}$
- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\text{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)

Borel resummation: The Φ^4_3 model

- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable
- $\mathrm{e}^{-\alpha X} = P(X) + F(X), \qquad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log \mathrm{e}^{-\alpha X - \beta Y - \gamma} = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \, \mathrm{e}^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$
- $F(X) = S_n + R_n, \qquad S_n = \sum_{p=4}^{n} \frac{(-\alpha)^p}{p!} X^p, \qquad R_n = \sum_{p=n+1}^{\infty} \frac{(-\alpha)^p}{p!} X^p$

  $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) S_n \asymp \sum_{p=4}^{n-1} p! \varepsilon^p, \qquad R_n = \frac{(-\alpha)^n}{n!} X^n \, \mathrm{e}^{-\alpha\theta X}$
- Control remainder by using
  - Moment bound
  - Sharp estimates on $(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) \Gamma_{pp}^{(k)}$ (Hairer 2018, B & Bruned 2019)