Directed Polymers in Random Medium

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Purpose.

Describe random paths which are not only weighted according to their lengths, but also according to random impurities which are met on the way

- Motivations:
  - Model for polymers: (i) irregular chains (ii) without self-intersections (iii) interacting with the environment
  - interface in random medium ($d = 1$),
  - random growth (KPZ class), . . .
  - non-zero temperature version of oriented percolation (last passage)
- Directed: our polymer leaves in dimension $d + 1$, and stretches in the first direction
  $\longrightarrow$ environment regenerates at each step, allows for martingales
- Discrete or continuous models

Medium: independent i.d. real r.v. $\eta(t,x),\, t \in \{1,2,\ldots\},\, x \in \mathbb{Z}^d$
"impurities" $\eta \sim Q$; $d \geq 1$: transverse dim. Assume $\forall \beta$

$$\exp \lambda(\beta) := Q[\exp \beta\eta(t,x)] < \infty$$

Path ω, P : simple random walk on $\mathbb{Z}^d$ (nearest neighbours)

Energy of path ω in time n: $H_n(\omega) = \sum_{t=1}^n \eta(t,\omega_t)$

Polymer measure = probability measure μ_n on path space

$$d\mu_n(\omega) = \frac{\exp\left(\beta H_n(\omega)\right)}{Z_n}\, dP(\omega)$$

with $\beta \in \mathbb{R}_+$, and $Z_n = P[\exp\left(\beta H_n(\omega)\right)]$.

The polymer ω is:

- attracted to locations (t,x) with $\eta(t,x) > 0$ (*rewards*)
- repelled by those with $\eta(t,x) < 0$ (*penalties, obstacles*)

more and more as $\beta \nearrow$ ($\beta \geq 0$).

Some Guidelines:

☞ $\beta = 0$: Simple Random Walk
$\beta = +\infty$: last passage, oriented percolation

☞ $\mathbb{Z}_+ \times \mathbb{Z}^d$ replaced by the tree: branching process

☞ *related, but more distant models:*
• RW in soft obstacles: Sznitman; Antal'95, Wüthrich'98
• heteropolymers near interface $H_n = \sum_{t \leq n} (\eta(t) + h) \mathrm{sign}(\omega_t)$
$d = 1$ Bolthausen, den Hollander, Biskup, Bodineau, Giacomin. . .

Outline.

Questions: for typical medium η, what is the polymer behavior under μ_n ? (n large)

1. Expand $\boxed{\ln Z_n \sim np}$; $\boxed{\mathrm{Var} \ln Z_n \asymp n^{2\chi}}$; $p, \chi(d, \beta, Q) =?$
2. Order of displacement: $\boxed{\mu^n(|\omega_n|) \asymp n^{\xi}}$
   Diffusivity or super-diffusivity ($\xi =$ or $> 1/2$)?
3. scaling identity between exponents (conjecture) $\boxed{\chi = 2\xi - 1}$

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Intuitive picture:

If the polymer does not feel too much the medium, it should behave like SRW

But if the disorder is strong enough, typical paths should be pinned down to favourable clouds (localization), which are at a distance (superdiffusivity); these clouds being small, thermodynamic quantities mostly depend on a few r.v. (large fluctuations)

What does "strong disorder" mean ?

η: Poisson field in $\mathbb{R}^+ \times \mathbb{R}^d$, with intensity $dtdx$
P : Wiener measure on $\mathbb{R}^d$
V_t : “tube”around the *graph* of the Brownian path ω,

$$V_t = V_t(\omega) = \{(s,x) \; ; \; s \in (0,t], \; x \in U(\omega_s)\},$$

with $U(x) \subset \mathbb{R}^d$ the closed ball with volume 1 and center x.

Polymer measure

$$\mu_t(d\omega) = \frac{\exp\left(\beta\eta(V_t)\right)}{Z_t} \; P(d\omega),$$

C-Yoshida’03

point-to-point partition function

$$Z_t(x) = P[e^{\beta H_t} : \omega_t = x] \,, \quad h_t(x) = \ln Z_t(x)$$

satisfies “formally” to a KPZ equation

$$dh_t(y) = \frac{1}{2}\left(\Delta h_t(y) + |\nabla h_t(y)|^2\right) dt + \beta \, \eta(dt \times U(y))$$

Phenomenological equation for growth models

Plan:

1. Thermodynamics of disordered systems
2. Z_n as a martingale
3. $\ln Z_n$ as a super-martingale
4. Strong disorder and localization
5. Continuous model

$$\lim_{n\to\infty} \frac{1}{n} Q[\ln Z_n] \overset{\text{sub-addit.}}{=:} p(\beta) \qquad \text{"}quenched\ pressure''$$

$$\overset{\text{concent.}}{=} \lim_{n\to\infty} \frac{1}{n} \ln Z_n \qquad Q - a.s$$

Standard concentration inequality (if $Q[e^{\delta \eta(t,x)^2}] < \infty$):

$$Q\left[\frac{1}{n} |\ln Z_n - Q[\ln Z_n]| \geq \varepsilon\right] \leq e^{-Cn\varepsilon^2} \qquad \text{hence} \qquad \boxed{\chi \leq 1/2}$$

Jensen's inequality $Q[\ln Z_n] \leq \ln Q[Z_n] = n\lambda$, hence $p \leq \lambda$.

Proposition 1: function $\beta \mapsto \lambda(\beta) - p(\beta)$ is non-decreasing on $\mathbb{R}_+$

Corollary: $\exists \beta_c^p \in [0, \infty]$ such that: $\qquad p(\beta) < \lambda(\beta) \iff \beta > \beta_c^p$

$$\text{Is} \quad \beta_c^p := \inf\{\beta \geq 0; p(\beta) < \lambda(\beta)\} \quad \text{finite?}$$

Adapting an argument for the tree case (eg, Kahane-Peyrière ’76):

$$(KP) \qquad \beta\lambda'(\beta) - \lambda(\beta) > \ln 2d \implies p(\beta) < \lambda(\beta)$$

Note: If the law of $\eta(t, x)$ has no mass at its maximum, condition (KP) holds for β large enough

β	0	β_c^p	∞	$\vert$
$\vert$	$-----------$	$\vert$	$--------------$	$\vert$
$\vert$	$p = \lambda$	$\vert$	$p < \lambda$	$\vert$
$\vert$	$-----------$	$\vert$	$--------------$	$\vert$

2- W_n as a Martingale.

$$W_n := Z_n e^{-n\lambda}$$

positive martingale w.r.t. $\mathcal{G}_n = \sigma\{\eta(t,x); 1 \le t \le n, x \in \mathbb{Z}^d\}$

[Bolthausen'89].

$$W_n \xrightarrow{\text{a.s.}} W_\infty \ , \quad \text{as } n \to \infty$$

with $\{W_\infty = 0\}$ tail event: By Kolmogorov's 0-1 law,

$$\begin{cases} \text{either} & W_\infty > 0 \text{ a.s.} \\ \text{or} & W_\infty = 0 \text{ a.s.} \end{cases}$$

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As in Prop. 1, monotonicity: for $\beta \leq \beta'$,

$$\text{(SD) at } \beta \quad \Rightarrow \quad \text{(SD) at } \beta'$$

$\longrightarrow$ Another phase diagram, with critical point

$$\beta_c = \inf\{\beta \geq 0; (\text{SD}) \text{ at } \beta\} \ldots$$

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... still of interest:

$$(\text{WD}) \iff Q \ln Z_n \sim n\lambda, \chi = 0 \Longrightarrow p = \lambda, \chi = 0 \;\text{ in Question1}-$$

$$\text{Clearly} \quad \beta_c \leq \beta_c^p \;. \qquad \text{Is it } = ?$$

Condition (L2): with $\mathrm{Escape} = \{\omega_n \neq 0 \; \forall n \geq 1\}$

$$\lambda(2\beta) - 2\lambda(\beta) < -\ln P(\mathrm{Escape})$$

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$$\lambda(2\beta) - 2\lambda(\beta) < -\ln P(\mathrm{Escape})$$

- (L2) holds when $d \geq 3$ provided β is small (for arbitrary Q)...
- ...but not necessarily:

In dimension $d \geq 3$, if $\eta \sim$ Bernoulli(p) with $p > P(\mathrm{Escape})$,

then (L2) holds for all $\beta \geq 0$.

$\longrightarrow$ Reminiscent of percolating regime.

Theorem 2 Assume condition (L2): Then,

1. (WD) holds
2. Diffusivity holds: central limit theorem for Q-a.e. η, invariance principle, local limit theorem
3. $\mu_n(H_n) - n\lambda'(\beta) \xrightarrow{a.s.} \frac{d}{d\beta} W_\infty / W_\infty$ □

Bolthausen$'89^{(1,2)}$, Imbrie-Spencer$'88^{(2)}$, Albeverio-Zhou$'96^{(2)}$, Sinaï$'95^{(2)}$, C-Yoshida$'04^{(3)}$, Birkner$'04^{(1)}$,...

- (KP) $\Rightarrow W_n = O(e^{-n\delta})$ a.s.
- Small dimension, $\forall \beta \neq 0$: $$W_n \begin{cases} = 0(e^{-\delta n^{1/3}}) \;, & d = 1 \\ \to 0 \;, & d = 2 \end{cases}$$

Estimate fractional moments $Q[W_t^\theta], \theta \in (0,1)$ *with a “differential” inequality*

Phase diagram, when η has no mass at the top of his support

β	0	β_c	∞
$d \geq 3$	(WD)		(SD)
$d = 1, 2$		(SD)	

3- $\ln W_n$ as a super-martingale.

Take two *replicas* $\omega, \widetilde{\omega}$ (=independent polymers in the same environment η), and define

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Theorem 3 For all $\beta \neq 0$

- criterium (WD) *versus* (SD) : $\quad W_\infty = 0 \overset{\text{a.s.}}{\iff} \sum_{n \geq 1} I_n = \infty$
- Then, $\quad -\ln W_n \asymp \sum_{t \leq n} I_t$ $\square$

Notation: $\quad f \asymp g \quad$ iff $\quad \left(\liminf_{t \to \infty} \frac{f(t)}{g(t)} > 0, \limsup \frac{f(t)}{g(t)} < \infty\right)$

Carmona-Hu'02, C-Shiga-Yoshida'03 *Quantitative statement !*

Doob's decomposition of supermartingale $\ln W_n = -A_n + M_n$

Write $\frac{W_n}{W_{n-1}} = 1 + U_n$ with $U_n = \mu_{n-1}[e^{\beta\eta(n,\omega_n)-\lambda} - 1]$ conditionnally centered

$$\begin{aligned} A_n - A_{n-1} &= -Q[\ln W_n - \ln W_{n-1} | \mathcal{F}_{n-1}] \;=\; -Q[\ln(1+U_n)|\mathcal{F}_{n-1}] \\ &\asymp -Q[U_n^2|\mathcal{F}_{n-1}] \\ &= -\mu_{n-1}^{\otimes 2} Q\left[(e^{\beta\eta(n,\omega_n)-\lambda} - 1)(e^{\beta\eta(n,\tilde{\omega}_n)-\lambda} - 1)|\mathcal{F}_{n-1}\right] \\ &\asymp -\mu_{n-1}^{\otimes 2}[\omega_n = \widetilde{\omega}_n] \;=\; -I_n \end{aligned}$$

Finally,

$$A_n \asymp \sum_{t \le n} I_t \;, \quad \langle M \rangle_n = O(\sum_{t \le n} I_t)$$

Theorem 3 follows from martingale Convergence Th. and L.L.N. □

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Theorem 4 (*weak* invariance principle) Assume (WD). $\forall F$ bounded continuous on the path space,

$$\lim_n \mu_n \left[F\left(\frac{\omega_{nt}}{\sqrt{n}} \right) \right] = \mathbf{E} F(B)$$

in Q-probability. B.M. with diffusion matrix $\frac{1}{d} Id$. □

Important step: the measure μ_n converges weakly to a Markov chain (time-inhomogeneous, depending on η) C.-Yoshida'06

4- (SD) and Localization.

$I_n = \sum_x \mu_{n-1}^{\otimes 2}(\omega_n = x)^2 \in (0,1]$ is all the closer to 1 as μ_{n-1} is localized:

$$\max_{x \in \mathbb{Z}^d} \mu_{n-1}[\omega_n = x]^2 \leq I_n \leq \max_{x \in \mathbb{Z}^d} \mu_{n-1}[\omega_n = x]$$

The maximizing x is the favourite “location" for ω_n of the polymer at time n (under μ_{n-1}); large maximum value means strong localization

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Theorem 5: localization transition

- p<$\lambda \iff \mathrm{Cesaro} - \lim_{n\to\infty} I_n > 0 \quad Q - \mathrm{a.s.}$
- (KP) or d=1 $\implies p < \lambda$
- $d = 2 \implies \limsup_n I_n \geq C$ a.s.
- (L2) $\implies I_n = O_Q(n^{-c}) \quad (c = d/2$?) □

..., Vargas’07, C.-Vargas’07

5- Exponents and Deviations.

Exponents (rough definitions) Under μ_t with t large,

$$|\omega_t| \sim t^{\xi(d)}\,, \quad \ln Z_t - Q[\ln Z_t] \sim t^{\chi(d)}$$

Conjectures: universal exponents (for low temperature),

$$\chi(1) = 1/3,\ \xi(1) = 2/3, \quad \chi(d) = 2\xi(d) - 1\ .$$

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Theorem 4 (Brownian model) Fix $\xi_0 > \frac{1+\chi(d)}{2}$. Then, the law of $t^{-\xi_0}\omega_t$ under μ_t satisfies an almost-sure large deviation principle with rate $I(x) = |x|^2/2$ and speed $t^{2\xi_0-1}$. In particular, for a.e. environment,

$$\mu_t(|\omega_t| \geq at^{\xi_0}) = \exp\{-t^{2\xi_0-1}(a^2/2 + o(1))\}$$

as $t \to \infty$ for all $a \geq 0$. □

Corollary:

$$\xi(d) \leq \frac{1+\chi(d)}{2} ,$$

and since $\chi(d) \leq 1/2$, this implies

$$\xi(d) \leq 3/4$$

$\square$

Piza'97, Newman-Piza'97, Wuthrich'98, Petermann'00, Mejane'04, Carmona-Hu'04

Proposition: $\chi(1) \geq 1/8$ *(in favor of superdiffusivity)*

□ of Theorem 4: Fix $t \geq 0$, define $\Theta_t : s \mapsto (s \wedge t)\theta$.
By Girsanov's formula, $\overline{\omega} = \omega - \Theta_t$ is a Brownian motion under $\overline{P}(d\omega) = \exp(\theta \cdot \omega_t - t|\theta|^2/2) P(d\omega)$. So,

$$\begin{aligned} P[e^{\beta\eta(V_t(\omega))} e^{\theta\cdot\omega_t - t|\theta|^2/2}] &=^{def.} \overline{P}[e^{\beta\eta(V_t(\overline{\omega}+\Theta_t))}] \\ &=^{Girs.} P[e^{\beta\eta(V_t(\omega+\Theta_t))}] \\ &= P[e^{\beta\eta(T_\theta V_t(\omega))}] \\ &= Z_t \circ T_{-\theta}(\eta) \\ &=^{law} Z_t(\eta) \ . \end{aligned}$$

Here, $T_\theta : (s, x) \mapsto (s, x + s\theta)$.

Now,

$$\begin{aligned}\ln \mu_t[e^{t^{\xi_0 - 1}\theta\cdot\omega_t}] &= t^{2\xi_0-1}|\theta|^2/2 + \ln Z_t \circ T^1_{-\theta t^{\xi_0-1}} - \ln Z_t \\ &= t^{2\xi_0-1}|\theta|^2/2 + \mathcal{O}(t^{\chi(d)})\end{aligned}$$

(same expectation $+$ def. of fluctuation exponent).

Now conclude by Gartner-Ellis. ■

so much is left!.

- Phase diagram: $\beta_c = \beta_c^p$ or not?
- $d = 1$: is $p < \lambda$ for $\beta \neq 0$?
- relations between exponents
- closer relation to percolation, “random geodesics” of Newman et al.
- $d = 1$ exact exponents and limit laws
  Baik-Deift-Johansson’99, Johansson’00, Prahofer-Spohn’01
  $\beta = +\infty, d = 1, \eta \sim$ exponential or geometric
- Universality