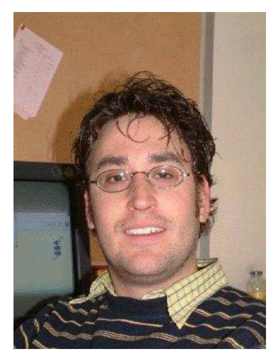
Entropy Production in Random Repeated Interaction Quantum Systems *

Alain JOYE

Institut Fourier Grenoble

* Joint work with



Laurent BRUNEAU (Université de Cergy) & Marco MERKLI (Memorial University)

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Quantum system $\mathcal{S}$:

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Interaction

- W_k operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k \geq 1$.

Repeated Interactions Quantum Systems

Evolution Let $\{\tau_k\}_{k\in\mathbb{N}^*}$, with $\inf_k \tau_k > 0$ be a set of durations.

For $t = \tau_1 + \tau_2 + \cdots + \tau_{m-1} + s, \ \ 0 \leq s < \tau_m$,

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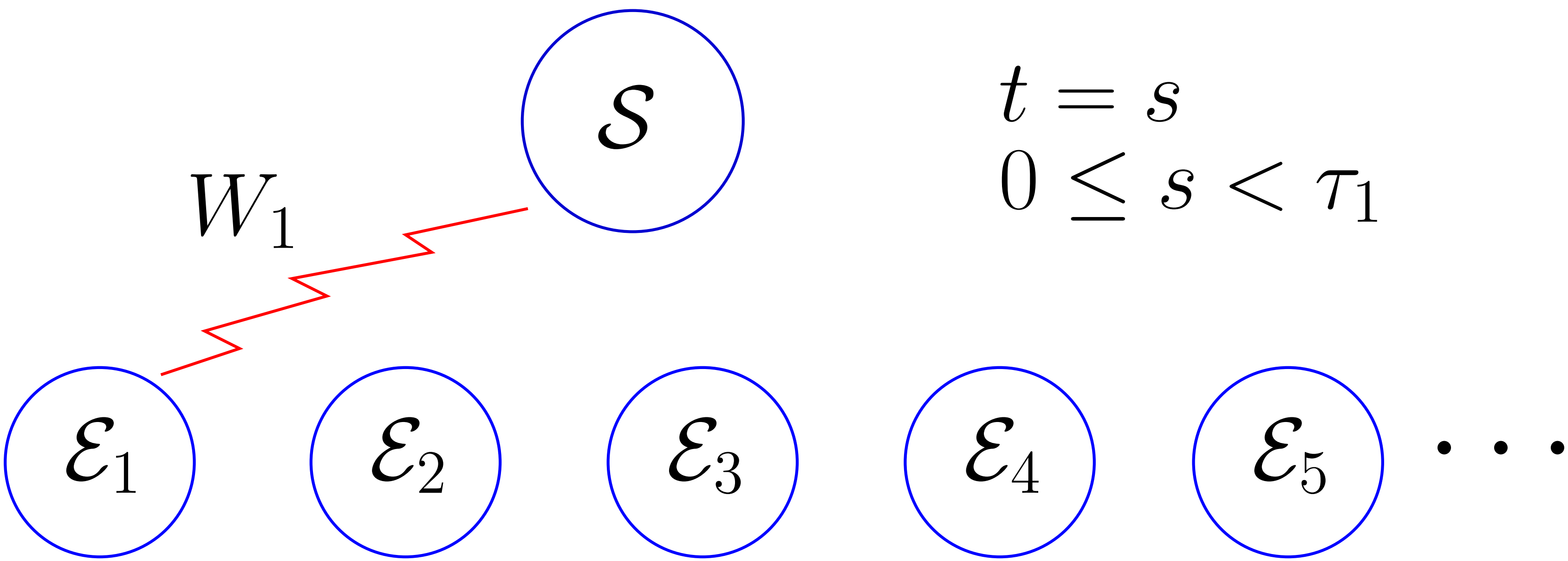

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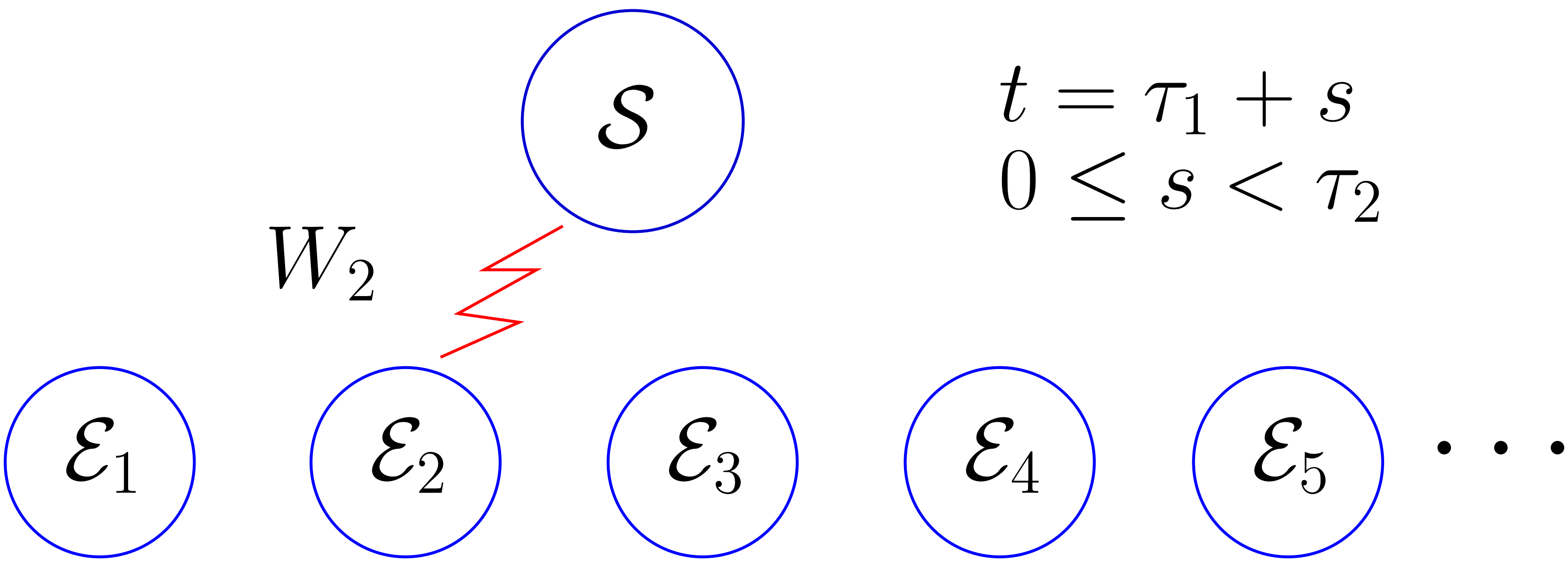

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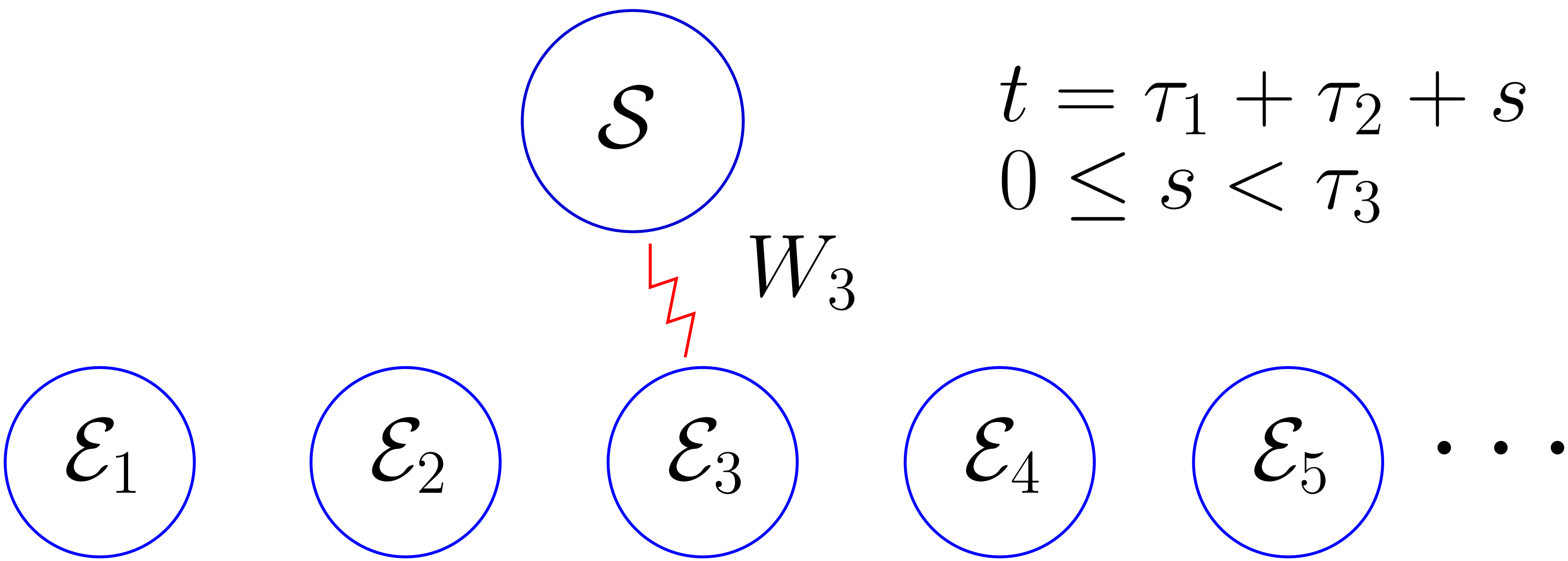

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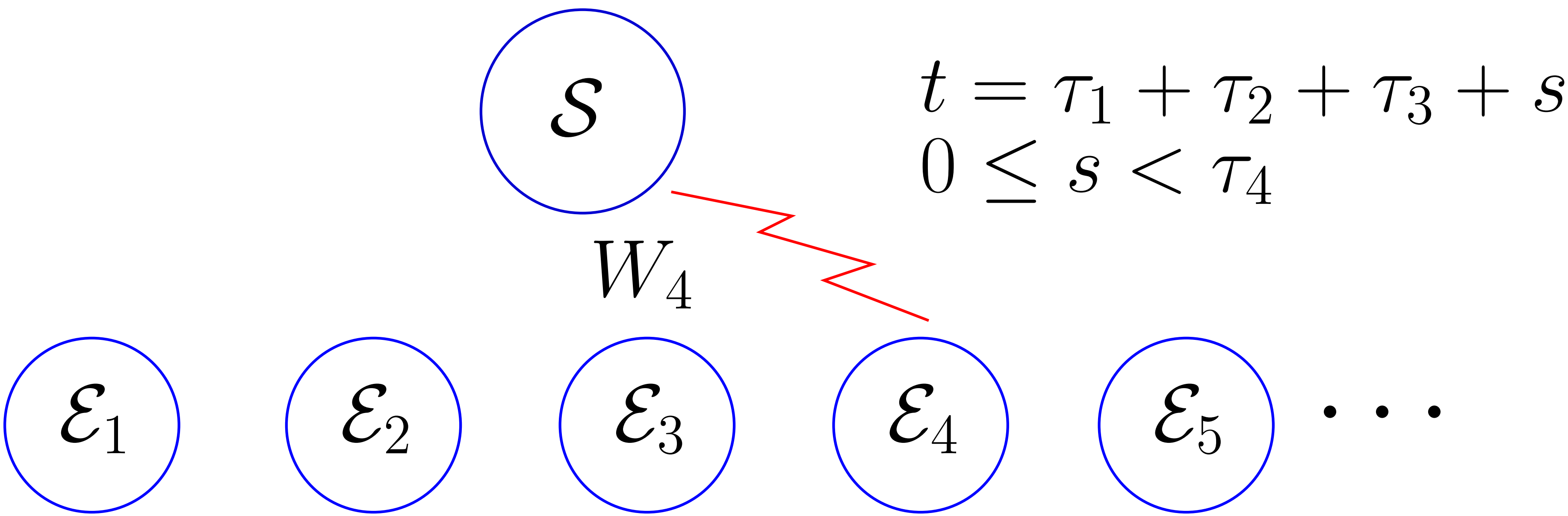

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{\mathcal{S}} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S}$ only

Let $\tau^t(A)$ be its Heisenberg evolution, at time $t = \tau_1 + \tau_2 + \cdots + \tau_m$

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- Non trivial entropy production and energy variation ?
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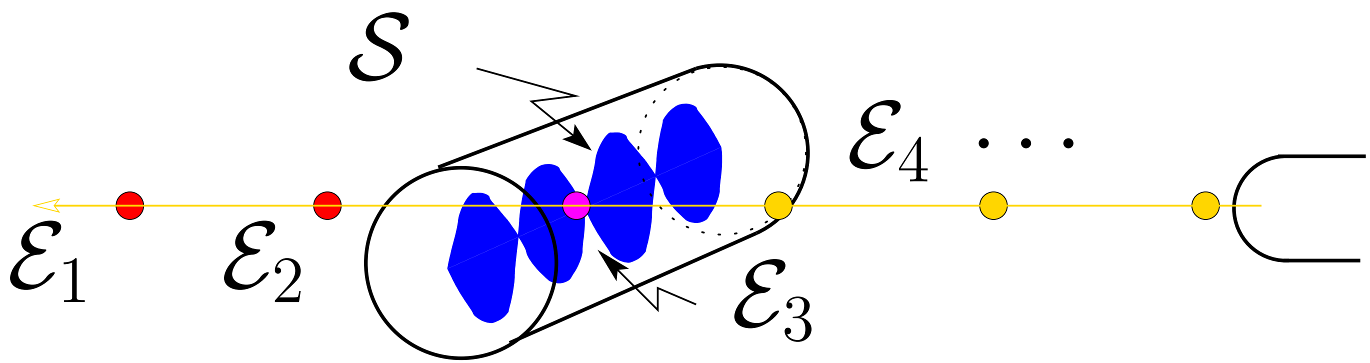
- Non-trivial examples ?

Motivations

Experimental reality

One-atom maser   Walther et al '85, Haroche et al '92



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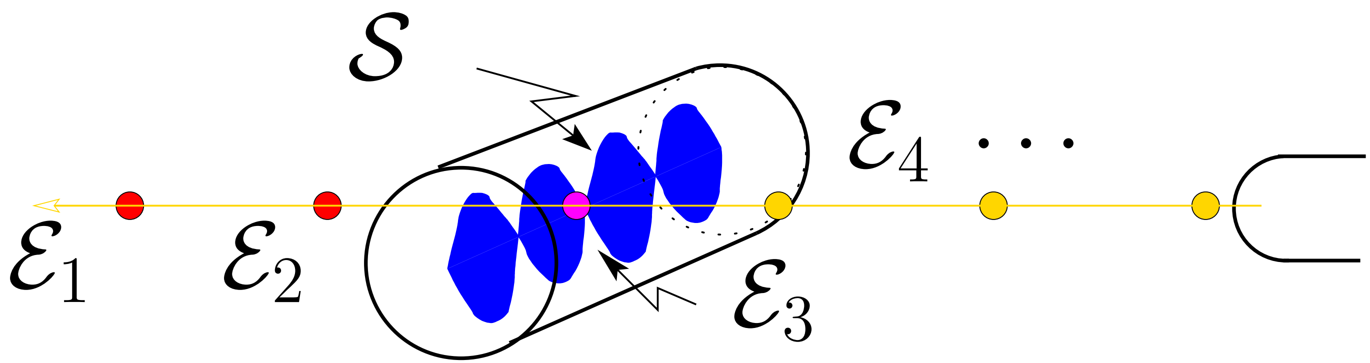

- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity

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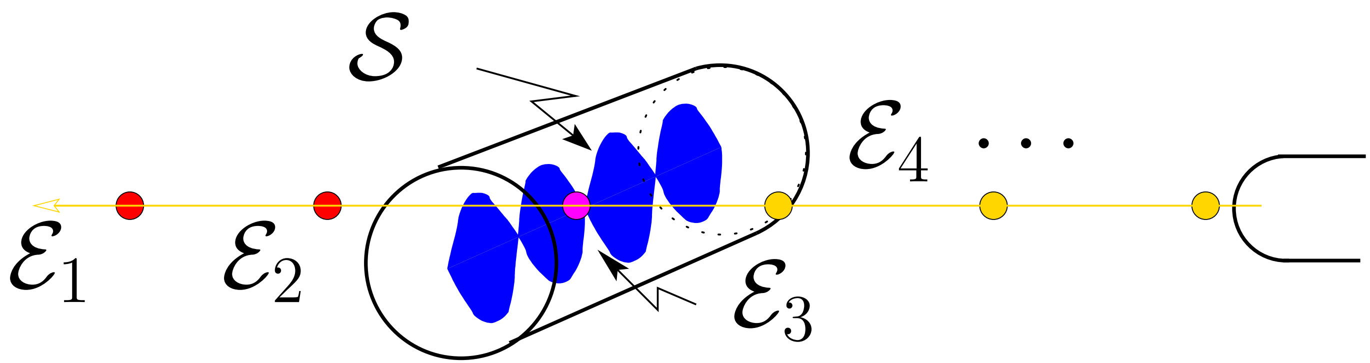

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Ideal RIQS used as models — Vogel et al '93, Wellens et al '00

$\Rightarrow$ Random RIQS model fluctuations w.r.t. ideal situation

Mathematical Framework

GNS representation

Let $\rho \in \mathcal{B}_1(\mathfrak{H})$ be a density matrix on $\mathfrak{H}$

$$0 < \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| \text{ and } \mathrm{Tr}\rho = 1$$

Mathematical Framework

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$\rho \;\longrightarrow\;$ pure state $|\Psi_\rho\rangle\langle\Psi_\rho|$ on enlarged Hilbert space

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- $\mathfrak{H} \rightarrow \mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$
- $\rho \in \mathcal{B}_1(\mathfrak{H}) \rightarrow \Psi_\rho = \sum_j \sqrt{\lambda_j}\varphi_j \otimes \varphi_j \in \mathcal{H}$
- $A \in \mathcal{B}(\mathfrak{H}) \rightarrow \Pi(A) = A \otimes \mathbb{I}_{\mathfrak{H}} \in \mathcal{B}(\mathcal{H})$

$$\Rightarrow \mathrm{Tr}_{\mathfrak{H}}(\rho A) = \langle\Psi_\rho|A \otimes \mathbb{I}_{\mathfrak{H}}\Psi_\rho\rangle_{\mathcal{H}} = \mathrm{Tr}_{\mathcal{H}}(|\Psi_\rho\rangle\langle\Psi_\rho|\Pi(A))$$

- $\mathbb{I}_{\mathfrak{H}} \otimes B \in \mathcal{B}(\mathcal{H})$ don't play any role
- For gases with ∞-ly many particles, GNS is required and non-trivial

Liouvillean

Evolution of observables and sates

$$\tau^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

$$\rho \in \mathcal{B}_1(\mathfrak{H}) \mapsto \rho \circ \tau^t = e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})$$

Invariant states

$$\mathsf{Tr}(e^{-itH} \rho e^{itH} A) = \mathsf{Tr}(\rho A), \;\; \forall A \in \mathcal{B}(\mathfrak{H})$$

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Liouville operator

Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

$$\begin{cases} \Pi(\tau^t(A)) &= e^{itL} \Pi(A) e^{-itL} \in \mathcal{H} \\ L\Psi_\rho &= 0 \end{cases}$$

Simple setup

$$L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H$$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_{\mathcal{S}} \subset \mathcal{B}(\mathcal{H}_{\mathcal{S}})$, $\mathfrak{M}_{\mathcal{E}_k} \subset \mathcal{B}(\mathcal{H}_{\mathcal{E}_k})$ and $\mathfrak{M}_{\mathcal{C}} \subset \mathcal{B}(\mathcal{H}_{\mathcal{C}})$
- States on $\mathcal{S}$, $\mathcal{E}_k$, $\mathcal{C}$ are density matrices on $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{E}_k}$, $\mathcal{H}_{\mathcal{C}}$
- Evolution of observables $A_{\mathcal{S}} \mapsto \tau_{\mathcal{S}}^t(A_{\mathcal{S}})$, $A_{\mathcal{E}_k} \mapsto \tau_{\mathcal{E}}^t(A_{\mathcal{E}_k})$ + tech. hyp.

- Assumption: $\exists$ invariant states (cyclic and separating)

$$\Psi_{\mathcal{S}} \in \mathcal{H}_{\mathcal{S}} \text{ and } \Psi_{\mathcal{E}_k} \in \mathcal{H}_{\mathcal{E}_k} \text{ s.t.}$$

$$\tau_{\#}^t(A_{\#}) = e^{itL_{\#}} A_{\#} e^{-itL_{\#}}, \quad \text{and} \quad L_{\#}\Psi_{\#} = 0, \qquad \text{where} \quad \# = \mathcal{S} \text{ or } \mathcal{E}_k$$

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Entire system

$$\mathcal{S} + \mathcal{C} \text{ on } \mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{C}}, \text{ driven by } L_{\text{free}} = L_{\mathcal{S}} + \sum_k L_{\mathcal{E}_k}$$

Interaction

$$V_k \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_k}, \text{ the GNS repres. of } W_k \qquad \text{+ tech. hyp.}$$

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = \tau_1 + \tau_2 + \cdots + \tau_m$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ_m is

$$\tilde{L}_m = \tau_m L_m + \tau_m \sum_{k \neq m} L_{\mathcal{E},k}$$

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$$\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{E}_m} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ L_{\mathcal{E},k} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$$

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To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}} \in \mathfrak{M}$ an observable on $\mathcal{S}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}} U(m)) \equiv \varrho(\alpha^m(A_{\mathcal{S}})), \quad \text{as} \quad m \to \infty$$

Reduction to a Product of Matrices

Special state

$\varrho_0 = \langle \Psi_0 | \cdot \Psi_0 \rangle$ where

$\Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{S}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{S}}$

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$C-$ Liouvillean — Given $L_{\mathcal{S}}$, $L_{\mathcal{E}_m}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t.} \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} \quad \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}} = 0. \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

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$P = \mathbb{I}_{\mathcal{H}_{\mathcal{S}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{S}}$

$C-$ Liouvillean Given $L_{\mathcal{S}}$, $L_{\mathcal{E}_m}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t.} \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} & \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}} = 0. & \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

$K_m = \tau_m(L_{\text{free}} + V_m - V_m'), \quad V_m' = J_m \Delta_m^{\frac{1}{2}} V_m \Delta_m^{-\frac{1}{2}} J_m$ Tomita-Takesaki '57

Reduction to a Product of Matrices

Evolution of ϱ_0

$$\begin{aligned}
\varrho_0(\alpha^m(A_S)) &= \langle \Psi_0 | e^{i\tilde{L}_1} \cdots e^{i\tilde{L}_m} A_S e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_S e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_S P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_S \Psi_0 \rangle \\
&\equiv \langle \Psi_S | M_1 M_2 \cdots M_m A_S \Psi_S \rangle
\end{aligned}$$

where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_S$.

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Reduced Dynamical Operators

$\{M_j \in L(\mathcal{H}_\mathcal{S})\}_{j\in\mathbb{N}}$ s.t. $\exists C < \infty$ and $\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$ with

$$
\begin{cases}
M_j \Psi_\mathcal{S} = \Psi_\mathcal{S} \\
\|M_{j_1} M_{j_2} \cdots M_{j_k}\| \leq C, \quad \forall \{j_i\}_{i\in\mathbb{N}}
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Facts: spec $M_j \subset \{|z|\le 1\}$ and $1\in$ spec $M_j \quad \forall j\in\mathbb{N}$,
In general, $M_1M_2\cdots M_n$ does not converge as $n\to\infty$!

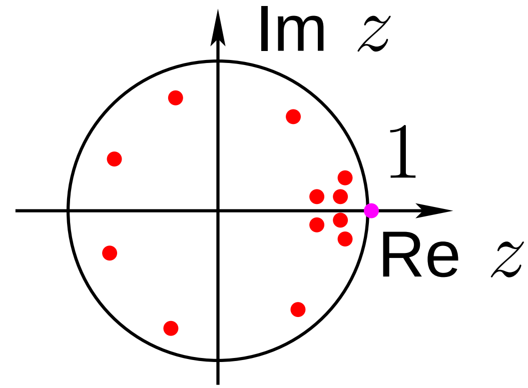
Ideal RIQS

Identical RDO's

$$M_j = M, \quad \forall j \in \mathbb{N}$$

Assumption (E)

$$\begin{cases} \text{spec } M \cap \mathbb{S}^1 = \{1\} \\ 1 \text{ is simple} \end{cases}$$



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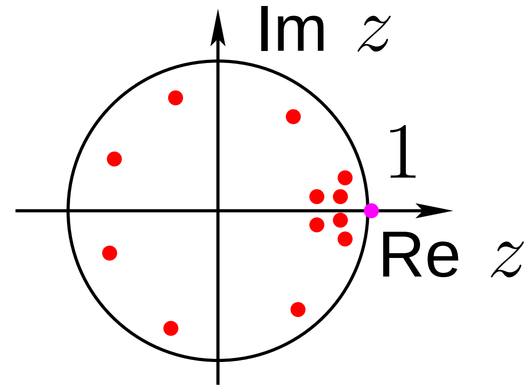

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Consequence

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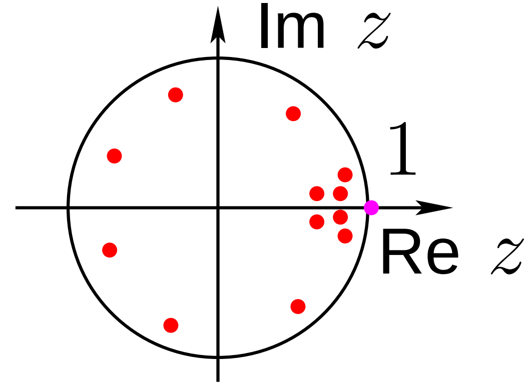

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Repeated Interaction Asymptotic State

Bruneau, J., Merkli '06

Under (E), for any state ϱ on $\mathfrak{M}$ and any observable $A_{\mathcal{S}}$ on $\mathcal{S}$

$$\lim_{m\to\infty} \varrho(\alpha^m(A_{\mathcal{S}})) = \varrho_+(A_{\mathcal{S}}), \qquad \varrho_+(\cdot) = \langle\psi| \cdot \Psi_{\mathcal{S}}\rangle \text{, a state on } \mathcal{H}_{\mathcal{S}}$$

Random Repeated Interaction Quantum Systems

Setup Let $(\Omega_0, \mathcal{F}, dp)$ be a probability space

Let $\Omega_0 \ni \omega_0 \mapsto M(\omega_0) \in \mathcal{H}$ be an RDO valued random variable:

$$\begin{cases} M(\omega_0)\Psi_{\mathcal{S}} = \Psi_{\mathcal{S}}, & \forall\, \omega_0 \in \Omega_0 \\ \|M(\omega_1)M(\omega_2)\cdots M(\omega_m)\| \leq C & \forall(\omega_1, \omega_2, \cdots, \omega_m) \in \Omega_0^m, \quad \forall m \in \mathbb{N}. \end{cases}$$

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Product of i.i.d. random matrices

Let $\Omega = \Omega_0^{\mathbb{N}}$, $d\mathbb{P} = \Pi_{j\geq 1} dp$ and $\omega = (\omega_1, \omega_2, \omega_3, \dots) \in \Omega$.

$\Rightarrow$

$$\chi(m, \omega) = M(\omega_1)M(\omega_2)\cdots M(\omega_m)A_{\mathcal{S}}, \text{ as } m \to \infty.$$

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Simplification: The RRDO $M(\omega_0)$ satisfies (E) on Ω_0 .

Thus

$$M(\omega_j) = M_j = |\Psi_S\rangle\langle\psi_j| + M_{Q_j}$$

where

$P_j = |\Psi_{\mathcal{S}}\rangle\langle\psi_j|$ spectral projector on 1, $Q_j = (\mathbb{I} - P_j)$

$M_{Q_j} = Q_j M_j Q_j$ s.t. spec$M_{Q_j} \subset \{|z| < 1\}$.

Main Results

Fluctuating + decaying parts

$$\left\{\begin{array}{rcl} \chi(n,\omega) & = & |\Psi_{\mathcal{S}}\rangle\langle\theta_n|A_{\mathcal{S}} + M_{Q_1}\cdots M_{Q_n}A_{\mathcal{S}} \\ \theta_n & = & \psi_n + M^*_{Q_n}\psi_{n-1} + M^*_{Q_n}M^*_{Q_{n-1}}\psi_{n-2} + \cdots + M^*_{Q_n}\cdots M^*_{Q_2}\psi_1 \\ & = & M^*_n M^*_{n-1}\cdots M^*_1\psi_1. \end{array}\right.$$

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Rem: If (E) not satisfied, $\psi_j := P^*_{1,M_j}\Psi_{\mathcal{S}}$ is OK.

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Fluctuations $\Rightarrow$ Ergodic/Cesaro limit

General hypothesis

$$p(M(\omega_0) \text{ satisfies } (E)) > 0.$$

Theorem (Decay) Bruneau, J., Merkli '07

$\exists\alpha > 0, \quad C > 0$ s.t.

i) $\|M_{Q_1}\cdots M_{Q_n}\| \leq Ce^{-\alpha n}, \quad \forall n \geq n_0(\omega),$ almost surely

ii) $\mathbb{E}(M)$ satisfies (E)

Main Results

Theorem (Fluctuation)

Recall $\chi(n,\omega) = M(\omega_1)M(\omega_2)\cdots M(\omega_n)A_{\mathcal{S}}$

i) $\lim_{\nu\to\infty} \frac{1}{\nu}\sum_{n=1}^{\nu} \chi(n,\omega) = |\Psi_{\mathcal{S}}\rangle\langle\bar{\theta}|A_{\mathcal{S}},$ almost surely

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Theorem (Ergodic limit of RRIQS)

For any state ϱ on $\mathfrak{M}$ and any observable $A_{\mathcal{S}}$ on $\mathcal{S}$,

$\lim_{\nu\to\infty} \frac{1}{\nu}\sum_{n=1}^{\nu} \varrho(\alpha_\omega^n(A_{\mathcal{S}})) = \langle\bar{\theta}|A_{\mathcal{S}}\Psi_{\mathcal{S}}\rangle,$ almost surely

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Notes:

- Beck-Schwarz '57 Existence of erg. lim. of $\chi(n,\omega)$, but no value
- Numerous results on pdts of random matrices (e.g. Guivarc'h, Kifer-Liu,...), but different focus, frameworks $\Rightarrow$ hypotheses don't fit.
- Fluctuations hard to study. $\exists$ partial results only, De Saporta et.al '04
- $\varrho_0 \mapsto \varrho$ via cyclicity and separability of Ψ_0

Instantaneous Observables

Time-dependent observables

$O = A_{\mathcal{S}} \otimes_{j=-l}^{r} B_m^{(j)}$ where $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}, B_m^{(j)} \in \mathfrak{M}_{\mathcal{E}_{m+j}}$

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Proposition

$$\langle \Psi_0 | \alpha^m(O) \Psi_0 \rangle = \langle \Psi_S | M_1 \cdots M_{m-l-1} N_m(A_S, B_m^{(-l)}, \cdots, B_m^{(r)}) \Psi_S \rangle$$

where

$$N_m(A_S, B_m^{(-l)}, \cdots, B_m^{(r)}) \Psi_0 = P \alpha^{m,m-l-1}(A_{\mathcal{S}} \otimes_{j=-l}^{r} B_m^{(j)}) P$$

$\alpha^{m,n}$ evolution from time n to m.

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$\alpha^{m,n}$ evolution from time n to m.

Probabilistic setting For all $m > l$

- $\omega_j \mapsto M(\omega_j)$ as above replaces M_j, $j = 1, 2, \cdots, m-l-1$,
- $N : \Omega_0^{r+l+1} \to M_d(\mathbb{C})$ uniformly bounded
- $\omega \mapsto N(\omega_{m-l}, \omega_{m-l+1}, \cdots, \omega_{m+r})$ replaces $N_m(A_S, B_m^{(-l)}, \cdots, B_m^{(r)})$

Modified Product of Random Matrices

To be considered as $m \to \infty$, with l, r fixed

$$\chi(m, \omega) = M(\omega_1)M(\omega_2)\cdots M(\omega_{m-l-1})N(\omega_{m-l}, \omega_{m-l+1}, \cdots, \omega_{m+r}).$$

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Theorem (Ergodic limit of $\chi(m,\omega)$)

i) $\lim_{\nu\to\infty} \frac{1}{\nu}\sum_{n=1}^{\nu} \chi(n,\omega) = |\Psi_{\mathcal{S}}\rangle\langle\bar{\theta}|\mathbb{E}(N),$ almost surely

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For any state ϱ on $\mathfrak{M}$ and any random instantanous observable O given by $N(\omega_1,\cdots,\omega_{l+r+1})$

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Energy

Instantaneous energy

The expression $\alpha^m(\tilde{L}_m)$ yields the “energy” for times $t \in [\tau_{m-1}, \tau_m)$.

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$$j(m) = \alpha^{m+1}(\tilde{L}_{m+1}) - \alpha^m(\tilde{L}_m) = \alpha^m(V_{m+1} - V_m)$$

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Asymptotic energy variation

$$dE_+ := \lim_{\nu \to \infty} \varrho\left(\frac{\Delta E(\nu)}{\nu}\right) = \langle \bar{\theta} | \mathbb{E}(P(V - \alpha^\tau(V))P)\Psi_{\mathcal{S}} \rangle$$

Entropy

KMS states

Let $\Psi_{\mathcal{S}}$ and $\Psi_{\mathcal{E}_j}$ correspond to Gibbs states at temperatures $\beta_{\mathcal{S}}$ and $\beta_{\mathcal{E}_j}$

Relative entropy ϱ and ϱ_0 are states on $\mathfrak{M}$, generalization of

$$Ent(\varrho|\varrho_0) = \text{Tr}\left(\rho(\ln\rho - \ln\rho_0)\right) \geq 0$$ Araki '75

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Araki '75

Variation of relative entropy

Jaksic, Pillet '03

Let ϱ_0 correspond to $\Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{C}}$ and ϱ be any state,

$$\begin{aligned} \Delta S(m) &:= Ent(\varrho \circ \alpha^m | \varrho_0) - Ent(\varrho|\varrho_0) \\ &= \varrho(\alpha^m[\beta_{\mathcal{S}} L_{\mathcal{S}} + \sum_{j=1}^{m} \beta_{\mathcal{E}_j} L_{\mathcal{E},j}] - \beta_{\mathcal{S}} L_{\mathcal{S}} + \sum_{j=1}^{m} \beta_{\mathcal{E}_j} L_{\mathcal{E},j}) \end{aligned}$$

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\end{aligned}$$

Asymptotic entropy production

$$dS_+ := \lim_{\nu\to\infty} \frac{\Delta S(\nu)}{\nu} = \langle\bar{\theta}|\mathbb{E}(\beta_{\mathcal{E}} P(L_{\mathcal{S}} + V - \alpha^{\tau}(L_{\mathcal{S}} + V))P)\Psi_{\mathcal{S}}\rangle$$

Consequences

Average Second Law of Thermodynamics

$$\begin{aligned} dE_+ &= \langle \bar{\theta} | \mathbb{E}(P(L_{\mathcal{S}} + V - \alpha^{\tau}(L_{\mathcal{S}} + V))P)\Psi_{\mathcal{S}} \rangle \\ dS_+ &= \langle \bar{\theta} | \mathbb{E}(\beta_{\mathcal{E}} P(L_{\mathcal{S}} + V - \alpha^{\tau}(L_{\mathcal{S}} + V))P)\Psi_{\mathcal{S}} \rangle \end{aligned}$$

If $\beta_{\mathcal{E}_k} = \beta_{\mathcal{E}}$ is not random, then

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For ideal RIQS, $\exists$ examples with

$$dS_+ > 0$$

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Average Second Law of Thermodynamics

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dS_+ &= \langle\bar{\theta}|\mathbb{E}(\beta_{\mathcal{E}} P(L_{\mathcal{S}} + V - \alpha^\tau(L_{\mathcal{S}} + V))P)\Psi_{\mathcal{S}}\rangle
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For ideal RIQS, $\exists$ examples with

$$dS_+ > 0$$

For RRIQS, $\exists$ examples with computable asymptotic state

$$\langle\bar{\theta}| \cdot \Psi_{\mathcal{S}}\rangle$$

in rigorous perturbation theory.

Applications

Spin-spin

- $\mathcal{S}$ and $\mathcal{E}_j$ spins with e.v. $\{0, E_S\}$, resp. $\{0, E_j\}$
- $W_j = \lambda(a_S \otimes a_j^* + a_S^* \otimes a_j)$
- Ψ_S tracial, $\Psi_{\mathcal{E}_j} \simeq \varrho_{\beta,\mathcal{E}_j} = e^{-\beta H_{\mathcal{E}_j}}/Z_{\beta,j}$

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Then, for any ϱ and a.s.,

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Ideal RIQS

$\tilde{\beta} = \beta E_\mathcal{E}/E_\mathcal{S}$ i.e. no thermalization

Applications

Spin-fermion Araki, Wyss '64

For $\mathcal{E}$ a Fermi gas at temperature β,

$$H_{\mathcal{S}} = \sigma_z \otimes \mathbb{I}, \quad L_{\mathcal{E}} = d\Gamma(h) \otimes \mathbb{I} - \mathbb{I} \otimes d\Gamma(h), \quad h \text{ a.c. mult. op. on } \mathfrak{h}$$

$$a_\beta(g) = a\left(\frac{e^{\beta g/2}}{\sqrt{1+e^{\beta g/2}}} g\right) \otimes \mathbb{I} + (-1)^N \otimes a^*\left(\frac{1}{\sqrt{1+e^{\beta} g/2}} \bar{g}\right)$$

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Perturbative result: For $0 < |\lambda| < \Lambda_0$, the a.s. asympt. state $\varrho_+(A_{\mathcal{S}})$ is s.t.

$$\varrho_+(A_{\mathcal{S}}) = q_\sigma \langle \uparrow |A_{\mathcal{S}}| \uparrow \rangle + (1 - q_\sigma) \langle \downarrow |A_{\mathcal{S}}| \downarrow \rangle + O(\|A_{\mathcal{S}}\|(\epsilon^3 + \lambda^2))$$

with explicit $q_\sigma = q_\sigma(\mathbb{E}(\sigma), \mathbb{E}(\sigma)^2) = O(\epsilon^2)$.