Random Poincaré maps and noise-induced mixed-mode-oscillation patterns

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The deterministic Koper model

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y,z) \ \ = y - x^3 + 3x \\ \dot{y} &= g_1(x,y,z) = kx - 2(y+\lambda) + z \\ \dot{z} &= g_2(x,y,z) = \rho(\lambda + y - z) \end{aligned}$$

▷ $0 < \varepsilon \ll 1$

▷ $k, \lambda, \rho \in \mathbb{R}$: control parameters

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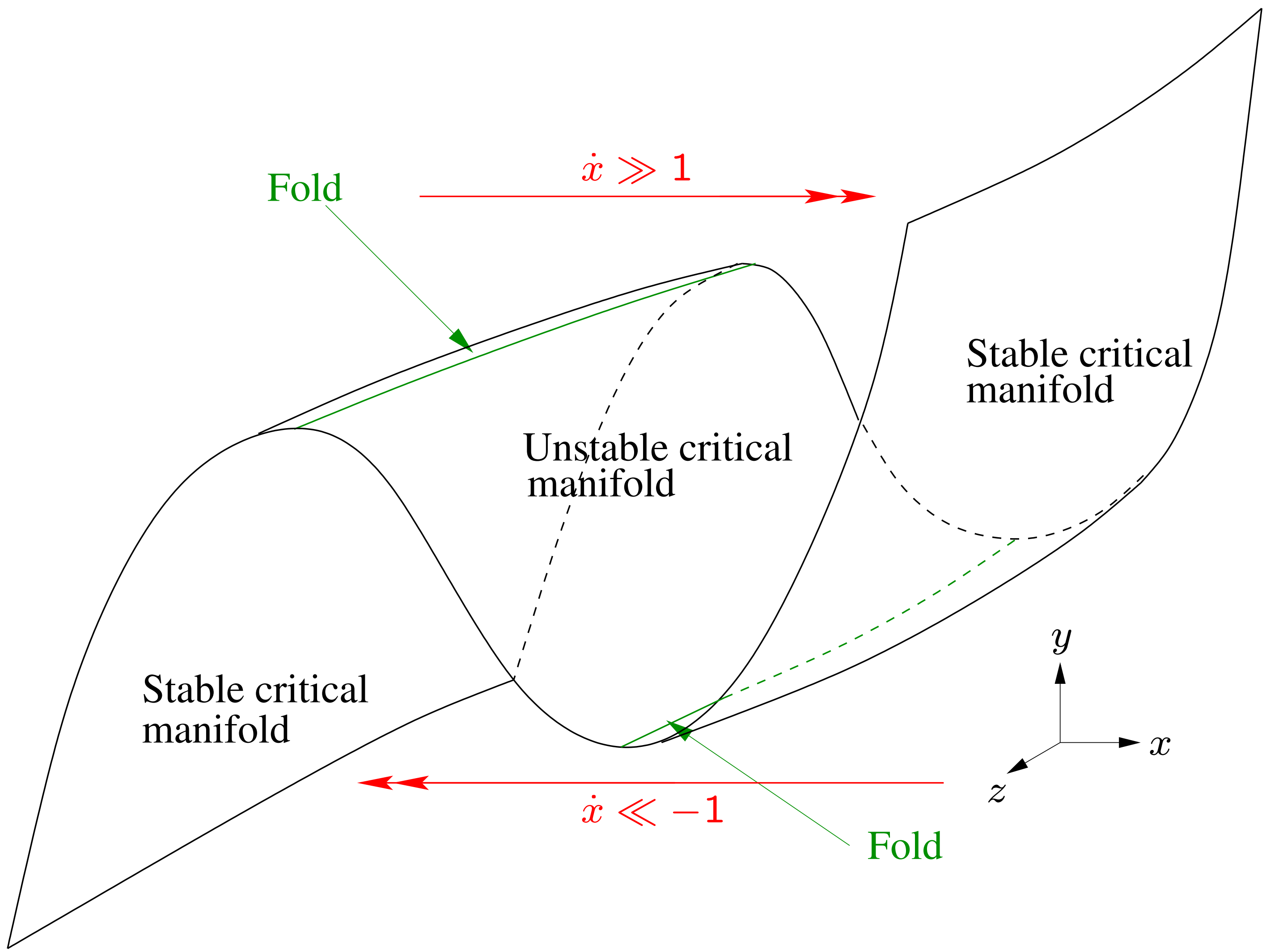
- $0 < \varepsilon \ll 1$
- $k, \lambda, \rho \in \mathbb{R}$: control parameters
- Critical manifold: $C_0 = \{f = 0\} = \{y = x^3 - 3x\}$
- Folds: $L = \{f = 0, \partial_x f = 0\} = \{y = x^3 - 3x, x = \pm 1\} = L^+ \cup L^-$

Critical manifold



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▷ $0 < \varepsilon \ll 1$

▷ $k, \lambda, \rho \in \mathbb{R}$: control parameters

▷ Critical manifold: $C_0 = \{f = 0\} = \{y = x^3 - 3x\}$

▷ Reduced flow on C_0 (Fenichel theory): eliminate y

$$\begin{aligned} \dot{x} &= \frac{kx - 2(x^3 - 3x + \lambda) + z}{3(x^2 - 1)} \\ \dot{z} &= \rho(\lambda + x^3 - 3x - z) \end{aligned}$$

⋈ Generic fold points: $\dot{x}$ diverges as $x \to \pm 1$

⋈ Folded node singularity: $\dot{x}$ finite, (desingularized) system has a node

Folded node singularity

Normal form [Benoît, Lobry '82, Szmolyan, Wechselberger '01]:

$$\begin{aligned} \epsilon\dot{x} &= y - x^2 \\ \dot{y} &= -(\mu+1)x - z \qquad\qquad (+\text{ higher-order terms}) \\ \dot{z} &= \frac{\mu}{2} \end{aligned}$$

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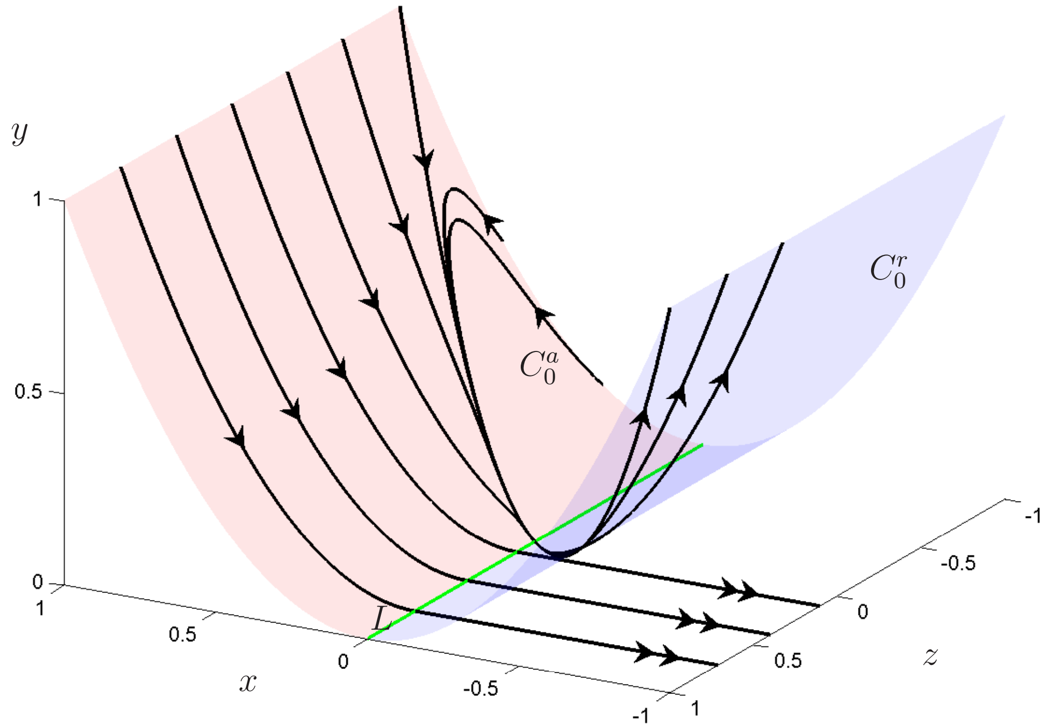
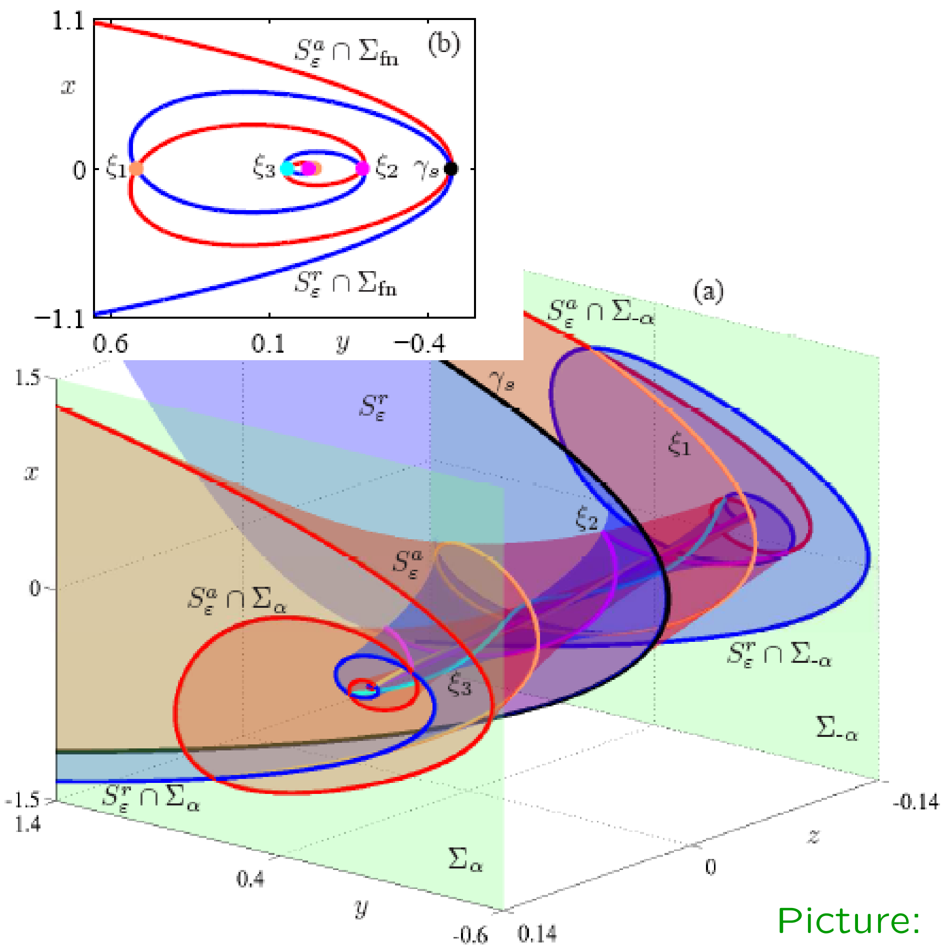
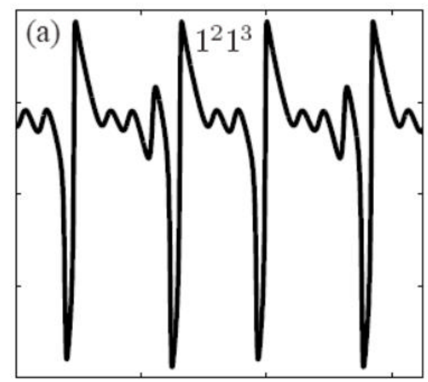


Folded node singularity

Theorem [Benoît, Lobry '82, Szmolyan, Wechselberger '01]:
For $2k+1 < \mu^{-1} < 2k+3$, the system admits k canard solutions
The j^{th} canard makes $(2j+1)/2$ oscillations

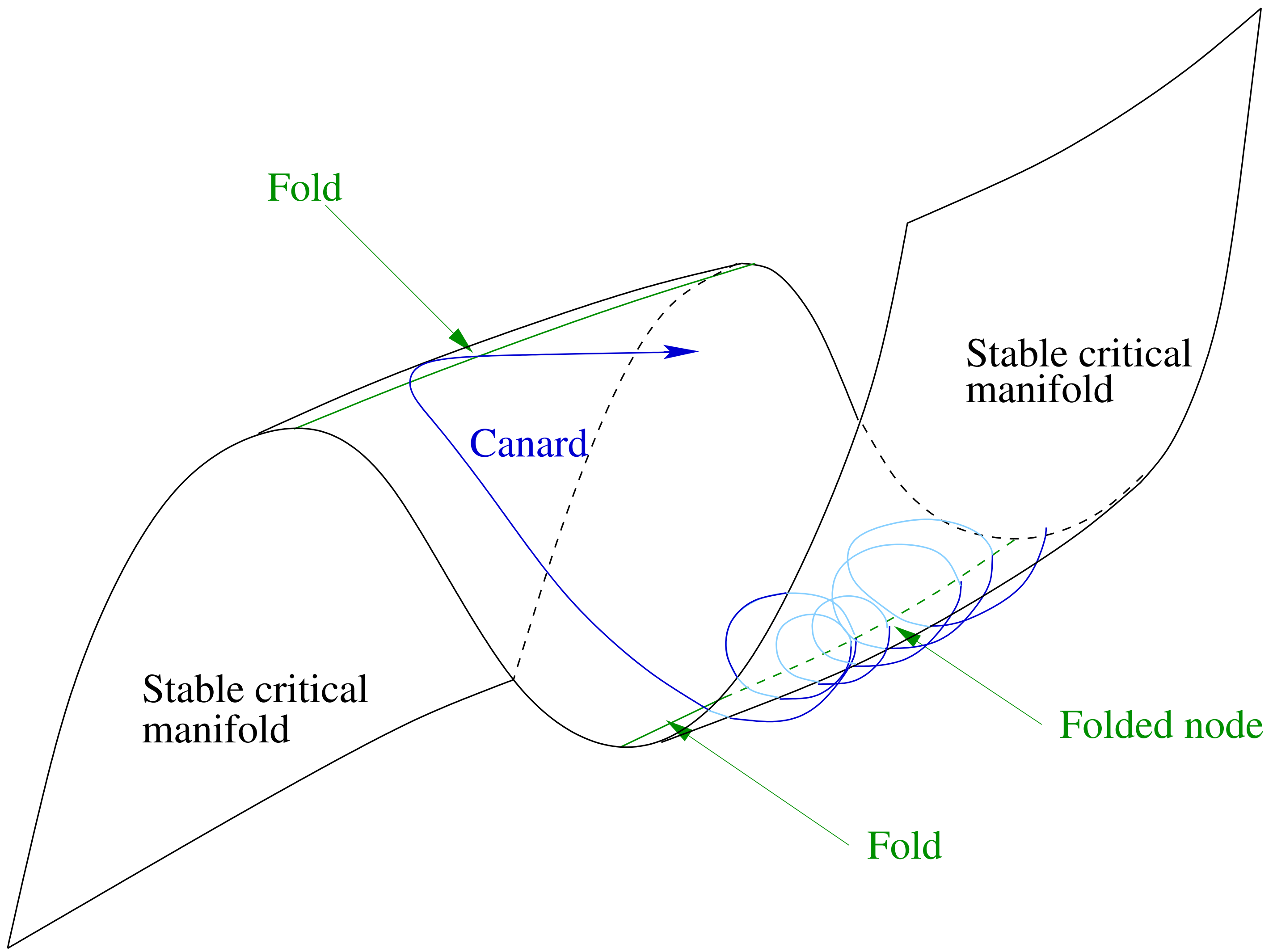




Mixed-mode oscillations (MMOs)

Picture: Mathieu Desroches

Global dynamics



▷ Canard orbits track unstable manifold (for some time)

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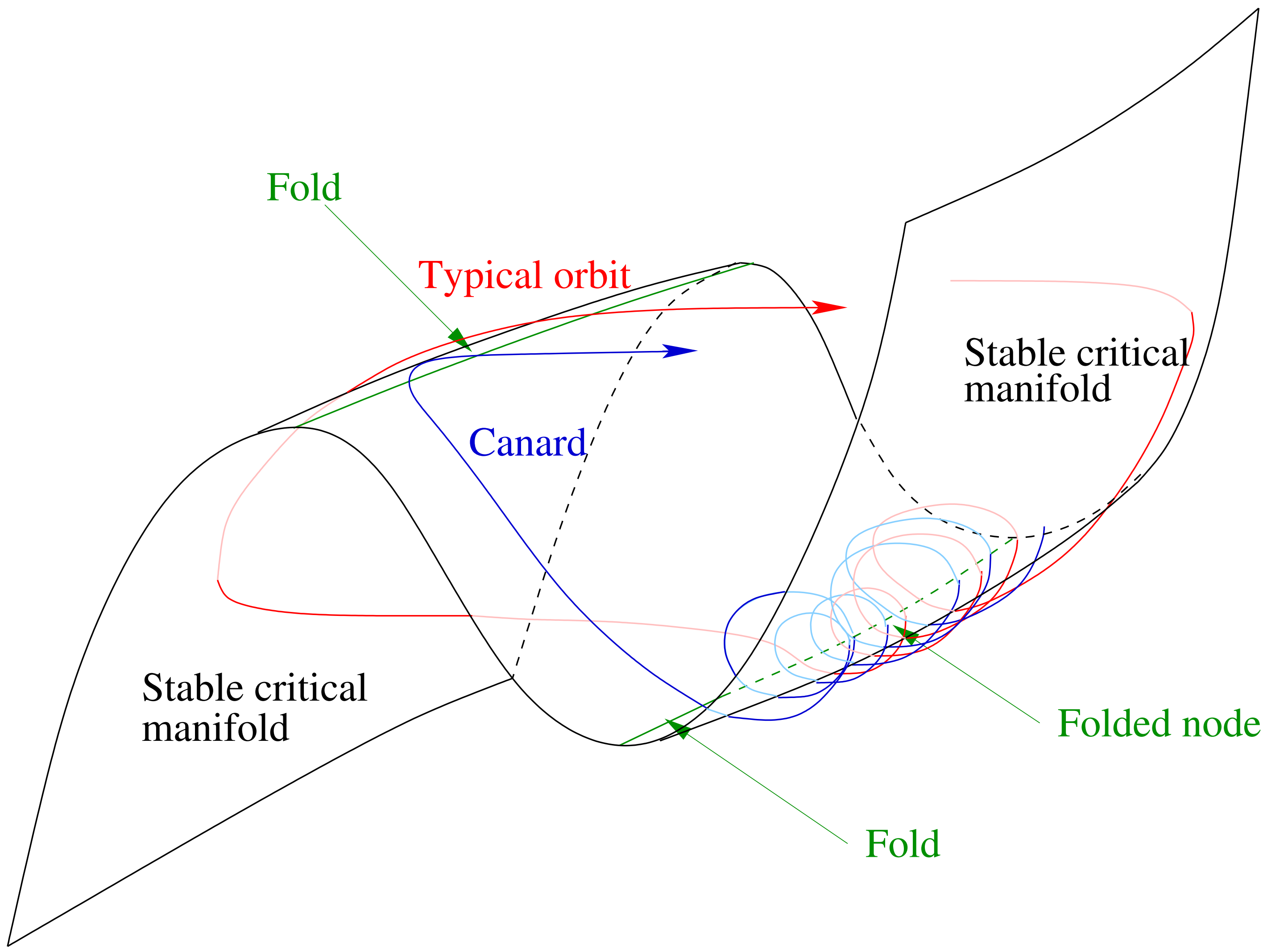
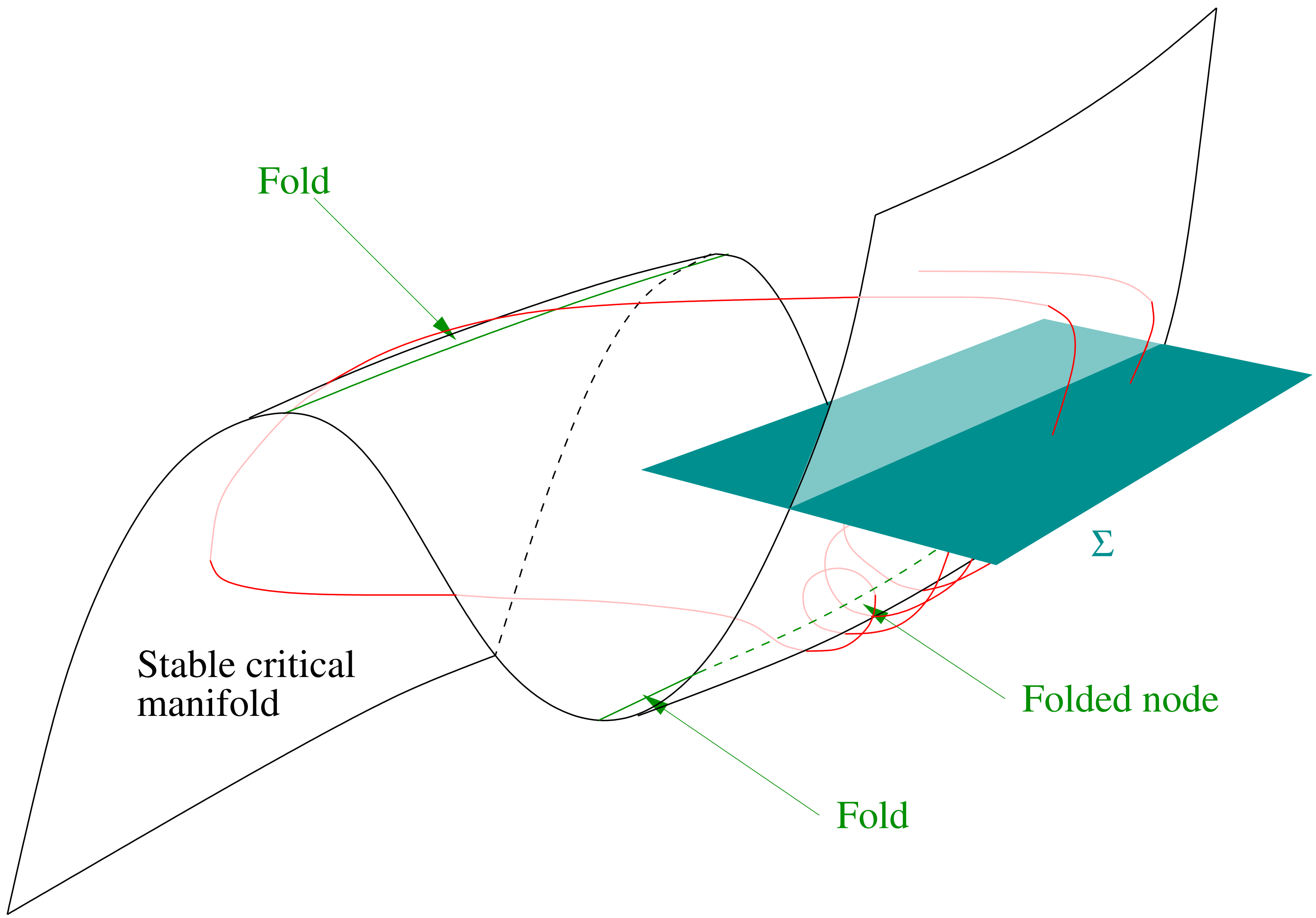


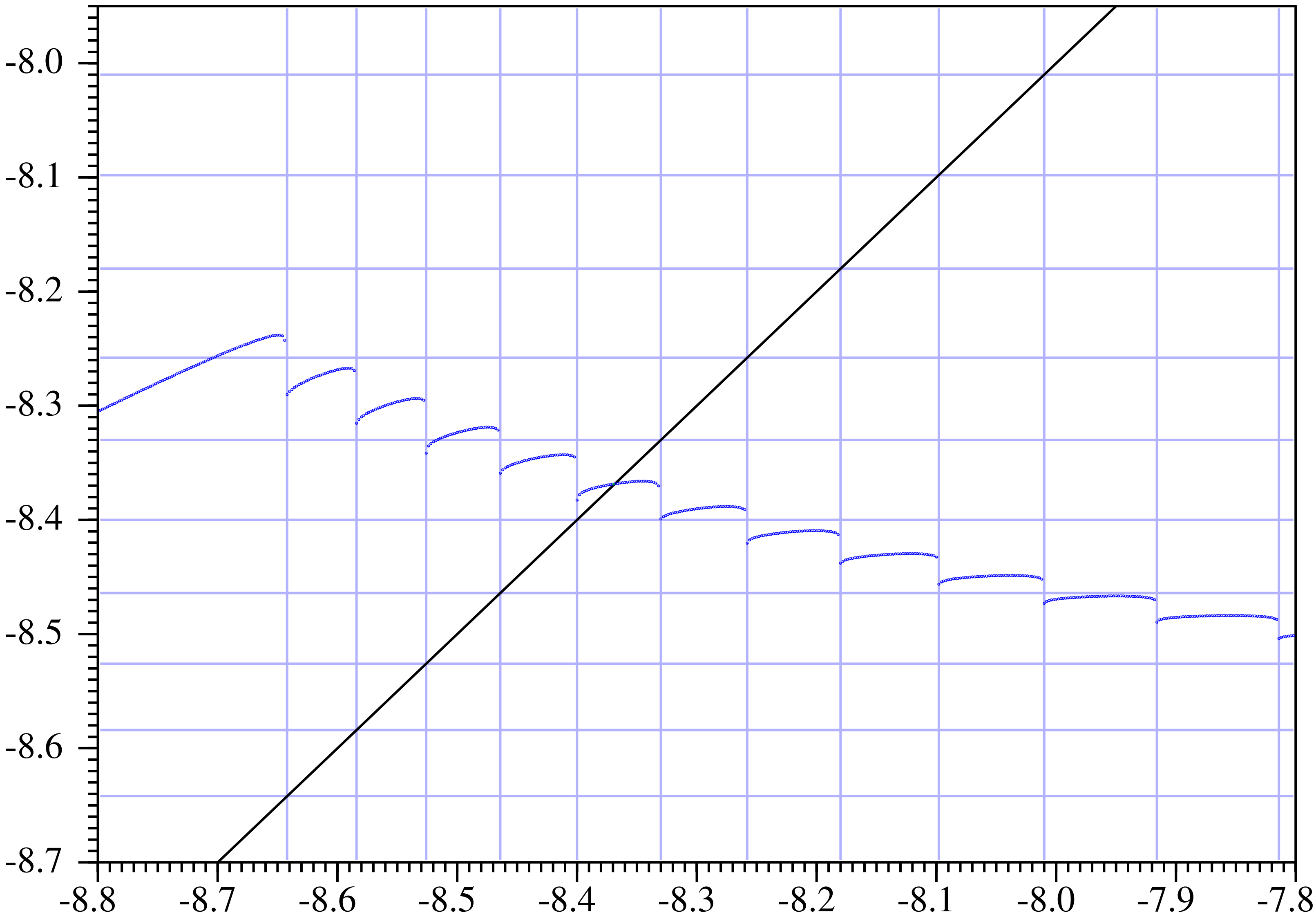
- ▷ Canard orbits track unstable manifold (for some time)
- ▷ Typical orbits may jump earlier to stable manifold

Poincaré map   c.f. e.g. [Guckenheimer, Chaos, 2008]



- ▷ Poincaré map $\Pi : \Sigma \to \Sigma$, invertible, 2-dimensional
- ▷ Due to contraction along C_0, close to 1d, non-invertible map

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.35, \rho = 0.7, \varepsilon = 0.01$

The stochastic Koper model

$$\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon} f(x_t, y_t, z_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= g_1(x_t, y_t, z_t)\,\mathrm{d}t + \sigma' G_1(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t &= g_2(x_t, y_t, z_t)\,\mathrm{d}t + \sigma' G_2(x_t, y_t, z_t)\,\mathrm{d}W_t
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▷ W_t: k-dimensional Brownian motion

▷ σ, σ': small parameters (may depend on ε)

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Random Poincaré map

In appropriate coordinates

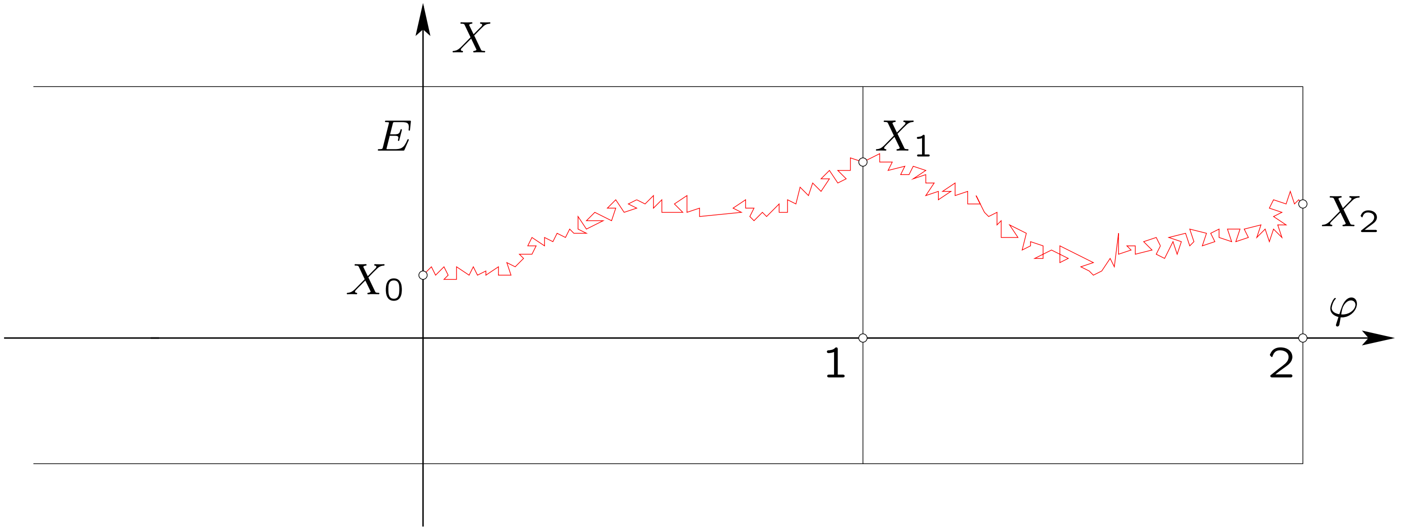
$$\begin{aligned}
\mathrm{d}\varphi_t &= \widehat{f}(\varphi_t, X_t)\,\mathrm{d}t + \widehat{\sigma}\widehat{F}(\varphi_t, X_t)\,\mathrm{d}W_t & \qquad \varphi &\in \mathbb{R} \\
\mathrm{d}X_t &= \widehat{g}(\varphi_t, X_t)\,\mathrm{d}t + \widehat{\sigma}\widehat{G}(\varphi_t, X_t)\,\mathrm{d}W_t & \qquad X &\in E \subset \Sigma
\end{aligned}$$

▷ all functions periodic in φ (say period 1)

▷ $\widehat{f} \geqslant c > 0$ and $\widehat{\sigma}$ small $\Rightarrow$ φ_t likely to increase

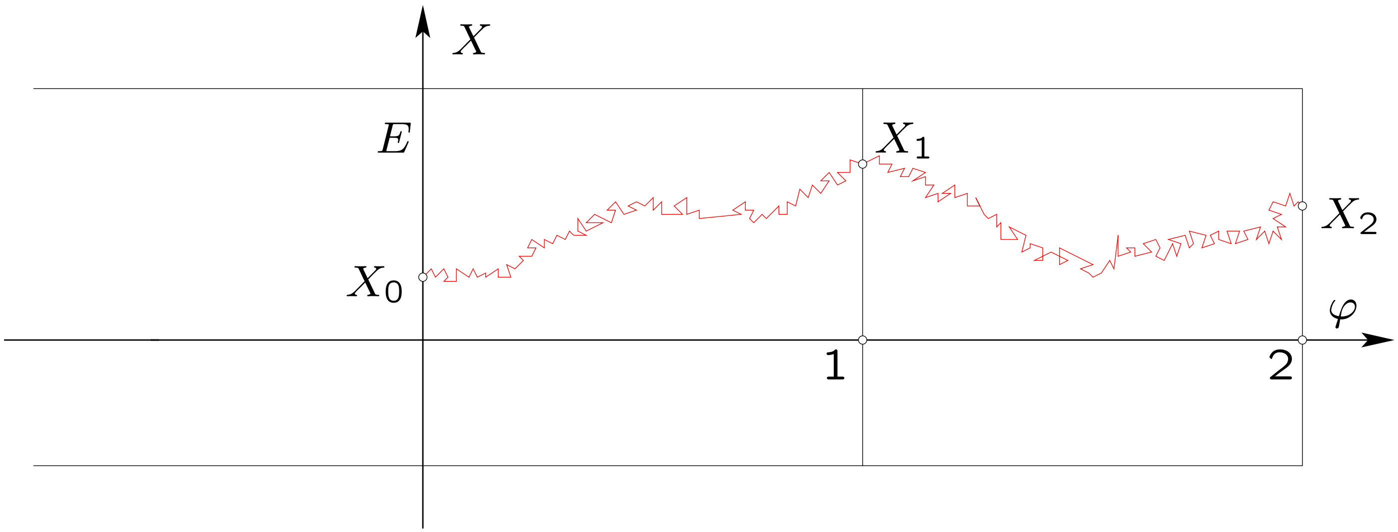
▷ process may be killed when X leaves E

Random Poincaré map



▷ $X_0, X_1, \ldots$ form (substochastic) Markov chain

Random Poincaré map



▷ $X_0, X_1, \dots$ form (substochastic) Markov chain

▷ τ: first-exit time of $Z_t = (\varphi_t, X_t)$ from $\mathcal{D} = (-M, 1) \times E$

▷ $\mu_Z(A) = \mathbb{P}^Z\{Z_\tau \in A\}$: harmonic measure (wrt generator $\mathcal{L}$)

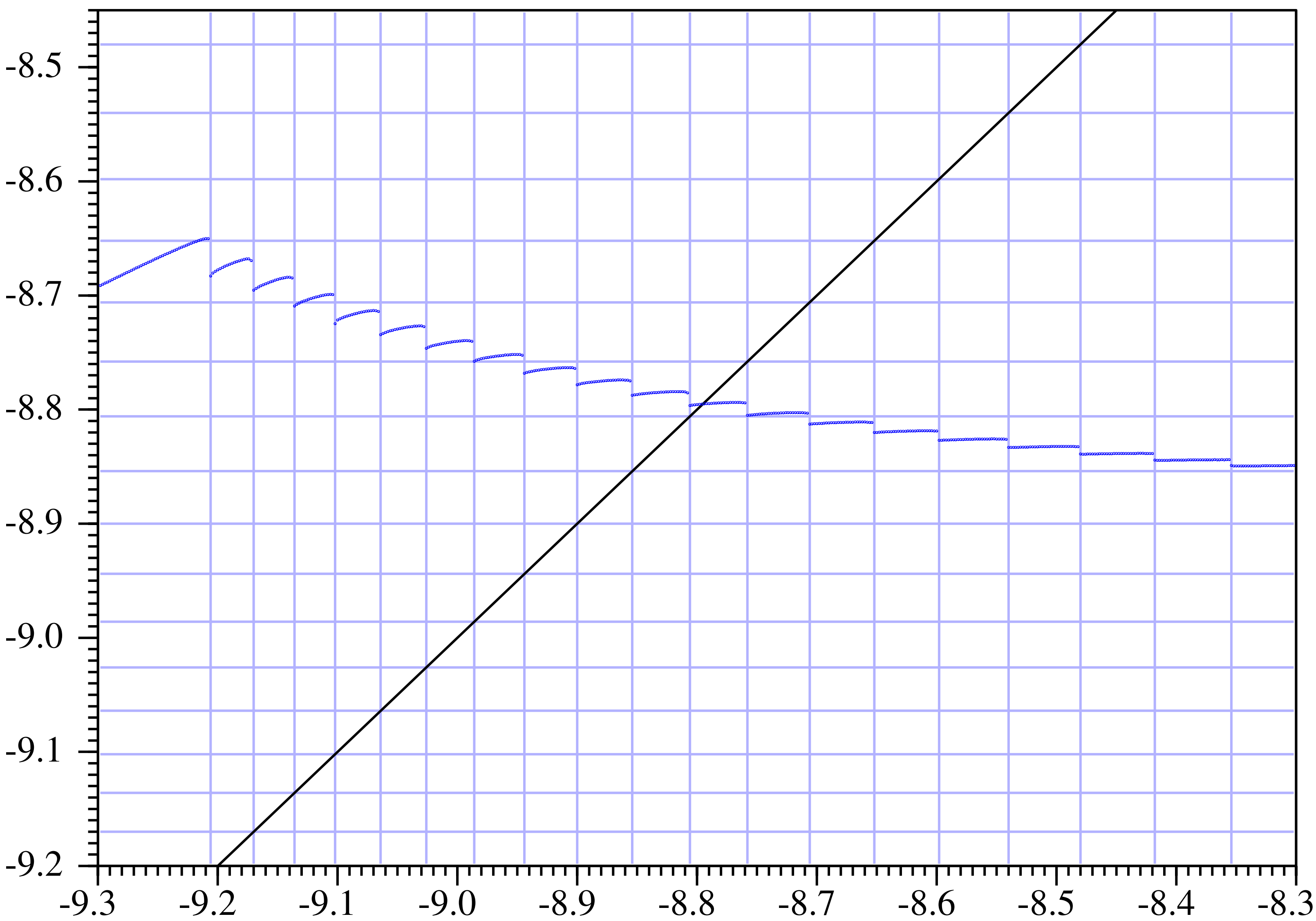
▷ [Ben Arous, Kusuoka, Stroock '84]: under hypoellipticity cond, μ_Z admits (smooth) density $h(Z,Y)$ wrt Lebesgue on $\partial\mathcal{D}$

▷ For $B \subset E$ Borel set

$$\mathbb{P}^{X_0}\{X_1 \in B\} = K(X_0, B) := \int_B K(X_0, \mathrm{d}y)$$

where $K(x, \mathrm{d}y) = h((0,x),(1,y))\,\mathrm{d}y =: k(x,y)\,\mathrm{d}y$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,\ \sigma = \sigma' = 0$

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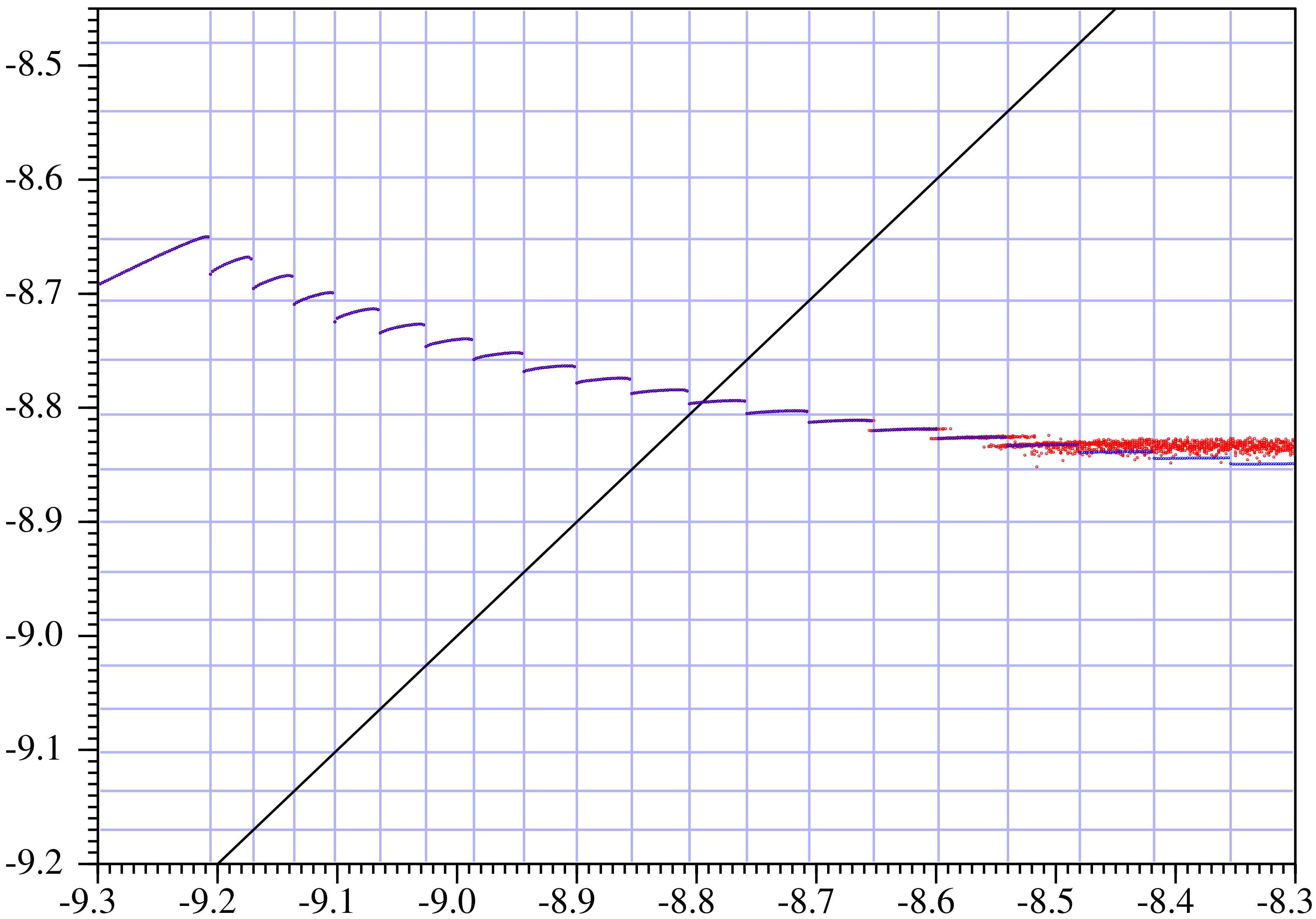

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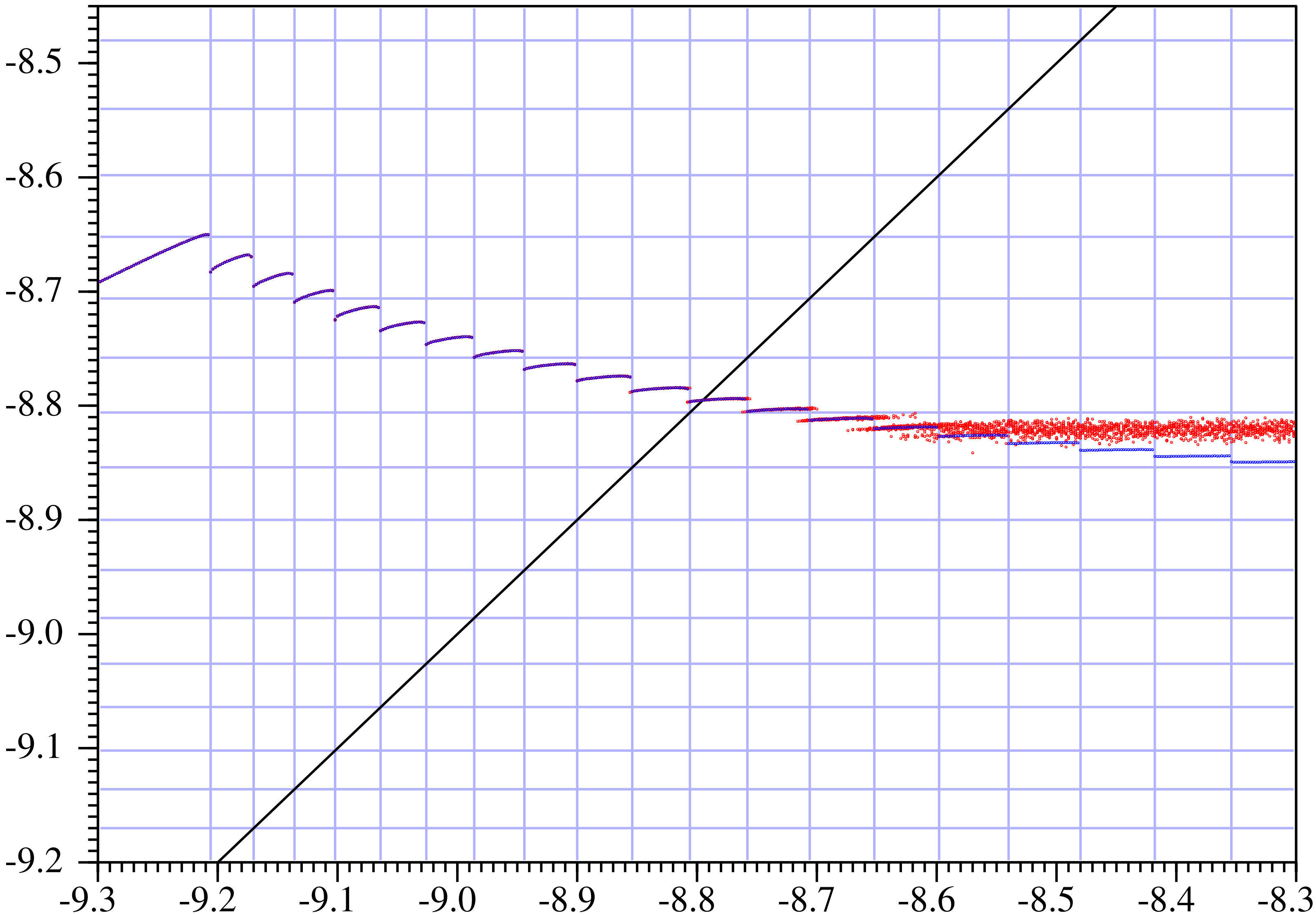

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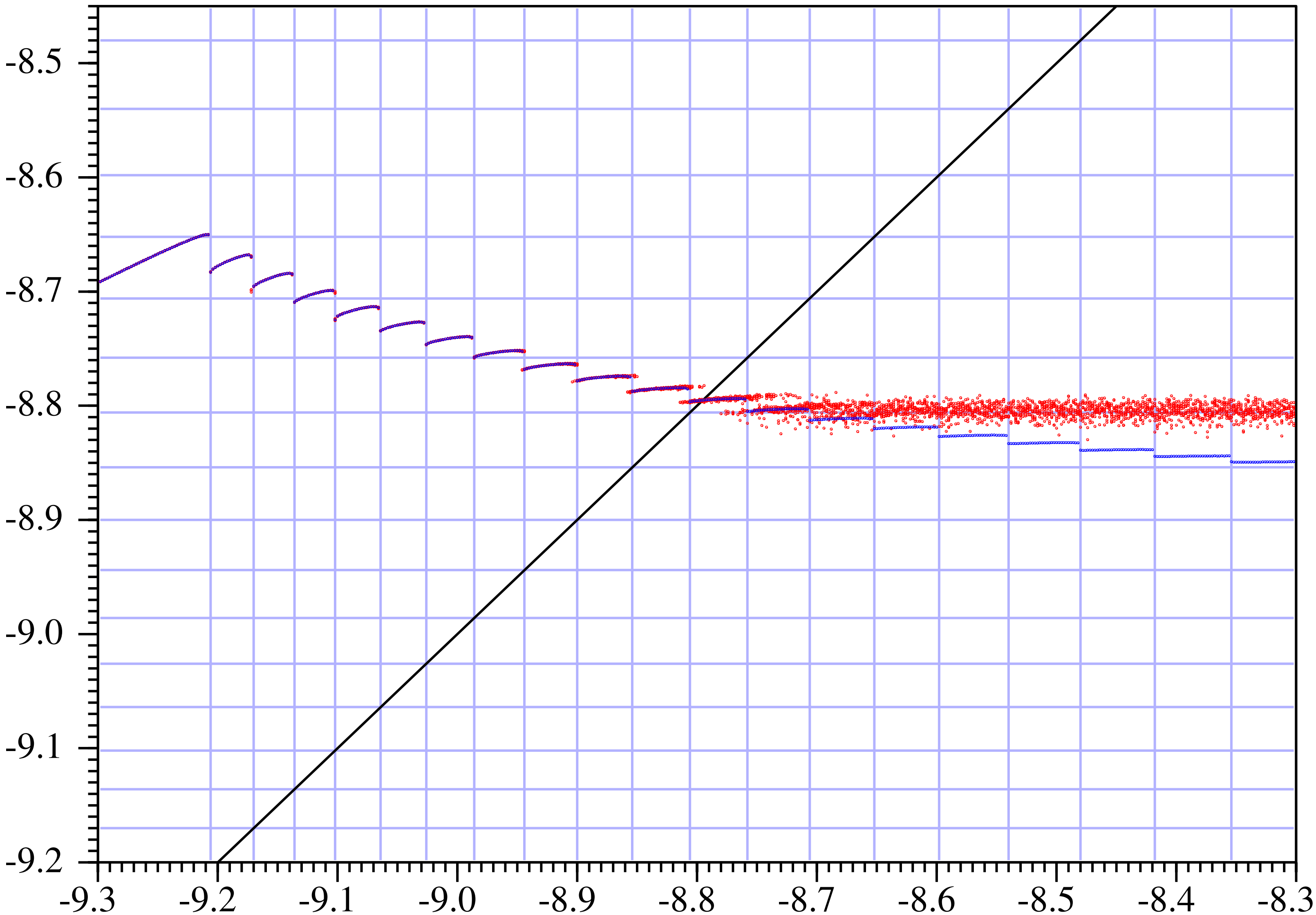

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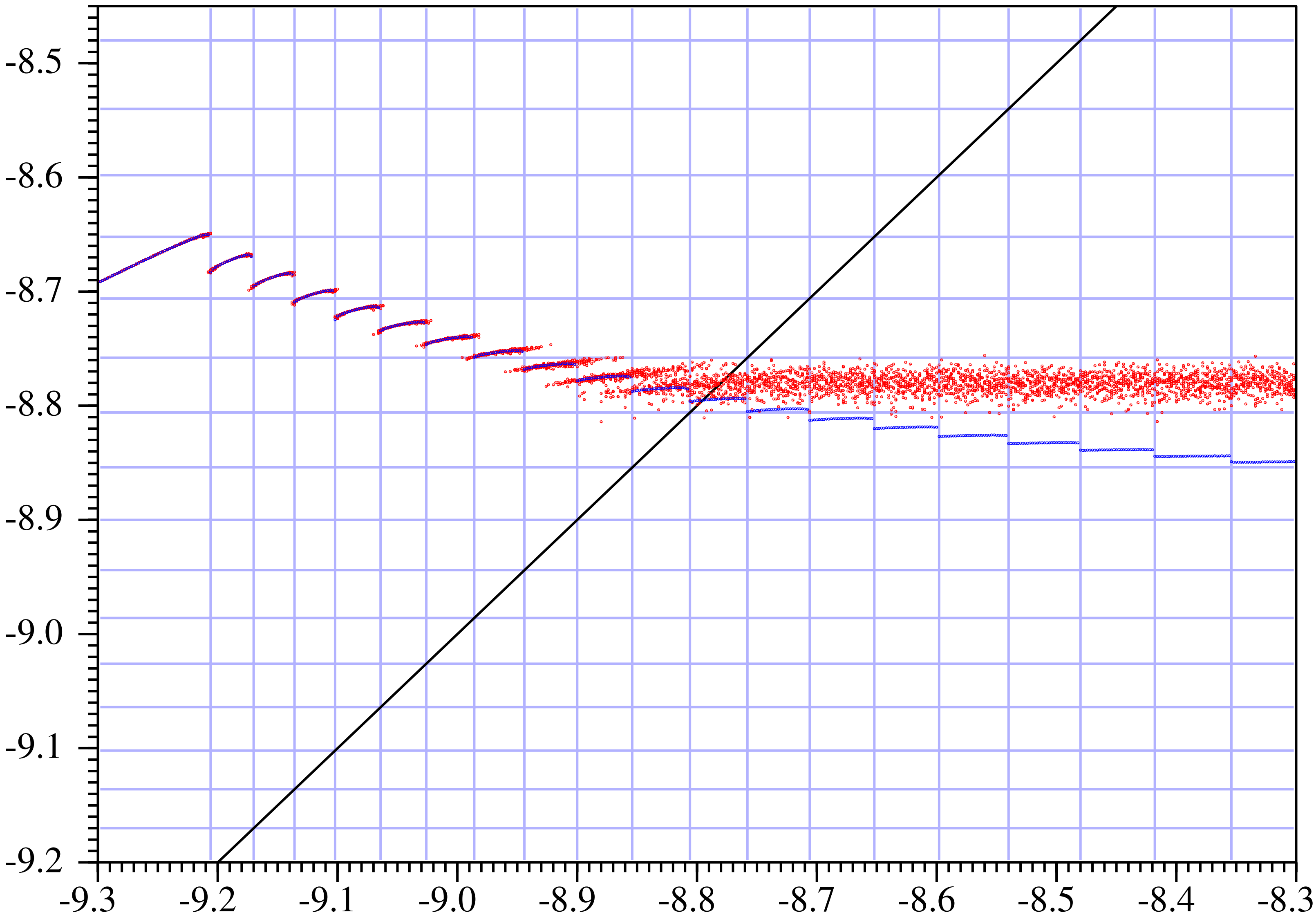

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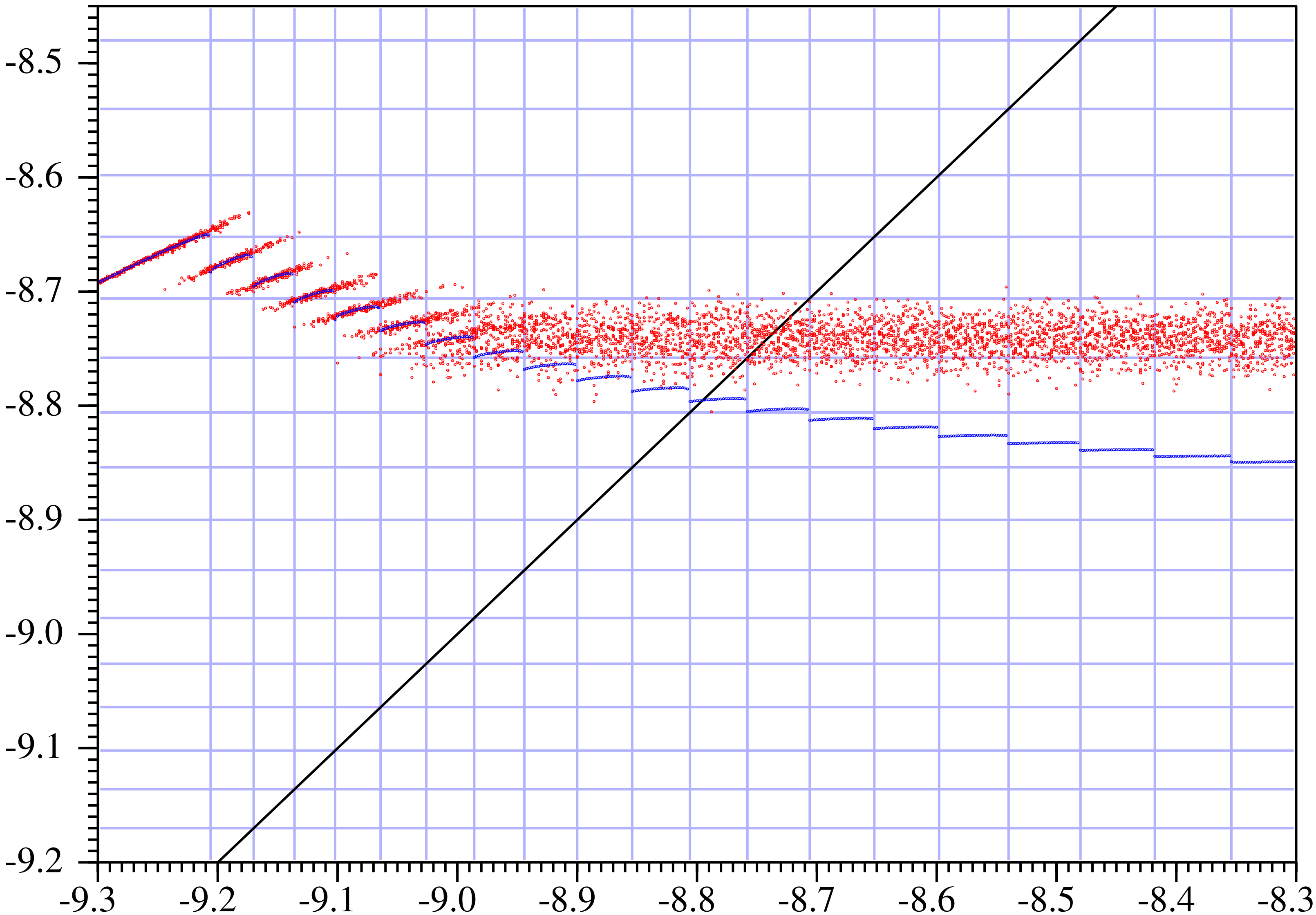

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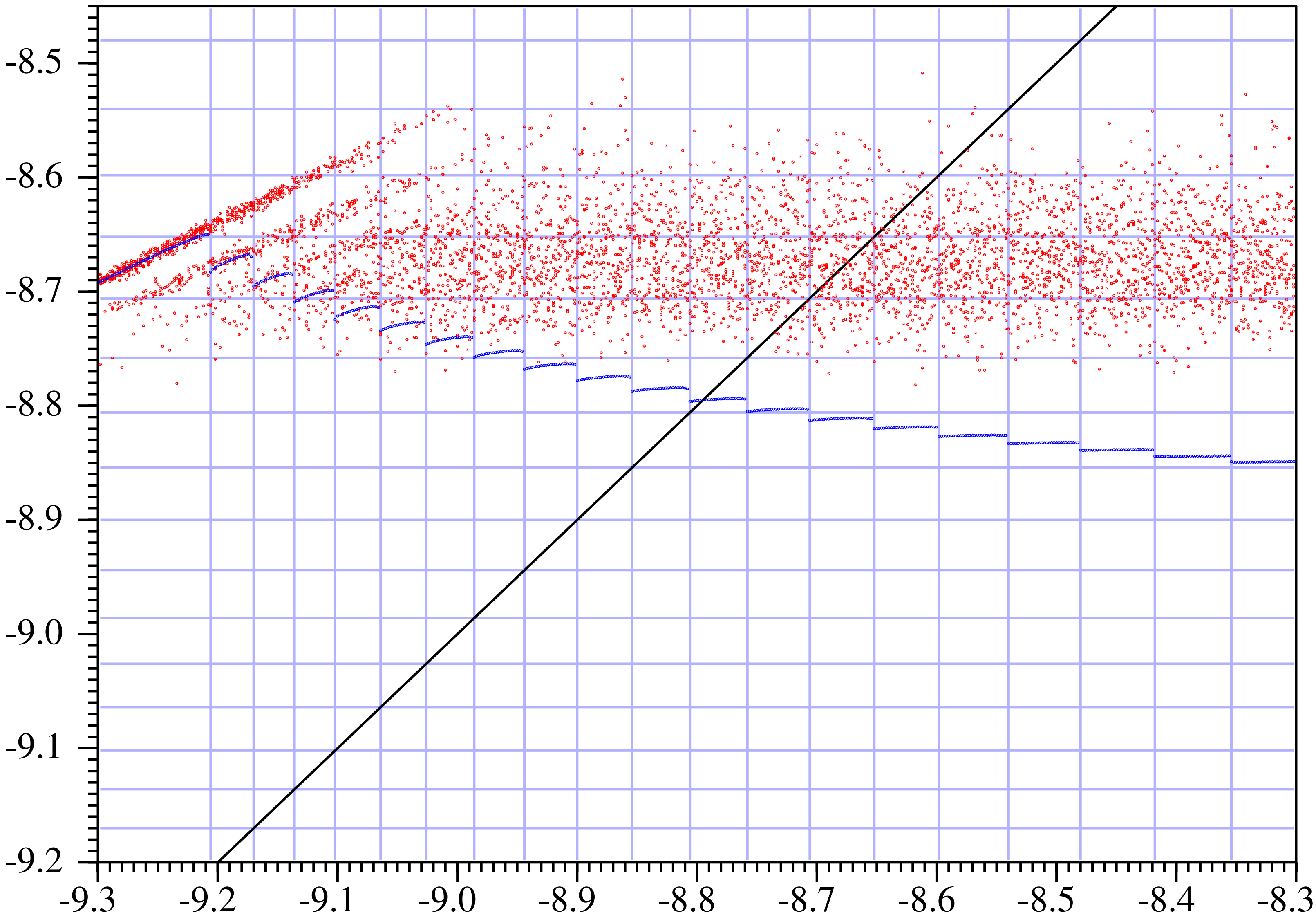

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Random Poincaré map

Observations:

- ▷ Size of fluctuations depends on noise intensity and canard number k: high order canards are more sensitive
- ▷ Saturation effect: constant distribution of z_{n+1} for $k > k_{\mathsf{c}}(\sigma, \sigma')$
- ▷ Consequence: if $k_{\mathsf{c}} < k^*_{\mathsf{det}}$, number of SAOs increases

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Questions:

- ▷ Prove saturation effect
- ▷ How does k_{c} depend on σ, σ'?
- ▷ How does size of fluctuations depend on σ, σ' and canard number k?
- ▷ In particular, size of fluctuations for $k > k_{\text{c}}$?

Size of noise-induced fluctuations

$\zeta_t = (x_t, y_t, z_t) - (x_t^{\text{det}}, y_t^{\text{det}}, z_t^{\text{det}})$

$$\mathrm{d}\zeta_t = \frac{1}{\varepsilon} A(t)\zeta_t \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathcal{F}(\zeta_t, t)\,\mathrm{d}W_t + \frac{1}{\varepsilon} \underbrace{b(\zeta_t, t)}_{=\mathcal{O}(\|\zeta_t\|^2)} \,\mathrm{d}t$$

$$\zeta_t = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t U(t,s)\mathcal{F}(\zeta_s, s)\,\mathrm{d}W_s + \frac{1}{\varepsilon} \int_0^t U(t,s) b(\zeta_s, s)\,\mathrm{d}s$$

where $U(t,s)$ principal solution of $\varepsilon\dot{\zeta} = A(t)\zeta$.

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Lemma (Bernstein-type estimate):

$$\mathbb{P}\left\{\sup_{0\leqslant s\leqslant t}\left\|\int_0^s \mathcal{G}(\zeta_u, u)\,\mathrm{d}W_u\right\| > h\right\} \leqslant 2n \exp\left\{-\frac{h^2}{2V(t)}\right\}$$

where $\int_0^s \mathcal{G}(\zeta_u, u)\mathcal{G}(\zeta_u, u)^T \,\mathrm{d}u \leqslant V(s)$ and $n = 3$

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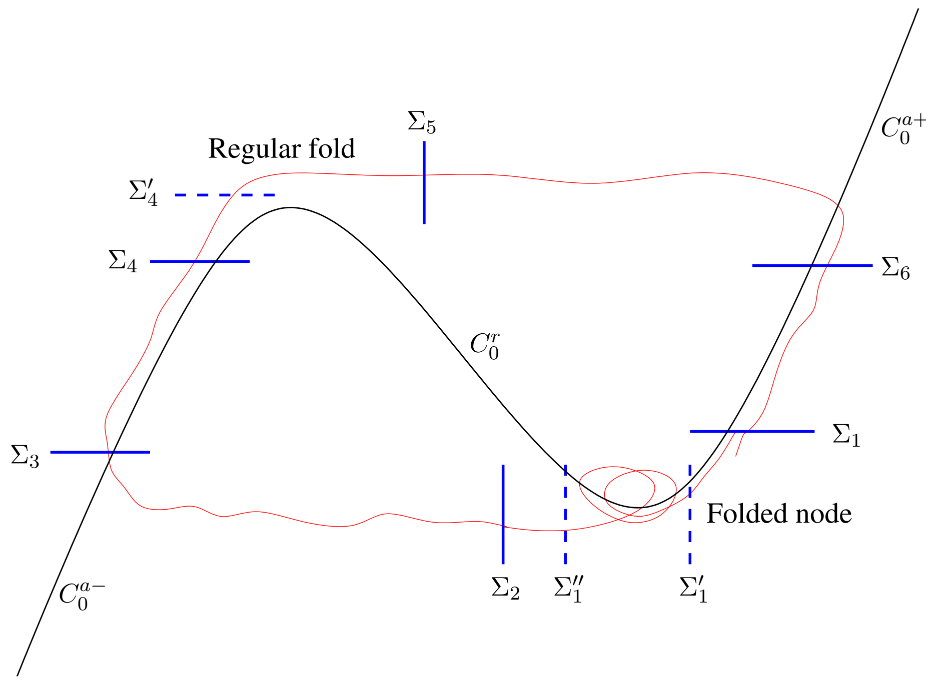
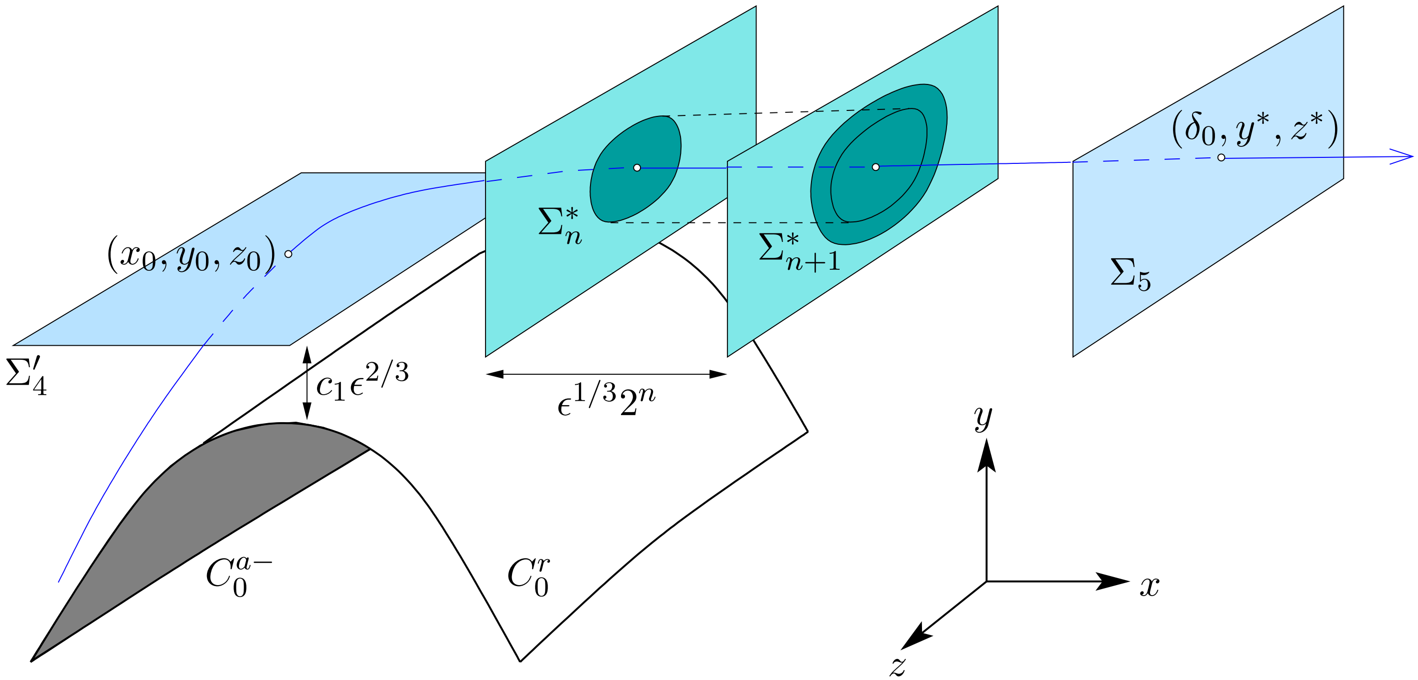
Remark: more precise results using ODE for covariance matrix of

$$\zeta_t^0 = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t U(t,s)\mathcal{F}(0,s)\,\mathrm{d}W_s$$



Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	σ		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	σ		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	σ		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	σ		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$(\sigma + \sigma')\varepsilon^{1/4}$

Example: Analysis near the regular fold



Proposition: For $h_1 = \mathcal{O}(\varepsilon^{2/3})$,

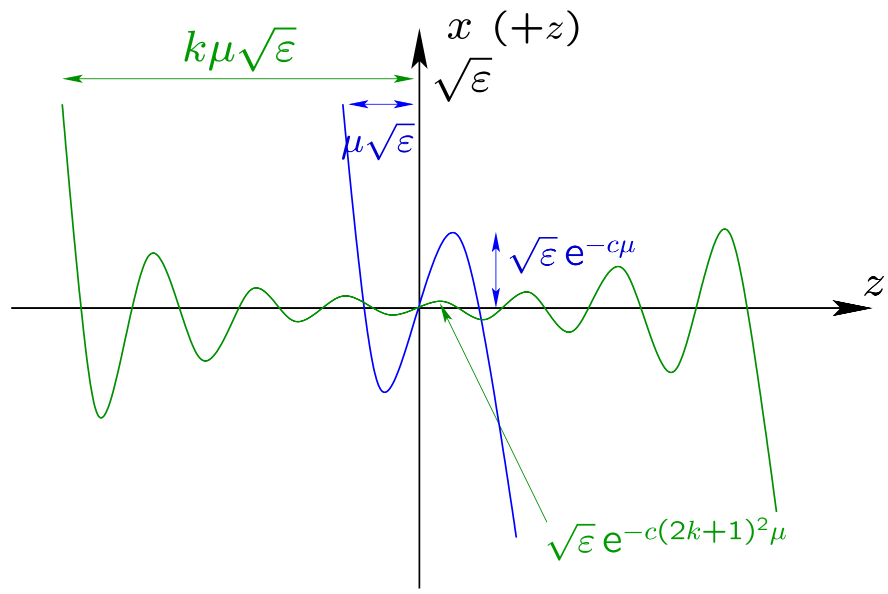
$$\mathbb{P}\Big\{\|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\| > h_1\Big\}$$

$$\leqslant C|\log \varepsilon|\bigg(\exp\bigg\{-\frac{\kappa h_1^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}}\bigg\} + \exp\bigg\{-\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon}\bigg\}\bigg)$$

Useful if $\sigma, \sigma' \ll \sqrt{\varepsilon}$

Local analysis near the folded node [B, Gentz, Kuehn, JDE 2012]

Thm 1: (Canard spacing)
For $z = 0$, the k^{th} canard lies at dist. $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



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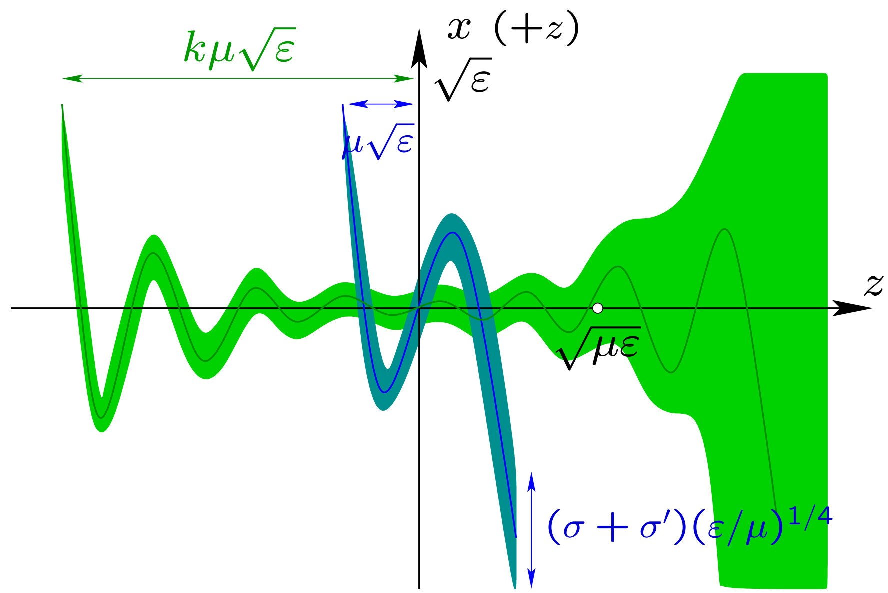
Thm 2: Size of fluctuations
$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$
$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



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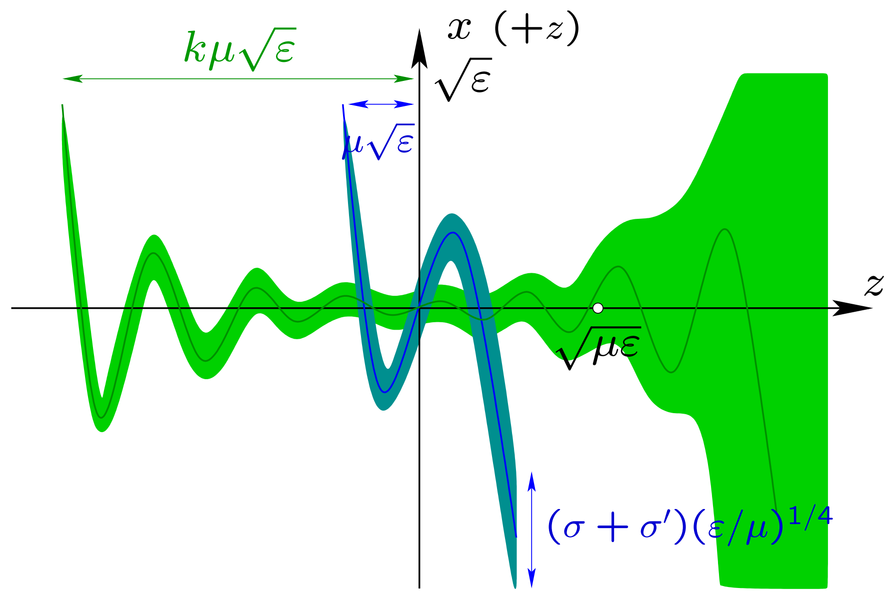
Thm 3: (Early escape)

Prob. to stay near det. solution

$\leqslant C|\log(\sigma+\sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$



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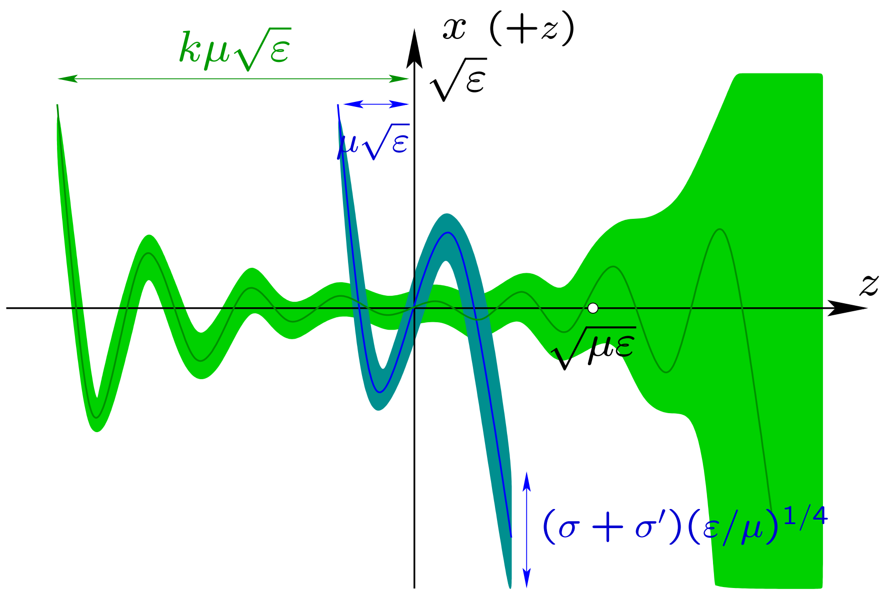
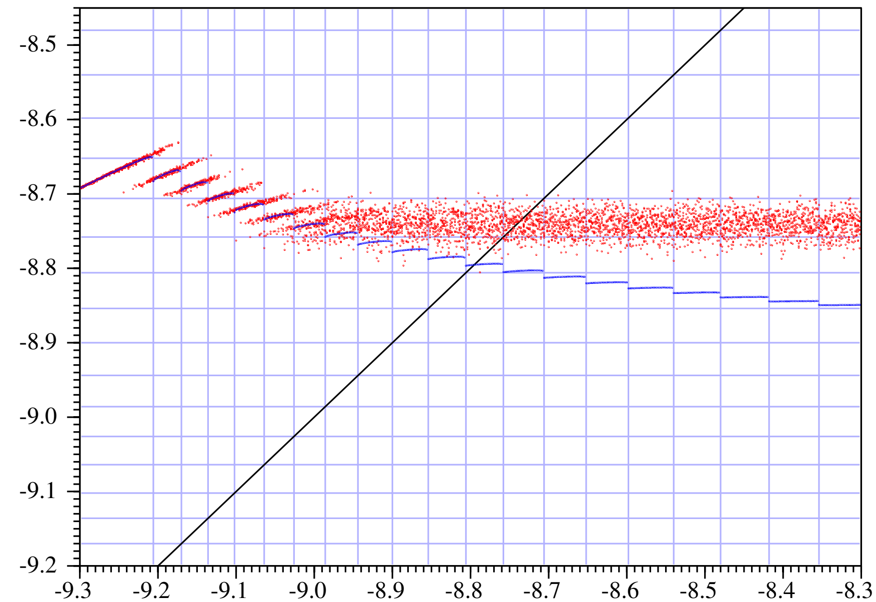


Consequence: Dichotomy

- ▷ Canards with $k \leqslant \sqrt{1/\mu}$: $\Delta z \asymp \sigma(\varepsilon/\mu)^{1/4} + \sigma'$ (assuming $\varepsilon \leqslant \mu$)
- ▷ Canards with $k > \sqrt{|\log(\sigma+\sigma')|/\mu}$: $\Delta z \leqslant \mathcal{O}\Big(\sqrt{\varepsilon\mu|\log(\sigma+\sigma')|}\Big)$

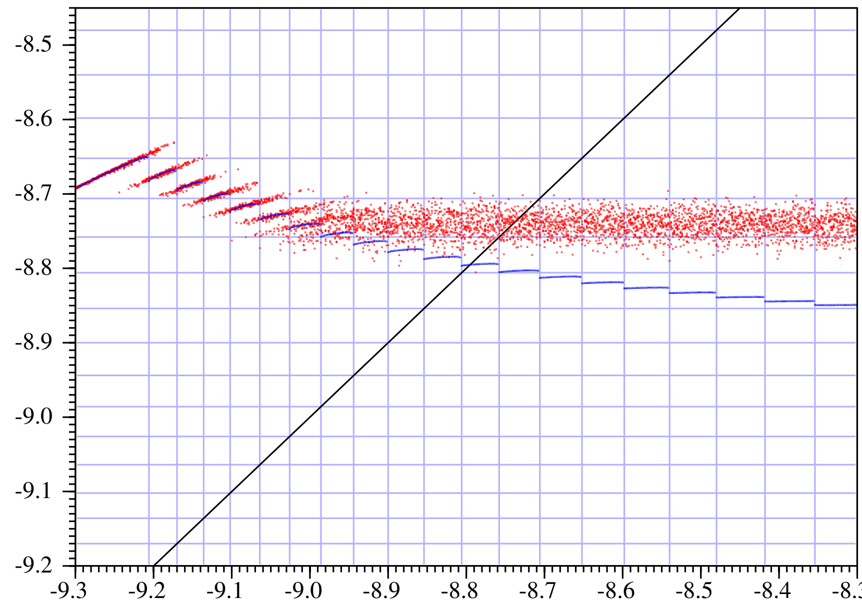
Summary

- ▷ $\sqrt{1/\mu} < k_{\mathsf{c}} < \sqrt{|\log(\sigma + \sigma')|/\mu}$
- ▷ For $k \leqslant \sqrt{1/\mu}$, dispersion $\Delta z \asymp \sigma(\varepsilon/\mu)^{1/4} + \sigma'$
- ▷ For $k > \sqrt{|\log(\sigma + \sigma')|/\mu}$ $\Delta z \leqslant \mathcal{O}\Big(\sqrt{\varepsilon\mu|\log(\sigma + \sigma')|}\Big)$
- ▷ If the deterministic system has MMO with k^* SAOs and $k^* < k_{\mathsf{c}}$ then noise increases number of SAOs



Summary

- $\sqrt{1/\mu} < k_c < \sqrt{|\log(\sigma + \sigma')|/\mu}$
- For $k \leqslant \sqrt{1/\mu}$, dispersion $\Delta z \asymp \sigma(\varepsilon/\mu)^{1/4} + \sigma'$
- For $k > \sqrt{|\log(\sigma + \sigma')|/\mu}$ $\Delta z \leqslant \mathcal{O}\Big(\sqrt{\varepsilon\mu|\log(\sigma + \sigma')|}\Big)$
- If the deterministic system has MMO with k^* SAOs and $k^* < k_c$ then noise increases number of SAOs



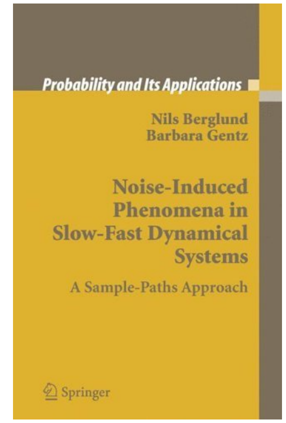
Further ways to analyse random Poincaré map

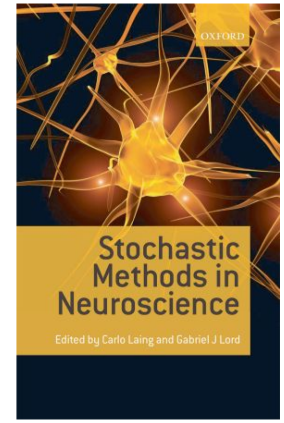
- Theory of singularly perturbed Markov chains
- For coexisting stable periodic orbits: Metastable transitions

Thanks for your attention – Further reading

N.B. and Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)

N.B. and Barbara Gentz, *Stochastic dynamic bifurcations and excitability*, in C. Laing and G. Lord, (Eds.), *Stochastic methods in Neuroscience*, p. 65-93, Oxford University Press (2009)





N.B., *Stochastic dynamical systems in neuroscience*, Oberwolfach Reports **8**:2290–2293 (2011)

N.B., Barbara Gentz and Christian Kuehn, *Hunting French Ducks in a Noisy Environment*, J. Differential Equations **252**:4786–4841 (2012). `arXiv:1011.3193`

N.B. and Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**:2303–2335 (2012). `arXiv:1105.1278`

N.B. and Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, preprint `arXiv:1208.2557`

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