Geometric singular perturbation theory for stochastic differential equations

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**Slow–fast systems: heuristics**

In fast time $s$:

\[
\begin{align*}
    x' &= f(x, y) & x \in \mathbb{R}^n, \text{ fast variable} \\
    y' &= \varepsilon g(x, y) & y \in \mathbb{R}^m, \text{ slow variable}
\end{align*}
\]

- Perturbation of $x' = f(x, \lambda)$, with slowly moving parameter $\lambda$
- Simplest case: $x^*(\lambda)$ asympt. stable equilibrium point

In slow time $t = \varepsilon s$:

\[
\begin{align*}
    \varepsilon \dot{x} &= f(x, y) & x \in \mathbb{R}^n, \text{ fast variable} \\
    \dot{y} &= g(x, y) & y \in \mathbb{R}^m, \text{ slow variable}
\end{align*}
\]

- Slow manifold: $f(x^*(y), y) = 0$ (for all $y$ in some domain)
- Reduced equation:

\[
    \dot{y} = g(x^*(y), y)
\]
Geometric singular perturbation theory

\[ \begin{align*}
\varepsilon \dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{align*} \]

\(x \in \mathbb{R}^n\), fast variable

\(y \in \mathbb{R}^m\), slow variable

• Slow manifold: \(f(x^*(y), y) = 0\) (for all \(y\) in some domain)

• Stability: Eigenvalues of \(\partial_x f(x^*(y), y)\) have negative real parts

**Theorem** [Tihonov '52, Fenichel '79]

\(\exists\) adiabatic manifold \(x = \bar{x}(y, \varepsilon)\)

s.t.

• \(\bar{x}(y, \varepsilon)\) is invariant

• \(\bar{x}(y, \varepsilon)\) attracts nearby solutions

• \(\bar{x}(y, \varepsilon) = x^*(y) + \mathcal{O}(\varepsilon)\)
Stochastic perturbation: one-dimensional case

\[ dx_t = \frac{1}{\varepsilon} f(x_t, t) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW_t \]

Slow–fast system with \( y_t = t \)

Stable equil. branch: \( f(x^*(t), t) = 0, \ a^*(t) = \partial_x f(x^*(t), t) \leq -a_0 \)

Adiabatic solution: \( \bar{x}(t, \varepsilon) = x^*(t) + O(\varepsilon) \)

\( \bar{a}(t, \varepsilon) = \partial_x f(\bar{x}(t, \varepsilon), t) = a^*(t) + O(\varepsilon) \)

\( \mathcal{B}(h) \): strip of width \( \simeq h/|\bar{a}(t, \varepsilon)| \) around \( \bar{x}(t, \varepsilon) \).
Stochastic perturbation: one-dimensional case

\[ dx_t = \frac{1}{\varepsilon} f(x_t, t) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW_t \]

**Theorem:** [B. & G., PTRF 2002]

\[ C(t, \varepsilon) e^{-\kappa_- h^2 / 2\sigma^2} \leq \mathbb{P}\{\text{leaving } B(h) \text{ before time } t\} \leq C(t, \varepsilon) e^{-\kappa_+ h^2 / 2\sigma^2} \]

\[ \kappa_{\pm} = 1 \pm O(h) \]

\[ C(t, \varepsilon) = \sqrt{\frac{2}{\pi \varepsilon}} \left| \int_0^t \bar{a}(s, \varepsilon) \, ds \right| \frac{h}{\sigma} \left[ 1 \pm \text{error} \right] \]
Stochastic perturbation: $n$-dimensional case

\[
\begin{cases}
  \frac{dx_t}{dt} = \frac{1}{\varepsilon} f(x_t, y_t) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t) \, dW_t & \text{(fast variables } \in \mathbb{R}^n) \\
  dy_t = g(x_t, y_t) \, dt + \sigma' G(x_t, y_t) \, dW_t & \text{(slow variables } \in \mathbb{R}^m)
\end{cases}
\]

Stable slow manifold: $f(x^*(y), y) = 0$, $A(y) = \partial_x f(x^*(y), y)$ stable

\[
\mathcal{B}(h) := \left\{ (x, y) : \left\langle \left[ x - \bar{x}(y, \varepsilon) \right], X^*(y)^{-1} \left[ x - \bar{x}(y, \varepsilon) \right] \right\rangle < h^2 \right\}
\]

$X^*(y)$ solution of $A(y)X^* + X^*A(y)^T + F(x^*, y)F(x^*, y)^T = 0$
Stochastic perturbation: $n$-dimensional case

\[
\begin{align*}
\frac{dx_t}{\varepsilon} &= \frac{1}{\varepsilon} f(x_t, y_t) \, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t) \, dW_t \\
\frac{dy_t}{\varepsilon} &= g(x_t, y_t) \, dt + \sigma' G(x_t, y_t) \, dW_t
\end{align*}
\]

(fast variables $\in \mathbb{R}^n$)

(slow variables $\in \mathbb{R}^m$)

**Theorem** [B. & G., JDE 2003]

- $\mathbb{P}\{\text{leaving } B(h) \text{ before time } t\} \simeq C(t, \varepsilon) e^{-\kappa h^2/2\sigma^2}$
  
  $\kappa = 1 - O(h) - O(\varepsilon)$.

- Projection on $\bar{x}(y, \varepsilon)$:
  
  \[
  \frac{dy^0_t}{\varepsilon} = g(\bar{x}(y^0_t, \varepsilon), y^0_t) \, dt + \sigma' G(\bar{x}(y^0_t, \varepsilon), y^0_t) \, dW_t
  \]

  $y^0_t$ approximates $y_t$ to order $\sigma \sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y} = g(\bar{x}(y, \varepsilon)y)$. 
Bifurcations

$x^*(y)$ slow manifold for $y \in D_0$

$A(y) = \partial_x f(x^*(y), y)$

Some ev of $A(y)$ cross imaginary axis as $y$ approaches $\partial D_0$

**Theorem** [B. & G., JDE 2003]

System can be approximated by projection on centre manifold.

- Saddle–node bifurcation: transitions through unstable manifold, relaxation oscillations, hysteresis
- (Avoided) transcritical bifurcation: stochastic resonance
- Pitchfork bifurcation: decrease of bifurcation delay
Saddle–node bifurcation

e.g. \( f(x, y) = -y - x^2 \)

\[ \sigma \ll \sqrt{\varepsilon} \]

\[ \sigma \gg \sqrt{\varepsilon} \]

Deterministic case \( \sigma = 0 \): Solutions stay at distance \( \varepsilon^{1/3} \) above bifurcation point until time \( \varepsilon^{2/3} \) after bifurcation.

**Theorem:** [B. & G., Nonlinearity 2002]

1. If \( \sigma \ll \sqrt{\varepsilon} \): Paths likely to stay in \( B(h) \) until time \( \varepsilon^{2/3} \) after bifurcation, maximal spreading \( \sigma \varepsilon^{-1/6} \).
2. If \( \sigma \gg \sqrt{\varepsilon} \): Paths likely to escape at time \( \sigma^{4/3} \) before bifurcation.
Avoided transcritical bifurcation

\[ f(x, y) = y^2 + \delta - x^2 \]

\[ \sigma \ll (\delta \vee \varepsilon)^{3/4} \]

\[ \sigma \gg (\delta \vee \varepsilon)^{3/4} \]

Minimal distance between branches = \( \delta^{1/2} \)

Det. case \( \sigma = 0 \): Solutions stay \((\delta \vee \varepsilon)^{1/2}\) above bif. point


1. If \( \sigma \ll (\delta \vee \varepsilon)^{3/4} \): Paths likely to stay in \( B(h) \), maximal spreading \( \sigma(\delta \vee \varepsilon)^{-1/4} \).
2. If \( \sigma \gg (\delta \vee \varepsilon)^{3/4} \): Paths likely to escape at time \( \sigma^{2/3} \) before avoided bifurcation.
Stochastic resonance

\[ \mathrm{d} x_t = \left[ x_t - x_t^3 + A \cos \varepsilon t \right] \mathrm{d} t + \sigma \, \mathrm{d} W_t \]

\[ = - \frac{\partial}{\partial x} V(x_t, t) \]

Potential: \[ V(x, t) = \frac{1}{4} x^4 - \frac{1}{2} x^2 - Ax \cos \varepsilon t. \]

\[ \sigma \gg (\delta \lor \varepsilon)^{3/4}, \, \delta = A - A_c, \, A_c = 2/3\sqrt{3}: \text{synchronisation} \]
Pitchfork bifurcation

\[ f(x, y) = yx - x^3 \]

**Theorem** [B. & G., PTRF 2002]

- Paths concentrated in \( B(h) \) up to time \( \sqrt{\varepsilon} \)  
  Typical spreading \( \sigma \varepsilon^{-1/4} \)
- Paths likely to leave \( D \) at time \( \sqrt{\varepsilon |\log \sigma|} \)
- Paths likely to stay in \( A^\tau(h) \) after leaving \( D \)
Application. North-Atlantic thermohaline circulation: Stommel’s Box Model ('61)

- $T_i$: temperatures
- $S_i$: salinities
- $F$: freshwater flux
- $Q(\Delta \rho)$: mass exchange
- $\Delta \rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$

\[
\frac{d}{ds} \Delta T = -\frac{1}{\tau_r} (\Delta T - \theta) - Q(\Delta \rho) \Delta T
\]
\[
\frac{d}{ds} \Delta S = \frac{S_0}{H} F - Q(\Delta \rho) \Delta S
\]

Model for $Q$ (Cessi): $Q(\Delta \rho) = \frac{1}{\tau_d} + \frac{q}{V} \Delta \rho^2$. 
Slow–fast system

Separation of time scales: \( \tau_r \ll \tau_d \)
Scaling: \( x = \Delta T/\theta, \ y = \Delta S\alpha_S/(\alpha_T\theta), \ s = \tau_d t, \ldots \)

\[
\begin{align*}
\varepsilon \dot{x} &= -(x - 1) - \varepsilon x \left[1 + \eta^2(x - y)^2\right] \\
\dot{y} &= \mu - y \left[1 + \eta^2(x - y)^2\right]
\end{align*}
\]

\( \varepsilon = \tau_r/\tau_d \ll 1 \)

Slow manifold: \( x = 1 + O(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0. \)
Reduced equation on slow manifold:

\[
\dot{y} = \mu - y \left[1 + \eta^2(1 - y)^2 + O(\varepsilon)\right]
\]

One or two stable equilibria, depending on \( \mu \) (and \( \eta \)).
Time-dependent freshwater flux

\[
dx_t = \frac{1}{\varepsilon_0} \left[ -(x_t - 1) - \varepsilon_0 x_t Q(x_t - y_t) \right] \, dt + \frac{\sigma}{\sqrt{\varepsilon_0}} \, dW_t^0
\]

\[
dy_t = \left[ z_t - y_t Q(x_t - y_t) \right] \, dt + \sigma_1 \, dW_t^1
\]

\[
dz_t = \varepsilon h(x_t, y_t, z_t) \, dt + \sqrt{\varepsilon} \sigma_2 \, dW_t^2
\]

Reduced equation, \( t \mapsto \varepsilon t \):

\[
dy_t = \frac{1}{\varepsilon} \left[ z_t - y_t Q(1 - y_t) \right] \, dt + \frac{\sigma_1}{\sqrt{\varepsilon}} \, dW_t^1
\]

\[
dz_t = h(1, y_t, z_t) \, dt + \sigma_2 \, dW_t^2
\]

Relaxation oscillations

\[ z = yQ(1 - y) \]

Excitability

\[ z = yQ(1 - y) \]
References


