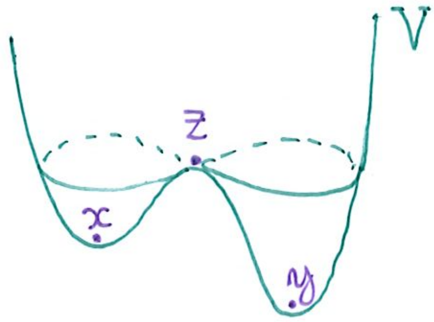
Metastability in stochastic Allen-Cahn PDEs

Based on: NB & Barbara Gentz, EJP 18(24): 1-58 (2013)
NB, Giacomo Di Gesù & Hendrik Weber, EJP 22(41): 1-27 (2017)

1. Metastability in gradient SDEs



$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$\pi(x) = \frac{1}{Z} e^{-V(x)/\varepsilon}$ reversible inv. density

$\tau := \inf\{t>0 : \|x_t - y\| < \delta\}$ $\qquad \mathbb{E}^x[\tau] = ?$

Arrhenius 1889: $\mathbb{E}^x[\tau] \simeq e^{[V(z)-V(x)]/\varepsilon}$ i.e. $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}^x[\tau] = V(z)-V(x)$

Proof: [Freidlin & Wentzell ~ 1970] (large deviations)

Eyring 1935, Kramers 1940: $\mathbb{E}^x[\tau] = \frac{2\pi}{|\lambda_0(z)|}\sqrt{\frac{|\det \operatorname{Hess} V(z)|}{\det \operatorname{Hess} V(x)}}\, e^{[V(z)-V(x)]/\varepsilon}\,[1+o_\varepsilon(1)]$

unique <0 ev of Hess $V(z)$

Proofs:
- WKB theory
- potential theory [Bovier, Eckhoff, Gayrard, Klein 2004]
- Witten Laplacian [Helffer, Klein, Nier 2005]

Potential theory: (~electric networks) $\quad A, B \subset \mathbb{R}^d$ smooth boundary $\quad A \cap B = \emptyset$

$$\boxed{\mathbb{E}^{\mu_{AB}}[\tau_B] = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} h_{AB}(x)\, e^{-V(x)/\varepsilon}\,dx}$$

- μ_{AB}: measure conc. on ∂A $\quad (\mu_{AB} \sim \partial_n h_{AB})$ $\qquad$ Harnack $\Rightarrow \mathbb{E}^{\mu_{AB}} \sim \mathbb{E}^x$
- $h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$ committor fct $\quad \Rightarrow \int_{B^c} h_{AB}(x) e^{-V(x)/\varepsilon}\,dx \sim \sqrt{\frac{(2\pi\varepsilon)^d}{\det \operatorname{Hess} V(x)}}\, e^{-V(x)/\varepsilon}$
- $\operatorname{cap}(A,B) = \Phi(h_{AB})$ $\qquad \Phi(h) = \varepsilon \int_{(A\cup B)^c} \|\nabla h\|^2 e^{-V(x)/\varepsilon}\,dx$

Dirichlet form $\quad \Phi(h) \sim \langle h, -\mathcal{L}h\rangle$

- Dirichlet principle:

$$\operatorname{cap}(A,B) = \inf_{\substack{h|_A = 1\\ h|_B = 0}} \Phi(h) \quad \Rightarrow \quad \operatorname{cap}(A,B) \sim \frac{|\lambda_0(z)|}{2\pi}\sqrt{\frac{(2\pi\varepsilon)^d}{|\det \operatorname{Hess} V(z)|}}$$

2. Allen-Cahn in dimension 1

$$\partial_t \phi = \Delta\phi + \phi - \phi^3 + \sqrt{2\varepsilon}\,\xi \longleftarrow$$ space-time white noise:

$\langle \xi, \varphi\rangle$ centred Gaussian r.v.

$$\mathbb{E}(\langle \xi,\varphi_1\rangle\langle\xi,\varphi_2\rangle) = \langle\varphi_1,\varphi_2\rangle$$

$\phi = \phi(t,x)$, $\quad x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $\quad t \geq 0$

Potential: $$V(\phi) = \int_0^L \Big[\tfrac{1}{2}\nabla\phi(x)^2 - \tfrac{1}{2}\phi(x)^2 + \tfrac{1}{4}\phi(x)^4\Big]\,dx$$

$$\lim_{h\to 0} \frac{V(\phi+h\psi)-V(\phi)}{h} = \int_0^L [\nabla\phi\nabla\psi - \phi\psi + \phi^3\psi]\,dx$$
$$= -\langle \Delta\phi + \phi - \phi^3, \psi\rangle$$

Stat. sols: $\Delta\phi = -\phi + \phi^3 \quad \Rightarrow \begin{cases} \phi_0(x) \equiv 0 \qquad \phi_\pm(x) \equiv \pm 1 \\ \text{additional non-const. sol if } L > 2\pi \end{cases}$

Fourier basis: $\phi(t,x) = \sum_{k\in\mathbb{Z}} z_k(t) e_k(x) \;\Rightarrow\; dz_t = -\nabla\hat{V}(z_t)\,dt + \sqrt{2\varepsilon}\,dW_t$

$(\varphi_i(t,x) = e_{k_i}(x)\, 1_{[0,t_i]}(t) \Rightarrow \langle\varphi_1,\varphi_2\rangle = (t_1\wedge t_2)\delta_{k_1k_2})$

Hessian at $\phi_0 \equiv 0$: $$V(\phi) = \frac{1}{2}\langle\phi, \underbrace{(-\Delta-1)}_{\text{Hess }V(\phi_0)}\phi\rangle + O(\phi^4)$$ $L<2\pi \Rightarrow -1 = \lambda_0 < 0 < \lambda_{\pm1} < \dots$

eigenvalues $\lambda_k = \left(\frac{2\pi}{L}k\right)^2 - 1$

Hessian at $\phi_- \equiv -1$: $$V(-1+\phi) = \frac{1}{2}\langle\phi, \underbrace{(-\Delta+2)}_{\text{Hess }V(\phi_-)}\phi\rangle + O(\phi^4)$$ ev $\lambda_k + 3$

Formally: $$\mathbb{E}^{\phi_-}[\tau_{\phi_+}] = \frac{2\pi}{|\lambda_0|}\underbrace{\sqrt{\frac{|\det(-\Delta-1)|}{\det(-\Delta+2)}}}_{\infty/\infty}\;\underbrace{e^{[V(\phi_0)-V(\phi_-)]/\varepsilon}}_{\text{c.f. Faris \& Jona-Lasinio '82}}\,[1+o_\varepsilon(1)]$$

[Maier & Stein]

$\Delta_\perp$: projection of Δ on $k \neq 0$

$$\det([-\Delta_\perp - 1][-\Delta_\perp+2]^{-1}) = \det([-\Delta_\perp+2-3][-\Delta_\perp+2]^{-1})$$
$$= \det(1 - 3[-\Delta_\perp+2]^{-1})$$ Fredholm det.

$$\log\det([-\Delta_\perp-1][-\Delta_\perp+2]^{-1}) = \operatorname{Tr}\log(1-3[-\Delta_\perp+2]^{-1})$$
$$= -\sum_{n\geq 1}\frac{3^n}{n}\operatorname{Tr}([-\Delta_\perp+2]^{-n})$$

$$\operatorname{Tr}([-\Delta_\perp+2]^{-n}) = \sum_{k\neq 0}\frac{1}{[(\frac{2\pi}{L}k)^2+2]^n} \leq \frac{\text{const}}{[(\frac{2\pi}{L})^2+2]^n}$$

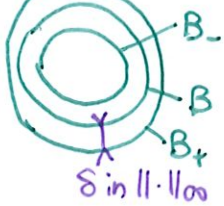
$\Rightarrow |\cdot| < \infty$ $\quad (L < 2\pi)$

Thm: [B & Gentz, EJP 2013] Result of formal comp. is correct

Rem: [Euler] $\Rightarrow \sqrt{\dfrac{\det(-\Delta-1)}{\det(-\Delta+2)}} = \dfrac{\sin L}{\sqrt{2}\sinh(\sqrt{2}L)}$

Proof: Spectral Galerkin: $\phi_N = \sum_{|k|\leq N} z_k(t)\, e_k(x)$

- E-K formula for τ_N with error terms uniform in N (Hausdorff-Young)
- [Blömker & Jentzen]: $\sup_{0\leq t\leq T} \|\phi_N(t,\omega) - \phi(t,\omega)\|_{L^\infty} < Z_T(\omega)\, N^{-\gamma}$ $0<\gamma<\frac{1}{2}$
- $\Omega_{K,N} = \left\{ \sup_{t\leq Ke^{\Delta V/\varepsilon}} \|\phi_N(t) - \phi(t)\|_{L^\infty} \leq \delta,\ \tau_{N,B_-} \leq Ke^{\Delta V/\varepsilon} \right\}$



$$\Rightarrow \limsup_{N\to\infty} \mathbb{P}(\Omega^c_{K,N}) \leq \frac{M(\varepsilon)}{K}$$

- $\mathbb{E}[\tau_B] = \underbrace{\mathbb{E}[\tau_B 1_{\Omega_{K,N}}]}_{\leq \mathbb{E}[\tau_{N,B_-}]} + \underbrace{\mathbb{E}[\tau_B 1_{\Omega^c_{K,N}}]}_{\leq (\mathbb{E}[\tau_B^2]\, \mathbb{P}(\Omega^c_{K,N}))^{1/2}}$
- Coupling argument [Martinelli, Olivieri, Scoppola] instead of Harnack □

Rem: • result also for $L \geq 2\pi$, more general potential, Neumann b.c.

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3. Allen-Cahn in dimension 2

$$\left.\begin{aligned}
\mathrm{Tr}[(-\Delta_\perp+2)^{-1}] &\sim \sum_{k\neq(0,0)} \frac{1}{\|k\|^2} \sim \int_1^\infty \frac{r\,dr}{r^2} = +\infty \\
\mathrm{Tr}[(-\Delta_\perp+2)^{-2}] &\sim \sum_{k\neq(0,0)} \frac{1}{\|k\|^4} \sim \int_1^\infty \frac{r\,dr}{r^4} < +\infty
\end{aligned}\right\}$$

$(-\Delta_\perp+2)^{-1}$ is not trace-class but Hilbert-Schmidt

Thm: [Da Prato, Debussche 2003]

$$\partial_t \phi_N = \Delta\phi_N + \phi_N - [\phi_N^3 - 3\varepsilon C_N \phi_N] + \sqrt{2\varepsilon}\,\xi_N$$

$\leftarrow$ mollified on scale $1/N$

with $C_N \sim \log N$ admits limit as $N\to\infty$

Spectral Galerkin: $dz_k = [-\lambda_k z_k + b_k(z)]\,dt + \sqrt{2\varepsilon}\, dW_k$ $\quad |k|\leq N$

$$dz_k = -\lambda_k z_k\, dt + \sqrt{2\varepsilon}\, dW_k \Rightarrow z_k(t) = z_k(0)e^{-\lambda_k t} + \sqrt{2\varepsilon}\int_0^t e^{-\lambda_k(t-s)}\, dW_s \quad \text{OU}$$

$$\xrightarrow[t\to\infty]{} \sqrt{2\varepsilon}\, \mathcal{N}\!\left(0, \frac{1}{2\lambda_k}\right) \longrightarrow \text{GFF}$$

$$\phi_N^{GFF}(x) = \sum_{|k|\leq N} \frac{\sqrt{2\varepsilon}\, Z_k}{\sqrt{2\lambda_k}}\, e_k(x) \Rightarrow \mathbb{E}\|\phi_N^{GFF}\|^2_{L^2} = \sum_{|k|\leq N} \frac{\varepsilon}{\lambda_k^2} = \varepsilon C_N \sim \varepsilon \log N$$

Wick powers:

$$\left.\begin{aligned} :\phi_N^2: &= \phi_N^2 - \varepsilon C_N \\ :\phi_N^3: &= \phi_N^3 - 3\varepsilon C_N \phi_N \\ :\phi_N^4: &= \phi_N^4 - 6\varepsilon C_N \phi_N^2 + 3\varepsilon^2 C_N^2 \end{aligned}\right\} \text{zero mean, bnd. moments}$$

Nelson estimate: $X \in n^{th}$ inhom. Wiener chaos

$$\Rightarrow \mathbb{E}[X^{2p}]^{1/2p} \leq C_n (2p-1)^{n/2} \, \mathbb{E}[X^2]^{1/2}$$

Renormalised potential: $V_N(\phi_N) = \frac{1}{2}\int_{\mathbb{T}^2} [\nabla\phi_N^2 - \phi_N^2]\,dx + \frac{1}{4}\int_{\mathbb{T}^2} :\phi_N^4:\,dx$

$\Rightarrow$ new prefactor $\quad \det\left([-\Delta_\perp - 1][\Delta_\perp + 2]^{-1} e^{3C_N}\right)$

$$= \det\left(1 - 3[-\Delta_\perp + 2]^{-1}\right) e^{3\,\mathrm{Tr}[-\Delta+2]^{-1}} \text{const}$$

Carleman-Fredholm determinant, $< \infty$

Thm: [B, Di Gesù, Weber, EJP 2017]

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}[\tau_B] \leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}} \frac{|\lambda_k|}{\lambda_k+3} e^{3/\lambda_k}}\; e^{[V(\phi_0)-V(\phi_-)]/\varepsilon}\,[1+O(\sqrt{\varepsilon})]$$

$$\liminf_{N\to\infty} \quad '' \quad \geq \quad '' \quad '' \quad [1-O(\varepsilon)]$$

proof: U.B on capacity

$$\mathrm{cap}(A,B) \leq \varepsilon \int_{(A\cup B)^c} \|\nabla h\|^2 e^{-V_N(z)/\varepsilon}\,dz \qquad h(z) = \begin{cases} 1 & z_0 \leq -\delta \\ \dfrac{\int_{z_0}^{\delta} e^{-t^2/2\varepsilon}dt}{\int_{-\delta}^{\delta} e^{-t^2/2\varepsilon}dt} & |z_0| \leq \delta \\ 0 & z \geq \delta \end{cases}$$

$$\cong \frac{1}{2\pi}\int e^{-[V_N(z)+z_0^2]/\varepsilon}\,dz$$

$$= \frac{1}{2\pi}\varepsilon^{(2N+1)^2/2}\int e^{-\underbrace{[V_N(\sqrt{\varepsilon}\bar{z}) + \varepsilon\bar{z}_0^2]/\varepsilon}_{= \frac{1}{2}\bar{z}_0^2 + \frac{1}{2}\sum_{0<|k|\leq N}\lambda_k \bar{z}_k^2 + \frac{\varepsilon}{4}\int :\bar{\phi}^4:\,dx}}\,d\bar{z}$$

$$= \sqrt{\frac{\varepsilon}{2\pi}} \prod_{0<|k|\leq N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\; \underbrace{\mathbb{E}^{\gamma}\left[e^{-\varepsilon\int :\phi^4:/4}\right]}_{= 1 + O(\varepsilon)}$$

Gaussian