Triangular billiards and generalized continued fractions

Pierre Arnoux

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Joint work with Thomas A. Schmidt
Geometric Interpretations of continued fractions: Classical continued fractions

The classical expansion of real numbers in continued fraction admits several interpretations:

- Arithmetic: approximation by rational numbers $\frac{p_n}{q_n}$
- Algebraic: matrices in $GL(2, \mathbb{Z})$
- Geometric: approximation of a line in the plane by elements of the lattice $\mathbb{Z}^2$
- Dynamic: geodesic flow on the modular surface $SL(2, \mathbb{Z})\backslash \mathbb{H}$, or equivalently Teichmüller flow for geometric structure for the torus.
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The Klein model
The classical domain for $SL(2, \mathbb{Z})$
Variants of the classical continued fraction

- Additive Continued Fraction
- Nearest Integer Continued Fraction
- Optimal Continued Fraction
- $\alpha$-Continued Fraction (*Japanese* continued fractions)

Occur naturally in various contexts. All linked to the group $SL(2, \mathbb{Z})$, lead essentially to the same results (infinite number of common approximants, for example). They can be analyzed as first return maps of the modular flow on various sections.
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Occur naturally in various contexts. All linked to the group $SL(2, \mathbb{Z})$, lead essentially to the same results (infinite number of common approximants, for example). They can be analyzed as first return maps of the modular flow on various sections. It is natural to look for such interpretations for generalized continued fractions.
An arithmetic generalization: Rosen Continued fraction

In the nearest integer continued fraction, take for the partial quotients multiples of a fixed number $\lambda$ instead of integers; i.e. write real numbers as:

$$x = n_0 \lambda + \frac{\epsilon_1}{n_1 \lambda + \frac{\epsilon_2}{n_2 \lambda + \frac{\epsilon_3}{n_3 \lambda + \ldots}}}$$

with $n_i \in \mathbb{Z}$, $\epsilon_i = \pm 1$
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Associated to the map:

$$x \mapsto \frac{1}{|x|} - k_x \lambda \quad \left[ -\frac{\lambda}{2}, \frac{\lambda}{2} \right] \mapsto \left[ -\frac{\lambda}{2}, \frac{\lambda}{2} \right]$$

where $k_x$ is the unique integer $k \in \mathbb{Z}$ such that $\frac{1}{|x|} - k \lambda \in \left[ -\frac{\lambda}{2}, \frac{\lambda}{2} \right]$
If \( \lambda = \lambda_q \), \( SL(2, \mathbb{Z}) \) is replaced by Hecke group \( H_q \), generated by:

\[
T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S_q = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}
\]

small problem with orientation, solved by symmetrization:

\[
x \mapsto -\frac{1}{x} - k_x \lambda
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written as \( x \mapsto (S^{-k}T).x \)

Problem: can we find a geometric interpretation of this generalized continued fraction, for example in terms of deformation of geometric structure (flow on moduli spaces)?
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The domain for $H_q$
Rosen symmetrized continued fraction
Rational triangular billiard

Billiard in a right triangle with one angle \( \frac{\pi}{n} \)

The group generated by reflexions on the sides has order \( 2n \)

By identifying sides, we obtain one or two regular polygon, depending on parity

Opposite sides are identified
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The billard with angle $\frac{\pi}{8}$
A geometric generalization: Veech Continued fraction

Regular polygon with \( q = 2n \) sides. By identification of opposite sides: Riemann surface with a canonical quadratic differential given by the natural flat structure out of the vertices of the polygon. This flat structure admits a non-trivial group of affine automorphisms, its Veech group \( V_q \), which preserves the affine structure.

Fix \( \mu_q = 2 \cot \frac{\pi}{q} \)

Both maps

\[
R_q = \begin{pmatrix}
\cos \frac{2\pi}{q} & -\sin \frac{2\pi}{q} \\
\sin \frac{2\pi}{q} & -\cos \frac{2\pi}{q}
\end{pmatrix}, \quad
P_q = \begin{pmatrix}
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Renormalization of the octogon
The domain of Veech group
The Veech Continued fraction

The Veech group acts on the lines through the origin; one defines the Veech continued fraction, on the slope of these lines, in the following way:

Line with slope in \([-\frac{\mu}{2}, \frac{\mu}{2}]\).
Act by \(R_q\) till the slope gets out of the interval. Then act by the parabolic element \(P_q\) till the slope comes back to the interval.
One obtains in this way a generalized continued fraction, the additive Veech Algorithm.

We will prove that The Veech and Rosen continued fraction are essentially the same.
More precisely: if we take a suitable conjugate the Rosen algorithm, then it has, for a.e. initial point, an infinite number of common iterates with Veech algorithm.
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Veech additive continued fraction
Sketch of the proof

- Find a model for the natural extension in terms of sections for the geodesic flow in $H \backslash PSL(2, \mathbb{R})$
- Characterize first-return maps
- Prove that, after taking the square of the Rosen map and conjugating, these two maps are first return maps of the geodesic Teichmüller flow on the Teichmüller disc of the regular polygon
- Consider their respective induction on the intersection of the definition domain: they are equal.
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Heuristics for models of natural extensions of piecewise homographies

Remark: everything is really done in $PSL(2, \mathbb{R}) = T_1 \mathbb{H}^2$; for simplicity, we work in $SL(2, \mathbb{R})$, allowing to replace $M$ by $-M$.

The Haar measure for $SL(2, \mathbb{R})$ has a simple expression.

In $SL(2, \mathbb{R})$ the set of matrices $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ has codimension 1, and Haar measure 0. Out of this set, one can take coordinates $(a, b, c)$ for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Lemma: In these coordinates, the Haar measure $m$, defined up to a constant, has expression:

$$dm = \frac{da \, db \, dc}{a}$$

Proof is a trivial computation.
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The diagonal parametrization

Using the diagonal flow $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, one can write almost all element of $PSL(2, \mathbb{R})$ as:

$$\begin{pmatrix} Xe^{t/2} & (XY - 1)e^{-t/2} \\ e^{t/2} & Ye^{-t/2} \end{pmatrix}$$

In these coordinates, the Haar measure admits the simple expression:

$$dm = dX \, dY \, dt$$
The Hyperbolic plane parametrization

One can write almost any element as

$$\begin{pmatrix}
\frac{Xe^{t/2}}{\sqrt{X-Y}} & \frac{Ye^{-t/2}}{\sqrt{X-Y}} \\
\frac{e^{t/2}}{\sqrt{X-Y}} & \frac{e^{-t/2}}{\sqrt{X-Y}}
\end{pmatrix}$$

In these coordinates, the Haar measure admits the expression:

$$\text{dm} = \frac{dX \ dY \ dt}{(X - Y)^2}$$

This formula admits a nice interpretation as Liouville measure on the unit tangent of the hyperbolic plane.
The arithmetic parametrization

Taking $Z = -\frac{1}{Y}$, one obtains an expression more common in arithmetics:

$$dm = \frac{dX \, dZ \, dt}{(1 + XZ)^2}$$
The search for a natural extension: numerical experiments.

\[ T : x \mapsto M_x \cdot x \text{ piecewise homography } T \text{ on an interval.} \]

\( M \) is a piecewise constant map to \( PSL(2, \mathbb{R}) \), the \( M_x \) generate a discrete subgroup \( \Gamma \) of \( PSL(2, \mathbb{R}) \). We look for an explicit model of the natural extension \( \tilde{T} \).

We look for this natural extension as a first-return map of the geodesic flow on \( \Gamma \backslash \mathbb{H} \) to the section \( \begin{pmatrix} x & (xy - 1) \\ 1 & y \end{pmatrix} \), using the diagonal parametrization and taking \( x \) as the first coordinate.

A computation shows that the natural extension must be of the form:

\[ (x, y) \mapsto \left( \frac{ax + b}{cx + d}, (cx + d)^2 y - c(cx + d) \right) \]
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Problem: one does not know the domain of the natural extension.
Solution: since we know the explicit form of the map, and since it is ergodic for Lebesgue measure, take a generic point and compute its orbit. It is then usually easy to find the domain of the natural extension.
The simplest example is the classical continued fraction, where one recovers the Gauss measure.
Some problems are encountered if there are indifferent fixed points: infinite measure. In this case, one needs to accelerate the algorithm near the fixed point.
The search for a natural extension: numerical experiments.

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Natural extension of Veech continued fraction.

This algorithm has two indifferent fixed points at the boundary of the interval; one accelerates the algorithm on the corresponding continuity interval.

define $\beta = \mu/2 \cdot (3\mu^2 - 4)/(5\mu^2 + 4)$ and $\gamma = \mu/2 \cdot (5\mu^2 + 4)/(3\mu^2 - 4)$.

The domain $\Omega_V$ of the model for the natural extension is bounded:

Above by
\[
\begin{cases}
y = 1/(x + \gamma) & \text{on } [-\mu/2, -\beta[;
\end{cases}
\]

Under by
\[
\begin{cases}
y = 1/(x + \mu/2) & \text{on } [-\beta, \mu/2[; \\
y = 1/(x - \mu/2) & \text{on } [-\mu/2, \beta[; \\
y = 1/(x - \gamma) & \text{on } [\beta, \mu/2[;
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The area of $\Omega_V$ is $c_V = 2 \log \left(8 \cos^2 \frac{\pi}{q}\right)$.
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\end{cases}
\]

The area of \( \Omega_V \) is \( c_V = 2 \log \left( 8 \cos^2 \frac{\pi}{q} \right) \).
Veech multiplicative continued fraction
Natural extension for Veech continued fraction
Natural extension of Rosen continued fraction

Using numerical experiments, one finds the domain of the natural extension; denoting $m = M.1$ its domain $\Omega_R$ is bounded by:

\[
\begin{align*}
\begin{cases}
y = \frac{1}{x-m} & \text{under } [-2/\mu, \mu/2] ; \\
y = \frac{1}{x+m} & \text{above } [-\mu/2, 2/\mu] ; \\
y = \frac{1}{x-R_{j}^{1/2}.m} & \text{under } [-\tan \left(\frac{(j+1)\pi}{q}\right), -\tan \frac{j\pi}{q}] ; \\
y = \frac{1}{x+R_{j}^{1/2}.m} & \text{above } [\tan \frac{j\pi}{q}, \tan \frac{(j+1)\pi}{q}] ,
\end{cases}
\end{align*}
\]

The conjugate algorithm is of first return type. It uses $R_q^{1/2}$; one checks that its square only uses $R_q$ and $P_q$, hence its associated natural extension is also of first return type.
Natural extension for Rosen continued fraction
Natural extension for Rosen continued fraction
Functions of first return type

Suppose that the map $T$ is a piecewise isometry whose associated matrices generate a discrete group $\Gamma$. Suppose that the natural extension $\tilde{T}$ is defined on a domain $\Omega$, which can be considered as a surface in $SL(2, \mathbb{R})$. We want to know whether $\tilde{T}$ is the first return map $F$ to $\Omega$ of the geodesic flow on $\Gamma$.

It is clear that one can write $\tilde{T}(P) = F^{nP}(P)$. Denote $\tau(x) = -\frac{1}{2} \log(cx + d)$. One has:

$$\int_{\Omega} \tau(x) \, dm \geq \pi \text{Vol}(\Gamma \setminus PSL(2, \mathbb{R}))$$
Functions of first return type

Theorem: \( \tilde{T} \) is the first return map to the section if and only if

\[
\int_{\Omega} \tau(x) \, dm = \pi \text{Vol}(\Gamma \backslash PSL(2, \mathbb{R}))
\]

One shows that the Veech algorithm is of first return type (not very surprising: it is defined in this way!)

Theorem: if \( T \) is of first return type, \( T^k \) is of first return type iff it generates a subgroup of index \( k \) of \( \Gamma \).
Conjugacy of the Rosen and Veech algorithms

Rosen and Veech algorithm correspond to different groups, and are defined on different intervals. BUT groups $H_q$ and $G_q$ are conjugate by

$$M = \frac{1}{\sqrt{\sin \pi/q}} \begin{pmatrix} 1 & \cos \pi/q \\ 0 & \sin \pi/q \end{pmatrix}$$

$M$ sends interval $[-\lambda_q/2, \lambda_q/2]$ to $[0, \mu_q]$. Explicit conjugacy between $H_q$ and $G_q$

One obtains in this way a conjugate Rosen algorithm on $[0, \mu_q]$; acting by $P_q$, one recovers an algorithm on $[-\mu_q/2, \mu_q/2]$. 
One has the identity:

\[ MTS_q M^{-1} = R_q^{1/2} \]

Using this identity, one computes the conjugate algorithm:

\[ k : \left[ -\mu/2, \mu/2 \right] \rightarrow \left[ -\mu/2, \mu/2 \right] \]

Which sends \( x \) to

\[
\begin{cases}
R^{-1/2} \cdot x & \text{si} \ 0 \leq x < R^{1/2} \cdot \frac{\mu}{2}; \\
P^{-k} R^{-1/2} \cdot x & \text{si} \ R^{1/2} P^{k} \cdot -\frac{\mu}{2} \leq x < R^{1/2} P^{k} \cdot \frac{\mu}{2}; \\
R^{1/2} \cdot x & \text{si} \ 0 > x \geq R^{-1/2} \cdot -\frac{\mu}{2}; \\
P^{l} R^{1/2} \cdot x & \text{si} \ R^{-1/2} P^{-l} \cdot -\frac{\mu}{2} \leq x < R^{-1/2} P^{-l} \cdot \frac{\mu}{2}.
\end{cases}
\]
Comparison of Rosen and Veech algorithms

The intersection $\Omega = \Omega_R \cap \Omega_V$ is defined by:

\[
\begin{cases}
  y = \frac{1}{x+M.1} & \text{over } x \in [-\mu/2, 2/\mu[ ; \\
  y = \frac{1}{x+\mu/2} & \text{over } x \in [2/\mu, \mu/2[ ; \\
  y = \frac{1}{x-\mu/2} & \text{under } x \in [-\mu/2, -2/\mu[ ; \\
  y = \frac{1}{x-M.1} & \text{under } x \in [-2/\mu, \mu/2[ .
\end{cases}
\]

From the preceding arguments, it is clear that the induced maps, on $\Omega$, of Veech algorithm and the square of the conjugate Rosen algorithm are equal. By ergodicity, for almost all starting point, the two orbits have infinite intersection.

Q.E.D.
Which are the points with no common approximants? (essentially, closed trajectories that do not intersect the common section)

It must be possible to give combinatorial rules: one of the viewpoints is that the two algorithms consist in taking a different set of generators for the group $V_q$.

There is another algorithm, using interval exchanges on 4 intervals (for octagons, $q = 8$). This is similar to the link between continued fractions and rotations. The relation of this algorithm to the previous two is unclear.

One should be in this setting able to use symbolic dynamics (like sturmian sequences for rotations). This should give some matrices in $SL(4, \mathbb{Z})$, corresponding to the subgroup $V_8$ of $SL(2, \mathbb{Q}(\sqrt{2}))$. 
Additional remarks

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