Geometrical Theory of Dynamical Systems

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Preface

This text is a slightly edited version of lecture notes for a course I gave at ETH, during the Winter term 2000-2001, to undergraduate Mathematics and Physics students. The choice of topics covered here is somewhat arbitrary, and was partly imposed by time limitations. Evidently, these notes do not intend to replace the many existing excellent textbooks on the subject, a few of which are listed in the bibliography, but they might provide a reasonably concise, albeit certainly biased introduction to this huge domain. The approach used here has probably been influenced by my first teacher in Dynamical Systems, Prof. Hervé Kunz. I also wish to acknowledge my student's contribution in mercilessly tracking down a substantial amount of typos.

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Chapter 1

Examples of Dynamical Systems

The last 30 years have witnessed a renewed interest in dynamical systems, partly due to the "discovery" of chaotic behaviour, and ongoing research has brought many new insights in their behaviour. What are dynamical systems, and what is their geometrical theory? Dynamical systems can be defined in a fairly abstract way, but we prefer to start with a few examples of historical importance before giving general definitions. This will allow us to specify the class of systems that we want to study, and to explain the differences between the geometrical approach and other approaches.

1.1 The Motion of the Moon

The problem of the Moon's motion is a particular case of the N-body problem, which gives a nice illustration of the historical evolution that led to the development of the theory of dynamical systems. This section follows mainly Gutzwiller's article [Gu98].

Everyone knows that the phases of the Moon follow a cycle of a bit less than 30 days. Other regularities in the Moon's motion were known to the Babylonians as early as 1000 B.C. One can look, for instance, at the time interval between Sunset and Moonrise at Full Moon. This interval is not constant, but follows a cycle over 19 years, including 235 Full Moons (the *Metonic Cycle*). Solar and Lunar Eclipses also follow a cycle with a period of 18 years and 11 days, containing 223 Full Moons (the *Saros Cycle*).

Greek astronomy started in the 5th century B.C. and initiated developments culminating in the work of Ptolemy in the second century A.D. In contrast with the Babylonians, who looked for regularities in long rows of numbers, the Greeks introduced geometrical models for their astronomical observations. To account for the various observed deviations from periodicity, they invented the model of *epicycles*. In modern notation, and assuming a planar motion with Cartesian coordinates $(x, y) \in \mathbb{R}^2$, the complex number $z = x + \mathrm{i}\, y \in \mathbb{C}$ evolves as a function of time t according to the law

$$z = a\, \mathrm{e}^{\mathrm{i}\,\omega_1 t}(1 + \varepsilon\, \mathrm{e}^{\mathrm{i}\,\omega_2 t}), \tag{1.1.1}$$

where a, ε, ω_1 and ω_2 are parameters which are fitted to experimental data.

The epicycle model was refined in subsequent centuries, with more terms being included into the sum (1.1.1) to explain the various "inequalities" (periodic deviations from the uniform motion of the Moon). Four inequalities were discovered by Tycho Brahe alone in the 16th century. These terms could be partly explained when Kepler discovered his three laws in 1609:

1. the trajectory of a planet follows an ellipse admitting the Sun as a focus,
2. equal areas, measured with respect to the Sun, are swept in equal time intervals,
3. when several planets orbit the Sun, the period of the motion squared is proportional to the third power of the semi-major axis of the ellipse.

Expanding the solution into Fourier series produces sums for which (1.1.1) is a first approximation. However, while these laws describe the motion of the planets quite accurately, they fail to fit the observations for the Moon in a satisfactory way.

A decisive new point of view was introduced by Newton when he postulated his law of Universal Gravitation (published in his *Principia* in 1687). A system with N planets is described by a set of *ordinary differential equations*

$$m_i \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = \sum_{\substack{j=1,\dots,N \\ j\neq i}} \frac{G m_i m_j (x_j - x_i)}{\|x_i - x_j\|^3}, \qquad i = 1, \dots, N. \tag{1.1.2}$$

Here the $x_i \in \mathbb{R}^3$ are vectors specifying the position of the planets, the m_i are positive scalars giving the masses of the particles, and G is a universal constant. Newton proved that for two bodies ($N = 2$), the equation (1.1.2) is equivalent to Kepler's first two laws. With three or more bodies, however, there is no simple solution to the equations of motion, and Kepler's third law is only valid approximately, when the interaction between planets is neglected.

The three-body problem initiated a huge amount of research in the following two hundred years. Newton himself invented several clever tricks allowing him to compute corrections to Kepler's laws in the motion of the Moon. He failed, however, to explain all the anomalies. Perturbation theory was subsequently systematized by mathematicians such as Laplace, Euler, Lagrange, Poisson and Hamilton, who developed the methods of analytical mechanics. As a first step, one can introduce the *Hamiltonian function*

$$\begin{aligned} H: \quad (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N &\to \mathbb{R} \\ (p, q) &\mapsto \sum_{i=1}^{N} \frac{p_i^2}{2m_i} - \sum_{i<j} \frac{G m_i m_j}{\|q_i - q_j\|}, \end{aligned} \tag{1.1.3}$$

where $p_i = m_i v_i \in \mathbb{R}^3$ are the momenta of the planets and $q_i = x_i \in \mathbb{R}^3$ for $i = 1, \dots, N$. The equation of motion (1.1.2) is then equivalent to the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}. \tag{1.1.4}$$

One advantage of this formulation is that all the information on the motion is contained in the scalar function H. The main advantage, however, is that the structure (1.1.4) of the equations of motion is preserved under special changes of variables, called *canonical transformations*. In the case of the two-body problem, a good set of coordinates is given by the *Delaunay variables* $(I, \varphi) \in \mathbb{R}^3 \times \mathbb{T}^3$ (actually, there are $6 + 6$ variables, but 6 of them correspond to the trivial motion of the center of mass of the system). The *action variables* I_1, I_2, I_3 are related to the semi-major axis, eccentricity and inclination of the Kepler ellipse, while the *angle variables* $\varphi_1, \varphi_2, \varphi_3$ describe the position of the planet and the spatial orientation of the ellipse. The two-body Hamiltonian takes the form

$$H(I, \varphi) = -\frac{\mu}{2I_1^2}, \tag{1.1.5}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$. The equations of motion are

$$\begin{aligned} \frac{\mathrm{d}\varphi_1}{\mathrm{d}t} &= \frac{\partial H}{\partial I_1} = \frac{\mu}{I_1^3}, & \frac{\mathrm{d}I_1}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi_1} = 0 \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}t} &= \frac{\partial H}{\partial I_2} = 0, & \frac{\mathrm{d}I_2}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi_2} = 0 \\ \frac{\mathrm{d}\varphi_3}{\mathrm{d}t} &= \frac{\partial H}{\partial I_3} = 0, & \frac{\mathrm{d}I_3}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi_3} = 0, \end{aligned} \tag{1.1.6}$$

describing the fact that the planet moves on an elliptical orbit with fixed dimensions and orientation.

In the case of the three-body problem Moon–Earth–Sun, one can use two sets of Delaunay variables (I, φ) and (J, ψ) describing, respectively, the motion of the system Moon–Earth, and the motion around the Sun of the center of mass of the system Moon–Earth. The Hamiltonian takes the form

$$H(I, J, \varphi, \psi) = H_0(I, J) + H_1(I, J, \varphi, \psi). \tag{1.1.7}$$

The unperturbed part of the motion is governed by the Hamiltonian

$$H_0(I, J) = -\frac{\mu}{2I_1^2} - \frac{\mu'}{2J_1^2}, \tag{1.1.8}$$

where $\mu' = \frac{(m_1+m_2)m_3}{m_1+m_2+m_3}$. Due to the special initial conditions of the system, the *perturbing function* H_1 has a small amplitude. It depends on several small parameters: the initial eccentricities $\varepsilon \simeq 1/18$ of the Moon and $\varepsilon' \simeq 1/60$ of the Earth, their inclinations i and i', and the ratio $a/a' \simeq 1/400$ of the semi-major axes of the two subsystems. All these quantities are functions of the actions I and J. The standard approach is to expand H_1 in a trigonometric series

$$H_1 = -\frac{G m_1 m_2 m_3}{m_1 + m_2} \frac{a^2}{a'^3} \sum_{j \in \mathbb{Z}^6} C_j \, \mathrm{e}^{\mathrm{i}(j_1\varphi_1 + j_2\varphi_2 + j_3\varphi_3 + j_4\psi_1 + j_5\psi_2 + j_6\psi_3)} \,. \tag{1.1.9}$$

The coefficients C_j are in turn expanded into Taylor series of the small parameters,

$$C_j = \sum_{k \in \mathbb{N}^5} c_{jk} \Big(\frac{a}{a'}\Big)^{k_1} \varepsilon^{k_2} \varepsilon'^{k_3} \gamma^{k_4} \gamma'^{k_5}, \tag{1.1.10}$$

where $\gamma = \sin i/2$ and $\gamma' = \sin i'/2$. The solutions can then be expanded into similar series, thus yielding a Fourier expansion of the form (1.1.1) (in fact, it is better to simplify the Hamiltonian by successive canonical transformations, but the results are equivalent). The most impressive achievement in this line of work is due to Delaunay, who published in 1860 and 1867 two volumes of over 900 pages. They contain expansions up to order 10, which are simplified with 505 transformations. The main result for the trajectory of the Moon is a series containing 460 terms, filling 53 pages.

At the turn of the century, these perturbative calculations were criticized by Poincaré, who questioned the convergence of the expansions. Indeed, although the magnitude of the first few orders decreases, he showed that this magnitude may become extremely large at sufficiently high order. This phenomenon is related to the problem of small divisors appearing in the expansion, which we will discuss in a simpler example in the next section.

Poincaré introduced a whole set of new methods to attack the problem from a geometric point of view. Instead of trying to compute the solution for a given initial condition, he wanted to understand the qualitative nature of solutions for all initial conditions, or, as we would say nowadays, the geometric structure of phase space. He thereby introduced concepts such as invariant points, curves and manifolds. He also provided examples where the solution cannot be written as a linear combination of periodic terms, a first encounter with chaotic motion.

The question of convergence of the perturbation series continued nonetheless to be investigated, and was finally solved in a series of theorems by Kolmogorov, Arnol'd and Moser (the so-called *KAM theory*) in the 1950s. They prove that the series converges for (very) small perturbations, for initial conditions living on a Cantor set.

This did not solve the question of the motion of the Moon completely, although fairly accurate ephemerides can be computed for relatively short time spans of a few decades. Using a combination of analytical and numerical methods, the existence of chaos in the Solar System was demonstrated by Laskar in 1989 [La89], implying that exact positions of the planets cannot be predicted for times more than a few hundred thousand years in the future.

1.2 The Standard Map

The standard map describes the motion of a "rotator" with one angular degree of freedom $q \in \mathbb{S}^1$ ($\mathbb{S}^1$ denotes the circle $\mathbb{R}/2\pi\mathbb{Z}$), which is periodically kicked by a pendulum-like force of intensity proportional to $-\sin q$. If q_n and p_n denote the position and momentum just before the n^{th} kick, one has

$$\begin{aligned} q_{n+1} &= q_n + p_{n+1} \qquad (\text{mod } 2\pi) \\ p_{n+1} &= p_n - \varepsilon \sin q_n. \end{aligned} \tag{1.2.1}$$

For $\varepsilon = 0$, the dynamics is very simple and one has explicitly

$$\begin{aligned} q_n &= q_0 + np_0 \qquad (\text{mod } 2\pi) \\ p_n &= p_0. \end{aligned} \tag{1.2.2}$$

Let us now analyse the iterated map (1.2.1) according to the perturbative method. The idea is to look for a change of variables $(q, p) \mapsto (\varphi, I)$ transforming the system into a similar one, but without the term $\varepsilon \sin q_n$. Let us write

$$\begin{aligned} q &= \varphi + f(\varphi, I) \\ p &= I + g(\varphi, I), \end{aligned} \tag{1.2.3}$$

where f and g are unknown functions, which are 2π-periodic in φ. We impose that this change of variables transforms the map (1.2.1) into the map

$$\begin{aligned} \varphi_{n+1} &= \varphi_n + I_{n+1} \qquad (\text{mod } 2\pi) \\ I_{n+1} &= I_n =: \omega. \end{aligned} \tag{1.2.4}$$

This is equivalent to requiring that f and g solve the functional equations

$$\begin{aligned} f(\varphi + \omega, \omega) &= f(\varphi, \omega) + g(\varphi + \omega, \omega) \\ g(\varphi + \omega, \omega) &= g(\varphi, \omega) - \varepsilon \sin(\varphi + f(\varphi, \omega)). \end{aligned} \tag{1.2.5}$$

One can try to solve these equations by expanding f and g into Taylor series in ε and Fourier series in φ:

$$f(\varphi,\omega)=\sum_{j=1}^{\infty}\varepsilon^j f_j(\varphi,\omega) \qquad f_j(\varphi,\omega)=\sum_{k=-\infty}^{\infty} a_{k,j}(\omega)\,\mathrm{e}^{\mathrm{i}\,k\varphi} \tag{1.2.6}$$

$$g(\varphi,\omega)=\sum_{j=1}^{\infty}\varepsilon^j g_j(\varphi,\omega) \qquad g_j(\varphi,\omega)=\sum_{k=-\infty}^{\infty} b_{k,j}(\omega)\,\mathrm{e}^{\mathrm{i}\,k\varphi}\,. \tag{1.2.7}$$

We will use the expansion

$$\sin(\varphi+\varepsilon f)=\sin\varphi+\varepsilon f_1\cos\varphi+\varepsilon^2\big(f_2\cos\varphi-\tfrac{1}{2}f_1^2\sin\varphi\big)+\mathcal{O}(\varepsilon^3). \tag{1.2.8}$$

At order ε, we have to solve the relations

$$\begin{aligned} f_1(\varphi+\omega,\omega)&=f_1(\varphi,\omega)+g_1(\varphi+\omega,\omega)\\ g_1(\varphi+\omega,\omega)&=g_1(\varphi,\omega)-\sin\varphi, \end{aligned} \tag{1.2.9}$$

which become, in Fourier components,

$$\begin{aligned} a_{k,1}\,\mathrm{e}^{\mathrm{i}\,k\omega}&=a_{k,1}+b_{k,1}\,\mathrm{e}^{\mathrm{i}\,k\omega}\\ b_{k,1}\,\mathrm{e}^{\mathrm{i}\,k\omega}&=b_{k,1}-c_{k,1}, \end{aligned} \tag{1.2.10}$$

where $c_{k,1}$ are the Fourier components of $\sin\varphi$, that is, $c_{1,1}=-c_{-1,1}=1/(2\,\mathrm{i})$ and all other components vanish. We thus get

$$\begin{aligned} g_1(\varphi,\omega)&=\frac{\mathrm{e}^{\mathrm{i}\,\varphi}}{2\,\mathrm{i}(1-\mathrm{e}^{i\omega})}-\frac{\mathrm{e}^{-\mathrm{i}\,\varphi}}{2\,\mathrm{i}(1-\mathrm{e}^{-i\omega})}=\frac{\cos(\varphi-\omega/2)}{2\sin(\omega/2)}\\ f_1(\varphi,\omega)&=-\frac{\mathrm{e}^{\mathrm{i}\,\omega}\,\mathrm{e}^{\mathrm{i}\,\varphi}}{2\,\mathrm{i}(1-\mathrm{e}^{i\omega})^2}+\frac{\mathrm{e}^{-\mathrm{i}\,\omega}\,\mathrm{e}^{-\mathrm{i}\,\varphi}}{2\,\mathrm{i}(1-\mathrm{e}^{-i\omega})^2}=\frac{\sin\varphi}{4\sin^2(\omega/2)}. \end{aligned} \tag{1.2.11}$$

Note that this is only possible for $\mathrm{e}^{\mathrm{i}\,\omega}\neq 1$, that is, $\omega\neq 0 \pmod{2\pi}$. At order ε^2 we obtain similar relations as (1.2.10), but now $c_{k,2}$ denotes the Fourier coefficients of $f_1(\varphi,\omega)\cos\varphi$, which are nonzero for $|k|=2$. Thus g_2 and f_2 only exist if $\mathrm{e}^{2\,\mathrm{i}\,\omega}\neq 1$, or $\omega\neq 0,\pi \pmod{2\pi}$. Similarly, we will find that g_j and f_j only exist if $\mathrm{e}^{j\,\mathrm{i}\,\omega}\neq 1$, so the equations (1.2.5) can only be solved for irrational $\omega/(2\pi)$. Even then, the expansions of f and g will contain small terms of the form $1-\mathrm{e}^{\mathrm{i}\,k\omega}$ in the denominators, so that the convergence of the series is not clear at all. In fact, the convergence has been proved by Moser for certain irrational ω called *Diophantine numbers* [Mo73].

Now let us turn to the geometric approach. We can consider (q,p) as coordinates in the plane (or on the cylinder because of the periodicity of q). For given (q_0,p_0), the set of points $\{(q_n,p_n)\}_{n\geqslant 0}$ is called the *orbit* with initial condition (q_0,p_0). We would like to know what the different orbits look like. The simplest case is the *fixed point*: if

$$\begin{aligned} q_{n+1}&=q_n\\ p_{n+1}&=p_n \end{aligned} \tag{1.2.12}$$

then the orbit will consist of a single point. The fixed points of the standard map are $(0,k)$ and (π,k) with $k\in\mathbb{Z}$. We can also have *periodic orbits*, consisting of m points, if

$$\begin{aligned} q_{n+m}&=q_n\\ p_{n+m}&=p_n. \end{aligned} \tag{1.2.13}$$

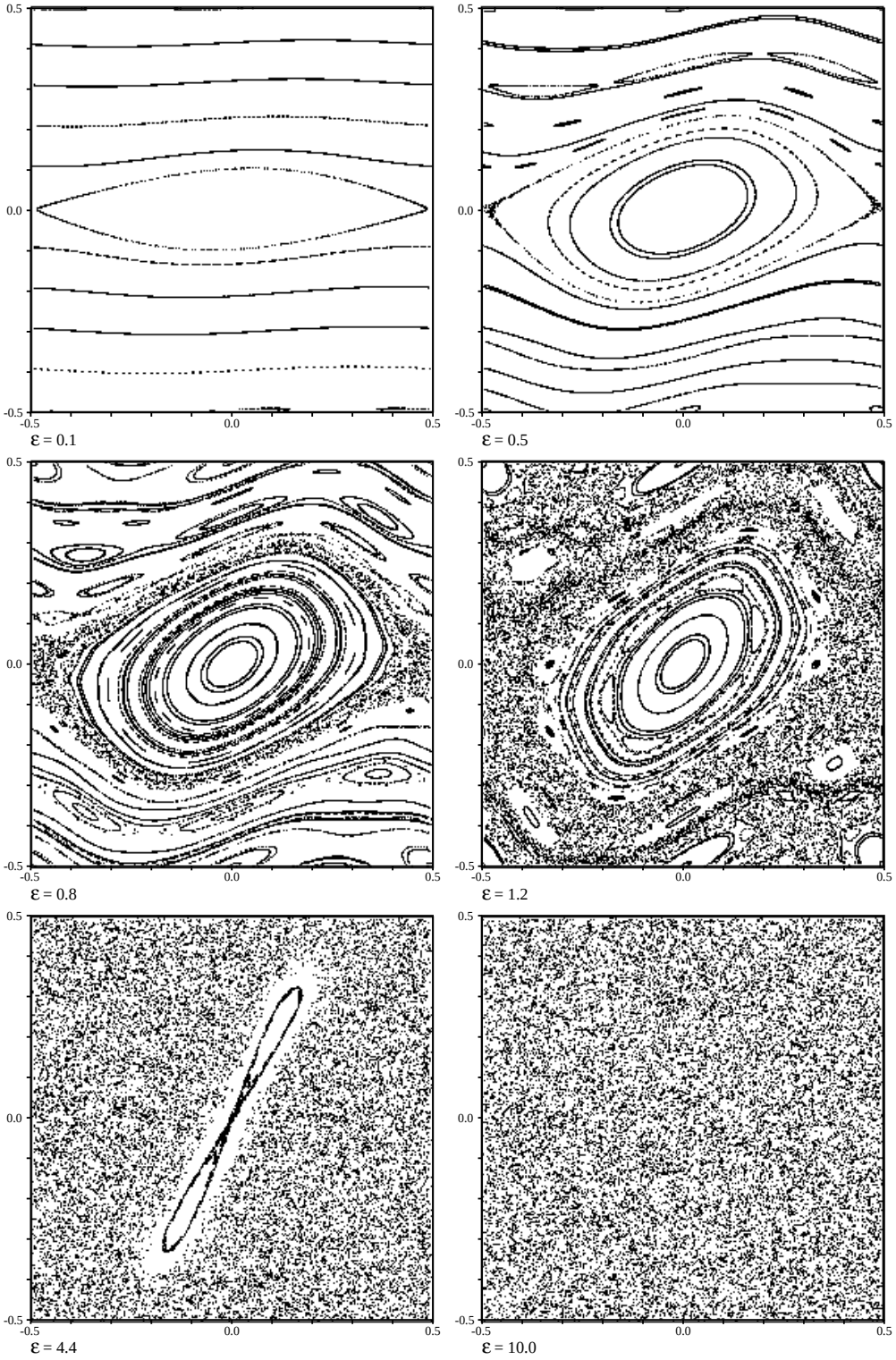

FIGURE 1.1. Phase portraits of the standard map, obtained by representing several orbits with different initial conditions, for increasing values of the perturbation ε. From left to right and top to bottom: $\varepsilon = 0.1$, $\varepsilon = 0.5$, $\varepsilon = 0.8$, $\varepsilon = 1.2$, $\varepsilon = 4.4$ and $\varepsilon = 10$.

Another possible orbit is the *invariant curve*. For instance if the equations (1.2.5) admit a solution, we can write

$$\begin{aligned} q_n &= \varphi_n + f(\varphi_n, \omega) \qquad (\text{mod } 2\pi) \\ p_n &= \omega + g(\varphi_n, \omega), \end{aligned} \tag{1.2.14}$$

where $\varphi_n = \varphi_0 + n\omega$. This is the parametric equation of a curve winding around the cylinder. Since ω is irrational, the points fill the curve densely. One can also analyse the dynamics in the vicinity of periodic orbits. It turns out that for this kind of map, most periodic orbits are of one of two types: elliptic orbits are surrounded by invariant curves, while hyperbolic orbits attract other orbits from one direction and expel them into another one. There are, however, much more exotic types of orbits. Some live on invariant Cantor sets, others densely fill regions of phase space with a positive surface.

The aim of the geometrical theory of dynamical systems is to classify the possible behaviours and to find ways to determine the most important qualitative features of the system. An important advantage is that large classes of dynamical systems have a similar qualitative behaviour. This does not mean that the perturbative approach is useless. But it is in general preferable to start by analysing the system from a qualitative point of view, and then, if necessary, use more sophisticated methods in order to obtain more detailed information.

1.3 The Lorenz Model

Convection is an important mechanism in the dynamics of the atmosphere: warm air has a lower density and therefore rises to higher altitudes, where it cools down and falls again, giving rise to patterns in the atmospheric currents.

This mechanism can be modeled in the laboratory, an experiment known as *Rayleigh-Bénard convection*. A fluid is contained between two horizontal plates, the upper one at temperature T_0 and the lower one at temperature $T_1 = T_0 + \Delta T > T_0$. The temperature difference ΔT is the *control parameter*, which can be modified.

For small values of ΔT, the fluid remains at rest, and the temperature decreases linearly in the vertical direction. At slightly larger ΔT, convection rolls appear (their shape depends on the geometry of the set-up). The flow is still stationary, that is, the fluid velocity at any given point does not change in time.

For still larger ΔT, the spatial arrangement of the rolls remains fixed, but their time dependence becomes more complex. Usually, it starts by getting periodic. Then different scenarios are observed, depending on the set-up. One of them is the *period doubling cascade*: the time-dependence of the velocity field has period $P, 2P, 4P, \ldots, 2^n P, \ldots$, where the n^{th} period doubling occurs for a temperature difference ΔT_n satisfying

$$\lim_{n\to\infty} \frac{\Delta T_n - \Delta T_{n-1}}{\Delta T_{n+1} - \Delta T_n} = \delta \simeq 4.4... \tag{1.3.1}$$

These ΔT_n accumulate at some finite ΔT_∞ for which the behaviour is no longer periodic, but displays temporal chaos. In this situation, the direction of rotation of the rolls changes erratically in time.

For very large ΔT, the behaviour can become *turbulent*: not only is the time dependence nonperiodic, but the spatial arrangement of the velocity field also changes.

RB convection has been modeled in the following way. For simplicity, one considers the two-dimensional case, with an infinite extension in the horizontal x_1-direction, while the vertical x_2-direction is bounded between $-\frac{1}{2}$ and $\frac{1}{2}$. Let $\mathcal{D} = \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]$. The state of the system is described by three fields

$$\begin{aligned} v &: \mathcal{D} \times \mathbb{R} \to \mathbb{R}^2 && \text{velocity,} \\ T &: \mathcal{D} \times \mathbb{R} \to \mathbb{R} && \text{temperature,} \\ p &: \mathcal{D} \times \mathbb{R} \to \mathbb{R} && \text{pressure.} \end{aligned} \tag{1.3.2}$$

The deviation $\theta(x,t)$ from the linear temperature profile is defined by

$$T(x,t) = T_0 + \Delta T\Big(\frac{1}{2} - x_2\Big) + T_1\theta(x,t). \tag{1.3.3}$$

The equations of hydrodynamics take the following form:

$$\begin{aligned} \frac{1}{\sigma}\Big[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\Big] &= \Delta v - \nabla p + (0,\theta)^T \\ \frac{\partial\theta}{\partial t} + (v\cdot\nabla)\theta &= \Delta\theta + Rv_2 \\ \nabla\cdot v &= 0. \end{aligned} \tag{1.3.4}$$

Here σ, the *Prandtl number*, is a constant related to physical properties of the fluid, while R, the *Reynolds number*, is proportional to ΔT. Furthermore,

$$\begin{aligned} \nabla p &= \Big(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}\Big)^T \\ \Delta\theta &= \frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} \\ \nabla\cdot v &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \\ (v\cdot\nabla)\theta &= v_1\frac{\partial\theta}{\partial x_1} + v_2\frac{\partial\theta}{\partial x_2}. \end{aligned}$$

The terms containing $(v\cdot\nabla)$ introduce the nonlinearity into the system. The boundary conditions require that θ, v_2 and $\partial v_1/\partial x_2$ should vanish for $x_2 = \pm\frac{1}{2}$. We thus have to solve four coupled nonlinear partial differential equations for the four fields v_1, v_2, θ, p. The continuity equation $\nabla\cdot v = 0$ can be satisfied by introducing the *vorticity* $\psi : \mathcal{D}\times\mathbb{R} \to \mathbb{R}$, such that

$$(v_1, v_2) = \Big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\Big). \tag{1.3.5}$$

It is also possible to eliminate the pressure from the two equations for $\partial v/\partial t$. We are left with two equations for ψ and θ. The problem can be further simplified by assuming a periodic dependence on x_1, of period $2\pi/q$. A possible approach (not the best one by modern standards, but historically important) is to expand the two fields into Fourier series (or "modes"):

$$\begin{aligned} \psi(x_1,x_2) &= \sum_{k\in\mathbb{Z}^2} a_k\, \mathrm{e}^{\mathrm{i}\,k_1 q x_1}\, \mathrm{e}^{\mathrm{i}\,k_2\pi x_2} \\ \theta(x_1,x_2) &= \sum_{k\in\mathbb{Z}^2} b_k\, \mathrm{e}^{\mathrm{i}\,k_1 q x_1}\, \mathrm{e}^{\mathrm{i}\,k_2\pi x_2} \end{aligned} \tag{1.3.6}$$

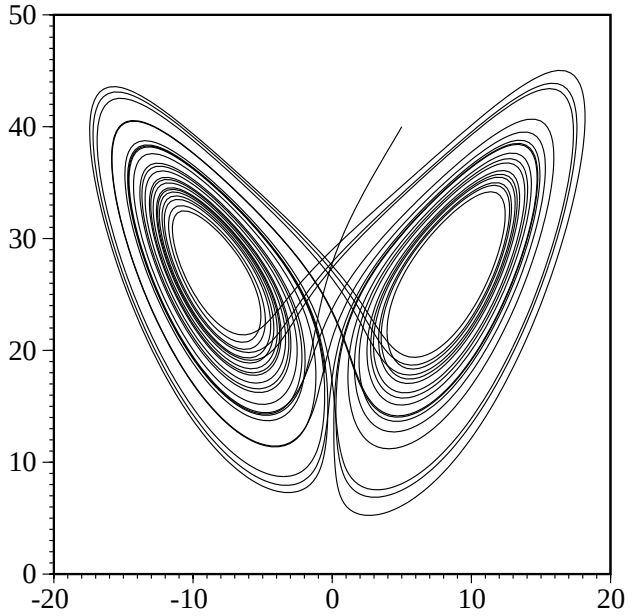

FIGURE 1.2. One trajectory of the Lorenz equations (1.3.9) for $\sigma = 10$, $b = 8/3$ and $r = 28$, projected on the (X, Z)-plane.

(where the boundary conditions impose some relations between Fourier coefficients of the same $|k_1|$ and $|k_2|$). Note that the terms of this sum are eigenfunctions of the linear operators in (1.3.4). Plugging these expansions into the equations, we obtain relations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_k(t) \\ b_k(t) \end{pmatrix} = L_k \begin{pmatrix} a_k(t) \\ b_k(t) \end{pmatrix} + N\big(\{a_{k'}, b_{k'}\}_{k' \in \mathbb{Z}}\big), \tag{1.3.7}$$

where L_k are 2×2 matrices and the term $N(\cdot)$ comes from the nonlinear terms in $(v \cdot \nabla)$ and may depend on all other k'. Without these nonlinear terms the problem would be easy to solve.

In 1962, Saltzmann considered approximations of the equations (1.3.7) with finitely many terms, and observed that the dynamics seemed to be dominated by three Fourier modes. In 1963, Lorenz decided to truncate the equations to these modes [Lo63], setting

$$\begin{aligned} \psi(x_1, x_2) &= \alpha_1 X(t) \sin qx_1 \cos \pi x_2 \\ \theta(x_1, x_2) &= \alpha_2 Y(t) \cos qx_1 \cos \pi x_2 + \alpha_3 Z(t) \sin 2\pi x_2. \end{aligned} \tag{1.3.8}$$

Here $\alpha_1 = \sqrt{2}(\pi^2 + q^2)/(\pi q)$, $\alpha_2 = \sqrt{2}\alpha_3$ and $\alpha_3 = (\pi^2 + q^2)^3/(\pi q^2)$ are constants introduced only in order to simplify the resulting equations. All other Fourier modes in the expansion (1.3.7) are set to zero, a rather drastic approximation. After scaling time by a factor $(\pi^2 + q^2)$, one gets the equations

$$\begin{aligned} \mathrm{d}X/\mathrm{d}t &= \sigma(Y - X) \\ \mathrm{d}Y/\mathrm{d}t &= rX - Y - XZ \\ \mathrm{d}Z/\mathrm{d}t &= -bZ + XY, \end{aligned} \tag{1.3.9}$$

where $b = 4\pi^2/(\pi^2 + q^2)$, and $r = Rq^2/(\pi^2 + q^2)^3$ is proportional to the control parameter R, and thus to ΔT. These so-called *Lorenz equations* are a very crude approximation of the equations (1.3.4), nevertheless they may exhibit very complicated dynamics.

In fact, for $0 \leqslant r \leqslant 1$, all solutions are attracted by the origin $X = Y = Z = 0$, corresponding to the fluid at rest. For $r > 1$, a pair of equilibria with $X \neq 0$ attracts the orbits, they correspond to convection rolls with the two possible directions of rotation. Increasing r produces a very complicated sequence of bifurcations, including period doubling cascades [Sp82]. For certain values of the parameters, a *strange attractor* is formed, in which case the convection rolls change their direction of rotation very erratically (Fig. 1.2), and the dynamics is very sensitive to small changes in the initial conditions (the *Butterfly effect*). The big surprise was that such a simple approximation, containing only three modes, could capture such complex behaviours.

1.4 The Logistic Map

Our last example is a famous map inspired by population dynamics. Consider a population of animals that reproduce once a year. Let P_n be the number of individuals in the year number n. The offspring being usually proportional to the number of adults, the simplest model for the evolution of the population from one year to the next is the linear equation

$$P_{n+1} = \lambda P_n \tag{1.4.1}$$

where λ is the natality rate (minus the mortality rate). This law leads to an exponential growth of the form

$$P_n = \lambda^n P_0 = \mathrm{e}^{n \ln \lambda} P_0 \tag{1.4.2}$$

(the *Malthus law*). This model becomes unrealistic when the number of individuals is so large that the limitation of resources becomes apparent. The simplest possibility to limit the growth is to introduce a quadratic term $-\beta P_n^2$, leading to the law

$$P_{n+1} = \lambda P_n - \beta P_n^2. \tag{1.4.3}$$

The rescaled variable $x = \beta P$ then obeys the equation

$$x_{n+1} = f_\lambda(x_n) \coloneqq \lambda x_n (1 - x_n). \tag{1.4.4}$$

The map f_λ is called the *logistic map*. Observe that for $0 \leqslant \lambda \leqslant 4$, f_λ maps the interval $[0, 1]$ into itself. The dynamics of the sequence x_n depends drastically on the value of λ.

For $0 \leqslant \lambda \leqslant 1$, all orbits converge to 0, which means that the population becomes extinct. For $1 < \lambda \leqslant 3$, all orbits starting at $x_0 > 0$ converge to $1 - 1/\lambda$, and thus the population reaches a stable equilibrium. For $3 < \lambda \leqslant 1 + \sqrt{6}$, the orbits converge to a cycle of period 2, so that the population asymptotically jumps back and forth between two values.

For $\lambda > 1+\sqrt{6}$, the system goes through a whole seqence of period doublings. Similarly as in RB convection, the values λ_n of the parameter for which the n^{th} period doubling occurs obey the law

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.669... \tag{1.4.5}$$

where δ is called the *Feigenbaum constant*. In 1978, Feigenbaum as well as Coullet and Tresser independently outlined an argument showing that such period doubling cascades

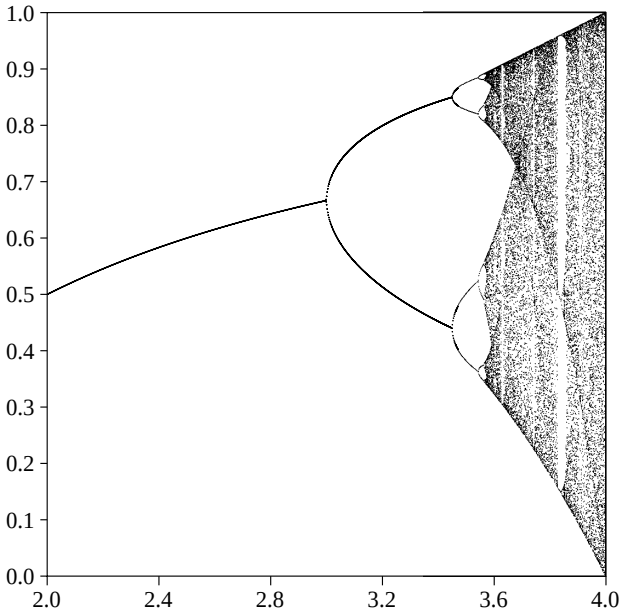

FIGURE 1.3. Bifurcation diagram of the logistic map. For each value of λ on the abscissa, the points x_{1001} through x_{1100} are represented on the ordinate, for an initial value $x_0 = \frac{1}{2}$.

should be observable for a large class of systems, and that the constant δ is universal. For instance, it also appears in the two-dimensional *Hénon map*

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 + y_n \\ y_{n+1} &= b x_n \qquad\qquad 0 < b < 1 \end{aligned} \tag{1.4.6}$$

Rigorous proofs of these properties were later worked out by Collet, Eckmann, Koch, Lanford and others [CE80].

For the logistic map, the period doublings accumulate at a value $\lambda_\infty = 3.56...$, beyond which the orbits become chaotic. For larger λ, there is a complicated interplay of regular and chaotic motion (Fig. 1.3), containing other period doubling cascades. Finally, for $\lambda = 4$, one can prove that the dynamics is as random as coin tossing.

Chapter 2

Stationary and Periodic Solutions

In Chapter 1, we have seen examples of two kinds of dynamical systems: ordinary differential equations (ODEs) and iterated maps. There are other types of dynamical systems, such as partial differential equations or cellular automata. These are in general more difficult to analyse, although some ideas developed for maps and ODEs can be carried over to their study. Here we will concentrate on ODEs and maps, by starting with the simplest kinds of dynamics: stationary and periodic.

2.1 Basic Concepts

2.1.1 Orbits and Flows

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open domain. One type of dynamical systems we will consider is given by a *map* $F : \mathcal{D} \to \mathcal{D}$. $\mathcal{D}$ is called the *phase space* of the system. It is possible to consider more general differentiable manifolds as phase space, such as the circle, the cylinder or the torus, but we will limit the discussion to Euclidean domains. Generalizations to other manifolds are usually straightforward.

Definition 2.1.1. *The* (positive) orbit *of F through a point $x_0 \in \mathcal{D}$ is the sequence $(x_k)_{k \geqslant 0}$ defined by $x_{k+1} = F(x_k)$ for all integers $k \geqslant 0$. We have thus*

$$x_k = F^k(x) \qquad \text{where } F^k = \underbrace{F \circ F \circ \cdots \circ F}_{k \text{ times}} \tag{2.1.1}$$

In case F is invertible, we can also define the negative orbit *of x_0 by the relations $F(x_k) = x_{k+1}$ for all $k < 0$, which are equivalent to* (2.1.1) *if we set $F^{-k} = (F^{-1})^k$ for all $k > 0$. The* orbit *of x_0 is then given by $(x_k)_{k \in \mathbb{Z}}$.*

Note the trivial relation

$$F^{k+l}(x_0) = F^k(F^l(x_0)) \tag{2.1.2}$$

for all positive integers k, l (and all integers if F is invertible), which will admit an analogue in the case of ODEs.

As particular cases of maps F, we have *homeomorphisms*, which are continuous maps admitting a continuous inverse, and *diffeomorphisms*, which are continuously differentiable maps admitting a continuously differentiable inverse. Similarly, for all $r \geqslant 1$, a *$\mathcal{C}^r$-diffeomorphism* is an invertible map F such that both F and F^{-1} admit continuous derivatives up to order r.

The *ordinary differential equations* we are going to consider are of the form

$$\dot{x} = f(x), \tag{2.1.3}$$

where $f : \mathcal{D} \to \mathbb{R}^n$, and $\dot{x}$ denotes $\frac{\mathrm{d}x}{\mathrm{d}t}$. Equivalently, we can write (2.1.3) as a system of equations for the components of x,

$$\dot{x}_i = f_i(x), \qquad i = 1, \dots, n. \tag{2.1.4}$$

$\mathcal{D} \subset \mathbb{R}^n$ is again called *phase space* and f is called a *vector field*.

To define the orbits of f, we have to treat the problem of existence and uniqueness a bit more carefully. The following results are assumed to be known from basic analysis (see for instance [Hal69], [Har64] or [HS74]).

Theorem 2.1.2 (Peano-Cauchy). *Let f be continuous. For every $x_0 \in \mathcal{D}$, there exists at least one local solution of (2.1.3) through x_0, that is, there is an open interval $I \ni 0$ and a function $x : I \to \mathcal{D}$ such that $x(0) = x_0$ and $\dot{x}(t) = f(x(t))$ for all $t \in I$.*

Theorem 2.1.3. *Every solution $x(t)$ with $x(0) = x_0$ can be continued to a maximal interval of existence $(t_1, t_2) \ni 0$. If $t_2 < \infty$ or $t_1 > -\infty$, then for any compact $\mathcal{K} \subset \mathcal{D}$, there exists a time $t \in (t_1, t_2)$ with $x(t) \notin \mathcal{K}$ (this means that solutions will diverge or reach $\partial\mathcal{D}$).*

Theorem 2.1.4 (Picard-Lindelöf). *Assume f is continuous and* locally Lipschitzian*, that is, for every compact $\mathcal{K} \subset \mathcal{D}$, there exists a constant $L_\mathcal{K}$ such that $\|f(x) - f(y)\| \leqslant L_\mathcal{K}\|x - y\|$ for all $x, y \in \mathcal{K}$. Then there is a unique solution $x(t)$ of (2.1.3) with $x(0) = x_0$ for every $x_0 \in \mathcal{D}$.*

Note in particular that if f is continuously differentiable, then it is locally Lipschitzian. We will usually consider vector fields which are at least once continuously differentiable.

Example 2.1.5. It is easy to give counterexamples to global existence and uniqueness. For instance,

$$\dot{x} = x^2 \qquad \Rightarrow \qquad x(t) = \frac{1}{\frac{1}{x_0} - t} \tag{2.1.5}$$

has a solution diverging for $t = \frac{1}{x_0}$. A physically interesting counterexample to uniqueness is the *leaky bucket equation*

$$\dot{x} = -\sqrt{|x|}. \tag{2.1.6}$$

Here x is proportional to the height of water in a bucket with a hole in the bottom, and (2.1.6) reflects the fact that the kinetic energy (proportional to $\dot{x}^2$) of the water leaving the bucket is equal to the potential energy of the water inside. For every c, (2.1.6) admits the solution

$$x(t) = \begin{cases} \frac{1}{4}(t - c)^2 & \text{for } t < c \\ 0 & \text{for } t \geqslant c. \end{cases} \tag{2.1.7}$$

In particular, for any $c \leqslant 0$, (2.1.7) is a solution of (2.1.6) such that $x(0) = 0$. This reflects the fact that if the bucket is empty at time 0, we do not know at what time it was full.

For simplicity, we will henceforth assume that the ODE (2.1.3) admits a unique global solution for all $x_0 \in \mathcal{D}$. This allows to introduce the following definitions:[1]

[1] In the case of $x(t)$ existing for all positive t but not necessarily for all negative t, the definition remains valid with orbit replaced by *positive orbit*, flow replaced by *semi-flow* and group replaced by *semi-group*.

Definition 2.1.6. *Let $x_0 \in \mathcal{D}$ and let $x(t)$ be the unique solution of* (2.1.3) *with* initial condition $x(0) = x_0$.

- *The* integral curve *through x_0 is the set* $\{(x,t) \in \mathcal{D} \times \mathbb{R} : x = x(t)\}$.
- *The* orbit *through x_0 is the set* $\{x \in \mathcal{D} \colon x = x(t), t \in \mathbb{R}\}$.
- *The* flow *of the equation* (2.1.3) *is the map*

$$\begin{array}{rccc} \varphi : & \mathcal{D} \times \mathbb{R} & \to & \mathcal{D} \\ & (x_0, t) & \mapsto & \varphi_t(x_0) = x(t) \end{array} \tag{2.1.8}$$

Geometrically speaking, the orbit is a curve in phase space containing x_0 such that the vector field $f(x)$ is tangent to the curve at any point x of the curve. Uniqueness of the solution means that there is only one orbit through any point in phase space.

By definition, we have $\varphi_0(x_0) = x_0$ for all $x_0 \in \mathcal{D}$, and uniqueness of solutions implies that $\varphi_t(\varphi_s(x_0)) = \varphi_{t+s}(x_0)$. These properties can be rewritten as

$$\varphi_0 = \mathrm{id} \qquad \varphi_t \circ \varphi_s = \varphi_{t+s} \tag{2.1.9}$$

which means that the family $\{\varphi_t\}_t$ forms a group. Note the similarity between this relation and the relation (2.1.2) for iterated maps.

Example 2.1.7. In the case $f(x) = -x$, $x \in \mathbb{R}$, we have

$$\varphi_t(x_0) = x_0 \, \mathrm{e}^{-t} \, . \tag{2.1.10}$$

The system admits three distinct orbits $(0, \infty)$, $(-\infty, 0)$ and $\{0\}$.

2.1.2 Evolution of Volumes

Let $\mathcal{M} \subset \mathcal{D}$ be a compact subset of phase space. We can define its *volume* by a usual Riemann integral:

$$\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} \mathrm{d}x, \qquad \mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{2.1.11}$$

The set $\mathcal{M}$ will evolve under the influence of the dynamics: we can define the sets $\mathcal{M}_k = F^k(\mathcal{M})$ or $\mathcal{M}(t) = \varphi_t(\mathcal{M})$. How does their volume evolve with time? The answer is actually quite simple.

Consider first the case of a map F. We assume that F is continuously differentiable and denote by

$$\frac{\partial F}{\partial x}(x) \tag{2.1.12}$$

the *Jacobian matrix* of F at x, which is the $n \times n$ matrix A with elements $a_{ij} = \frac{\partial F_i}{\partial x_j}(x)$.

Proposition 2.1.8. *Assume F is a diffeomorphism and let $V_k = \mathrm{Vol}(\mathcal{M}_k)$. Then*

$$V_{k+1} = \int_{\mathcal{M}_k} \Big| \det\Big(\frac{\partial F}{\partial x}(x) \Big) \Big| \, \mathrm{d}x. \tag{2.1.13}$$

Proof: This is a simple application of the formula for a change of variables in an integral:

$$V_{k+1} = \int_{\mathcal{M}_{k+1}} \mathrm{d}y = \int_{\mathcal{M}_k} \Big| \det\Big(\frac{\partial y}{\partial x}(x) \Big) \Big| \, \mathrm{d}x = \int_{\mathcal{M}_k} \Big| \det\Big(\frac{\partial F}{\partial x}(x) \Big) \Big| \, \mathrm{d}x.$$

□

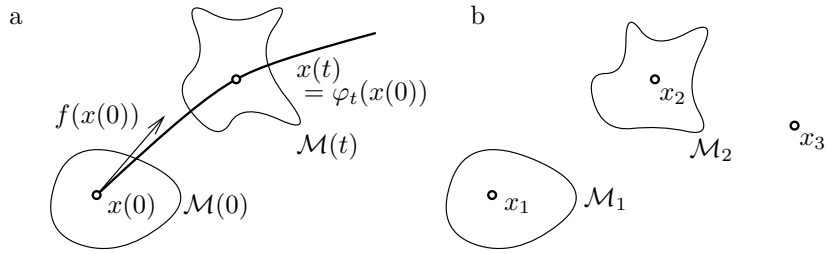

FIGURE 2.1. (a) Evolution of a volume with the flow, (b) evolution with an iterated map.

Definition 2.1.9. *The map F is called* conservative *if*

$$\left|\det\Big(\frac{\partial F}{\partial x}(x)\Big)\right| = 1 \qquad \forall x \in \mathcal{D} \tag{2.1.14}$$

The map F is called dissipative *if*

$$\left|\det\Big(\frac{\partial F}{\partial x}(x)\Big)\right| < 1 \qquad \forall x \in \mathcal{D}. \tag{2.1.15}$$

Proposition 2.1.8 implies that $V_{k+1} = V_k$ if F is conservative and $V_{k+1} < V_k$ if F is dissipative. More generally, if $|\det \frac{\partial F}{\partial x}(x)| \leqslant \lambda$ for some constant λ and all $x \in \mathcal{D}$, then $V_k \leqslant \lambda^k V_0$.

For differential equations, the result is the following:

Proposition 2.1.10. *Assume f is continuously differentiable and let $V(t) = \mathrm{Vol}(\mathcal{M}(t))$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \int_{\mathcal{M}(t)} \nabla \cdot f(x)\,\mathrm{d}x, \tag{2.1.16}$$

where $\nabla \cdot f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$ is the divergence.

PROOF: We have

$$V(t) = \int_{\mathcal{M}(t)} \mathrm{d}y = \int_{\mathcal{M}} \Big|\det \frac{\partial}{\partial x}\varphi_t(x)\Big|\,\mathrm{d}x.$$

Let us fix $x \in \mathcal{M}$, let $y(t) = \varphi_t(x)$ and set

$$J(t) := \frac{\partial}{\partial x}\varphi_t(x), \qquad A(t) := \frac{\partial f}{\partial x}(y(t)).$$

Note that by definition of φ_t, $J(0) = \mathbb{1}$ is the identity matrix. Now we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}J(t) = \frac{\partial}{\partial x}\dot{y}(t) = \frac{\partial}{\partial x}f(\varphi_t(x)) = \frac{\partial f}{\partial x}(\varphi_t(x))\frac{\partial}{\partial x}\varphi_t(x),$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}J(t) = A(t)J(t), \qquad J(0) = \mathbb{1}.$$

This is a linear, time-dependent differential equation for $J(t)$, which is known to admit a unique global solution. This implies in particular that $\det J(t) \neq 0\ \forall t$, since otherwise $J(t)$ would not be surjective, contradicting uniqueness. Since $\det J(0) = 1$, continuity implies

that $\det J(t) > 0\ \forall t$. Now let us determine the evolution of $\det J(t)$. By Taylor's formula, there exists $\theta \in [0,1]$ such that

$$\begin{aligned} J(t+\varepsilon) &= J(t) + \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} J(t+\theta\varepsilon) \\ &= J(t)\big[\mathbb{1} + \varepsilon J(t)^{-1} A(t+\theta\varepsilon) J(t+\theta\varepsilon)\big]. \end{aligned}$$

From linear algebra, we know that for any $n \times n$ matrix B,

$$\det(\mathbb{1} + \varepsilon B) = 1 + \varepsilon \operatorname{Tr} B + r(\varepsilon)$$

with $\lim_{\varepsilon\to 0} r(\varepsilon)/\varepsilon = 0$ (this is a consequence of the definition of the determinant as a sum over permutations). Using $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, this leads to

$$\det J(t+\varepsilon) = \det J(t)\big[1 + \varepsilon \operatorname{Tr}\big(A(t+\theta\varepsilon) J(t+\theta\varepsilon) J(t)^{-1}\big) + r(\varepsilon)\big],$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \det J(t) = \lim_{\varepsilon\to 0} \frac{\det J(t+\varepsilon) - \det J(t)}{\varepsilon} = \operatorname{Tr}(A(t)) \det J(t).$$

Taking the derivative of $V(t)$ we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t) &= \int_{\mathcal{M}} \frac{\mathrm{d}}{\mathrm{d}t} \det J(t)\, \mathrm{d}x \\ &= \int_{\mathcal{M}} \operatorname{Tr}\Big(\frac{\partial f}{\partial x}(y(t))\Big) \det J(t)\, \mathrm{d}x \\ &= \int_{\mathcal{M}(t)} \operatorname{Tr}\Big(\frac{\partial f}{\partial x}(y)\Big)\, \mathrm{d}y, \end{aligned}$$

and the conclusion follows from the fact that $\operatorname{Tr} \frac{\partial f}{\partial x} = \nabla \cdot f$. □

Definition 2.1.11. *The vector field f is called* conservative *if*

$$\nabla \cdot f(x) = 0 \qquad \forall x \in \mathcal{D} \tag{2.1.17}$$

The vector field f is called dissipative *if*

$$\nabla \cdot f(x) < 0 \qquad \forall x \in \mathcal{D}. \tag{2.1.18}$$

Proposition 2.1.10 implies that $V(t)$ is constant if f is conservative, and monotonously decreasing when f is dissipative. More generally, if $\nabla \cdot f(x) \leqslant c\ \forall x \in \mathcal{D}$, then $V(t) \leqslant V(0)\, \mathrm{e}^{ct}$.

Of course, one can easily write down dynamical systems which are neither conservative nor dissipative, but the conservative and dissipative situations are very common in applications.

Example 2.1.12. Consider a Hamiltonian system, with Hamiltonian $H \in \mathcal{C}^2(\mathbb{R}^{2m}, \mathbb{R})$. Then $x = (q,p) \in \mathbb{R}^m \times \mathbb{R}^m$ and the equations (1.1.4) take the form

$$f_i(x) = \begin{cases} \dfrac{\partial H}{\partial p_i} & i = 1, \dots m \\ -\dfrac{\partial H}{\partial q_{i-m}} & i = m+1, \dots 2m. \end{cases} \tag{2.1.19}$$

This implies that

$$\nabla \cdot f = \sum_{i=1}^{m} \frac{\partial}{\partial q_i}\Big(\frac{\partial H}{\partial p_i}\Big) + \sum_{i=1}^{m} \frac{\partial}{\partial p_i}\Big(-\frac{\partial H}{\partial q_i}\Big) = 0. \tag{2.1.20}$$

Thus all (sufficiently smooth) Hamiltonian systems are conservative.

Exercise 2.1. Determine whether the following systems are conservative, dissipative, or none of the above:

- the standard map (1.2.1);
- the Lorenz model (1.3.9);
- the logistic map (1.4.4);
- the Hénon map (1.4.6).

2.2 Stationary Solutions

A *stationary solution* of a dynamical system is a solution that does not change in time. We thus define

Definition 2.2.1.

- *A* fixed point *of the map F is a point $x^\star \in \mathcal{D}$ such that*

$$F(x^\star) = x^\star. \tag{2.2.1}$$

- *A* singular point *of the vector field f is a point $x^\star \in \mathcal{D}$ such that*

$$f(x^\star) = 0. \tag{2.2.2}$$

In both cases, $x^\star$ is also called *equilibrium point*. Its orbit is simply $\{x^\star\}$ and is called a *stationary orbit*. Note that a singular point of f is also a fixed point of the flow, and therefore sometimes abusively called a "fixed point of f".

We are now interested in the behaviour near an equilibrium point. In this section, we will always assume that f and F are twice continuously differentiable. If $x^\star$ is a singular point of f, the change of variables $x = x^\star + y$ leads to the equation

$$\begin{aligned} \dot{y} &= f(x^\star + y) \\ &= Ay + g(y), \end{aligned} \tag{2.2.3}$$

where we have introduced the Jacobian matrix

$$A = \frac{\partial f}{\partial x}(x^\star). \tag{2.2.4}$$

Taylor's formula implies that there exists a neighbourhood $\mathcal{N}$ of 0 and a constant $M > 0$ such that

$$\|g(y)\| \leqslant M\|y\|^2 \qquad \forall y \in \mathcal{N}. \tag{2.2.5}$$

Similarly, the change of variables $x_k = x^\star + y_k$ transforms an iterated map into

$$\begin{aligned} y_{k+1} &= F(x^\star + y_k) - x^\star \\ &= By_k + G(y_k), \end{aligned} \tag{2.2.6}$$

where

$$B = \frac{\partial F}{\partial x}(x^\star), \qquad \|G(y)\| \leqslant M\|y\|^2 \quad \forall y \in \mathcal{N}. \tag{2.2.7}$$

2.2.1 Linear Case

Let us start by analysing the equations (2.2.3) and (2.2.6) in the linear case, that is without the terms $g(x)$ and $G(x)$. Consider first the ODE

$$\dot{y} = Ay. \tag{2.2.8}$$

The solution can be written as

$$y(t) = \mathrm{e}^{At}\, y(0), \tag{2.2.9}$$

where the *exponential of* A is defined by the absolutely convergent series

$$\mathrm{e}^{At} \equiv \exp(At) := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{2.2.10}$$

In order to understand the behaviour of e^{At}, let us recall some facts from linear algebra. We can write the characteristic polynomial of A as

$$c_A(\lambda) = \det(\lambda \mathbb{1} - A) = \prod_{j=1}^{m} (\lambda - a_j)^{m_j}, \tag{2.2.11}$$

where $a_1, \ldots, a_m \in \mathbb{C}$ are distinct eigenvalues of A, and m_j are their *algebraic multiplicities*. The *geometric multiplicity* g_j of a_j is defined as the number of independent eigenvectors associated with a_j, and satisfies $1 \leqslant g_j \leqslant m_j$.

The results on decomposition of matrices leading to the Jordan canonical form can be formulated as follows [HS74]. The matrix A can be decomposed as

$$A = S + N, \qquad SN = NS. \tag{2.2.12}$$

Here S, the *semisimple part*, can be written as

$$S = \sum_{j=1}^{m} a_j P_j, \tag{2.2.13}$$

where the P_j are projectors on the eigenspaces of A, satisfying $P_j P_k = \delta_{jk} P_j$, $\sum_j P_j = \mathbb{1}$ and $m_j = \dim(P_j \mathbb{R}^n)$. The *nilpotent part* N can be written as

$$N = \sum_{j=1}^{m} N_j, \tag{2.2.14}$$

where the N_j satisfy the relations

$$N_j^{m_j} = 0, \qquad N_j N_k = 0 \quad \text{for } j \neq k, \qquad P_j N_k = N_k P_j = \delta_{jk} N_j. \tag{2.2.15}$$

In an appropriate basis, each N_j is block-diagonal, with g_j blocks of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}. \tag{2.2.16}$$

In fact, $N_j = 0$ unless $g_j < m_j$.

Lemma 2.2.2. *With the above notations*

$$\mathrm{e}^{At} = \sum_{j=1}^{m} \mathrm{e}^{a_j t}\, P_j \Big(\mathbb{1} + N_j t + \cdots + \frac{1}{(m_j-1)!} N_j^{m_j-1} t^{m_j-1} \Big) \tag{2.2.17}$$

Proof: We use the fact that $e^{At}\, \mathrm{e}^{Bt} = \mathrm{e}^{(A+B)t}$ whenever $AB = BA$, which can be checked by a direct calculation. Then $\mathrm{e}^{At} = \mathrm{e}^{St}\, \mathrm{e}^{Nt}$ with

$$\mathrm{e}^{St} = \prod_{j=1}^{m} \mathrm{e}^{a_j P_j t} = \prod_{j=1}^{m} \big(\mathbb{1} + (\mathrm{e}^{a_j t} - 1) P_j \big) = \mathbb{1} + \sum_{j=1}^{m} (\mathrm{e}^{a_j t} - 1) P_j = \sum_{j=1}^{m} \mathrm{e}^{a_j t}\, P_j,$$
$$\mathrm{e}^{Nt} = \prod_{j=1}^{m} \mathrm{e}^{N_j t} = \mathbb{1} + \sum_{j=1}^{m} (\mathrm{e}^{N_j t} - \mathbb{1}).$$

The result follows from the facts that $P_j(\mathrm{e}^{N_k t} - \mathbb{1}) = 0$ for $j \neq k$, and that $\mathrm{e}^{N_j t}$ contains only finitely many terms, being nilpotent. □

The expression (2.2.17) shows that the long-time behaviour is determined by the real parts of the eigenvalues a_j, while the nilpotent terms, when present, influence the short time behaviour. This motivates the following terminology:

Definition 2.2.3. *The* unstable, stable *and* center subspace *of the singular point $x^\star$ are defined, respectively, by*

$$\begin{aligned} E^+ &:= P^+ \mathbb{R}^n = \big\{ y \colon \lim_{t \to -\infty} \mathrm{e}^{At}\, y = 0 \big\}, & P^+ &:= \sum_{j : \operatorname{Re} a_j > 0} P_j, \\ E^- &:= P^- \mathbb{R}^n = \big\{ y \colon \lim_{t \to +\infty} \mathrm{e}^{At}\, y = 0 \big\}, & P^- &:= \sum_{j : \operatorname{Re} a_j < 0} P_j, \\ E^0 &:= P^0 \mathbb{R}^n, & P^0 &:= \sum_{j : \operatorname{Re} a_j = 0} P_j. \end{aligned} \tag{2.2.18}$$

The subspaces are invariant subspaces of e^{At}, that is, $\mathrm{e}^{At}\, E^+ \subset E^+$, $\mathrm{e}^{At}\, E^- \subset E^-$ and $\mathrm{e}^{At}\, E^0 \subset E^0$. The fixed point is called

- *a* sink *if $E^+ = E^0 = \{0\}$,*
- *a* source *if $E^- = E^0 = \{0\}$,*
- *a* hyperbolic point *if $E^0 = \{0\}$,*
- *an* elliptic point *if $E^+ = E^- = \{0\}$.*

Example 2.2.4. Let $n = 2$, and let A be in Jordan canonical form, with $\det A \neq 0$. Then we can distinguish between the following behaviours, depending on the eigenvalues a_1, a_2 of A (see Fig. 2.2).

1. $a_1 \neq a_2$
   (a) If $a_1, a_2 \in \mathbb{R}$, then $A = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)$ and

$$\mathrm{e}^{At} = \begin{pmatrix} \mathrm{e}^{a_1 t} & 0 \\ 0 & \mathrm{e}^{a_2 t} \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} y_1(t) &= \mathrm{e}^{a_1 t}\, y_1(0) \\ y_2(t) &= \mathrm{e}^{a_2 t}\, y_2(0) \end{aligned}$$

   The orbits are curves of the form $y_2 = c y_1^{a_2/a_1}$. $x^\star$ is called a *node* if $a_1 a_2 > 0$, and a *saddle* if $a_1 a_2 < 0$.

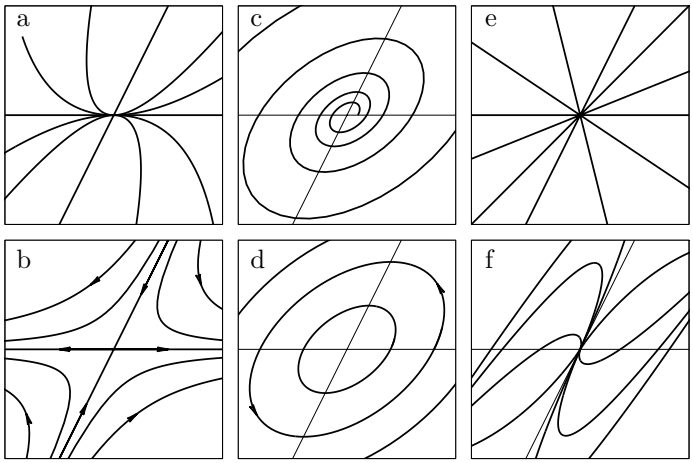

FIGURE 2.2. Phase portraits of a linear two–dimensional system: (a) node, (b) saddle, (c) focus, (d) center, (e) degenerate node, (f) improper node.

(b) If $a_1 = \overline{a_2} = a + \mathrm{i}\,\omega \in \mathbb{C}$, then the real canonical form of A is $A = \left(\begin{smallmatrix} a & -\omega \\ \omega & a \end{smallmatrix}\right)$ and

$$\mathrm{e}^{At} = \mathrm{e}^{at} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} y_1(t) &= \mathrm{e}^{at}(y_1(0)\cos\omega t - y_2(0)\sin\omega t) \\ y_2(t) &= \mathrm{e}^{at}(y_1(0)\sin\omega t + y_2(0)\cos\omega t) \end{aligned}$$

$x^\star$ is called a *focus* if $a \neq 0$, and a *center* if $a = 0$. The orbits are spirals or ellipses.

2. $a_1 = a_2 =: a$
   (a) If a has geometric multiplicity 2, then $A = a\mathbb{1}$ and $\mathrm{e}^{At} = \mathrm{e}^{at}\,\mathbb{1}$; $x^\star$ is called a *degenerate node*.
   (b) If a has geometric multiplicity 1, then $A = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ and

$$\mathrm{e}^{At} = \mathrm{e}^{at} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} y_1(t) &= \mathrm{e}^{at}(y_1(0) + y_2(0)t) \\ y_2(t) &= \mathrm{e}^{at}\, y_2(0) \end{aligned}$$

   $x^\star$ is called an *improper node*.

Let us now turn to the case of the linear iterated map

$$y_{k+1} = By_k \tag{2.2.19}$$

which admits the solution

$$y_k = B^k y_0. \tag{2.2.20}$$

Using a similar decomposition $B = S + N$ into the semisimple and nilpotent part, we arrive at

Lemma 2.2.5. *Let b_i be the eigenvalues of B, and P_i, N_i the associated projectors and nilpotent matrices. Then*

$$B^k = \sum_{i=1}^{m} P_i \sum_{j=0}^{\min\{k, m_i - 1\}} \binom{k}{j} b_i^{k-j} N_i^j. \tag{2.2.21}$$

PROOF: The main point is to observe that

$$B^k = \Big(\sum_{i=1}^{m}(b_iP_i + N_i)\Big)^k = \sum_{i=1}^{m}(b_iP_i + N_i)^k$$

because all cross-terms vanish. Then one applies the binomial formula. □

For large k, the behaviour of B^k is dictated by the terms $b_i^{k-m_i+1}$. This leads to the following equivalent of Definition 2.2.3:

Definition 2.2.6. *The* unstable, stable *and* center subspace *of the fixed point $x^\star$ are defined, respectively, by*

$$\begin{aligned} E^+ &:= P^+\mathbb{R}^n = \big\{y\colon \lim_{k\to-\infty} B^k y = 0\big\}, & P^+ &:= \sum_{j:|b_j|>1} P_j, \\ E^- &:= P^-\mathbb{R}^n = \big\{y\colon \lim_{k\to+\infty} B^k y = 0\big\}, & P^- &:= \sum_{j:|b_j|<1} P_j, \\ E^0 &:= P^0\mathbb{R}^n, & P^0 &:= \sum_{j:|b_j|=1} P_j. \end{aligned} \tag{2.2.22}$$

These subspaces are invariant under B. The remaining terminology on sinks, sources, hyperbolic and elliptic points is unchanged.

Exercise 2.2. Find the equilibrium points of the standard map (1.2.1) and the Lorenz equations (1.3.9). Give the dimensions of their stable, unstable and center subspaces. *Hint:* To determine the sign of the real parts of the eigenvalues of a 3×3 matrix, one can apply the Viète formula to its characteristic polynomial.

2.2.2 Stability and Liapunov Functions

Definition 2.2.7. *Let $x^\star$ be an equilibrium point of the system $\dot x = f(x)$.*

- *$x^\star$ is called* stable *if for any $\varepsilon > 0$, one can find a $\delta = \delta(\varepsilon) > 0$ such that whenever $\|x_0 - x^\star\| < \delta$, one has $\|\varphi_t(x_0) - x^\star\| < \varepsilon$ for all $t \geqslant 0$.*
- *$x^\star$ is called* asymptotically stable *if it is stable, and there is a $\delta_0 > 0$ such that $\lim_{t\to\infty}\varphi_t(x_0) = x^\star$ for all x_0 such that $\|x_0 - x^\star\| < \delta_0$.*
- *The* basin of attraction *of an asymptotically stable equilibrium $x^\star$ is the set*

$$\big\{x\in\mathcal{D}\colon \lim_{t\to\infty}\varphi_t(x) = x^\star\big\}. \tag{2.2.23}$$

- *$x^\star$ is called* unstable *if it is not stable.*

If $x^\star$ is a fixed point of the map F, similar definitions hold with $\varphi_t(\cdot)$ replaced by $F^k(\cdot)$.

The linearization of the system $\dot x = f(x)$ around an equilibrium $x^\star$ is the equation $\dot y = Ay$ with $A = \frac{\partial f}{\partial x}(x^\star)$. $x^\star$ is called *linearly stable* if $y = 0$ is a stable equilibrium of its linearization, and similarly in the asymptotically stable and unstable cases. Lemma 2.2.2 shows that

- $x^\star$ is linearly asymptotically stable if and only if all eigenvalues of A have a strictly negative real part;

- $x^\star$ is linearly stable if and only if no eigenvalue of A has positive real part, and all purely imaginary eigenvalues have equal algebraic and geometric multiplicities.

The problem is now to determine relations between linear and nonlinear stability. A useful method to do this is due to Liapunov. Here we will limit the discussion to differential equations, although similar results can be obtained for maps.

Theorem 2.2.8 (Liapunov). *Let $x^\star$ be a singular point of f, let $\mathcal{U}$ be a neighbourhood of $x^\star$ and set $\mathcal{U}_0 := \mathcal{U} \setminus \{x^\star\}$. Assume there exists a continuous function $V : \mathcal{U} \to \mathbb{R}$, continuously differentiable on $\mathcal{U}_0$, such that*

1. *$V(x) > V(x^\star)$ for all $x \in \mathcal{U}_0$;*
2. *the derivative of V along orbits is negative in $\mathcal{U}_0$, that is,*

$$\dot{V}(x) := \frac{\mathrm{d}}{\mathrm{d}t} V(\varphi_t(x))\Big|_{t=0} = \nabla V(x) \cdot f(x) \leqslant 0 \qquad \forall x \in \mathcal{U}_0. \tag{2.2.24}$$

Then $x^\star$ is stable. If, furthermore,

3. *the derivative of V along orbits is strictly negative,*

$$\dot{V}(x) = \nabla V(x) \cdot f(x) < 0 \qquad \forall x \in \mathcal{U}_0, \tag{2.2.25}$$

then $x^\star$ is asymptotically stable.

Proof: Pick $\varepsilon > 0$ small enough that the closed ball $\overline{\mathcal{B}}(x^\star, \varepsilon)$ with center $x^\star$ and radius ε, is contained in $\mathcal{U}$. Let $\mathcal{S} = \partial\overline{\mathcal{B}}(x^\star, \varepsilon)$ be the sphere of radius ε centered in $x^\star$. $\mathcal{S}$ being compact, V admits a minimum on $\mathcal{S}$, that we call β. Consider the open set

$$\mathcal{W} = \big\{x \in \overline{\mathcal{B}}(x^\star, \varepsilon) \colon V(x) < \beta\big\}.$$

$x^\star \in \mathcal{W}$ by Condition 1., and thus there exists $\delta > 0$ such that the open ball $\mathcal{B}(x^\star, \delta)$ is contained in $\mathcal{W}$. For any $x_0 \in \mathcal{B}(x^\star, \delta)$, we have $V(\varphi_t(x_0)) < \beta$ for all $t \geqslant 0$, and thus $\varphi_t(x_0) \in \mathcal{W}$ for all $t \geqslant 0$ by Condition 2., which proves that $x^\star$ is stable.

Assume now that (2.2.25) holds. Since the positive orbit of $x_0 \in \mathcal{W}$ is bounded, it admits a convergent subsequence $(x_n)_{n\geqslant 0} = (\varphi_{t_n}(x_0))_{n\geqslant 0} \to y^\star \in \overline{\mathcal{W}}$, $t_n \to \infty$. Consider the function $t \mapsto V(\varphi_t(x_0))$. It is continuously differentiable, monotonously decreasing, and admits a subsequence converging to $V(y^\star)$. Thus $V(\varphi_t(x_0))$ must converge to $V(y^\star)$ as $t \to \infty$. Let $\delta > 0$ be a small constant and define the compact set

$$\mathcal{K} = \big\{x \in \overline{\mathcal{W}} \colon V(y^\star) \leqslant V(x) \leqslant V(y^\star) + \delta\big\}.$$

If $y^\star \neq x^\star$, then $x^\star \notin \mathcal{K}$, and thus the maximum of $\dot{V}$ on $\mathcal{K}$ is a strictly negative constant c. Take n large enough that $x_n \in \mathcal{K}$. Then $\varphi_t(x_n) \in \mathcal{K}$ for all $t \geqslant 0$. But this implies that $V(\varphi_t(x_n)) \leqslant V(x_n) + ct$ which becomes smaller than $V(y^\star)$ for t large enough, which is impossible. Thus $y^\star = x^\star$, and all orbits starting in $\mathcal{W}$ converge to $x^\star$. □

The interpretation of (2.2.24) is that the vector field crosses all level sets of V in the same direction (Fig. 2.3). V is called a *Liapunov function* for $x^\star$, and a *strict Liapunov function* if (2.2.25) holds. In fact, the proof also shows that if V is a strict Liapunov function on $\mathcal{U}$, and $\mathcal{W}$ is a set of the form $\mathcal{W} = \{x \colon V(x) < \beta\}$ contained in $\mathcal{U}$, then $\mathcal{W}$ is contained in the basin of attraction of $x^\star$. Thus Liapunov functions can be used to estimate such basins of attraction.

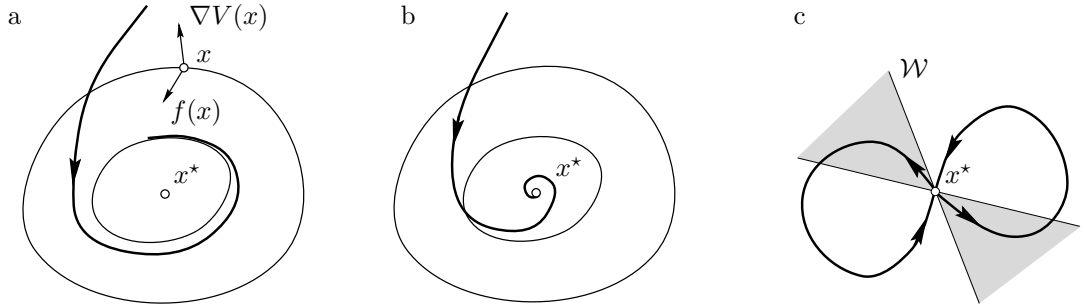

FIGURE 2.3. (a) Stable fixed point with level curves of a Liapunov function. (b) Asymptotically stable fixed point, here the vector field must cross all level curves in the same direction. (c) Example of an unstable fixed point with the set $\mathcal{W}$ of Četaev's Theorem.

Exercise 2.3. Give sufficient conditions on the parameters of the Lorenz equations 1.3.9 for *all* orbits to converge to the point $(0,0,0)$ (the origin is said to be *globally asymptotically stable*). *Hint:* Try Liapunov functions of the form $V(X,Y,Z) = \alpha X^2 + \beta Y^2 + \gamma Z^2$, where $\alpha, \beta, \gamma > 0$ are constants.

The advantage of the Liapunov method is that one does not need to solve the differential equation. However, the method is not constructive, and the form of V has to be guessed in each case. In the linearly asymptotically stable case, such a V can always be constructed, which leads to the following result.

Corollary 2.2.9. *Assume $x^\star$ is a linearly asymptotically stable equilibrium, that is, all eigenvalues of $A = \frac{\partial f}{\partial x}(x^\star)$ have strictly negative real parts. Then $x^\star$ is asymptotically stable.*

PROOF: There are many different constructions of strict Liapunov functions. We will give one of them. In order to satisfy Condition 1. of the theorem, we will look for a quadratic form

$$V(x) = (x - x^\star) \cdot Q(x - x^\star)$$

where Q is a symmetric, positive definite matrix. By assumption, there is a constant $a_0 > 0$ such that $\operatorname{Re} a_i \leqslant -a_0$ for all eigenvalues a_i of A. Thus Lemma 2.2.2 implies that

$$\|\mathrm{e}^{At}\, y\| \leqslant p(t)\, \mathrm{e}^{-a_0 t} \|y\| \qquad \forall y,$$

where p is a polynomial of degree less than n. This implies that the function

$$V(x) = \int_0^\infty \|\mathrm{e}^{As}(x - x^\star)\|^2 \, \mathrm{d}s$$

exists. $V(x)$ is of the above form with

$$Q = \int_0^\infty \mathrm{e}^{A^T s}\, \mathrm{e}^{As} \, \mathrm{d}s.$$

Q is clearly symmetric, positive definite and bounded, thus there is a $K > 0$ such that $\|Qy\| \leqslant K\|y\|$ for all y. Now we calculate the following expression in two different ways:

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{A^T s}\, \mathrm{e}^{As}\big)\, \mathrm{d}s = \lim_{t\to\infty} \mathrm{e}^{A^T t}\, \mathrm{e}^{At} - \mathbb{1} = -\mathbb{1}$$

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{A^T s}\, \mathrm{e}^{As}\big)\, \mathrm{d}s = \int_0^\infty \big(A^T \mathrm{e}^{A^T s}\, \mathrm{e}^{As} + \mathrm{e}^{A^T s}\, \mathrm{e}^{As}\, A\big)\, \mathrm{d}s = A^T Q + QA.$$

We have thus proved that

$$A^TQ + QA = -\mathbb{1}.$$

Now if $y(t) = \varphi_t(x) - x^\star$, we have

$$\dot V = \frac{\mathrm{d}}{\mathrm{d}t}V(\varphi_t(x)) = \frac{\mathrm{d}}{\mathrm{d}t}\big(y(t)\cdot Qy(t)\big) = \dot y(t)\cdot Qy(t) + y(t)\cdot Q\dot y(t).$$

Inserting $\dot y = Ay + g(y)$ produces two terms. The first one is

$$Ay\cdot Qy + y\cdot QAy = y\cdot(A^TQ + QA)y = -\|y\|^2,$$

and the second one gives

$$g(y)\cdot Qy + y\cdot Qg(y) = 2g(y)\cdot Qy \leqslant 2\|g(y)\|\|Qy\| \leqslant 2MK\|y\|^3.$$

Hence $\dot V \leqslant -\|y\|^2 + 2MK\|y\|^3$, which shows that V is a strict Liapunov function for $\|x - x^\star\| < \frac{1}{2MK}$, and the result is proved. □

There exists a characterization of unstable equilibria based on similar ideas:

Theorem 2.2.10 (Četaev). *Let $x^\star$ be a singular point of f, $\mathcal{U}$ a neighbourhood of $x^\star$ and $\mathcal{U}_0 = \mathcal{U}\setminus\{x^\star\}$. Assume there exists an open set $\mathcal{W}$, containing $x^\star$ in its closure, and a continuous function $V : \mathcal{U}\to\mathbb{R}$, which is continuously differentiable on $\mathcal{U}_0$ and satisfies*

1. *$V(x) > 0$ for all $x\in\mathcal{U}_0\cap\mathcal{W}$;*
2. *$\dot V(x) > 0$ for all $x\in\mathcal{U}_0\cap\mathcal{W}$;*
3. *$V(x) = 0$ for all $x\in\mathcal{U}_0\cap\partial\mathcal{W}$.*

Then $x^\star$ is unstable.

Proof: First observe that the definition of an unstable point can be stated as follows: there exists $\varepsilon > 0$ such that, for any $\delta > 0$, one can find x_0 with $\|x_0 - x^\star\| < \delta$ and $T > 0$ such that $\|\varphi_T(x_0) - x^\star\| \geqslant \varepsilon$.

Now take $\varepsilon > 0$ sufficiently small that $\mathcal{B}(x^\star,\varepsilon)\subset\mathcal{U}$. For any $\delta > 0$, $\mathcal{B}(x^\star,\delta)\cap\mathcal{W}\neq\emptyset$. We can thus take an $x_0\in\mathcal{U}_0\cap\mathcal{W}$ such that $\|x_0 - x^\star\| < \delta$, and by Condition 1. $V(x_0) > 0$. Now assume by contradiction that $\|\varphi_t(x_0) - x^\star\| < \varepsilon$ for all $t\geqslant 0$. $\varphi_t(x_0)$ must stay in $\mathcal{U}_0\cap\mathcal{W}$ for all t, because it cannot reach the boundary of $\mathcal{W}$ where $V = 0$. Thus there exists a sequence $x_n = \varphi_{t_n}(x_0)$ converging to some $x_1\in\mathcal{U}_0\cup\mathcal{W}$. But this contradicts the fact that $\dot V(x_1) > 0$, as in the proof of Theorem 2.2.8, and thus $\|\varphi_t(x_0) - x^\star\|$ must become larger than ε. □

Corollary 2.2.11. *Assume $x^\star$ is an equilibrium point such that $A = \frac{\partial f}{\partial x}(x^\star)$ has at least one eigenvalue with positive real part. Then $x^\star$ is unstable.*

Proof: Consider first the case of A having no purely imaginary eigenvalues. We can choose a coordinate system along the unstable and stable subspaces of $x^\star$, in which the dynamics is described by the equation

$$\begin{aligned}\dot y_+ &= A_+y_+ + g_+(y)\\ \dot y_- &= A_-y_- + g_-(y),\end{aligned}$$

where all eigenvalues of A_+ have a strictly positive real part, all eigenvalues of A_- have a strictly negative real part, $y = (y_+, y_-)$ and the terms $g_\pm$ are bounded in norm by a positive constant M times $\|y\|^2$. We can define the matrices

$$Q_- = \int_0^\infty \mathrm{e}^{A_-^T s}\, \mathrm{e}^{A_- s}\, \mathrm{d}s, \qquad Q_+ = \int_0^\infty \mathrm{e}^{-A_+^T s}\, \mathrm{e}^{-A_+ s}\, \mathrm{d}s,$$

which are bounded, symmetric, positive definite, and satisfy $A_-^T Q_- + Q_- A_- = -\mathbb{1}$ and $A_+^T Q_+ + Q_+ A_+ = \mathbb{1}$. Define the quadratic form

$$V(y) = y_+ \cdot Q_+ y_+ - y_- \cdot Q_- y_-.$$

The cone $\mathcal{W} = \{y\colon V(y) > 0\}$ is non-empty, because it contains an eigenvector of A corresponding to an eigenvalue with positive real part. Proceeding similarly as in the proof of Corollary 2.2.9, we find

$$\begin{aligned}\dot V(y) &= \|y_+\|^2 + 2g_+(y)\cdot Q_+ y_+ + \|y_-\|^2 - 2g_-(y)\cdot Q_- y_- \\ &\geqslant \|y\|^2 - 2KM\|y\|^3.\end{aligned}$$

Thus Četaev's theorem can be applied to show that $x^\star$ is unstable. If A also has purely imaginary eigenvalues, we obtain the additional equation

$$\dot y_0 = A_0 y_0 + g_0(y),$$

where all eigenvalues of A_0 are purely imaginary. Let S be an invertible complex matrix of the same dimension as A_0 and consider the function

$$V(y) = y_+ \cdot Q_+ y_+ - y_- \cdot Q_- y_- - \|S y_0\|^2,$$

where $u \cdot v = \sum \overline{u_i} v_i$ for complex vectors u, v. Proceeding as above, we obtain that

$$\begin{aligned}\dot V(y) \geqslant \|y_+\|^2 + \|y_-\|^2 - 2KM\|y\|^2(\|y_+\| + \|y_-\|) \\ - 2\,\mathrm{Re}(Sy_0 \cdot A_0 S y_0) - 2\,\mathrm{Re}(S g_0(y)\cdot S y_0).\end{aligned}$$

We shall prove below that for any $\varepsilon > 0$, one can construct a matrix $S(\varepsilon)$ such that $\mathrm{Re}(Sy_0 \cdot A_0 S y_0) \leqslant \varepsilon \|S y_0\|^2$. We now take $\mathcal{U} = \{y\colon \|y\| < \delta\}$, where $\delta > 0$ has to be determined, and $\mathcal{W} = \{y\colon V(y) > 0\}$. If $y \in \mathcal{W}$, we have

$$\mathrm{Re}(Sy_0 \cdot A_0 S y_0) \leqslant \varepsilon \|S y_0\|^2 < \varepsilon y_+ \cdot Q_+ Y_+ < \varepsilon K \|y_+\|^2.$$

We introduce the constants

$$C(\varepsilon) = \sup_{y_0 \neq 0} \frac{\|S y_0\|}{\|y_0\|}, \qquad c(\varepsilon) = \sup_{y_0 \neq 0} \frac{\|y_0\|}{\|S y_0\|}.$$

Then we have

$$\begin{gathered}\mathrm{Re}(S g_0(y)\cdot S y_0) \leqslant C\|g_0(y)\|\|S y_0\| \leqslant CMK^{1/2}\|y\|^2\|y_+\| \\ \|y\|^2 \leqslant \|y_+\|^2 + \|y_-\|^2 + c^2\|S y_0\|^2 < (1 + c^2 K)(\|y_+\|^2 + \|y_-\|^2).\end{gathered}$$

Putting everything together, we obtain for all $y \in \mathcal{U} \cap \mathcal{W}$

$$\begin{aligned}\dot V(y) &> (1 - \varepsilon K)(\|y_+\|^2 + \|y_-\|^2) - 2KM\|y\|^2\big[(1 + CK^{-1/2})\|y_+\| + \|y_-\|\big] \\ &> (\|y_+\|^2 + \|y_-\|^2)\big[1 - \varepsilon K - 2KM(1 + c(\varepsilon)^2 K)(2 + C(\varepsilon)K^{-1/2})\delta\big].\end{aligned}$$

Taking $\varepsilon < 1/K^2$ and then δ small enough, we can guarantee that this quantity is positive for all $y \in \mathcal{U}_0 \cap \mathcal{W}$, and the corollary is proved. □

	ODE $\dot{x} = f(x)$	Map $x_{k+1} = F(x_k)$
Conservative	$\nabla \cdot f = 0$	$\left\lvert \det \frac{\partial F}{\partial x} \right\rvert = 1$
Dissipative	$\nabla \cdot f < 0$	$\left\lvert \det \frac{\partial F}{\partial x} \right\rvert < 1$
Equilibrium	$f(x^\star) = 0$	$F(x^\star) = x^\star$
Asympt. stable if	$\operatorname{Re} a_i < 0 \ \forall i$	$\lvert b_i \rvert < 1 \ \forall i$
Unstable if	$\exists i : \operatorname{Re} a_i > 0$	$\exists i : \lvert b_i \rvert > 1$

TABLE 2.1. Comparison of some properties of ordinary differential equations and iterated maps. Here a_i and b_i are eigenvalues of the matrices $A = \frac{\partial f}{\partial x}(x^\star)$ and $B = \frac{\partial F}{\partial x}(x^\star)$.

In the proof we have used the following result.

Lemma 2.2.12. *Assume all the eigenvalues of the $m \times m$ matrix A_0 are purely imaginary. For every $\varepsilon > 0$, there exists a complex invertible matrix S such that*

$$\left\lvert \operatorname{Re}(Sz \cdot SA_0 z) \right\rvert \leqslant \varepsilon \|Sz\|^2 \qquad \forall z \in \mathbb{C}^m. \tag{2.2.26}$$

PROOF: Let S_0 be such that $B = S_0 A_0 S_0^{-1}$ is in complex Jordan canonical form, that is, the diagonal elements $b_{jj} = \mathrm{i}\,\lambda_j$ of B are purely imaginary, $b_{jj+1} = \sigma_j$ is either zero or one, and all other elements of B are zero. Let S_1 be the diagonal matrix with entries $1, \varepsilon^{-1}, \dots, \varepsilon^{1-m}$. Then

$$C \coloneqq S_1 B S_1^{-1} = \begin{pmatrix} \mathrm{i}\,\lambda_1 & \varepsilon\sigma_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon\sigma_{m-1} \\ 0 & & & \mathrm{i}\,\lambda_m \end{pmatrix}$$

Let $S = S_1 S_0$ and $u = Sz$. Then

$$\left| \operatorname{Re} \frac{Sz \cdot SA_0 z}{\|Sz\|^2} \right| = \left| \operatorname{Re} \frac{u \cdot Cu}{\|u\|^2} \right| = \left| \operatorname{Re} \frac{\sum \mathrm{i}\,\lambda_j |u_j|^2 + \varepsilon \sum \sigma_j \overline{u_j} u_{j+1}}{\sum |u_j|^2} \right| \leqslant \varepsilon \frac{\sum |u_j||u_{j+1}|}{\sum |u_j|^2} \leqslant \varepsilon,$$

where we used the fact that the upper sum has $m-1$ terms and the lower one m terms. □

The two corollaries of this section can be stated as follows: if $x^\star$ is a hyperbolic equilibrium, then it has the same type of stability as the linearized system. The same properties are valid for maps (see Table 2.1). This is in general not true for non-hyperbolic equilibria. This situation will be studied in more detail in Chapter 3.

2.2.3 Invariant Manifolds

For hyperbolic equilibrium points, the analogies between nonlinear and linear systems can be pushed further. One of them has to do with invariant manifolds, which generalize the invariant subspaces of the linear case. We assume in this section that f is of class $\mathcal{C}^r$ with $r \geqslant 2$.

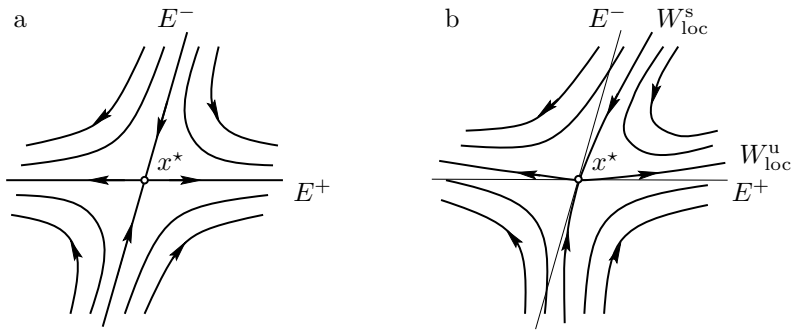

FIGURE 2.4. Orbits near a hyperbolic fixed point: (a) orbits of the linearized system, (b) orbits of the nonlinear system with local stable and unstable manifolds.

Definition 2.2.13. *Let $x^\star$ be a singular point of the system $\dot{x} = f(x)$, and let $\mathcal{U}$ be a neighbourhood of $x^\star$. The* local stable and unstable manifolds *of $x^\star$ in $\mathcal{U}$ are defined, respectively, by*

$$\begin{aligned} W^{\rm s}_{\rm loc}(x^\star) &:= \bigl\{ x \in \mathcal{U} \colon \lim_{t\to\infty} \varphi_t(x) = x^\star \text{ and } \varphi_t(x) \in \mathcal{U}\ \forall t \geqslant 0 \bigr\} \\ W^{\rm u}_{\rm loc}(x^\star) &:= \bigl\{ x \in \mathcal{U} \colon \lim_{t\to-\infty} \varphi_t(x) = x^\star \text{ and } \varphi_t(x) \in \mathcal{U}\ \forall t \leqslant 0 \bigr\}. \end{aligned} \tag{2.2.27}$$

The global stable and unstable manifolds *of $x^\star$ are defined by*

$$\begin{aligned} W^{\rm s}(x^\star) &= \bigcup_{t\leqslant 0} \varphi_t(W^{\rm s}_{\rm loc}(x^\star)), \\ W^{\rm u}(x^\star) &= \bigcup_{t\geqslant 0} \varphi_t(W^{\rm u}_{\rm loc}(x^\star)). \end{aligned} \tag{2.2.28}$$

Similar definitions can be made for maps. Global invariant manifolds can have a very complicated structure, and may return infinitely often to a neighbourhood of the equilibrium point. This is why one prefers to define separately local and global invariant manifolds. The following theorem states that local invariant manifolds have a nice structure.

Theorem 2.2.14 (Stable manifold theorem). *Let $x^\star$ be a hyperbolic equilibrium point of the system $\dot{x} = f(x)$, such that the matrix $\frac{\partial f}{\partial x}(x^\star)$ has n_+ eigenvalues with positive real parts and n_- eigenvalues with negative real parts, with $n_+, n_- \geqslant 1$. Then $x^\star$ admits, in a neighbourhood $\mathcal{U}$,*

- *a local stable manifold $W^{\rm s}_{\rm loc}(x^\star)$, which is a differentiable manifold of class $\mathcal{C}^r$ and dimension n_-, tangent to the stable subspace E^- at $x^\star$, and which can be represented as a graph;*
- *a local unstable manifold $W^{\rm u}_{\rm loc}(x^\star)$, which is a differentiable manifold of class $\mathcal{C}^r$ and dimension n_+, tangent to the unstable subspace E^+ at $x^\star$, and which can be represented as a graph.*

We will omit the proof of this result, but we will give in Chapter 3 the proof of the center manifold theorem, which relies on similar ideas. Let us now explain a bit more precisely what this result means. The geometric interpretation is shown in Fig. 2.4. To explain the meaning of "a differentiable manifold of class $\mathcal{C}^r$ tangent to $E^\pm$ and representable

as a graph", let us introduce a coordinate system along the invariant subspaces of the linearization. The vector field near $x^\star$ can be written as

$$\begin{aligned} \dot{y}_+ &= A_+ y_+ + g_+(y_+, y_-) \\ \dot{y}_- &= A_- y_- + g_-(y_+, y_-), \end{aligned} \tag{2.2.29}$$

where A_+ is a $n_+ \times n_+$ matrix, which has only eigenvalues with positive real parts, and A_- is a $n_- \times n_-$ matrix, which has only eigenvalues with negative real parts. The terms $g_\pm$ are nonlinear and satisfy $\|g_\pm(y_+, y_-)\| \leqslant M\|y\|^2$ in $\mathcal{U}$, where M is a positive constant. The theorem implies the existence of a function of class $\mathcal{C}^r$

$$h^{\mathrm{u}} : \mathcal{U}_+ \to \mathbb{R}^{n_-}, \qquad h^{\mathrm{u}}(0) = 0, \qquad \frac{\partial h^{\mathrm{u}}}{\partial y_+}(0) = 0, \tag{2.2.30}$$

where $\mathcal{U}_+$ is a neighbourhood of the origin in $\mathbb{R}^{n_+}$, such that the local unstable manifold is given by the equation

$$y_- = h^{\mathrm{u}}(y_+). \tag{2.2.31}$$

Similar relations hold for the stable manifold.

In order to determine the function h^{u}, let us compute $\dot{y}_-$ in two ways, for a given orbit on the unstable manifold:

$$\begin{aligned} \dot{y}_- &= A_- h^{\mathrm{u}}(y_+) + g_-(y_+, h^{\mathrm{u}}(y_+)) \\ \dot{y}_- &= \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\dot{y}_+ = \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\Big[A_+ y_+ + g_+(y_+, h^{\mathrm{u}}(y_+))\Big]. \end{aligned} \tag{2.2.32}$$

Since both expressions must be equal, we obtain that h^{u} must satisfy the partial differential equation

$$A_- h^{\mathrm{u}}(y_+) + g_-(y_+, h^{\mathrm{u}}(y_+)) = \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\Big[A_+ y_+ + g_+(y_+, h^{\mathrm{u}}(y_+))\Big]. \tag{2.2.33}$$

This equation is difficult to solve in general. However, since we know by the theorem that h^{u} is of class $\mathcal{C}^r$, we can compute h^{u} perturbatively, by inserting its Taylor expansion into (2.2.33) and solving order by order.

Example 2.2.15. Consider, for $n = 2$, the system

$$\begin{aligned} \dot{y}_1 &= y_1 \\ \dot{y}_2 &= -y_2 + y_1^2. \end{aligned} \tag{2.2.34}$$

Then the equation (2.2.33) reduces to

$$-h^{\mathrm{u}}(y_1) + y_1^2 = h^{\mathrm{u}\prime}(y_1) y_1, \tag{2.2.35}$$

which admits the solution

$$h^{\mathrm{u}}(y_1) = \frac{1}{3} y_1^2. \tag{2.2.36}$$

This is confirmed by the explicit solution of (2.2.34),

$$\begin{aligned} y_1(t) &= y_1(0)\, \mathrm{e}^t \\ y_2(t) &= \Big(y_2(0) - \frac{1}{3} y_1(0)^2\Big)\, \mathrm{e}^{-t} + \frac{1}{3} y_1(0)^2\, \mathrm{e}^{2t}\,. \end{aligned} \tag{2.2.37}$$

2.2.4 Normal Forms

In this section we will examine some further connections between the flow near an equilibrium point and its linearization. We will assume that $x^\star$ is a singular point of the system $\dot{x} = f(x)$, and that f is of class $\mathcal{C}^r$, $r \geqslant 2$, in a neighbourhood of $x^\star$.

Definition 2.2.16. *Let $\mathcal{U}$ and $\mathcal{W}$ be open sets in $\mathbb{R}^n$, and let $f : \mathcal{U} \to \mathbb{R}^n$ and $g : \mathcal{W} \to \mathbb{R}^n$ be two vector fields of class $\mathcal{C}^r$. These vector fields are called*

- topologically equivalent *if there exists a homeomorphism $h : \mathcal{U} \to \mathcal{W}$ taking the orbits of $\dot{x} = f(x)$ to the orbits of $\dot{y} = g(y)$ and preserving the sense of time;*
- differentiably equivalent *if there exists a diffeomorphism $h : \mathcal{U} \to \mathcal{W}$ taking the orbits of $\dot{x} = f(x)$ to the orbits of $\dot{y} = g(y)$ and preserving the sense of time.*

If, in addition, h preserves parametrization of the orbits by time, the vector fields are called conjugate.

Equivalence means that if φ_t and ψ_t are the flows of the two systems, then

$$\psi_t \circ h = h \circ \varphi_{\tau(t)} \tag{2.2.38}$$

on $\mathcal{U}$, where τ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. If $\tau(t) = t$ for all t, the systems are conjugate.

Theorem 2.2.17 (Hartman-Grobman). *Let $x^\star$ be a hyperbolic equilibrium point of $\dot{x} = f(x)$, that is, the matrix $A = \frac{\partial f}{\partial x}(x^\star)$ has no eigenvalue with zero real part. Then, in a sufficiently small neighbourhood of $x^\star$, f is topologically conjugate to the linearization $\dot{y} = Ay$.*

Note, however, that topological equivalence is not a very strong property, since h need not be differentiable. In fact, one can show that all linear systems with the same number of eigenvalues with positive and negative real parts are topologically equivalent (see for instance [HK91]). So for instance, the node and focus in Fig. 2.2 are topologically equivalent. On the other hand, differentiable equivalence is harder to achieve as shows the following example.

Example 2.2.18. Consider the following vector field and its linearization:

$$\begin{aligned} \dot{y_1} &= 2y_1 + y_2^2 & \qquad \dot{z_1} &= 2z_1 \\ \dot{y_2} &= y_2 & \qquad \dot{z_2} &= z_2. \end{aligned} \tag{2.2.39}$$

The orbits can be found by solving the differential equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}y_2} = 2\frac{y_1}{y_2} + y_2, \qquad \frac{\mathrm{d}z_1}{\mathrm{d}z_2} = 2\frac{z_1}{z_2}, \tag{2.2.40}$$

which admit the solutions

$$y_1 = \big[c + \log|y_2|\big] y_2^2, \qquad z_1 = cz_2^2. \tag{2.2.41}$$

Because of the logarithm, the two flows are $\mathcal{C}^1$- but not $\mathcal{C}^2$-conjugate.

The theory of normal forms allows to explain these phenomena, and to obtain conditions for the existence of such $\mathcal{C}^r$-conjugacies. Consider the system for $y = x - x^\star$,

$$\dot{y} = Ay + g(y). \tag{2.2.42}$$

We can try to simplify the nonlinear term by a change of coordinates $y = z + h(z)$, which leads to

$$\dot{z} + \frac{\partial h}{\partial z}(z)\dot{z} = Az + Ah(z) + g(z + h(z)). \tag{2.2.43}$$

Assume that $h(z)$ solves the partial differential equation

$$\frac{\partial h}{\partial z}(z)Az - Ah(z) = g(z + h(z)). \tag{2.2.44}$$

Then we obtain for z the linear equation

$$\dot{z} = Az. \tag{2.2.45}$$

Unfortunately, we do not know how to solve the equation (2.2.44) in general. One can, however, work with Taylor series. To this end, we rewrite the system (2.2.42) as

$$\dot{y} = Ay + g_2(y) + g_3(y) + \cdots + g_{r-1}(y) + \mathcal{O}(\|y\|^r). \tag{2.2.46}$$

Here the last term is bounded in norm by a constant times $\|y\|^r$, and the terms $g_k(y)$ are homogeneous polynomial maps of degree k from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($g_k(\lambda y) = \lambda^k g_k(y)\ \forall \lambda \in \mathbb{R}$). Let us denote by $\mathcal{H}_k$ the set of all such maps. $\mathcal{H}_k$ is a vector space for the usual addition and multiplication by scalars. For instance, when $n = 2$, $\mathcal{H}_2$ admits the basis vectors

$$\begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix}. \tag{2.2.47}$$

We now define a linear map from $\mathcal{H}_k$ to itself given by

$$\begin{array}{rccl} \mathrm{ad}_k A: & \mathcal{H}_k & \to & \mathcal{H}_k \\ & h(y) & \mapsto & \dfrac{\partial h}{\partial y}(y)Ay - Ah(y). \end{array} \tag{2.2.48}$$

The fundamental result of normal form theory is the following:

Proposition 2.2.19. *For each k, $2 \leqslant k < r$, choose a complementary space $\mathcal{G}_k$ of the image of $\mathrm{ad}_k A$, that is, such that $\mathcal{G}_k \oplus \mathrm{ad}_k A(\mathcal{H}_k) = \mathcal{H}_k$. Then there exists, in a neighbourhood of the origin, an analytic (polynomial) change of variables $y = z + h(z)$ transforming* (2.2.46) *into*

$$\dot{z} = Az + g_2^{\mathrm{res}}(z) + g_3^{\mathrm{res}}(z) + \cdots + g_{r-1}^{\mathrm{res}}(z) + \mathcal{O}(\|z\|^r), \tag{2.2.49}$$

where $g_k^{\mathrm{res}} \in \mathcal{G}_k$, $2 \leqslant k < r$.

Proof: The proof proceeds by induction. Assume that for some k, $2 \leqslant k \leqslant r-1$, we have obtained an equation of the form

$$\dot{y} = Ay + \sum_{j=2}^{k-1} g_j^{\mathrm{res}}(y) + g_k(y) + \mathcal{O}(\|y\|^{k+1}).$$

We decompose the term $g_k(y)$ into a resonant and a non-resonant part,

$$g_k(y) = g_k^{\rm res}(y) + g_k^0(y), \qquad g_k^0 \in {\rm ad}_k\, A(\mathcal{H}_k), \quad g_k^{\rm res} \in \mathcal{G}_k.$$

There exists $h_k \in \mathcal{H}_k$ satisfying

$${\rm ad}_k\, A(h_k(z)) := \frac{\partial h_k}{\partial z}(z)Az - Ah_k(z) = g_k^0(z).$$

Observe that $g_j^{\rm res}(z + h_k(z)) = g_j^{\rm res}(z) + \mathcal{O}(\|z\|^{k+1})$ for all j, and a similar relation holds for g_k. Thus the change of variables $y = z + h(z)$ yields the equation

$$\begin{aligned}\dot y = \Big[\mathbb{1} + \frac{\partial h_k}{\partial z}(z)\Big]\dot z &= Az + Ah_k(z) + \sum_{j=2}^{k-1} g_j^{\rm res}(z) + g_k(z) + \mathcal{O}(\|z\|^{k+1}) \\ &= \Big[\mathbb{1} + \frac{\partial h_k}{\partial z}(z)\Big]Az + \sum_{j=2}^{k} g_j^{\rm res}(z) + \mathcal{O}(\|z\|^{k+1}),\end{aligned}$$

where we have used the definition of h to get the second line. Now for sufficiently small z, the matrix $\mathbb{1} + \frac{\partial h_k}{\partial z}$ admits an inverse $\mathbb{1} + \mathcal{O}(\|z\|^{k-1})$. Multiplying the above identity on the left by this inverse, we have proved the induction step. □

The terms $g_k^{\rm res}(z)$ are called *resonant* and (2.2.49) is called the *normal form* of (2.2.46). The equation ${\rm ad}_k\, A(h_k(z)) = g_k^0(z)$ that has to be satisfied to eliminate the non-resonant terms of order k is called the *homological equation*. Whether a term is resonant or not is a problem of linear algebra, which depends only on the matrix A. While it can be difficult to determine the coefficients of the resonant terms in a particular case, it is in general quite easy to find which terms can be eliminated. In particular, the following result holds:

Lemma 2.2.20. *Let $(a_1, \dots, a_n)$ be the eigenvalues of A, counting multiplicity. Assume that for each j, $1 \leqslant j \leqslant n$, we have*

$$p_1 a_1 + \cdots + p_n a_n \neq a_j \tag{2.2.50}$$

for all n-tuples of non-negative integers $(p_1, \dots, p_n)$ satisfying $p_1 + \cdots + p_n = k$. Then ${\rm ad}_k\, A$ is invertible, and thus there are no resonant terms of order k.

Proof: We can assume that A is in Jordan canonical form. Consider first the case of a diagonal A. Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$. For $\mathcal{H}_k$ we choose the basis vectors $z_1^{p_1} \dots z_n^{p_n} e_j$, where $p_1 + \cdots + p_n = k$. Then an explicit calculation shows that

$${\rm ad}_k\, A(z_1^{p_1} \dots z_n^{p_n} e_j) = (p_1 a_1 + \cdots + p_n a_n - a_j) z_1^{p_1} \dots z_n^{p_n} e_j.$$

By assumption, the term in brackets is different from zero. Thus the linear operator ${\rm ad}_k\, A$ is diagonal in the chosen basis, with nonzero elements on the diagonal, which shows that it is invertible.

Consider now the case of a matrix A that is not diagonal, but has elements of the form $a_{jj+1} = 1$. Then ${\rm ad}_k\, A$ applied to a basis vector will contain additional, off-diagonal terms. One of them is proportional to $z_1^{p_1} \dots z_n^{p_n} e_{j-1}$, while the others are of the form $z_{k+1}\partial_{z_k}(z_1^{p_1} \dots z_n^{p_n})e_j$. One can show that the basis vector of $\mathcal{H}_k$ can be ordered in such a way that ${\rm ad}_k\, A$ is represented by a triangular matrix, with the same diagonal elements as in the case of a diagonal A, thus the conclusion is unchanged. □

The non-resonance condition (2.2.50) is called a *Diophantine condition*, since it involves integer coefficients. Thus resonant terms can exist only when the eigenvalues of A satisfy a relation of the form $p_1a_1+\cdots+p_na_n = a_j$, which is called a resonance of order $p_1+\cdots+p_n$. In Example 2.2.18, the relation $2a_2 = a_1$ induces a resonance of order 2, which makes it impossible to eliminate the term y_2^2 by a polynomial change of coordinates.

In order to solve the question of differentiable equivalence, the really difficult problem is to eliminate the remainder $\mathcal{O}(\|z\|^r)$ in the normal form (2.2.49). This problem was solved by Poincaré for sources and sinks, and by Sternberg and Chen for general hyperbolic equilibria (see for instance [Har64]). We state here the main result without proof.

Theorem 2.2.21 (Poincaré-Sternberg-Chen). *Let A be a $n\times n$ matrix with no eigenvalues on the imaginary axis. Consider the two equations*

$$\begin{aligned} \dot{x} &= Ax + b_1(x) \\ \dot{y} &= Ay + b_2(y). \end{aligned} \tag{2.2.51}$$

We assume that b_1 and b_2 are of class $\mathcal{C}^r$, $r \geqslant 2$, in a neighbourhood of the origin, that $b_1(x) = \mathcal{O}(\|x\|^2)$, $b_2(x) = \mathcal{O}(\|x\|^2)$, and $b_1(x) - b_2(x) = \mathcal{O}(\|x\|^r)$. Then for every $k \geqslant 2$, there is an integer $N = N(k, n, A) \geqslant k$ such that, if $r \geqslant N$, there exists a map h of class $\mathcal{C}^k$ such that the two systems can be transformed into one another by the transformation $x = y + h(y)$.

This result implies that for the system $\dot{y} = Ay + g(y)$ to be $\mathcal{C}^k$-conjugate to its linearization $\dot{z} = Az$, it must be sufficiently smooth and satisfy non-resonance conditions up to sufficiently high order N, where this order and the conditions depend only on k, n and the eigenvalues of A. In the special case of all eigenvalues of A having real parts with the same sign (source or sink), Poincaré showed that $N = k$. For general hyperbolic equilibria, N can be much larger than k.

Another, even more important consequence, is that vector fields near singular points can be classified by their normal forms, each normal form being a representative of an equivalence class (with respect to $\mathcal{C}^k$-conjugacy). This property plays an important role in bifurcation theory, as we shall see in Chapter 3.

Let us finally remark that in the much more difficult case of non-hyperbolic equilibria, certain results on $\mathcal{C}^k$-conjugacy have been obtained by Siegel, Moser, Takens and others.

2.3 Periodic Solutions

2.3.1 Periodic Orbits of Maps

Definition 2.3.1. *Let $p \geqslant 1$ be an integer. A* periodic orbit of period p *of the map F is a set of points $\{x_1^\star, \ldots, x_p^\star\}$ such that*

$$F(x_1^\star) = x_2^\star, \quad \ldots \quad F(x_{p-1}^\star) = x_p^\star, \quad F(x_p^\star) = x_1^\star. \tag{2.3.1}$$

Each point of the orbit is called a periodic point of period p. *Thus a periodic point $x^\star$ of period p is also a fixed point of F^p. p is called the* least period *of $x^\star$ if $F^j(x^\star) \neq x^\star$ for $1 \leqslant j < p$.*

To find periodic orbits of an iterated map, it is thus sufficient to find the fixed points of F^p, $p = 1, 2, \ldots$. Unfortunately, this becomes usually extremely difficult with increasing

p. Moreover, the number of periodic orbits of period p often grows very quickly with p. Methods that simplify the search for periodic orbits are known for special classes of maps. For instance, for two-dimensional conservative maps, there exists a variational method: periodic orbits of period p correspond to stationary points of some function of $\mathbb{R}^p$ to $\mathbb{R}$.

Once a periodic orbit has been found, the problem of its linear stability is rather easily solved. Indeed, it is sufficient to find the eigenvalues of the matrix

$$\begin{aligned}\frac{\partial F^p}{\partial x}(x_1^\star) &= \frac{\partial F}{\partial x}(F^{p-1}(x_1^\star))\frac{\partial F}{\partial x}(F^{p-2}(x_1^\star))\dots\frac{\partial F}{\partial x}(x_1^\star)\\ &= \frac{\partial F}{\partial x}(x_p^\star)\frac{\partial F}{\partial x}(x_{p-1}^\star)\dots\frac{\partial F}{\partial x}(x_1^\star).\end{aligned} \tag{2.3.2}$$

Note that the result has to be invariant under cyclic permutations of the matrices.

The dynamics near any point of the periodic orbit can be inferred from the dynamics near one of them, considered as a fixed point of F^p. Thus periodic orbits can also be classified into sinks, sources, hyperbolic and elliptic orbits, and the concepts of nonlinear stability, invariant manifolds and normal forms can be carried over from fixed points to periodic orbits.

2.3.2 Periodic Orbits of Flows and Poincaré Sections

Definition 2.3.2. *Let f be a vector field, φ_t its flow and $T > 0$ a constant. A* periodic solution of period T *of f is a function $\gamma(t)$ such that*

$$\dot\gamma(t) = f(\gamma(t)) \text{ and } \gamma(t+T) = \gamma(t) \qquad \forall t. \tag{2.3.3}$$

The corresponding closed curve $\Gamma = \{\gamma(t)\colon 0 \leqslant t \leqslant T\}$ is called a periodic orbit of period T. *Thus each point x of this orbit is a fixed point of φ_T. T is called the* least period *of the orbit if $\varphi_t(x) \neq x$ for $0 < t < T$.*

Finding periodic orbits of differential equations is even more difficult than for maps. There exist methods which help to find periodic orbits in a number of particular cases, such as two-dimensional flows, or systems admitting constants of the motion and small perturbations of them.

Let us now assume that we have found a periodic solution $\gamma(t)$. We would like to discuss its stability. The difference $y(t) = x(t) - \gamma(t)$ between an arbitrary solution and the periodic solution satisfies the equation

$$\dot y = f(\gamma(t) + y) - f(\gamma(t)). \tag{2.3.4}$$

If f is twice continuously differentiable and y is small, we may expand f into Taylor series, which yields

$$\dot y = A(t)y + g(y,t), \qquad A(t) = \frac{\partial f}{\partial x}(\gamma(t)), \quad \|g(y,t)\| \leqslant M\|y\|^2. \tag{2.3.5}$$

Let us examine the linearization of this equation, given by

$$\dot y = A(t)y. \tag{2.3.6}$$

Note that a similar equation already appeared in the proof of Proposition 2.1.10. This equation admits a unique global solution, which can be represented, because of linearity, as

$$y(t) = U(t)y(0), \tag{2.3.7}$$

where $U(t)$ is an $n \times n$ matrix-valued function solving the equation

$$\dot{U}(t) = A(t)U(t), \qquad U(0) = \mathbb{1}. \tag{2.3.8}$$

The function $U(t)$ is called the *principal solution* of the equation (2.3.6). It should be clear that the linear stability of Γ is related to the asymptotic behaviour of the eigenvalues of $U(t)$. Unfortunately, there is no general method to determine these eigenvalues. Note, however, that $A(t)$ is periodic in t, and in this case we can say more.

Theorem 2.3.3 (Floquet). *Let $A(t) = A(t+T)$ for all t. Then the principal solution of $\dot{y} = A(t)y$ can be written as*

$$U(t) = P(t)\,\mathrm{e}^{Bt}, \tag{2.3.9}$$

where $P(t+T) = P(t)$ for all t, $P(0) = \mathbb{1}$, and B is a constant matrix.

Proof: The matrix $V(t) = U(t+T)$ satisfies the equation

$$\dot{V}(t) = \dot{U}(t+T) = A(t+T)U(t+T) = A(t)V(t),$$

which is the same as (2.3.8), except for the initial value $V(0) = U(T)$. We already saw in Proposition 2.1.10 that $\det U(t) \neq 0$ for all t. Thus the matrix $V(t)U(T)^{-1}$ exists and satisfies (2.3.8), including the initial condition. By uniqueness of the solution, it must be equal to $U(t)$:

$$V(t)U(T)^{-1} = U(t) \qquad \Rightarrow \qquad U(t+T) = U(t)U(T).$$

We claim that there exists a matrix B such that $U(T) = \mathrm{e}^{BT}$. To see this, let $\lambda_i \neq 0$ and m_i, $i = 1, \ldots, m$ be the eigenvalues of $U(T)$ and their algebraic multiplicities. Let $U(T) = \sum_{i=1}^{m}(\lambda_i P_i + N_i)$ be the decomposition of $U(T)$ into its semisimple and nilpotent parts. Then, using Lemma 2.2.2, it is easy to check that

$$B = \frac{1}{T}\sum_{i=1}^{m}\left(\log(\lambda_i)P_i - \sum_{j=1}^{m_i}\frac{(-N_i)^j}{j\lambda_i^j}\right)$$

satisfies $\mathrm{e}^{BT} = U(T)$. Here the sum over j is simply the Taylor expansion of $\log(\mathbb{1}+N_i/\lambda_i)$. B is unique up to the determination of the logarithms. We now define

$$P(t) = U(t)\,\mathrm{e}^{-Bt}\,.$$

Then we have for all t

$$P(t+T) = U(t+T)\,\mathrm{e}^{-B(t+T)} = U(t)\,\mathrm{e}^{BT}\,\mathrm{e}^{-B(t+T)} = P(t).$$

Finally, $P(0) = U(0) = \mathbb{1}$, which completes the proof. □

Floquet's theorem shows that the solution of (2.3.6) can be written as

$$y(t) = P(t)\,\mathrm{e}^{Bt}\,y(0). \tag{2.3.10}$$

Since $P(t)$ is periodic, the long-time behaviour depends only on B. The eigenvalues of B are called the *characteristic exponents* of the equation. e^{BT} is called the *monodromy matrix*, and its eigenvalues, called the *characteristic multipliers*, are exponentials of the characteristic exponents times T. Computing the characteristic exponents is difficult in general, but the existence of the representation (2.3.10) is already useful to classify the possible behaviours near a periodic orbit.

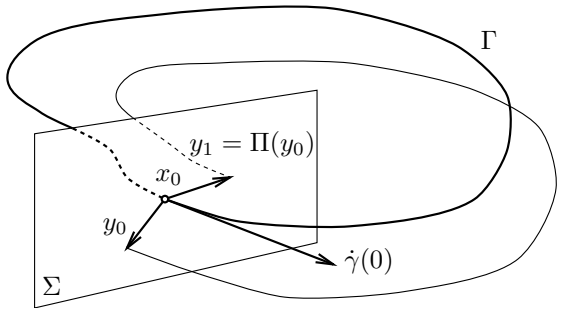

FIGURE 2.5. Definition of the Poincaré map associated with the periodic orbit $\gamma(t)$.

Exercise 2.4. Consider the Hill equation

$$\ddot{x} = -\omega(t)^2 x, \qquad \omega(t) = \begin{cases} \Omega & \text{for } nT < t < (n+\frac{1}{2})T \\ 1 & \text{for } (n+\frac{1}{2})T < t < (n+1)T \end{cases} \quad \forall n \in \mathbb{Z}\,,$$

where T, Ω are positive parameters. Write this system in the form (2.3.6), and compute the monodromy matrix and the characteristic exponents. Plot the exponents as a function of Ω for $T = \pi$. *Hint:* The eigenvalues of a 2×2 matrix can be expressed as a function of its determinant and its trace.

Once we have determined the linear stability of the periodic orbit, we could proceed in a similar way as in the case of a stationary point, in order to determine the nonlinear stability, the existence of invariant manifolds, and similar properties. However, Poincaré invented a remarkable method, which allows to shortcut all these steps by reducing the problem to a simpler one, which has already been studied. Appropriately enough, this method is called the *Poincaré section*.

Definition 2.3.4. *Let $\gamma(t)$ be a periodic solution of period T, $x_0 = \gamma(0)$, and let Σ be a hyperplane transverse to the orbit at x_0 (see Fig. 2.5). By continuity of the flow, there is a neighbourhood $\mathcal{U}$ of x_0 in Σ such that for all $x = x_0 + y \in \mathcal{U}$, we can define a continuous map $\tau(y)$, $\tau(0) = T$, such that $\varphi_t(x)$ returns for the first time to Σ in a vicinity of x_0 at $t = \tau(y)$. The* Poincaré map Π *associated with the periodic orbit is defined by*

$$x_0 + \Pi(y) := \varphi_{\tau(y)}(x_0 + y). \tag{2.3.11}$$

Proposition 2.3.5. *The Poincaré map is as smooth as the vector field in a neighbourhood of the origin. The characteristic multipliers of the periodic orbit are given by* 1 *and the $n-1$ eigenvalues of the Jacobian matrix $\frac{\partial \Pi}{\partial y}(0)$.*

PROOF: The smoothness of Π follows directly from the smoothness of the flow and the implicit function theorem. Let us now observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \frac{\partial f}{\partial x}(\gamma(t))\dot\gamma(t) = A(t)\dot\gamma(t).$$

Thus by Floquet's theorem, we can write

$$\dot\gamma(t) = P(t)\, \mathrm{e}^{Bt}\, \dot\gamma(0),$$

and, in particular,

$$\dot\gamma(0) = \dot\gamma(T) = P(T)\,\mathrm{e}^{BT}\,\dot\gamma(0) = \mathrm{e}^{BT}\,\dot\gamma(0).$$

This shows that $\dot\gamma(0)$ is an eigenvector of e^{BT} with eigenvalue 1. Let $(e_1, \dots, e_{n-1}, \dot\gamma(0))$, $e_1, \dots, e_{n-1} \in \Sigma$, be a basis of $\mathbb{R}^n$. In this basis,

$$\mathrm{e}^{BT} = \begin{pmatrix} \mathrm{e}^{B_\Sigma T} & 0 \\ \dots & 1 \end{pmatrix},$$

where B_Σ is the restriction of B to Σ, and the dots denote arbitrary entries. Now, if we consider momentarily y as a vector in $\mathbb{R}^n$ instead of Σ, linearization of (2.3.11) gives

$$\frac{\partial \Pi}{\partial y}(0) = \frac{\partial}{\partial y}\Big(\varphi_{\tau(y)}(x_0 + y)\Big)\Big|_{y=0} = \frac{\partial \varphi_T}{\partial t}(x_0)\frac{\partial \tau}{\partial y}(0) + \frac{\partial \varphi_T}{\partial x}(x_0) = \dot\gamma(0)\frac{\partial \tau}{\partial y}(0) + \mathrm{e}^{BT}.$$

The first term is a matrix with zero entries except on the last line, so that $\partial_y \Pi(0)$ has the same representation as e^{BT}, save for the entries marked by dots. In particular, when y is restricted to Σ, $\frac{\partial \Pi}{\partial y}(0)$ has the same eigenvalues as $\mathrm{e}^{B_\Sigma T}$. □

The consequence of this result is that, by studying the Poincaré map, we obtain a complete characterization of the dynamics in a neighbourhood of the periodic orbit. In particular, if $y = 0$, considered as a fixed point of the Poincaré map, admits invariant manifolds, they can be interpreted as the intersection of Σ and invariant manifolds of the periodic orbit.

Chapter 3

Local Bifurcations

Up to now, we have obtained quite a precise picture of the dynamics near hyperbolic equilibrium solutions. One might wonder whether it is of any interest to examine the case of nonhyperbolic equilibria, since a matrix chosen at random will have eigenvalues on the imaginary axis with probability zero. This argument no longer works, however, if the dynamical system depends on a parameter:

$$\dot{x} = f(x, \lambda) \qquad \text{or} \qquad x_{k+1} = F(x_k, \lambda), \tag{3.0.1}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{R}^p$). By changing λ, it is quite possible to encounter nonhyperbolic equilibria.

An important result in this connection is the implicit function theorem:

Theorem 3.0.1. *Let $\mathcal{N}$ be a neighbourhood of $(x^\star, y^\star)$ in $\mathbb{R}^n \times \mathbb{R}^m$. Let $f : \mathcal{N} \to \mathbb{R}^n$ be of class $\mathcal{C}^r$, $r \geqslant 1$, and satisfy*

$$f(x^\star, y^\star) = 0, \tag{3.0.2}$$

$$\det \frac{\partial f}{\partial x}(x^\star, y^\star) \neq 0. \tag{3.0.3}$$

Then there exists a neighbourhood $\mathcal{U}$ of $y^\star$ in $\mathbb{R}^m$ and a unique function $\varphi : \mathcal{U} \to \mathbb{R}^n$ of class $\mathcal{C}^r$ such that

$$\varphi(y^\star) = x^\star, \tag{3.0.4}$$

$$f(\varphi(y), y) = 0 \qquad \textit{for all } y \in \mathcal{U}. \tag{3.0.5}$$

This result tells us under which conditions the equation $f(x, y) = 0$ "can be solved with respect to x". Assume $x^\star$ is an equilibrium point of $f(x, \lambda_0)$ and let A be the linearization $\frac{\partial f}{\partial x}(x^\star, \lambda_0)$. Then the following situations can occur:

- If A has no eigenvalues with zero real part, then f will admit equilibrium points $x^\star(\lambda)$ for all λ in a neighbourhood of λ_0. By continuity of the eigenvalues of a matrix-valued function, $x^\star(\lambda)$ will be hyperbolic near λ_0. The curve $x^\star(\lambda)$ is usually called an *equilibrium branch* of f.
- If A has one or several eigenvalues equal to zero, then the implicit function theorem can no longer be applied, and various interesting phenomena can occur. For instance, the number of equilibrium points of f may change at $\lambda = \lambda_0$. Such a situation is called a *bifurcation*, and $(x^\star, \lambda_0)$ is called a *bifurcation point* of f.

- If A has purely imaginary, nonzero eigenvalues, the implicit function theorem can still be applied to show the existence of an equilibrium branch $x^\star(\lambda)$, but its stability may change at $\lambda = \lambda_0$. This situation is also called a bifurcation.

Let us point out that we consider here *local* bifurcations, that is, changes of the orbit structure in a small neighbourhood of equilibria. We will not discuss in any depth *global* bifurcations, which involve simultaneous changes in a larger region of phase space.

3.1 Center Manifolds

3.1.1 Existence of Center Manifolds

One of the most useful methods to study the flow near a bifurcation point is the center manifold theorem, which generalizes the stable manifold theorem (Theorem 2.2.14) to nonhyperbolic equilibrium points.

Definition 3.1.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set. Let $\mathcal{S} \subset \mathbb{R}^n$ have the structure of a differentiable manifold. For $x \in \mathcal{U}$, let $(t_1^x, t_2^x) \ni 0$ be the maximal interval such that $\varphi_t(x) \in \mathcal{U}$ for all $t \in (t_1^x, t_2^x)$. $\mathcal{S}$ is called a* local invariant manifold *if $\varphi_t(x) \in \mathcal{S}$ for all $x \in \mathcal{S} \cap \mathcal{U}$ and all $t \in (t_1^x, t_2^x)$.*

Theorem 3.1.2. *Let $x^\star$ be a singular point of f, where f is of class $\mathcal{C}^r$, $r \geqslant 2$, in a neighbourhood of $x^\star$. Let $A = \frac{\partial f}{\partial x}(x^\star)$ have, respectively, n_+, n_0 and n_- eigenvalues with positive, zero and negative real parts, where $n_0 > 0$. Then there exist, in a neighbourhood of $x^\star$, local invariant $\mathcal{C}^r$ manifolds $W^{\mathrm{u}}_{\mathrm{loc}}$, $W^{\mathrm{c}}_{\mathrm{loc}}$ and $W^{\mathrm{s}}_{\mathrm{loc}}$, of respective dimension n_+, n_0 and n_-, and such that*

- *$W^{\mathrm{u}}_{\mathrm{loc}}$ is the unique local invariant manifold tangent to E_+ at $x^\star$, and $\varphi_t(x) \to x^\star$ as $t \to -\infty$ for all $x \in W^{\mathrm{u}}_{\mathrm{loc}}$.*
- *$W^{\mathrm{s}}_{\mathrm{loc}}$ is the unique local invariant manifold tangent to E_- at $x^\star$, and $\varphi_t(x) \to x^\star$ as $t \to \infty$ for all $x \in W^{\mathrm{s}}_{\mathrm{loc}}$.*
- *$W^{\mathrm{c}}_{\mathrm{loc}}$ is tangent to E_0, but not necessarily unique.*

Before giving a (partial) proof of this result, we shall introduce a useful lemma from the theory of differential inequalities.

Lemma 3.1.3 (Gronwall's inequality). *Let φ, α and β be continuous and real-valued on $[a, b]$, with β non-negative, and assume that*

$$\varphi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\varphi(s)\,\mathrm{d}s \qquad \forall t \in [a, b]. \tag{3.1.1}$$

Then

$$\varphi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\alpha(s)\,\mathrm{e}^{\int_s^t \beta(u)\,\mathrm{d}u}\,\mathrm{d}s \qquad \forall t \in [a, b]. \tag{3.1.2}$$

Proof: Let

$$R(t) = \int_a^t \beta(s)\varphi(s)\,\mathrm{d}s.$$

Then $\varphi(t) \leqslant \alpha(t) + R(t)$ for all $t \in [a, b]$ and, since $\beta(t) \geqslant 0$,

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = \beta(t)\varphi(t) \leqslant \beta(t)\alpha(t) + \beta(t)R(t).$$

Let $K(s) = \int_a^s \beta(u)\,\mathrm{d}u$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-K(s)}\,R(s) = \big[R'(s) - \beta(s)R(s)\big]\,\mathrm{e}^{-K(s)} \leqslant \beta(s)\alpha(s)\,\mathrm{e}^{-K(s)}$$

and thus, integrating from a to t,

$$\mathrm{e}^{-K(t)}\,R(t) \leqslant \int_a^t \beta(s)\alpha(s)\,\mathrm{e}^{-K(s)}\,\mathrm{d}s.$$

We obtain the conclusion by multiplying this expression by $\mathrm{e}^{K(t)}$ and inserting the result into (3.1.1). □

There exist various generalizations of this result, for instance to functions $\beta(t)$ that are only integrable.

Let us now proceed to the proof of the center manifold theorem. There exist more or less sophisticated proofs. We will give here a rather straightforward one taken from [Ca81]. For simplicity, we consider the case $n_+ = 0$, and we will only prove the existence of a Lipschitz continuous center manifold.

Proof of Theorem 3.1.2. We write the system near the equilibrium point as

$$\begin{aligned}\dot y &= By + g_-(y,z)\\ \dot z &= Cz + g_0(y,z),\end{aligned}$$

where all eigenvalues of B have strictly negative real parts, all eigenvalues of C have zero real parts, and $\|g_-(y,z)\|, \|g_0(y,z)\| \leqslant M(\|y\|^2 + \|z\|^2)$ in a neighbourhood of the origin. We shall prove the existence of a local center manifold for a modified equation, that agrees with the present equation in a small neighbourhood of the equilibrium. Let $\psi : \mathbb{R}^{n_0} \to [0,1]$ be a $\mathcal{C}^\infty$ function such that $\psi(x) = 1$ when $\|x\| \leqslant 1$ and $\psi(x) = 0$ when $\|x\| \geqslant 2$. We introduce the functions

$$G_-(y,z) = g_-\big(y, z\psi(\frac{z}{\varepsilon})\big), \qquad G_0(y,z) = g_0\big(y, z\psi(\frac{z}{\varepsilon})\big),$$

Then the system

$$\begin{aligned}\dot y &= By + G_-(y,z)\\ \dot z &= Cz + G_0(y,z),\end{aligned}$$

agrees with the original system for $\|z\| \leqslant \varepsilon$. We look for a center manifold with equation $y = h(z)$, where h is in a well-chosen function space, in which we want to apply Banach's fixed point theorem. Let $\rho > 0$ and $\kappa > 0$ be constants, and let $\mathcal{X}$ be the set of Lipschitz continuous functions $h : \mathbb{R}^{n_0} \to \mathbb{R}^{n_-}$ with Lipschitz constant κ, $\|h(z)\| \leqslant \rho$ for all $z \in \mathbb{R}^{n_0}$ and $h(0) = 0$. $\mathcal{X}$ is a complete space with the supremum norm $|\cdot|$.

For $h \in \mathcal{X}$, we denote by $\varphi_t(\cdot, h)$ the flow of the differential equation

$$\dot z = Cz + G_0(h(z), z).$$

For any solution $(y(s), z(s))_{t_0 \leqslant s \leqslant t}$, the relation

$$y(t) = \mathrm{e}^{B(t-t_0)}\,y(t_0) + \int_{t_0}^t \mathrm{e}^{B(t-s)}\,G_-(y(s), z(s))\,\mathrm{d}s$$

is satisfied, as is easily checked by differentiation. Among all possible solutions, we want to select a class of solutions that are tangent to the center subspace $y = 0$ at the origin. In the linear case, the above solutions satisfy this property only if $\mathrm{e}^{-Bt_0}\, y(t_0) = 0$. We will thus consider the class of particular solutions satisfying

$$y(t) = \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}\, G_-(y(s), z(s))\, \mathrm{d}s$$

If we set $t = 0$ and require that $y(s) = h(z(s))$ for all s, we arrive at the equality

$$h(z(0)) = \int_{-\infty}^{0} \mathrm{e}^{-Bs}\, G_-\big(h(z(s)), z(s)\big)\, \mathrm{d}s,$$

where $z(s) = \varphi_s(z(0), h)$. Thus we conclude that if h is a fixed point of the operator $T : \mathcal{X} \to \mathcal{X}$, defined by

$$(Th)(z) = \int_{-\infty}^{0} \mathrm{e}^{-Bs}\, G_-\big(h(\varphi_s(z,h)), \varphi_s(z,h)\big)\, \mathrm{d}s,$$

then h is a center manifold of the equation. Note that there may be center manifolds that do not satisfy this equation, and thus we will not be proving uniqueness.

Now we want to show that T is a contraction on $\mathcal{X}$ for an appropriate choice of ε, κ and ρ. Observe first that since all eigenvalues of B have a strictly negative real part, Lemma 2.2.2 implies the existence of positive constants β and K such that

$$\|\mathrm{e}^{-Bs}\, y\| \leqslant K\, \mathrm{e}^{\beta s} \|y\| \qquad \forall y \in \mathbb{R}^{n_-},\ \forall s \leqslant 0.$$

Since the eigenvalues of C have zero real parts, the same lemma implies that $\|\mathrm{e}^{Cs}\, z\|$ is a polynomial in s. Hence, for every $\nu > 0$, there exists a constant $Q(\nu)$ such that

$$\|\mathrm{e}^{Cs}\, z\| \leqslant Q(\nu)\, \mathrm{e}^{\nu|s|} \|z\| \qquad \forall z \in \mathbb{R}^{n_0},\ \forall s \in \mathbb{R}\,.$$

It is possible that $Q(\nu) \to \infty$ as $\nu \to 0$.

We first need to show that $T\mathcal{X} \subset \mathcal{X}$. Assume from now on that $\rho \leqslant \varepsilon$, so that $\|h(z)\| \leqslant \varepsilon$. The definition of G_- implies that

$$\|G_-(h(z), z)\| \leqslant M'\varepsilon^2$$

for a constant $M' > 0$ depending on M. Moreover, the derivatives of G_0 are of order ε, so that there is a constant $M'' > 0$ depending on M with

$$\begin{aligned} \|G_0(h(z_1), z_1) - G_0(h(z_2), z_2)\| &\leqslant M''\varepsilon\big[\|h(z_1) - h(z_2)\| + \|z_1 - z_2\|\big] \\ &\leqslant M''\varepsilon(1+\kappa)\|z_1 - z_2\|. \end{aligned}$$

A similar relation holds for G_-. The bound on $\|G_-\|$ implies that

$$\|Th(z)\| \leqslant \int_{-\infty}^{0} K\, \mathrm{e}^{\beta s}\, M'\varepsilon^2\, \mathrm{d}s \leqslant \varepsilon^2 \frac{KM'}{\beta},$$

and hence $\|Th(z)\| \leqslant \rho$ provided $\varepsilon \leqslant (\beta/KM')(\rho/\varepsilon)$. Next we want to estimate the Lipschitz constant of T. Let $z_1, z_2 \in \mathbb{R}^{n_0}$. By definition of the flow $\varphi_t(\cdot, h)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t(z_1,h) - \varphi_t(z_2,h)\big) &= C\big(\varphi_t(z_1,h) - \varphi_t(z_2,h)\big) \\ &\quad + G_0\big(h(\varphi_t(z_1,h)), \varphi_t(z_1,h)\big) - G_0\big(h(\varphi_t(z_2,h)), \varphi_t(z_2,h)\big). \end{aligned}$$

Taking into account the fact that $\varphi_0(z,h) = z$, we get

$$\begin{aligned}\varphi_t(z_1,h) - \varphi_t(z_2,h) = {} & \mathrm{e}^{Ct}\big(z_1 - z_2\big) \\ & + \int_0^t \mathrm{e}^{C(t-s)}\Big[G_0\big(h(\varphi_s(z_1,h)),\varphi_s(z_1,h)\big) - G_0\big(h(\varphi_s(z_2,h)),\varphi_s(z_2,h)\big)\Big]\,\mathrm{d}s.\end{aligned}$$

For $t \leqslant 0$, we obtain with the properties of G_0

$$\begin{aligned}\|\varphi_t(z_1,h) - \varphi_t(z_2,h)\| \leqslant {} & Q(\nu)\,\mathrm{e}^{-\nu t}\|z_1 - z_2\| \\ & + \int_t^0 Q(\nu)\,\mathrm{e}^{-\nu(t-s)}\,M''\varepsilon(1+\kappa)\|\varphi_s(z_1,h) - \varphi_s(z_2,h)\|\,\mathrm{d}s.\end{aligned}$$

We can now apply Gronwall's inequality to $\psi(t) = \mathrm{e}^{-\nu t}\|\varphi_{-t}(z_1,h) - \varphi_{-t}(z_2,h)\|$, with the result

$$\|\varphi_t(z_1,h) - \varphi_t(z_2,h)\| \leqslant Q(\nu)\,\mathrm{e}^{-\gamma t}\|z_1 - z_2\|,$$

where $\gamma = \nu + Q(\nu)M''\varepsilon(1+\kappa)$. We can arrange that $\gamma < \beta$, taking for instance $\nu = \beta/2$ and ε small enough. We thus obtain

$$\begin{aligned}\|Th(z_1) - Th(z_2)\| &\leqslant \int_{-\infty}^0 K\,\mathrm{e}^{\beta s}\,M''\varepsilon(1+\kappa)\|\varphi_s(z_1,h) - \varphi_s(z_2,h)\|\,\mathrm{d}s \\ &\leqslant \frac{KM''\varepsilon(1+\kappa)Q(\nu)}{\beta-\gamma}\|z_1 - z_2\|.\end{aligned}$$

For any given κ and ν, we can find ε small enough that $\|Th(z_1) - Th(z_2)\| \leqslant \kappa\|z_1 - z_2\|$. This completes the proof that $T\mathcal{X} \subset \mathcal{X}$.

Finally, we want to show that T is a contraction. For $h_1, h_2 \in \mathcal{X}$, using

$$\begin{aligned}\|G_0\big(h_1(\varphi_s(z,h_1)),\varphi_s(z,h_1)\big) - G_0\big(h_2(\varphi_s(z,h_2)),\varphi_s(z,h_2)\big)\| & \\ \leqslant M''\varepsilon\big[(1+\kappa)\|\varphi_s(z,h_1) - \varphi_s(z,h_2)\| + |h_2 - h_1|\big],&\end{aligned}$$

we obtain in a similar way by Gronwall's inequality that

$$\|\varphi_t(z,h_1) - \varphi_t(z,h_2)\| \leqslant 2\frac{Q(\nu)}{\nu}\varepsilon M''|h_2 - h_1|$$

(this is a rough estimate, where we have thrown away some t-dependent terms). This leads to the bound

$$\|Th_1(z) - Th_2(z)\| \leqslant \frac{K}{\beta}M''\varepsilon\Big[1 + 2M''\varepsilon(1+\kappa)\frac{Q(\nu)}{\nu}\Big]|h_2 - h_1|.$$

Again, taking ε small enough, we can achieve that $\|Th_1(z) - Th_2(z)\| \leqslant \lambda|h_2 - h_1|$ for some $\lambda < 1$ and for all $z \in \mathbb{R}^{n_0}$. This shows that T is a contraction, and we have proved the existence of a Lipschitz continuous center manifold by Banach's fixed point theorem. One can proceed in a similar way to show that T is a contraction in a space of Lipschitz differentiable functions. □

One can also prove that if f is of class $\mathcal{C}^r$, $r \geqslant 2$, then h is also of class $\mathcal{C}^r$. If f is analytic or $\mathcal{C}^\infty$, however, then h will be $\mathcal{C}^r$ for all $r \geqslant 1$, but in general it will not be analytic, and not even $\mathcal{C}^\infty$. In fact, the size of the domain in which h is $\mathcal{C}^r$ may become smaller and smaller as r goes to infinity, see [Ca81] for examples.

As pointed out in the proof of the theorem, the center manifold is not necessarily unique. It is easy to give examples of systems admitting a continuous family of center manifolds. However, as we shall see, these manifolds have to approach each other extremely fast near the equilibrium point, and thus the dynamics will be qualitatively the same on all center manifolds.

Example 3.1.4. The system

$$\begin{aligned} \dot{y} &= -y \\ \dot{z} &= -z^3 \end{aligned} \tag{3.1.3}$$

admits a two-parameter family of center manifolds

$$y = h(z, c_1, c_2) = \begin{cases} c_1 \, \mathrm{e}^{-1/2z^2} & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ c_2 \, \mathrm{e}^{-1/2z^2} & \text{for } z < 0. \end{cases} \tag{3.1.4}$$

The operator T in the proof of Theorem 3.1.2 admits a unique fixed point $h(z) \equiv 0$, but there exist other center manifolds which are not fixed points of T. Note, however, that all functions $h(z, c_1, c_2)$ have identically zero Taylor expansions at $z = 0$.

3.1.2 Properties of Center Manifolds

We assume in this section that $x^\star$ is a non-hyperbolic equilibrium point of f, such that $A = \frac{\partial f}{\partial x}(x^\star)$ has $n_0 \geqslant 1$ eigenvalues with zero real parts and $n_- \geqslant 1$ eigenvalues with negative real parts. In appropriate coordinates, we can write

$$\begin{aligned} \dot{y} &= By + g_-(y, z) \\ \dot{z} &= Cz + g_0(y, z), \end{aligned} \tag{3.1.5}$$

where all eigenvalues of B have strictly negative real parts, all eigenvalues of C have zero real parts, and $g_-(y, z)$, $g_0(y, z)$ are nonlinear terms. Theorem 3.1.2 shows the existence of a local center manifold with parametric equation $y = h(z)$. The dynamics on this locally invariant manifold is governed by the equation

$$\dot{u} = Cu + g_0(h(u), u). \tag{3.1.6}$$

Equation (3.1.6) has the advantage to be of lower dimension than (3.1.5), and thus easier to analyse. The following result shows that (3.1.6) is a good approximation of (3.1.5).

Theorem 3.1.5. *If the origin of* (3.1.6) *is stable (asymptotically stable, unstable), then the origin of* (3.1.5) *is stable (asymptotically stable, unstable).*

Assume that the origin of (3.1.6) *is stable. For any solution* $(y(t), z(t))$ *of* (3.1.5) *with* $(y(0), z(0))$ *sufficiently small, there exist a solution* $u(t)$ *of* (3.1.6) *and a constant* $\gamma > 0$ *such that*

$$\begin{aligned} y(t) &= h(u(t)) + \mathcal{O}(\mathrm{e}^{-\gamma t}) \\ z(t) &= u(t) + \mathcal{O}(\mathrm{e}^{-\gamma t}) \end{aligned} \tag{3.1.7}$$

as $t \to \infty$.

For a proof, see [Ca81]. The center manifold is said to be *locally attractive*. Note that if the center manifold is not unique, then (3.1.7) holds for any center manifold.

Let us now discuss how to compute center manifolds. In the proof of Theorem 3.1.2, we used the fact that h is a fixed point of a functional operator T. While being useful to prove existence of a center manifold, T is not very helpful for the computation of h, but there exists another operator for this purpose. Replacing y by $h(z)$ in (3.1.5), we obtain

$$\frac{\partial h}{\partial z}(z)\big[Cz + g_0(h(z), z)\big] = Bh(z) + g_-(h(z), z). \tag{3.1.8}$$

For functions $\phi : \mathbb{R}^{n_0} \to \mathbb{R}^{n_-}$ which are continuously differentiable in a neighbourhood of the origin, let us define

$$(L\phi)(z) = \frac{\partial \phi}{\partial z}(z)\big[Cz + g_0(\phi(z), z)\big] - B\phi(z) - g_-(\phi(z), z). \tag{3.1.9}$$

Then (3.1.8) implies that $(Lh)(z) = 0$ for any center manifold of (3.1.5). This equation is impossible to solve in general. However, its solutions can be computed perturbatively, and the approximation procedure is justified by the following result, which is also proved in [Ca81].

Theorem 3.1.6. *Let $\mathcal{U}$ be a neighbourhood of the origin in $\mathbb{R}^{n_0}$ and let $\phi \in \mathcal{C}^1(\mathcal{U}, \mathbb{R}^{n_-})$ satisfy $\phi(0) = 0$ and $\frac{\partial \phi}{\partial z}(0) = 0$. If there is a $q > 1$ such that $(L\phi)(z) = \mathcal{O}(\|z\|^q)$ as $z \to 0$, then $\|h(z) - \phi(z)\| = \mathcal{O}(\|z\|^q)$ as $z \to 0$ for any center manifold h.*

An important consequence of this result is that if h_1 and h_2 are two different center manifolds of $x^\star$, then one must have $h_1(z) - h_2(z) = \mathcal{O}(\|z\|^q)$ for all $q > 1$, i.e., all center manifolds have the same Taylor expansion at $z = 0$.

Example 3.1.7. Consider the two-dimensional system

$$\begin{aligned} \dot{y} &= -y + cz^2 \\ \dot{z} &= yz - z^3, \end{aligned} \tag{3.1.10}$$

where c is a real parameter. We want to determine the stability of the origin. Naively, one might think that since the first equation suggests that $y(t)$ converges to zero as $t \to \infty$, the dynamics can be approximated by projecting on the line $y = 0$. This would lead to the conclusion that the origin is asymptotically stable, because $\dot{z} = -z^3$ when $y = 0$. We will now compute the center manifold in order to find the correct answer to the question of stability. The operator (3.1.9) has the form

$$(L\phi)(z) = \phi'(z)\big[z\phi(z) - z^3\big] + \phi(z) - cz^2. \tag{3.1.11}$$

Theorem 3.1.6 allows us to solve the equation $(Lh)(z) = 0$ perturbatively, by an Ansatz of the form

$$h(z) = h_2 z^2 + h_3 z^3 + h_4 z^4 + \mathcal{O}(z^5). \tag{3.1.12}$$

The equation $(Lh)(z) = 0$ becomes

$$(Lh)(z) = (h_2 - c)z^2 + h_3 z^3 + \big[h_4 + 2h_2(h_2 - 1)\big]z^4 + \mathcal{O}(z^5) = 0, \tag{3.1.13}$$

which requires $h_2 = c$, $h_3 = 0$ and $h_4 = -2c(c-1)$. Hence the center manifold has a Taylor expansion of the form

$$h(z) = cz^2 - 2c(c-1)z^4 + \mathcal{O}(z^5), \tag{3.1.14}$$

and the motion on the center manifold is governed by the equation

$$\dot{u} = u\,h(u) - u^3 = (c-1)u^3 - 2c(c-1)u^5 + \mathcal{O}(u^6). \tag{3.1.15}$$

It is easy to show (using, for instance, u^2 as a Liapunov function) that the equilibrium point $u = 0$ is asymptotically stable if $c < 1$ and unstable if $c > 1$. By Theorem 3.1.5, we conclude that the origin of the system (3.1.10) is asymptotically stable if $c < 1$ and unstable if $c > 1$, which contradicts the naive approach when $c > 1$.

The case $c = 1$ is special. In this case, the function $h(z) = z^2$ is an exact solution of the equation $(Lh)(z) = 0$, and the curve $y = z^2$ is the unique center manifold of (3.1.10), which has the particularity to consist only of equilibrium points. The origin is thus stable.

3.2 Bifurcations of Differential Equations

We consider in this section parameter-dependent ordinary differential equations of the form

$$\dot{x} = f(x, \lambda), \tag{3.2.1}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$, $\lambda \in \Lambda \subset \mathbb{R}^p$ and f is of class $\mathcal{C}^r$ for some $r \geqslant 2$. We assume that $(x^\star, 0)$ is a bifurcation point of (3.2.1), which means that

$$\begin{aligned} f(x^\star, 0) &= 0 \\ \frac{\partial f}{\partial x}(x^\star, 0) &= A, \end{aligned} \tag{3.2.2}$$

where the matrix A has $n_0 \geqslant 1$ eigenvalues on the imaginary axis. When $\lambda = 0$, the equilibrium point $x^\star$ admits a center manifold. We would like, however, to examine the dynamics of (3.2.1) for all λ in a neighbourhood of 0, where the center manifold theorem cannot be applied directly. There is, however, an elegant trick to solve this problem. In the enlarged phase space $\mathcal{D} \times \Lambda$, consider the system

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{3.2.3}$$

It admits $(x^\star, 0)$ as a non-hyperbolic equilibrium point. The linearization of (3.2.3) around this point is a matrix of the form

$$\begin{pmatrix} A & \frac{\partial f}{\partial \lambda}(x^\star, 0) \\ 0 & 0 \end{pmatrix}, \tag{3.2.4}$$

which has $n_0 + p$ eigenvalues on the imaginary axis, including the (possibly multiple) eigenvalue zero. This matrix can be made block-diagonal by a linear change of variables, where one of the blocks contains all eigenvalues with zero real part. In these variables, the system (3.2.3) becomes

$$\begin{aligned} \dot{y} &= By + g_-(y, z, \lambda) \\ \dot{z} &= Cz + D\lambda + g_0(y, z, \lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{3.2.5}$$

Here the $(n-n_0)\times(n-n_0)$ matrix B has only eigenvalues with nonzero real parts, the $n_0\times n_0$ matrix C has all eigenvalues on the imaginary axis, D is a matrix of size $n_0\times p$, and

$$\|g_-(y,z,\lambda)\|,\ \|g_0(y,z,\lambda)\|\leqslant M\bigl(\|y\|^2+\|z\|^2+\|\lambda\|^2\bigr) \tag{3.2.6}$$

in a neighbourhood of the origin, for some positive constant M. We can thus apply Theorem 3.1.2, which shows the existence of a local invariant center manifold of the form $y=h(z,\lambda)$. The dynamics on this manifold is governed by the n_0-dimensional equation

$$\dot u = Cu + D\lambda + g_0(h(u,\lambda),u,\lambda). \tag{3.2.7}$$

If A has no eigenvalues with positive real part, Theorem 3.1.5 shows that this equation gives a good approximation to the dynamics of (3.2.1) for small λ and near $x=x^\star$. The big advantage is that generically, the number of eigenvalues on the imaginary axis at the bifurcation point is small, and thus the reduced equation (3.2.7) is of low dimension.

With these preliminaries, it becomes possible to investigate bifurcations in a systematic way. Recall that A is a real matrix, and thus its eigenvalues are either real or appear in complex conjugate pairs. Thus the two simplest bifurcations, which we will consider below, involve either a single zero eigenvalue, or a pair of conjugate imaginary eigenvalues. More complicated bifurcations correspond to a double zero eigenvalue, a zero eigenvalue and two conjugate imaginary ones, and so on. These cases are, however, less "generic", and we will not discuss them here.

Exercise 3.1. Show that the Lorenz equations (1.3.9) admit $(X,Y,Z)=(0,0,0)$, $r=1$ as a bifurcation point. Compute an approximation to second order of the center manifold in the extended phase space, and deduce the equation governing the dynamics on the center manifold. *Hint:* let $R^2=(\sigma-1)^2+4r\sigma$. To put the system into the form (3.2.5), use the transformation $X=\sigma(z-y_2)$, $Y=\frac12(1-\sigma)(y_2-z)+\frac12R(y_2+z)$, $Z=y_1$, and $r=1+\lambda$.

3.2.1 One-Dimensional Center Manifold

We first discuss bifurcations involving a single eigenvalue equal to zero, i.e. $n_0=1$ and $C=0$. The dynamics on the one-dimensional center manifold is governed by an equation of the form

$$\dot u = F(u,\lambda),\qquad u\in\mathbb{R}\,. \tag{3.2.8}$$

We will also assume that $\lambda\in\mathbb{R}$. The origin $(0,0)$ is a bifurcation point of (3.2.8), meaning

$$F(0,0)=0,\qquad \frac{\partial F}{\partial u}(0,0)=0. \tag{3.2.9}$$

In order to understand the dynamics for small u and λ, we need in particular to determine the singular points of F, that is, we have to solve the equation $F(u,\lambda)=0$ in a neighbourhood of the origin. Note that the second condition in (3.2.9) implies that we cannot apply the implicit function theorem.

Let us start by expanding F in Taylor series,

$$F(u,\lambda)=\sum_{\substack{p+q\leqslant r\\ p,q\geqslant0}} c_{pq}u^p\lambda^q+\sum_{\substack{p+q=r\\ p,q\geqslant0}} u^p\lambda^qR_{pq}(u,\lambda), \tag{3.2.10}$$

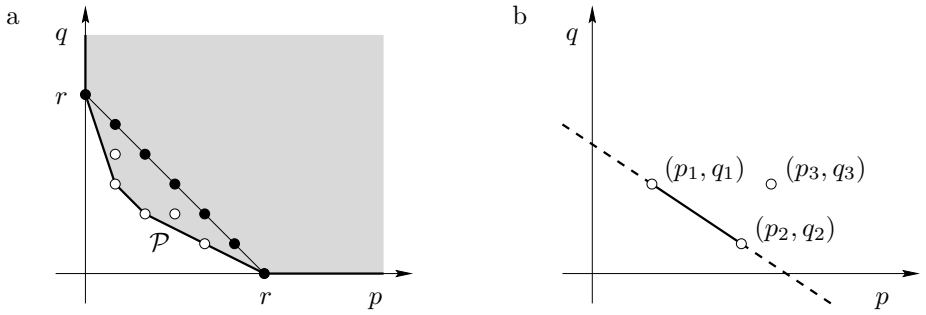

FIGURE 3.1. (a) Definition of the Newton polygon. White circles correspond to points (p, q) such that $c_{pq} \neq 0$, black circles to points (p, q) such that $p + q = r$. The slopes of full lines correspond to possible exponents of equilibrium branches. (b) The setting of the proof of Proposition 3.2.2.

where the functions R_{pq} are continuous near the origin, $R_{pq}(0, 0) = 0$ and

$$c_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q} F}{\partial u^p \partial \lambda^q}(0, 0). \tag{3.2.11}$$

The bifurcation conditions (3.2.9) amount to $c_{00} = c_{10} = 0$. An elegant way to describe the solutions of $F(u, \lambda) = 0$ is based on *Newton's polygon*.

Definition 3.2.1. *Consider the set*

$$\mathcal{A} = \big\{(p, q) \in \mathbb{N}^2 \colon p + q \leqslant r \text{ and } c_{pq} \neq 0\big\}. \tag{3.2.12}$$

(In case F is analytic, we simply drop the condition $p + q \leqslant r$). For each $(p, q) \in \mathcal{A}$, we construct the sector $\{(x, y) \in \mathbb{R}^2 \colon x \geqslant p \text{ and } y \geqslant q\}$. The Newton polygon $\mathcal{P}$ *of* (3.2.10) *is the broken line in $\mathbb{R}^2$ defined by the convex envelope of the union of all these sectors, see Fig. 3.1a.*

Proposition 3.2.2. *Assume for simplicity*[1] *that $c_{pq} \neq 0$ whenever $p + q = r$. Assume further that the equation $F(u, \lambda) = 0$ admits a solution of the form $u = C|\lambda|^\mu(1 + \rho(\lambda))$ for small λ, where $C \neq 0$ and $\rho(\lambda) \to 0$ continuously as $\lambda \to 0$. Then Newton's polygon must have a segment of slope $-\mu$.*

PROOF: It is sufficient to consider a function F of the form

$$F(u, \lambda) = \sum_{i=1}^{3} c_i u^{p_i} \lambda^{q_i}.$$

Indeed, if the expansion (3.2.10) contains only two terms, the result is immediate, and if it has more than three terms, one can proceed by induction. The hypothesis implies

$$\sum_{i=1}^{3} \sigma_i c_i C^{p_i} |\lambda|^{\mu p_i + q_i} (1 + \rho(\lambda))^{p_i} = 0,$$

[1]This assumption allows to neglect all remainders $R_{pq}(u, \lambda)$, and to avoid pathological situations such as $F(u, \lambda) = u^2 - \lambda^{5/2}$.

where $\sigma_i = \pm 1$. Assume for definiteness that $p_1\mu + q_1 \leqslant p_2\mu + q_2 \leqslant p_3\mu + q_3$. Consider first the case $p_1\mu + q_1 < p_2\mu + q_2$. Then division be $|\lambda|^{\mu p_1 + q_1}$ gives

$$\sigma_1 c_1 C^{p_1}(1+\rho(\lambda))^{p_1} + \sum_{i=2}^{3} \sigma_i c_i C^{p_i} |\lambda|^{\mu p_i + q_i - \mu p_1 - q_1}(1+\rho(\lambda))^{p_i} = 0.$$

The exponent of $|\lambda|$ is strictly positive. Thus taking the limit $\lambda \to 0$, we obtain $C^{p_1} = 0$, a contradiction. We conclude that we must have $p_1\mu + q_1 = p_2\mu + q_2$. Graphically, the relation $p_1\mu + q_1 = p_2\mu + q_2 \leqslant p_3\mu + q_3$ means that $\mu = \frac{q_2 - q_1}{p_1 - p_2}$ is minus the slope of the segment from (p_1, q_1) to (p_2, q_2), and that (p_3, q_3) lies above, see Fig. 3.1b. □

This result does not prove the existence of equilibrium branches $u = u^\star(\lambda)$, but it tells us where to look. By drawing Newton's polygon, we obtain the possible values of μ. By inserting the Ansatz $u^\star(\lambda) = C|\lambda|^\mu(1+\rho)$ into the equation $F(u,\lambda) = 0$, we can determine whether of not such a branch exists. This is mainly a matter of signs of the coefficients c_{pq}. For instance, the equation $u^2 + \lambda^2 = 0$ admits no solution other than $(0,0)$, while the equation $u^2 - \lambda^2 = 0$ does admits solutions $u = \pm\lambda$.

Saddle-Node Bifurcation

We now illustrate the procedure of computing equilibrium branches in the generic case $r = 2$, $c_{20} \neq 0$, $c_{01} \neq 0$. Then

$$F(u,\lambda) = c_{01}\lambda + c_{20}u^2 + c_{11}u\lambda + c_{02}\lambda^2 + \sum_{p+q=2} u^p \lambda^q R_{pq}(u,\lambda), \tag{3.2.13}$$

and Newton's polygon has two vertices $(0,1)$ and $(2,0)$, connected by a segment with slope $-1/2$. Proposition 3.2.2 tells us that if there is an equilibrium branch, then it must be of the form $u = C|\lambda|^{1/2}(1+\rho(\lambda))$, where $\lim_{\lambda\to 0}\rho(\lambda) = 0$. In fact, it turns out to be easier to express λ as a function of u:

Lemma 3.2.3. *In a neighbourhood of the origin, there exists a continuous function $\bar\rho(u)$ with $\bar\rho(0) = 0$ such that $F(u,\lambda) = 0$ if and only if*

$$\lambda = -\frac{c_{20}}{c_{01}}u^2(1+\bar\rho(u)). \tag{3.2.14}$$

Proof: Proposition 3.2.2 indicates that any equilibrium branch must be of the form $\lambda = \overline{C}u^2(1+\bar\rho(u)))$. Define the function

$$G(\bar\rho, u) = \frac{1}{u^2}F(u, \overline{C}u^2(1+\bar\rho)).$$

Using the expansion (3.2.13) of F, it is easy to see that

$$\lim_{u\to 0} G(0,u) = c_{01}\overline{C} + c_{20},$$

and thus $G(0,0) = 0$ if and only if $\overline{C} = -c_{20}/c_{01}$. Moreover, using a Taylor expansion to first order of $\frac{\partial F}{\partial\lambda}$, one obtains

$$\lim_{u\to 0}\frac{\partial G}{\partial\bar\rho}(0,u) = c_{01}\overline{C} \neq 0.$$

Thus, by the implicit function theorem, there exists, for small u, a unique function $\bar\rho(u)$ such that $\bar\rho(0) = 0$ and $G(\bar\rho(u), u) = 0$. □

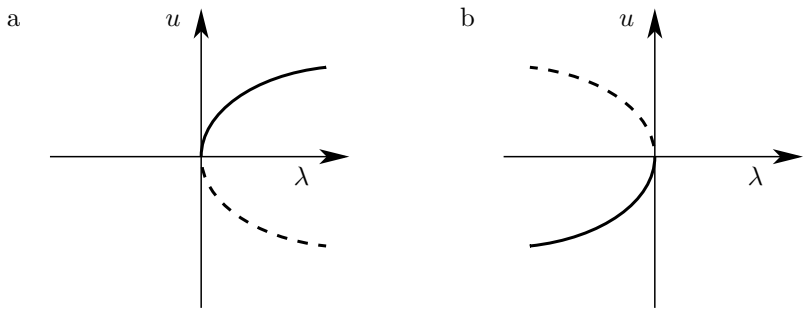

FIGURE 3.2. Saddle-node bifurcation, (a) in the case $c_{01} > 0$, $c_{20} < 0$ (direct bifurcation), and (b) in the case $c_{01} > 0$, $c_{20} > 0$ (indirect bifurcation). The other cases are similar, with stable and unstable branches interchanged. Full curves indicate stable equilibrium branches, while dashed curves indicate unstable equilibrium branches.

Expressing u as a function of λ, we find the existence of two equilibrium branches

$$u = u_{\pm}^{\star}(\lambda) = \pm\sqrt{-\frac{c_{01}}{c_{20}}\lambda}\,\big[1 + \rho(\lambda)\big], \tag{3.2.15}$$

which exist only for $\operatorname{sign}\lambda = -\operatorname{sign}(c_{01}/c_{20})$. Their stability can be determined by using the Taylor expansion

$$\frac{\partial F}{\partial u}(u,\lambda) = 2c_{20}u + c_{11}\lambda + \sum_{p+q=1} u^p\lambda^q\overline{R}_{pq}(u,\lambda), \tag{3.2.16}$$

where $\overline{R}_{pq}$ are some continuous functions vanishing at the origin. Inserting (3.2.15), we get

$$\frac{\partial F}{\partial u}(u_{\pm}^{\star}(\lambda),\lambda) = \pm 2c_{20}\sqrt{-\frac{c_{01}}{c_{20}}\lambda} + \mathcal{O}\big(\sqrt{|\lambda|}\big). \tag{3.2.17}$$

We thus obtain the following cases, depending on the signs of the coefficients:

1. If $c_{01} > 0$ and $c_{20} < 0$, the branches exist for $\lambda > 0$, $u_+^\star$ is stable and $u_-^\star$ is unstable;
2. if $c_{01} < 0$ and $c_{20} > 0$, the branches exist for $\lambda > 0$, $u_+^\star$ is unstable and $u_-^\star$ is stable;
3. if $c_{01} < 0$ and $c_{20} < 0$, the branches exist for $\lambda < 0$, $u_+^\star$ is stable and $u_-^\star$ is unstable;
4. if $c_{01} > 0$ and $c_{20} > 0$, the branches exist for $\lambda < 0$, $u_+^\star$ is unstable and $u_-^\star$ is stable.

These bifurcations are called *saddle-node bifurcations*, because when considering the full system instead of its restriction to the center manifold, they involve a saddle and a node. If the branches exist for $\lambda > 0$, the bifurcation is called *direct*, and if they exist for $\lambda < 0$, it is called *indirect* (Fig. 3.2).

It is important to observe that the qualitative behaviour depends only on those coefficients in the Taylor series which correspond to vertices of Newton's polygon. Thus we could have thrown away all other terms, to consider only the truncated equation, or *normal form*,

$$\dot{u} = c_{20}u^2 + c_{01}\lambda. \tag{3.2.18}$$

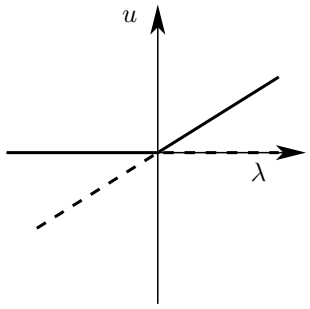

FIGURE 3.3. Transcritical bifurcation, in the case of equation (3.2.21) with $c_{20} < 0$ and $c_{11} > 0$.

Transcritical Bifurcation

Let us consider next the slightly less generic case where $c_{01} = 0$, but $c_{20} \neq 0$, $c_{11} \neq 0$, $c_{02} \neq 0$. Then the Taylor expansion of F takes the form

$$F(u,\lambda) = c_{20}u^2 + c_{11}u\lambda + c_{02}\lambda^2 + \sum_{p+q=2} u^p\lambda^q R_{pq}(u,\lambda), \tag{3.2.19}$$

and Newton's polygon has three vertices $(0,2)$, $(1,1)$ and $(2,0)$, connected by segments of slope -1. Proposition 3.2.2 tells us to look for equilibrium branches of the form $u = C\lambda(1+\rho(\lambda))$. Proceeding in a similar way as in Lemma 3.2.3, we obtain the conditions

$$\begin{aligned} c_{20}C^2 + c_{11}C + c_{02} &= 0 \\ 2c_{20}C^2 + c_{11}C &\neq 0 \end{aligned} \tag{3.2.20}$$

for the existence of a unique equilibrium branch of this form. We thus conclude that if $c_{11}^2 - 4c_{20}c_{02} > 0$, there are two intersecting equilibrium branches. It is easy to see that the linearization of f around such a branch is $(2c_{20}C + c_{11})\lambda + o(\lambda)$, and thus one of the branches is stable, the other is unstable, and they exchange stability at the bifurcation point. This bifurcation is called *transcritical*.

If $c_{11}^2 - 4c_{20}c_{02} < 0$, there are no equilibrium branches near the origin. Finally, if $c_{11}^2 - 4c_{20}c_{02} = 0$ there may be several branches with the same slope through the origin.

Let us point out that the condition $c_{02} \neq 0$ is not essential. In fact, one often carries out a change of variables taking one of the equilibrium branches to the λ-axis. The resulting normal form is

$$\dot{u} = c_{20}u^2 + c_{11}u\lambda. \tag{3.2.21}$$

One of the equilibrium branches is $u \equiv 0$. The slope of the other branch and the stability depend on the signs of c_{20} and c_{11} (Fig. 3.3).

Pitchfork Bifurcation

One can go on like that for ever, considering cases with more coefficients in the Taylor series equal to zero, which is not especially interesting unless one has to do with a concrete problem. However, sometimes symmetries of the differential equation may cause many terms in the Taylor expansion to vanish. Consider the case $F \in \mathcal{C}^3$ satisfying

$$F(-u,\lambda) = -F(u,\lambda) \qquad \forall (u,\lambda). \tag{3.2.22}$$

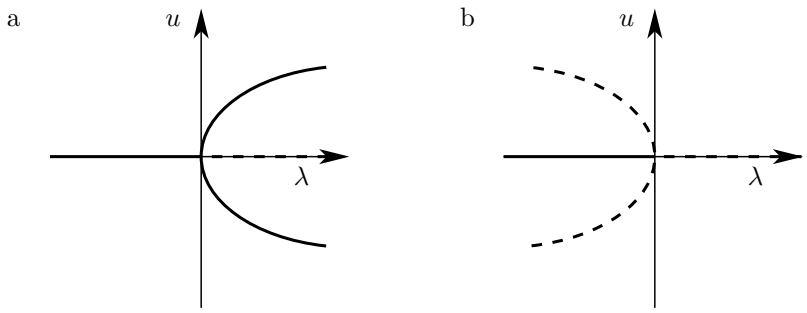

FIGURE 3.4. Pitchfork bifurcation, (a) in the case $c_{11} > 0$, $c_{30} < 0$ (supercritical bifurcation), and (b) in the case $c_{11} > 0$, $c_{30} > 0$ (subcritical bifurcation).

Then $c_{pq} = 0$ for even p. In fact, $F(u, \lambda)/u$ is $\mathcal{C}^2$ near the origin and thus we can write

$$F(u, \lambda) = u\Big[c_{11}\lambda + c_{30}u^2 + c_{12}\lambda^2 + \sum_{p+q=2} u^p \lambda^q \overline{R}_{pq}(u, \lambda)\Big]. \tag{3.2.23}$$

The term in brackets is similar to the expansion for the saddle-node bifurcation, and thus we obtain similar equilibrium branches. In addition, there is the equilibrium branch $u \equiv 0$. Depending on the signs of the coefficients, we have the following cases:

1. If $c_{11} > 0$ and $c_{30} < 0$, the branch $u = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$, and two additional stable branches exist for $\lambda > 0$;
2. if $c_{11} < 0$ and $c_{30} > 0$, the branch $u = 0$ is unstable for $\lambda < 0$ and stable for $\lambda > 0$, and two additional unstable branches exist for $\lambda > 0$;
3. if $c_{11} > 0$ and $c_{30} > 0$, the branch $u = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$, and two additional unstable branches exist for $\lambda < 0$;
4. if $c_{11} < 0$ and $c_{30} < 0$, the branch $u = 0$ is unstable for $\lambda < 0$ and stable for $\lambda > 0$, and two additional stable branches exist for $\lambda < 0$.

This situation in called a *pitchfork bifurcation*, which is said to be *supercritical* if stable equilibrium branches are created, and *subcritical* is unstable equilibrium branches are destroyed (Fig. 3.4). The normal form of the pitchfork bifurcation is

$$\dot{u} = c_{11}\lambda u + c_{30}u^3. \tag{3.2.24}$$

In Physics, this equation is often written in the form

$$\dot{u} = -\frac{\partial V}{\partial u}(u, \lambda), \qquad V(u, \lambda) = -\frac{c_{11}}{2}\lambda u^2 - \frac{c_{30}}{4}u^4. \tag{3.2.25}$$

If $c_{30} < 0$, the function $V(u, \lambda)$ has one or two minima, and in the latter case it is called a *double-well potential*.

Exercise 3.2. Consider the case $F \in \mathcal{C}^3$ with $c_{20} = c_{11} = 0$ and $c_{30}, c_{11}, c_{02} \neq 0$, without the symmetry assumption (3.2.22) and discuss the shape and stability of equilibrium branches.

3.2.2 Two-Dimensional Center Manifold: Hopf Bifurcation

We consider now the case $n_0 = 2$, with C having eigenvalues $\pm \mathrm{i}\,\omega_0$, where $\omega_0 \neq 0$. The dynamics on the center manifold is governed by a two-dimensional system of the form

$$\dot{u} = F(u, \lambda), \tag{3.2.26}$$

where we shall assume that $\lambda \in \mathbb{R}$ and $F \in \mathcal{C}^3$. Since $\frac{\partial F}{\partial u}(0,0) = C$ is invertible, the implicit function theorem (Theorem 3.0.1) shows the existence, near $\lambda = 0$, of a unique equilibrium branch $u^\star(\lambda)$, with $u(0) = 0$ and $F(u^\star(\lambda), \lambda) = 0$. By continuity of the eigenvalues of a matrix-valued function, the linearization of F around this branch has eigenvalues $a(\lambda) \pm \mathrm{i}\,\omega(\lambda)$, where $\omega(0) = \omega_0$ and $a(0) = 0$. A translation of $-u^\star(\lambda)$, followed by a linear change of variables, puts the system (3.2.26) into the form

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} a(\lambda) & -\omega(\lambda) \\ \omega(\lambda) & a(\lambda) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} g_1(u_1, u_2, \lambda) \\ g_2(u_1, u_2, \lambda) \end{pmatrix}, \tag{3.2.27}$$

where the g_i are nonlinear terms satisfying $\|g_i(u_1, u_2, \lambda)\| \leqslant M(u_1^2 + u_2^2)$ for small u_1 and u_2 and some constant $M > 0$.

Our strategy is now going to be to simplify the nonlinear terms as much as possible, following the theory of normal forms developed in Subsection 2.2.4. It turns out to be useful to introduce the complex variable $z = u_1 + \mathrm{i}\,u_2$ (an idea going back to Poincaré), which satisfies an equation of the form

$$\dot{z} = \big[a(\lambda) + \mathrm{i}\,\omega(\lambda)\big] z + g(z, \overline{z}, \lambda), \tag{3.2.28}$$

where $\overline{z}$ is the complex conjugate of z. This should actually be considered as a two-dimensional system for the independent variables z and $\overline{z}$:

$$\begin{aligned} \dot{z} &= \big[a(\lambda) + \mathrm{i}\,\omega(\lambda)\big] z + g(z, \overline{z}, \lambda), \\ \dot{\overline{z}} &= \big[a(\lambda) - \mathrm{i}\,\omega(\lambda)\big] \overline{z} + \overline{g}(z, \overline{z}, \lambda). \end{aligned} \tag{3.2.29}$$

Lemma 2.2.20 shows that monomials of the form $c_p z^{p_1} \overline{z}^{p_2}$ in the nonlinear term g can be eliminated by a nonlinear change of variables, provided the non-resonance condition (2.2.50) is satisfied. In the present case, this condition has the form

$$(p_1 + p_2 - 1)a(\lambda) + (p_1 - p_2 \mp 1)\,\mathrm{i}\,\omega(\lambda) \neq 0, \tag{3.2.30}$$

where the signs $\mp$ refer, respectively, to the first and second equation in (3.2.29). This condition can also be checked directly by carrying out the transformation $z = \zeta + h_p \zeta^{p_1} \overline{\zeta}^{p_2}$ in (3.2.29). Condition (3.2.30) is hardest to satisfy for $\lambda = 0$, where it becomes

$$(p_1 - p_2 \mp 1)\,\mathrm{i}\,\omega_0 \neq 0. \tag{3.2.31}$$

Since $\omega_0 \neq 0$ by assumption, this relation always holds for $p_1 + p_2 = 2$, so quadratic terms can always be eliminated. The only resonant term of order 3 in g is $z^2\overline{z} = |z|^2 z$, corresponding to $(p_1, p_2) = (2, 1)$. (Likewise, the term $z\overline{z}^2 = |z|^2\overline{z}$ is resonant in $\overline{g}$.) We conclude from Proposition 2.2.19 that there exists a polynomial change of variables $z = \zeta + h(\zeta, \overline{\zeta})$, transforming (3.2.28) into

$$\dot{\zeta} = \big[a(\lambda) + \mathrm{i}\,\omega(\lambda)\big]\zeta + c(\lambda)|\zeta|^2\zeta + R(\zeta, \overline{\zeta}, \lambda), \tag{3.2.32}$$

where $c(\lambda) \in \mathbb{C}$ and $R(\zeta, \overline{\zeta}, \lambda) = \mathcal{O}(|\zeta|^3)$ (meaning that $\lim_{\zeta \to 0} |\zeta|^{-3} R(\zeta, \overline{\zeta}, \lambda) = 0$). Equation (3.2.32) is the normal form of our bifurcation. To analyse it further, we introduce polar coordinates $\zeta = r\,\mathrm{e}^{\mathrm{i}\,\varphi}$, in which the system becomes

$$\begin{aligned} \dot{r} &= a(\lambda) r + \mathrm{Re}\, c(\lambda) r^3 + R_1(r, \varphi, \lambda) \\ \dot{\varphi} &= \omega(\lambda) + \mathrm{Im}\, c(\lambda) r^2 + R_2(r, \varphi, \lambda), \end{aligned} \tag{3.2.33}$$

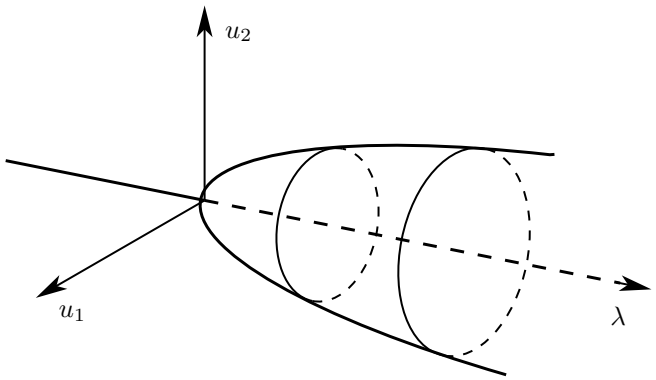

FIGURE 3.5. Supercritical Hopf bifurcation. The stationary solution $(u_1, u_2) = (0, 0)$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. For positive λ, a stable periodic orbit close to a circle of radius $\sqrt{\lambda}$ appears.

where $R_1 = \mathcal{O}(r^3)$ and $R_2 = \mathcal{O}(r^2)$. We henceforth assume that $a'(0) \neq 0$, and changing λ into $-\lambda$ if necessary, we may assume that $a'(0) > 0$. If we discard the remainder R_1, the first equation in (3.2.33) describes a pitchfork bifurcation for r, which is supercritical if $\operatorname{Re} c(0) < 0$, and subcritical if $\operatorname{Re} c(0) > 0$. The first case corresponds to the appearance of a stable periodic orbit, of amplitude $\sqrt{-a(\lambda)/\operatorname{Re} c(\lambda)}$ (Fig. 3.5), the second to the destruction of an unstable periodic orbit. The rotation frequency on this orbit is $\omega_0 + \mathcal{O}(\lambda)$.

It remains to show that this picture is not destroyed by the remainders R_1 and R_2. Note that in the present case we cannot apply the Poincaré-Sternberg-Chen theorem (Theorem 2.2.21) because the linear part is not hyperbolic.

Theorem 3.2.4 (Andronov-Hopf). *Assume that the system $\dot{x} = f(x, \lambda)$ admits an equilibrium branch $x^\star(\lambda)$ such that the linearization of f at $x^\star(\lambda)$ has two eigenvalues $a(\lambda) \pm \mathrm{i}\,\omega(\lambda)$, with $a(0) = 0$, $a'(0) > 0$ and $\omega(0) \neq 0$, and all other eigenvalues have strictly negative real parts. If the coefficient $c(\lambda)$ in the normal form* (3.2.32) *satisfies $\operatorname{Re} c(0) \neq 0$, then*

- *if $\operatorname{Re} c(0) < 0$, the system admits a stable isolated periodic orbit for small positive λ, close to a circle with radius proportional to $\sqrt{\lambda}$ (supercritical case);*
- *if $\operatorname{Re} c(0) > 0$, the system admits an unstable isolated periodic orbit for small negative λ, close to a circle with radius proportional to $\sqrt{-\lambda}$ (subcritical case).*

PROOF: There exist various proofs of this result. One of them is based on the method of averaging, another one on the Poincaré-Bendixson theorem. Since we did not introduce these methods, we will give a straightforward geometrical proof. The main idea is to consider the set $\{(r, \varphi) \colon \varphi = 0, r > 0\}$ as a Poincaré section, and to examine the associated Poincaré map. Consider the case $\operatorname{Re} c(0) < 0$. If we assume that $0 < \lambda \ll 1$, set $r = \sqrt{\lambda}\rho$ in (3.2.33), and expand the λ-dependent terms, we arrive at the equivalent system

$$\begin{aligned} \dot{\rho} &= \rho\big[a'(0)\lambda + \operatorname{Re} c(0)\lambda\rho^2 + \mathcal{O}(\lambda)\big] \\ \dot{\varphi} &= \omega_0 + \lambda\big[\operatorname{Im} c(0)\rho^2 + \omega'(0)\big] + \mathcal{O}(\lambda). \end{aligned}$$

Let $\rho_0 = \sqrt{-a'(0)/\operatorname{Re} c(0)}$. We shall consider this system in the disc $\rho \leqslant 2\rho_0$. The second equation tells us that $\dot{\varphi} \neq 0$ (say, $\dot{\varphi} > 0$) for λ small enough. Under this condition, we can

use φ instead of t as new time variable, and write the system in the form

$$\begin{aligned}\dot\rho &= \frac{a'(0)}{\omega_0}\rho\Big[\lambda - \frac{\rho^2}{\rho_0^2}\lambda + \mathcal{O}(\lambda)\Big]\\ \dot\varphi &= 1.\end{aligned}$$

We define a Poincaré map P by the fact that the solution starting at $(\rho, 0)$ passes, after one revolution, through the point $(P(\rho), 2\pi)$. $P(\rho)$ is monotonous by uniqueness of solutions. Fix a constant $\delta \in (0,1)$. The term $\mathcal{O}(\lambda)$ may depend on φ, but by taking λ small enough, it can be made smaller in absolute value than $(\delta/2)\lambda$ (uniformly in ρ for $\rho \leqslant 2\rho_0$). It follows that

$$\begin{aligned}&\text{for } 0 < \rho^2 \leqslant (1-\delta)\rho_0^2\text{:} && \dot\rho \geqslant \frac{a'(0)}{\omega_0}\rho\lambda\Big(1-(1-\delta)-\frac{\delta}{2}\Big) > 0\\ &\text{for } (1+\delta)\rho_0^2 \leqslant \rho^2 \leqslant (2\rho_0)^2\text{:} && \dot\rho \leqslant \frac{a'(0)}{\omega_0}\rho\lambda\Big(1-(1+\delta)+\frac{\delta}{2}\Big) < 0.\end{aligned}$$

This means that $\rho(t)$ is monotonous outside the annulus $(1-\delta)\rho_0^2 \leqslant \rho^2 \leqslant (1+\delta)\rho_0^2$, and thus P cannot have fixed points outside the interval $I = [\sqrt{1-\delta}\,\rho_0, \sqrt{1+\delta}\,\rho_0]$. Furthermore, since the vector field enters the annulus, P maps the interval I into itself, and must admit a fixed point $\rho^\star$. Returning to the original variables, we see that this fixed point corresponds to the desired periodic orbit.

Finally, it is also possible to show that $\rho^\star$ is the only fixed point of P in $(0, 2\rho_0)$. We know that there can be no fixed points outside I (except 0). By examining the error term $\mathcal{O}(\lambda)$ a bit more carefully, one finds that $\dot\rho$ is a decreasing function of ρ near $\rho = \rho_0$. Thus, if $\rho_1 < \rho_2$ belong to I, with δ small enough, the orbits starting in ρ_1 and ρ_2 must approach each other as time increases. This means that P is contracting in I, and thus its fixed point is unique. □

This bifurcation is called *Poincaré-Andronov-Hopf bifurcation*. One should note that it is the first time we prove the existence of periodic orbits in any generality. Bifurcations with eigenvalues crossing the imaginary axis are common in parameter-dependent differential equations, and thus periodic orbits are frequent in these systems.

The nature of the bifurcation depends crucially on the sign of $\operatorname{Re} c(0)$. This quantity can be determined by a straightforward, though rather tedious computation, and is given, for instance, in [GH83, p. 152].

Exercise 3.3. Show that the van der Pol oscillator

$$\begin{aligned}\dot x_1 &= x_2 + \lambda x_1 - \frac{1}{3}x_1^3\\ \dot x_2 &= -x_1\end{aligned} \tag{3.2.34}$$

displays a Hopf bifurcation at the origin, and determine whether it is subcritical or supercritical.

3.3 Bifurcations of Maps

We turn now to parameter-dependent iterated maps of the form

$$x_{k+1} = F(x_k, \lambda), \tag{3.3.1}$$

with $x \in \mathcal{D} \subset \mathbb{R}^n$, $\lambda \in \Lambda \subset \mathbb{R}^p$ and $F \in \mathcal{C}^r$ for some $r \geqslant 2$. We assume again that $(x^\star, 0)$ is a bifurcation point of (3.3.1), which means that

$$\begin{aligned} F(x^\star, 0) &= x^\star \\ \frac{\partial F}{\partial x}(x^\star, 0) &= A, \end{aligned} \tag{3.3.2}$$

where A has $n_0 \geqslant 0$ eigenvalues of module 1. For simplicity, we assume that all other eigenvalues of A have a module strictly smaller than 1. In appropriate coordinates, we can thus write this system as

$$\begin{aligned} y_{k+1} &= By_k + g_-(y_k, z_k, \lambda_k) \\ z_{k+1} &= Cz_k + D\lambda_k + g_0(y_k, z_k, \lambda_k) \\ \lambda_{k+1} &= \lambda_k, \end{aligned} \tag{3.3.3}$$

where all eigenvalues of B are inside the unit circle, all eigenvalues of C are on the unit circle, and g_- and g_0 are nonlinear terms. One can prove, in much the same way we used for differential equations, the existence of a local invariant center manifold $y = h(z, \lambda)$. This manifold has similar properties as in the ODE case: it is locally attractive, and can be computed by solving approximately the equation

$$h\big(Cz + D\lambda + g_0(h(z,\lambda), z, \lambda)\big) = Bh(z,\lambda) + g_-(h(z,\lambda), z, \lambda). \tag{3.3.4}$$

The dynamics on this manifold is governed by the n_0-dimensional map

$$u_{k+1} = Cu_k + D\lambda + g_0(h(u_k, \lambda), u_k, \lambda). \tag{3.3.5}$$

Since C is a real matrix with all eigenvalues on the unit circle, the most generic cases are the following:

1. one eigenvalue equal to 1: $n_0 = 1$ and $C = 1$;
2. one eigenvalue equal to -1: $n_0 = 1$ and $C = -1$;
3. two complex conjugate eigenvalues of module 1: $n_0 = 2$ and C having eigenvalues $\mathrm{e}^{\pm 2\pi \mathrm{i}\, \theta_0}$ with $2\theta_0 \notin \mathbb{Z}$.

The first case is easily dealt with, because extremely similar to the case of differential equations. Indeed, the dynamics on the one-dimensional center manifold is governed by the equation

$$u_{k+1} = u_k + G(u_k, \lambda), \qquad G(0,0) = 0, \quad \frac{\partial G}{\partial u}(0,0) = 0. \tag{3.3.6}$$

The fixed points are obtained by solving the equation $G(u, \lambda) = 0$, which behaves exactly as the equation $F(u, \lambda) = 0$ in Subsection 3.2.1. Let us simply indicate the normal forms of the most common bifurcations. For the saddle-node bifurcation, we have

$$u_{k+1} = u_k + c_{01}\lambda + c_{20}u_k^2; \tag{3.3.7}$$

for the transcritical bifurcation, one can reduce the equation to

$$u_{k+1} = u_k + c_{11}\lambda u_k + c_{20}u_k^2; \tag{3.3.8}$$

and for the pitchfork bifurcation, the normal form is given by

$$u_{k+1} = u_k + c_{11}\lambda u_k + c_{30}u_k^3. \tag{3.3.9}$$

Exercise 3.4. Find the fixed points of the above three maps and determine their stability. Show that higher order terms do not affect the behaviour near the origin. If the map is a Poincaré map associated with a periodic orbit Γ, what is the meaning of these bifurcations for the flow?

3.3.1 Period-Doubling Bifurcation

We turn now to the case $n_0 = 1$, $C = -1$, and assume that the map is of class $\mathcal{C}^3$. The map restricted to the center manifold has the form

$$\begin{aligned} u_{k+1} &= -u_k + G(u_k, \lambda), \\ G(u, \lambda) &= c_{01}\lambda + c_{02}\lambda^2 + c_{11}u\lambda + c_{20}u^2 + c_{30}u^3 + \cdots \end{aligned} \tag{3.3.10}$$

We first note that the implicit function theorem can be applied to the equation $u_{k+1} = u_k$, and yields the existence of a unique equilibrium branch through the origin. It has the form

$$u = u^\star(\lambda) = \frac{c_{01}}{2}\lambda + \mathcal{O}(\lambda) \tag{3.3.11}$$

and changes stability as λ passes through 0 if $c_{11} + c_{20}c_{01} \neq 0$. Thus something must happen to nearby orbits. To understand what is going on, it is useful to determine the second iterates. A straightforward computation gives

$$\begin{aligned} u_{k+2} &= -u_{k+1} + G(u_{k+1}, \lambda) \\ &= u_k - G(u_k, \lambda) + G(-u_k + G(u_k, \lambda), \lambda) \\ &= u_k + c_{01}(c_{11} + c_{01}c_{20})\lambda^2 - 2(c_{11} + c_{01}c_{20})u_k\lambda - 2(c_{30} + c_{20}^2)u_k^3 + \cdots \end{aligned} \tag{3.3.12}$$

Now let us consider the equation $u_{k+2} = u_k$, the solutions of which yield orbits of period 2. Using the method of Newton's polygon, we find that in addition to the solution $u = u^\star(\lambda)$, there exist solutions of the form

$$u^2 = -\frac{c_{11} + c_{20}c_{01}}{c_{30} + c_{20}^2}\lambda + \mathcal{O}(\lambda), \tag{3.3.13}$$

provided $c_{30} + c_{20}^2 \neq 0$ and $c_{11} + c_{20}c_{01} \neq 0$. In other words, the map $u_k \mapsto u_{k+2}$ undergoes a pitchfork bifurcation. The equilibrium branch (3.3.13) does not correspond to fixed points of (3.3.10), but to an orbit of least period 2. We have thus obtained the following result:

Theorem 3.3.1. *Let $F(\cdot, \lambda) : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of maps of the form* (3.3.10)*, i.e., such that $F(\cdot, 0)$ admits 0 as a fixed point with linearization -1. Assume*

$$\begin{aligned} c_{11} + c_{20}c_{01} &\neq 0 \\ c_{30} + c_{20}^2 &\neq 0. \end{aligned} \tag{3.3.14}$$

Then there exists a curve of fixed points $u = u^\star(\lambda)$ passing through the origin, which changes stability at $\lambda = 0$. In addition, there is a curve of the form (3.3.13)*, tangent to the u-axis, consisting of points of period 2. The orbit of period 2 is stable if $c_{30} + c_{20}^2 > 0$ and unstable if $c_{30} + c_{20}^2 < 0$.*

This bifurcation is called a *period doubling*, *flip* or *subharmonic bifurcation*. It is called *supercritical* if a stable cycle of period 2 is created, and *subcritical* if an unstable orbit of period 2 is destroyed.

Remark 3.3.2. If f is a function of class $\mathcal{C}^3$, its *Schwartzian derivative* is defined as

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{3.3.15}$$

The bifurcation is supercritical if the Schwartzian derivative of the map at $(0, 0)$ with respect to u is negative, and subcritical if this derivative is positive.

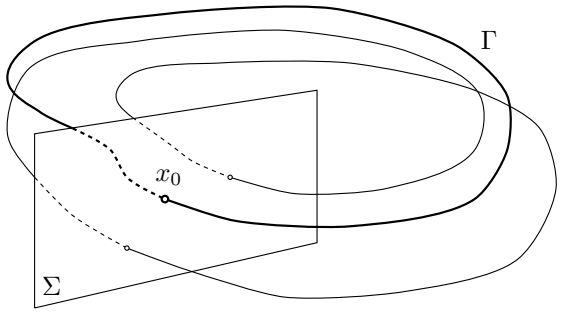

FIGURE 3.6. The supercritical flip bifurcation of a Poincaré map, associated with a periodic Γ, corresponds to the appearance of a new periodic orbit with twice the period.

The period doubling bifurcation has an interesting consequence if F is the Poincaré map associated with a periodic orbit Γ. Consider for instance the supercritical case. For $\lambda < 0$, the Poincaré map has a stable fixed point, which means that the periodic orbit Γ is stable. For $\lambda > 0$, the fixed point is unstable and there exists a stable orbit of period two. In phase space, this means that the periodic orbit Γ has become unstable, but a new stable periodic orbit has appeared (Fig. 3.6). If Γ has period T for $\lambda = 0$, the new orbit has a period close to $2T$, or half the frequency (which accounts for the name subharmonic given to the bifurcation). Note that this bifurcation requires a phase space with dimension at least 3.

3.3.2 Hopf Bifurcation and Invariant Tori

We finally consider what happens when two eigenvalues cross the unit circle in complex plane. Then we have to study the map

$$u_{k+1} = Cu_k + G(u_k, \lambda), \tag{3.3.16}$$

where $u \in \mathbb{R}^2$, and C has eigenvalues $\mathrm{e}^{\pm 2\pi \mathrm{i}\,\theta_0}$ with $2\theta_0 \notin \mathbb{Z}$. We shall assume that $\lambda \in \mathbb{R}$ and $G \in \mathcal{C}^3$. As in the case of differential equations, the implicit function theorem shows the existence of an equilibrium branch $u^\star(\lambda)$ through the origin. The linearization around this branch has eigenvalues $\mu(\lambda) = \rho(\lambda)\,\mathrm{e}^{2\pi \mathrm{i}\,\theta(\lambda)}$ and $\overline{\mu}(\lambda)$, with $\rho(0) = 1$ and $\theta(0) = \theta_0$. An appropriate linear change of variables casts the system (3.3.16) into the form

$$z_{k+1} = \mu(\lambda) z_k + g(z_k, \overline{z}_k, \lambda), \tag{3.3.17}$$

where g is a nonlinear term, which we will try to simplify by normal form theory. We can assume that

$$g(z, \overline{z}, \lambda) = g_2(z, \overline{z}, \lambda) + g_3(z, \overline{z}, \lambda) + \mathcal{O}(|z|^3), \tag{3.3.18}$$

where g_2 is a homogeneous polynomial of degree 2 in z, $\overline{z}$, and g_3 is of degree 3. Normal form theory for maps is very similar to normal form theory for differential equations. In fact, if $h_2(z, \overline{z}, \lambda)$ satisfies the *homological equation*

$$h_2(\mu(\lambda)z, \overline{\mu}(\lambda)\overline{z}, \lambda) - \mu(\lambda) h_2(z, \overline{z}, \lambda) = g_2(z, \overline{z}, \lambda), \tag{3.3.19}$$

then it is easy to see that the transformation $z = \zeta + h_2(\zeta, \overline{\zeta}, \lambda)$ eliminates terms of order 2 from (3.3.17). A similar transformation can be used to eliminate terms of order 3. Let

us now try to solve the homological equation, assuming

$$h_2(z,\overline{z},\lambda) = \sum_{p+q=2} h_{pq}(\lambda) z^p \overline{z}^q, \qquad g_2(z,\overline{z},\lambda) = \sum_{p+q=2} c_{pq}(\lambda) z^p \overline{z}^q. \tag{3.3.20}$$

Substitution into (3.3.19) shows that

$$h_{pq}(\lambda) = \frac{c_{pq}(\lambda)}{\mu(\lambda)^p \overline{\mu}(\lambda)^q - \mu(\lambda)}. \tag{3.3.21}$$

In particular, for $\lambda = 0$, we obtain

$$h_{pq}(0) = \frac{c_{pq}(0)}{\mathrm{e}^{2\pi\,\mathrm{i}\,\theta_0}\left[\mathrm{e}^{2\pi\,\mathrm{i}\,\theta_0(p-q-1)} - 1\right]}. \tag{3.3.22}$$

This equation can only be solved under the non-resonance condition

$$\mathrm{e}^{2\pi\,\mathrm{i}\,\theta_0(p-q-1)} \neq 1. \tag{3.3.23}$$

Since $2\theta_0 \notin \mathbb{Z}$, this condition is always satisfied for $(p,q) = (2,0)$ and $(1,1)$. Thus terms of the form z^2 and $z\overline{z}$ can always be eliminated. Terms of the form $\overline{z}^2$, however, can only be eliminated if $3\theta_0$ is not an integer. Similarly, third order terms of the form z^3 and $z\overline{z}^2$ can always be eliminated, terms of the form $\overline{z}^3$ can only be eliminated if $4\theta_0$ is not an integer, while the term $z^2\overline{z}$ can never be removed. We conclude that if $\mathrm{e}^{2\pi\,\mathrm{i}\,\theta_0}$ is neither a cubic nor a quartic root of unity, there exists a polynomial change of variables transforming (3.3.17) into its normal form

$$\zeta_{k+1} = \mu(\lambda)\zeta_k + c(\lambda)|\zeta_k|^2\zeta_k + \mathcal{O}(|\zeta_k|^3). \tag{3.3.24}$$

Using polar coordinates $\zeta_k = r_k\,\mathrm{e}^{\mathrm{i}\,\varphi_k}$, one obtains a map of the form

$$\begin{aligned} r_{k+1} &= \rho(\lambda) r_k + \alpha(\lambda) r_k^3 + \mathcal{O}(r_k^3) \\ \varphi_{k+1} &= \varphi_k + 2\pi\theta(\lambda) + \beta(\lambda) r_k^2 + \mathcal{O}(r_k^2). \end{aligned} \tag{3.3.25}$$

An explicit calculation shows that if $c(\lambda) = |c(\lambda)|\,\mathrm{e}^{2\pi\,\mathrm{i}\,\psi(\lambda)}$, then the coefficients α and β are given by

$$\alpha(\lambda) = |c(\lambda)| \cos 2\pi(\psi(\lambda) - \theta(\lambda)), \qquad \beta(\lambda) = \frac{|c(\lambda)|}{\rho(\lambda)} \sin 2\pi(\psi(\lambda) - \theta(\lambda)). \tag{3.3.26}$$

If we neglect the remainders, the map (3.3.25) describes a pitchfork bifurcation for the radial variable r. Depending on the signs of $\rho'(0)$ and $\alpha(0)$, there will be creation or destruction of an invariant circle. This can be proved to remain true when the $\mathcal{O}(\cdot)$ terms are present, but the proof is more difficult than in the case of differential equations.

Theorem 3.3.3 (Ruelle). *Let $F(\cdot,\lambda) : \mathbb{R}^2 \to \mathbb{R}^2$ be a one-parameter family of maps, admitting a smooth curve of fixed points $u^\star(\lambda)$. Assume the linearization around the fixed points has complex conjugate eigenvalues $\mu(\lambda)$ and $\overline{\mu}(\lambda)$ such that*

$$|\mu(0)| = 1 \quad \textit{but } \mu(0)^j \neq 1 \textit{ for } j = 1,2,3,4, \tag{3.3.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|\mu(\lambda)|\Big|_{\lambda=0} \neq 0. \tag{3.3.28}$$

Then there is a smooth change of coordinates transforming the map F into (3.3.25). If $\alpha(0) \neq 0$, then the map admits, either for small positive λ or for small negative λ, an invariant curve close to a circle of radius $\sqrt{|\lambda|}$.

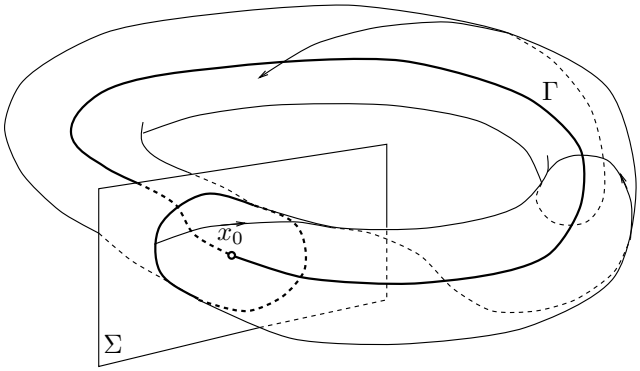

FIGURE 3.7. If the Poincaré map of a periodic orbit Γ undergoes supercritical Hopf bifurcation, an attracting invariant torus is created.

In the strongly resonant cases $\mu(0)^j = 1$ for $j = 1, 2, 3$ or 4, the situation is more complicated, and there is no invariant curve in general. If such an invariant curve exists, the dynamics on this curve is described by a *circle map*. The theory of circle maps is a huge subject in itself. Roughly speaking, they can be characterized by a *rotation number*, measuring the average angle of rotation per iteration. Two cases can occur:

- if the rotation number is rational, there exists a periodic orbit, which usually attracts most orbits;
- if the rotation number is irrational, all orbits are dense, and under suitable smoothness assumptions, the map is conjugate to a rotation.

If the bifurcating map is the Poincaré map of a periodic orbit Γ, the invariant circle will be the intersection of an *invariant torus* with the Poincaré section Σ (Fig. 3.7). If the rotation number is irrational, orbits fill this torus in a dense way. They are called *quasiperiodic* (with two frequencies), which means that any solution $x(t)$ on the invariant torus can be written as

$$x(t) = H(\omega t, \omega' t), \qquad H(u+1, v) = H(u, v+1) = H(u, v), \tag{3.3.29}$$

where ω/ω' is equal to the irrational rotation number. Besides invariant points, curves, and manifolds, we have thus found a new kind of invariant set appearing quite commonly in dynamical systems.

Chapter 4

Introduction to Chaotic Dynamics

The characterization of chaotic dynamics is a large subject, with many recent developments, and ramifications in several domains of Mathematics. We will discuss here only a few selected topics, the major objective being to provide an idea of what we call chaotic motion.

The definition of chaos may vary from system to system, but usually at least one of the following elements is present:

- The time dependence of solutions is more complicated than stationary, periodic or quasiperiodic.
- The motion is very sensitive to variations in the initial conditions: nearby solutions diverge exponentially fast.
- The asymptotic motion takes place on a geometrically complicated object (often a fractal), called a *strange attractor*.
- Chaotic orbits coexist with a (countable) infinity of unstable periodic orbits; the number of orbits of period less or equal T grows exponentially fast with T.
- As time goes by, the images under the flow of any two subsets of phase space get entangled in a complicated way.

These properties do not necessarily all occur at once: for instance, strange attractors cannot occur in conservative systems, but are quite typical in chaotic dissipative systems. Often, there are subtle relations between the above properties, many of which can be characterized quantitatively (by Liapunov exponents, topological entropy, ...).

Our approach in this introduction to chaotic dynamics will be to start with some very simple examples, where many of these chaotic properties can be proven to hold. Then we will show how similar properties can be proven to exist for more realistic systems.

4.1 Symbolic Dynamics

Symbolic dynamics is a very useful technique for characterizing dynamical systems. It consists in associating with every orbit a sequence of symbols, related to a partition of phase space, and describing in which order the orbit of x visits the elements of the partition. In this way, one can sometimes reduce the problem to a combinatorial one, by transposing the dynamics to the space of allowed symbolic sequences.

We will start by illustrating the method on a very simple one-dimensional map called the tent map, and later discuss a less trivial two-dimensional map which is already useful in proving existence of chaotic orbits for a more general class of systems.

4.1.1 The Tent Map

In Section 1.4, we claimed that the logistic map

$$f_\lambda(y) = \lambda y(1-y) \tag{4.1.1}$$

is as random as coin tossing when $\lambda = 4$. Let us now explain what we meant. It is easy to check that the transformation $y = h(x) = \frac{1}{2}(1 - \cos \pi x)$ transforms the logistic map f_4 into the map

$$g_2(x) = h^{-1} \circ f_4 \circ h(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2} \\ 2 - 2x & \text{for } \frac{1}{2} \leqslant x \leqslant 1. \end{cases} \tag{4.1.2}$$

This map is called the *tent map* because of its triangular shape (one can define more general g_μ with slope $\mu \neq 2$). In the transformation, we have lost differentiability at $x = \frac{1}{2}$, but the piecewise linearity will be useful to classify the orbits.

First of all, we note that the map (4.1.2) has two unstable fixed points, at 0 and 2/3. We also observe that 1 is mapped to 0 and 1/2 is mapped to 1, so there exist "transient" orbits, ending at 0 after finitely many iterations.

In order to understand the other orbits, it turns out to be a good idea to write x in binary expansion. We will write

$$x = \sum_{i=0}^{\infty} b_i 2^{-i}, \quad b_i \in \{0,1\}, \qquad \Rightarrow \qquad x = b_0.b_1b_2b_3\ldots \tag{4.1.3}$$

This decomposition is not unique. Indeed,

$$\sum_{i=1}^{\infty} 2^{-i} = 1 \qquad \Rightarrow \qquad 0.1^\infty = 1.0^\infty, \tag{4.1.4}$$

where the superscript $^\infty$ means that the symbol is repeated indefinitely. This, however, is the only kind of degeneracy. For our purposes, the following convention will be useful. We denote by $\mathcal{B}$ the set of symbolic sequences $\underline{b} = 0.b_1b_2\ldots$ which are *not* terminated by 0^∞, except that we include the sequence 0.0^∞. Then (4.1.3) defines a bijection $\underline{b} : [0,1] \to \mathcal{B}$. In fact, $\mathcal{B}$ is a metric space with the distance

$$d(\underline{b}, \underline{b}') = \sum_{i=1}^{\infty} (1 - \delta_{b_i b_i'}) 2^{-i}, \tag{4.1.5}$$

where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise, and $\underline{b}$ is continuous in the resulting topology.

How does the tent map act on binary expansions? We first observe that $x \leqslant 1/2$ if and only if $b_1 = 0$. In this case,

$$g_2(0.0b_2b_3\ldots) = 2\sum_{i=2}^{\infty} b_i 2^{-i} = \sum_{i=1}^{\infty} b_{i+1} 2^{-i} = 0.b_2b_3\ldots \tag{4.1.6}$$

Otherwise, we have $x > 1/2$, $b_1 = 1$ and

$$\begin{aligned} g_2(0.1b_2b_3\ldots) &= 2 - 2\Big[\frac{1}{2} + \sum_{i=2}^{\infty} b_i 2^{-i}\Big] = 1 - \sum_{i=1}^{\infty} b_{i+1} 2^{-i} \\ &= \sum_{i=1}^{\infty} (1 - b_{i+1}) 2^{-i} = 0.(1-b_2)(1-b_3)\ldots \end{aligned} \tag{4.1.7}$$

Hence g_2 induces a map τ from $\mathcal{B}$ to itself, defined by the following rules:

- shift all digits of $\underline{b}$ one unit to the left, discarding b_0;
- if the first digit is 1, reverse all digits;
- replace the sequence 10^∞, if present, by 01^∞.

We can now analyse the dynamics in $\mathcal{B}$ instead of $[0,1]$, which is easier due to the relatively simple form of τ. In fact, g_2 has an even simpler representation. With every $\underline{b} \in \mathcal{B}$, we associate a sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots)$ defined by

$$\varepsilon_j = (-1)^{b_{j-1}+b_j}, \qquad j = 1, 2, \dots \tag{4.1.8}$$

The elements of ε are -1 if adjacent digits of $\underline{b}$ are different, and $+1$ if they are equal. The set Σ of sequences $\underline{\varepsilon}$ constructed in this way consists of $\{-1,+1\}^{\mathbb{N}}$, from which we exclude sequences ending with $(+1)^\infty$ and containing an even number of -1. The correspondence (4.1.8) admits an inverse defined by

$$b_0 = 0, \qquad (-1)^{b_j} = \varepsilon_j (-1)^{b_{j-1}} = \prod_{i=1}^{j} \varepsilon_i, \quad b_j \in \{0,1\}, \quad j \geqslant 1. \tag{4.1.9}$$

Σ can be endowed with a similar distance as (4.1.5), and then (4.1.8) defines a homeomorphism between $\mathcal{B}$ and Σ, and, by composition with $\underline{b}$, a homeomorphism between $[0,1]$ and Σ.

Treating separately the cases $b_1 = 0$ and $b_1 = 1$, it is easy to see that the tent map induces a dynamics in Σ given by

$$\sigma : (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) \mapsto (\varepsilon_2, \varepsilon_3, \dots) \tag{4.1.10}$$

which is called the *shift map*. The sequence $\underline{\varepsilon}$ has a very simple interpretation. Indeed,

$$x \in [0, \tfrac{1}{2}] \quad \Leftrightarrow \quad b_1 = 0 \quad \Leftrightarrow \quad \varepsilon_1 = +1, \tag{4.1.11}$$

so that

$$\varepsilon_j = +1 \quad \Leftrightarrow \quad \sigma^j(\underline{\varepsilon})_1 = +1 \quad \Leftrightarrow \quad g_2^j(x) \in [0, \tfrac{1}{2}]. \tag{4.1.12}$$

We have thus shown that the sequence of ε_j indicates whether the j^{th} iterate of x is to the left or to the right of $1/2$, and that this information is encoded in the binary expansion of $\underline{b}(x)$. The sequence $\underline{\varepsilon}$ associated with x is called its *itinerary*.

Let us now examine the different possible orbits.

1. **Periodic orbits:** The itinerary of a periodic orbit must be periodic. Conversely, if an itinerary $\underline{\varepsilon}$ is periodic with period $p \geqslant 1$, then $\sigma^p(\underline{\varepsilon}) = \underline{\varepsilon}$, and by bijectivity the corresponding x is a fixed point of g_2^p. Thus x is a point of period p if and only if its itinerary is of the form $\underline{\varepsilon} = A^\infty$, with A a finite sequence of length p. There are exactly 2^p such sequences, and thus $2^p/p$ orbits of period p (the number of orbits of least period p can be a bit smaller). For instance, we obtain again the fixed points of g_2:

$$\begin{aligned} &\underline{\varepsilon} = (+1)^\infty \quad \Leftrightarrow \quad \underline{b} = 0.0^\infty \quad \Leftrightarrow \quad x = 0 \\ &\underline{\varepsilon} = (-1)^\infty \quad \Leftrightarrow \quad \underline{b} = 0.(10)^\infty \quad \Leftrightarrow \quad x = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{32} + \dots = \tfrac{2}{3}. \end{aligned} \tag{4.1.13}$$

2. **Transient orbits:** Itineraries of the form $\underline{\varepsilon} = BA^\infty$ with B a finite sequence of length q correspond to orbits that reach a periodic orbit after a finite number of steps.

Exercise 4.1. Find all periodic orbits of periods up to 3 of the tent map.

Let us call *eventually periodic* orbits which are either periodic, or reach a periodic orbit after a finite number of iterations. Their itineraries are of the form $\underline{\varepsilon} = A^\infty$ or $\underline{\varepsilon} = BA^\infty$. It is easy to see that the orbit of $x \in [0,1]$ is eventually periodic if and only if x is rational.[1] $\mathbb{Q}$ being dense in $\mathbb{R}$, the union of all eventually periodic orbits is dense in $[0,1]$. However, this set is countable and has zero Lebesgue measure.

3. **Chaotic orbits:** All irrational initial conditions, by contrast, admit itineraries which are not periodic. The corresponding orbits will typically look quite random.

The following properties are direct consequences of the symbolic representation:

- For every sequence $\underline{\varepsilon} \in \Sigma$, there exists an $x \in [0,1]$ with itinerary $\underline{\varepsilon}$. We may thus choose the initial condition in such a way that the orbit passes left and right of $1/2$ in any prescribed order.
- The dynamics is sensitive to initial conditions. If we only know x with finite precision δ, we are incapable of making any prediction on its orbit after n iterations, whenever $2^n \geqslant 1/\delta$.
- However, simply by looking whether successive iterates of x lie to the left or right of $1/2$, we are able to determine the binary expansion of x (even though g_2 is not injective).
- There exists a dense orbit. Indeed, choose x in such a way that its itinerary

$$\underline{\varepsilon} = (\underbrace{+1}_{\text{period 1}}, \underbrace{-1}_{\text{period 1}}, \underbrace{+1,+1}_{\text{period 2}}, \underbrace{+1,-1}_{\text{period 2}}, \underbrace{-1,+1}_{\text{period 2}}, \underbrace{-1,-1}_{\text{period 2}}, \underbrace{+1,+1,+1}_{\text{period 3}}, \dots, \dots) \tag{4.1.14}$$

  contains all possible finite sequences, ordered by increasing length. Given any $\delta > 0$ and $y \in [0,1]$, one can find $n \in \mathbb{N}$ such that $|g_2^n(x) - y| < \delta$. It suffices to take n in such a way that the shifted sequence $\sigma^n \underline{\varepsilon}$ and the itinerary of y agree for the first $\lceil |\log \delta| / \log 2 \rceil$ bits.

Remark 4.1.1. The tent map g_2 has other interesting properties, from the point of view of measure theory. The main property is that its unique invariant measure which is absolutely continuous with respect to the Lebesgue measure is the uniform measure. This measure is ergodic, meaning that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(g_2^n(x)) = \int_0^1 \varphi(x)\,\mathrm{d}x \tag{4.1.15}$$

for every continuous test function φ, and for Lebesgue-almost all $x \in [0,1]$. Thus from a probabilistic perspective, almost all orbits will be uniformly distributed over the interval.

Itineraries can be defined for more general maps, just by choosing some partition of phase space. In general, however, there will be no simple relation between a point and its itinerary, and the correspondence need not be one-to-one. Some symbolic sequences

[1] If x has an eventually periodic orbit, its itinerary, and hence its binary expansion become eventually periodic, which implies that $2^{p+q}x - 2^q x \in \mathbb{Z}$ for some $p, q \geqslant 1$, and thus $x \in \mathbb{Q}$. Conversely, let $x = n/m \in \mathbb{Q}$. The map $x \mapsto \{2x\}$, where $\{\cdot\}$ denotes the fractional part, shifts the bits of $\underline{b}(x)$ one unit to the left. It also maps the set $\{0, 1/m, 2/m, \dots, (m-1)/m\}$ into itself. Thus the orbit of x under this map is eventually periodic, and so are its binary expansion and its itinerary.

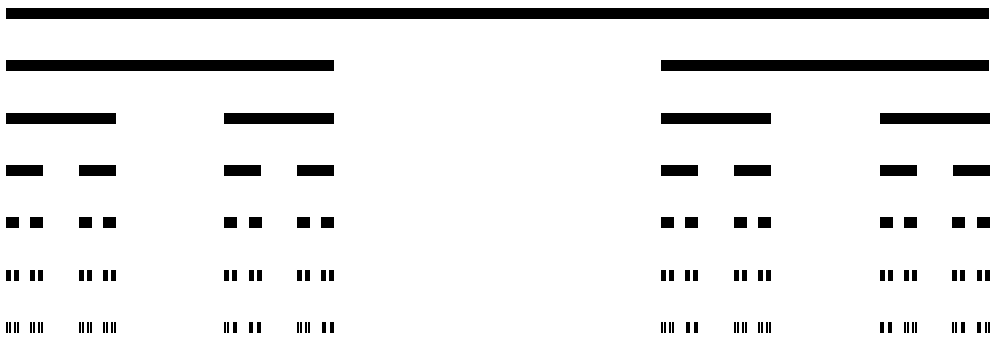

FIGURE 4.1. The first steps of the construction of the Cantor set Λ.

may never occur, and several initial conditions may have the same itinerary, for instance if they are in the basin of attraction of a stable equilibrium.

Let us now consider the following variant of the tent map:

$$g_3(x) = \begin{cases} 3x & \text{for } x \leqslant \frac{1}{2} \\ 3 - 3x & \text{for } x \geqslant \frac{1}{2}. \end{cases} \tag{4.1.16}$$

Since the interval $[0,1]$ is not invariant, we define g_3 on all of $\mathbb{R}$. We first observe that

- if $x_0 < 0$, the orbit of x_0 converges to $-\infty$;
- if $x_0 > 1$, then $x_1 = g_3(x_0) < 0$;
- if $x_0 \in (1/3, 2/3)$, then $x_1 = g_3(x_0) > 1$ and thus $x_2 < 0$.

Hence all orbits which leave the interval $[0,1]$ eventually converge to $-\infty$. One could suspect that all orbits leave the interval after a certain number of iterations. This is not the case since, for instance, orbits starting in multiples of 3^{-n} reach the fixed point 0 after finitely many iterations. But there exists a more subtle nontrivial invariant set. In order to describe it, we use a ternary (base 3) representation of $x \in [0,1]$:

$$x = \sum_{i=0}^{\infty} b_i 3^{-i}, \quad b_i \in \{0,1,2\}, \qquad \Rightarrow \qquad x = b_0.b_1b_2b_3\ldots \tag{4.1.17}$$

Again, this representation is not unique. We can make it unique by replacing 10^∞ with 02^∞ and 12^∞ with 20^∞ if applicable. Now we observe that

$$\begin{aligned} g_3(0.0b_2b_3\ldots) &= 0.b_2b_3\ldots \\ g_3(0.2b_2b_3\ldots) &= 0.(2-b_2)(2-b_3)\ldots \end{aligned} \tag{4.1.18}$$

Points of the form $0.1b_2b_3\ldots$ belong to $(1/3, 2/3)$, and we already know that these leave the interval $[0,1]$. It is thus immediate that the orbit of x never leaves the interval $[0,1]$ if and only if its ternary expansion $\underline{b}(x)$ does not contain the symbol 1. The largest invariant subset of $[0,1]$ is thus

$$\Lambda = \big\{x \in [0,1] \colon b_i(x) \neq 1 \ \forall i \geqslant 1\big\}. \tag{4.1.19}$$

Λ is obtained by removing from $[0,1]$ the open intervals $(1/3, 2/3)$, $(1/9, 2/9)$, $(7/9, 8/9)$, $(1/27, 2/27)$ and so on (Fig. 4.1). The resulting set is called a *Cantor set*:

Definition 4.1.2. *A set Λ is called a* Cantor set *if it is closed, its interior is empty, and all its points are accumulation points.*

Exercise 4.2. Prove that Λ defined in (4.1.19) is a Cantor set.

The Cantor set Λ is an example of *fractal*.

Definition 4.1.3. *Let $\mathcal{M}$ be a subset of $\mathbb{R}^d$. Assume that for any $\varepsilon > 0$, $\mathcal{M}$ can be covered by a finite number of hypercubes of side length ε. Let $N(\varepsilon)$ be the smallest possible number of such cubes. Then the* box-counting dimension *of $\mathcal{M}$ is defined by*

$$D_0(\mathcal{M}) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \tag{4.1.20}$$

One easily shows that for "usual" sets $\mathcal{M} \subset \mathbb{R}^d$, such as a d-dimensional hypercube, $D_0 = d$. For the Cantor set (4.1.19), however, we find that $N(\varepsilon) = 2^n$ if $3^{-n} \leqslant \varepsilon < 3^{-n+1}$, and thus

$$D_0(\Lambda) = \frac{\log 2}{\log 3}. \tag{4.1.21}$$

The map g_3 restricted to Λ is conjugate to the tent map g_2. Indeed, with any point $x = 0.b_1b_2b_3\ldots$ in Λ, we can associate $h(x) = 0.(b_1/2)(b_2/2)(b_3/2)\ldots$ in $[0,1]$, and the relations (4.1.18) and (4.1.6), (4.1.7) show that

$$h \circ g_3|_\Lambda \circ h^{-1} = g_2. \tag{4.1.22}$$

Thus we can easily compute itineraries of points $x \in \Lambda$, and the conclusions on the behaviour of orbits of g_2 can be transposed to $g_3|_\Lambda$. Λ is the simplest example of what is called a *hyperbolic invariant set*.

Similar properties can be seen to hold for nonlinear perturbations of the tent map. Indeed, the ternary representation is a useful tool, but not essential for the existence of orbits with all possible symbolic representations. It is, in fact, sufficient that f maps two disjoint intervals $I_1, I_2 \subset [0,1]$ onto $[0,1]$, and maps all other points outside $[0,1]$. Then the symbolic representation corresponds to the sequence of intervals I_1 or I_2 visited by the orbit.

4.1.2 Homoclinic Tangles and Smale's Horseshoe Map

One-dimensional maps are rather special cases of dynamical systems, but it turns out that some of their properties can often be transposed to more "realistic" systems. As a motivating example, let us consider the equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3, \end{aligned} \tag{4.1.23}$$

called the (undamped) unforced *Duffing oscillator*. This is a Hamiltonian system, with Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2. \tag{4.1.24}$$

H is a constant of the motion, and thus orbits of (4.1.23) belong to level curves of H (Fig. 4.2a). The point $(0,0)$ is a hyperbolic equilibrium, with the particularity that its unstable and stable manifolds W^{u} and W^{s} are identical: they form the curve $H = 0$, and are called *homoclinic loops*.

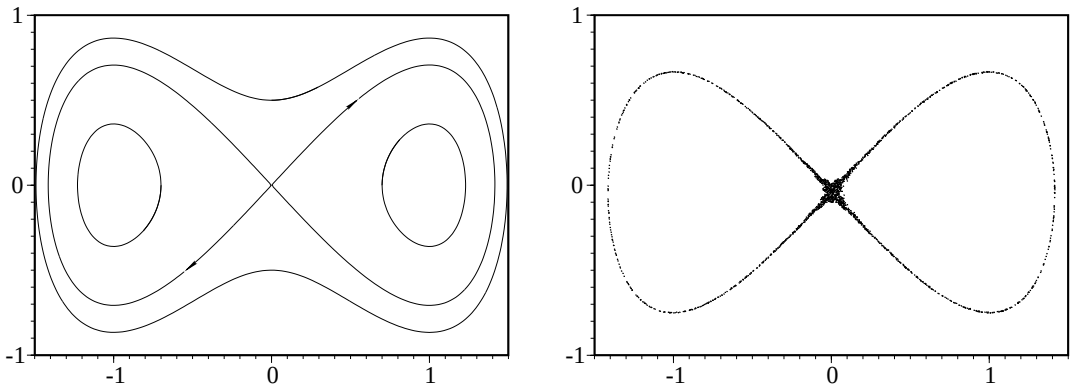

FIGURE 4.2. (a) Orbits of Duffing's equation for $\varepsilon = 0$, (b) one orbit of the Poincaré map obtained for $\varepsilon = 0.25$.

Let us now perturb (4.1.23) by a periodic forcing,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 + \varepsilon \sin(2\pi t). \end{aligned} \tag{4.1.25}$$

We can introduce $x_3 = t$ as additional variable to obtain the autonomous system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 + \varepsilon \sin(2\pi x_3) \\ \dot{x}_3 &= 1. \end{aligned} \tag{4.1.26}$$

Here $x_3 \in \mathbb{S}^1$ should be considered as a periodic variable. We can thus use the surface $x_3 = 0$ as a Poincaré section. Fig. 4.2b shows an orbit of the Poincaré map, which has a complicated structure: it seems to fill a two-dimensional region, in which its dynamics is quite random.

This phenomenon can be explained at least partially, by analysing the properties of the Poincaré map P_ε. First observe that the system (4.1.26) is conservative. The section being perpendicular to the flow, the Poincaré map is also conservative. When $\varepsilon = 0$, $P_0 = \varphi_1$ is the time-one flow of (4.1.23), and thus P_0 admits the origin as hyperbolic fixed point. The implicit function theorem implies that for small ε, P_ε admits an isolated hyperbolic fixed point $x^\star(\varepsilon)$ near $(0,0)$. It is unlikely that the stable and unstable manifolds of $x^\star(\varepsilon)$ still form a loop when $\varepsilon > 0$, and if not, they must intersect transversally at some point x_0 because of area conservation. x_0 is called a *homoclinic point*. A method due to Melnikov allows to prove that such a transverse intersection indeed exists.

Consider now the successive images $x_n = P_\varepsilon^n(x_0)$. Since x_0 belongs to the stable manifold W^{s} of $x^\star(\varepsilon)$, all x_n must also belong to W^{s}, and they must accumulate at $x^\star$ for $n \to \infty$. Similarly, since x_0 belongs to the unstable manifold W^{u}, all x_n must belong to W^{u} and accumulate at $x^\star$ for $n \to -\infty$. Thus W^{s} and W^{u} must intersect infinitely often. Because the map is area preserving, the area between W^{s}, W^{u}, and any consecutive intersection points must be the same, and thus W^{u} has to oscillate with increasing amplitude when approaching $x^\star$ (Fig. 4.3). W^{s} has a similar behaviour. This complicated geometrical structure, which was first described by Poincaré, is called the *homoclinic tangle*.

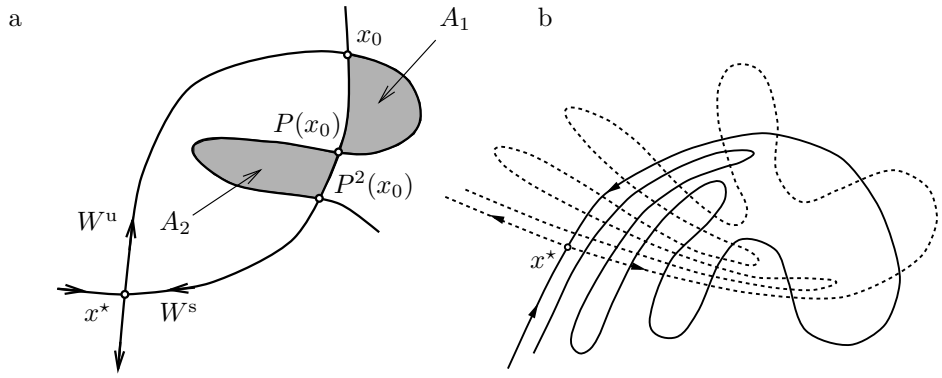

FIGURE 4.3. If the stable and unstable manifolds of a conservative map intersect transversally, they must intersect infinitely often, forming a homoclinic tangle.

The dynamics near the homoclinic tangle can be described as follows. Let Q be a small rectangle, containing a piece of unstable manifold W^{u} and two points x_{-m}, x_{-m+1} of the homoclinic orbit (Fig. 4.4). At least during the first iterations of P_ε, Q will be stretched in the unstable direction, and contracted in the stable one. After m iterations, the image $P_\varepsilon^m(Q)$ contains x_0 and x_1. Let n be sufficiently large that the points x_n and x_{n+1} of the homoclinic orbit are close to $x^\star$. Due to area conservation, one can arrange that the piece of W^{u} between x_n and x_{n+1} crosses Q at least twice, and so does the image $P_\varepsilon^{m+n}(Q)$.

This behaviour is reproduced qualitatively by the *Smale horseshoe map* (Fig. 4.5). This map T takes a square $[0,1]\times[0,1]$, stretches it vertically by a factor $\mu > 2$, and contracts it horizontally by a factor $\lambda < 1/2$. Then it bends the resulting rectangle in the shape of a horseshoe, and superimposes it with the initial square. Two horizontal strips H_+ and H_-, of size $1\times\mu^{-1}$, are mapped, respectively, to two vertical strips V_+ and V_-, of size $\lambda\times 1$. For simplicity, the map restricted to $V_+\cup V_-$ is assumed to be linear, but its qualitative features can be shown to remain unchanged by small nonlinear perturbations.

The aim is now to construct an invariant set Λ of T. To this end, we observe that $T(x)\in Q$ if and only if $x\in H_-\cup H_+$. Since $T(x)\in V_-\cup V_+$, $T^2(x)\in Q$ if and only if $T(x)$ belongs to one of the four rectangles $\Gamma_{\varepsilon_0\varepsilon_1} = V_{\varepsilon_0}\cap H_{\varepsilon_1}$, where $\varepsilon_0,\varepsilon_1\in\{-,+\}$. These rectangles have size $\lambda\times\mu^{-1}$. The preimage of each $\Gamma_{\varepsilon_0\varepsilon_1}$ is a rectangle $H_{\varepsilon_0\varepsilon_1}\subset H_{\varepsilon_0}$, of size $1\times\mu^{-2}$. The image of $\Gamma_{\varepsilon_0\varepsilon_1}$ is a rectangle $V_{\varepsilon_0\varepsilon_1}\subset V_{\varepsilon_1}$, of size $\lambda^2\times 1$. Thus T^2 maps $H_{\varepsilon_0\varepsilon_1}$ to $V_{\varepsilon_0\varepsilon_1}$.

More generally, we can define

$$H_{\varepsilon_0\varepsilon_1\dots\varepsilon_n} = \bigl\{x\in Q\colon x\in H_{\varepsilon_0}, T(x)\in H_{\varepsilon_1},\dots,T^n(x)\in H_{\varepsilon_n}\bigr\}. \tag{4.1.27}$$

Note that $H_{\varepsilon_0\varepsilon_1\dots\varepsilon_n}\subset H_{\varepsilon_0\dots\varepsilon_{n-1}}$, and that

$$H_{\varepsilon_0\varepsilon_1\dots\varepsilon_n} = \bigl\{x\in H_{\varepsilon_0}\colon T(x)\in H_{\varepsilon_1\dots\varepsilon_n}\bigr\}. \tag{4.1.28}$$

By induction, it is straightforward to show that $H_{\varepsilon_0\varepsilon_1\dots\varepsilon_n}$ is a rectangle of size $1\times\mu^{-n}$. Similarly, we introduce

$$\begin{aligned} V_{\varepsilon_{-m}\dots\varepsilon_{-1}} &= \bigl\{x\in Q\colon T^{-1}(x)\in H_{\varepsilon_{-1}},\dots,T^{-m}(x)\in H_{\varepsilon_{-m}}\bigr\} \\ &= \bigl\{x\in V_{\varepsilon_{-1}}\colon T^{-1}(x)\in V_{\varepsilon_{-m}\dots\varepsilon_{-2}}\bigr\}, \end{aligned} \tag{4.1.29}$$

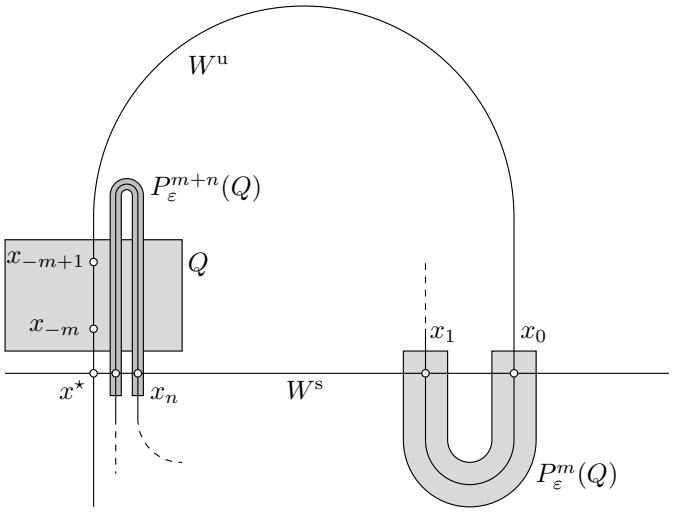

FIGURE 4.4. Schematic representation of the homoclinic tangle. The homoclinic orbit $\{x_n\}_{n\in\mathbb{Z}}$ is asymptotic to $x^\star$ for $n \to \pm\infty$. One can choose m and n in such a way that the image of a rectangle Q after $m+n$ iterations intersects Q twice.

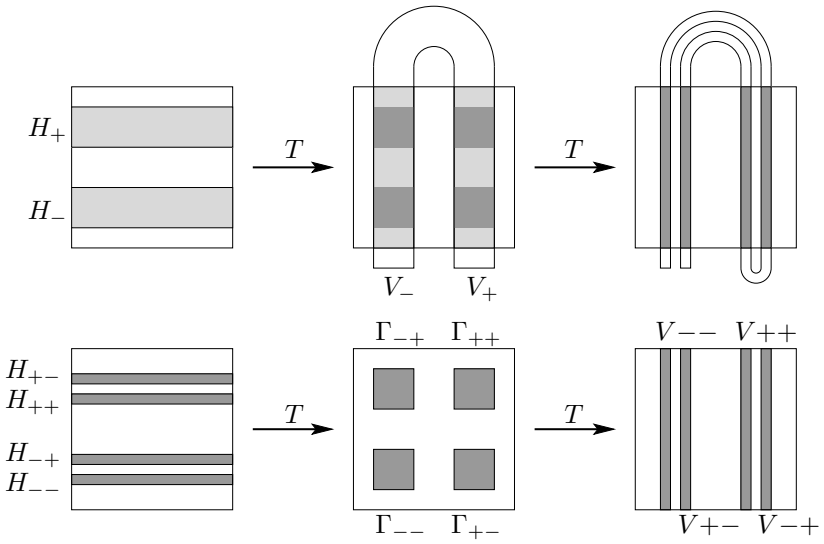

FIGURE 4.5. The horseshoe map T maps the rectangles $H_\pm$ of the unit square Q to the rectangles $V_\pm$. If $\Gamma_{\varepsilon_0\varepsilon_1} = V_{\varepsilon_0} \cap H_{\varepsilon_1}$, one also has $T(H_{\varepsilon_0\varepsilon_1}) = \Gamma_{\varepsilon_0\varepsilon_1}$ and $T(\Gamma_{\varepsilon_0\varepsilon_1}) = V_{\varepsilon_0\varepsilon_1}$.

which is a rectangle of size $\lambda^m \times 1$ contained in $V_{\varepsilon_{-m+1}\dots\varepsilon_{-1}}$. It follows that

$$\begin{aligned}\Gamma_{\varepsilon_{-m}\dots\varepsilon_{-1},\varepsilon_0\varepsilon_1\dots\varepsilon_n} &= V_{\varepsilon_{-m}\dots\varepsilon_{-1}} \cap H_{\varepsilon_0\varepsilon_1\dots\varepsilon_n} \\ &= \bigl\{x \in Q\colon T^j(x) \in H_{\varepsilon_j}, -m \leqslant j \leqslant n\bigr\} \qquad (4.1.30)\end{aligned}$$

is a rectangle of size $\lambda^m \times \mu^{-n}$. By construction, $T^j(x) \in Q$ for $-m \leqslant j \leqslant n+1$ if and only if $x \in \Gamma_{\varepsilon_{-m}\dots,\varepsilon_0\dots\varepsilon_n}$ for some sequence $\varepsilon_{-m}\dots,\varepsilon_0\dots\varepsilon_n$ of symbols -1 and $+1$.

Now let n, m go to ∞. Let Σ be the set of bi-infinite sequences $\underline{\varepsilon} = \dots\varepsilon_{-1},\varepsilon_0\varepsilon_1\dots$ of

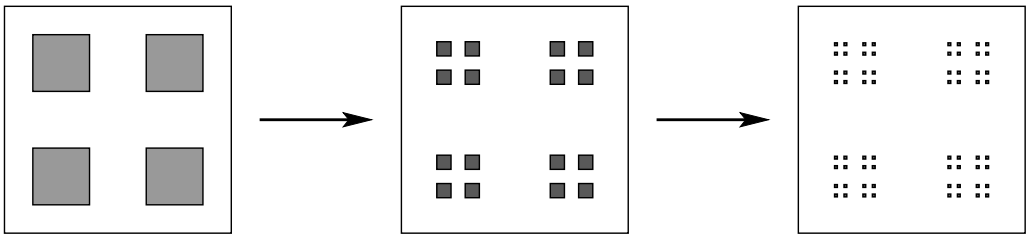

FIGURE 4.6. Successive approximations of the invariant set Λ.

symbols -1 and $+1$. It is a metric space for the distance

$$d(\underline{\varepsilon}, \underline{\varepsilon}') = \sum_{i=-\infty}^{\infty} (1 - \delta_{\varepsilon_i \varepsilon_i'}) 2^{-|i|}. \tag{4.1.31}$$

From (4.1.30) we conclude that the largest subset of Q invariant under T is

$$\Lambda = \bigcap_{n=-\infty}^{+\infty} T^n(Q) = \big\{\Gamma_{\underline{\varepsilon}} \colon \underline{\varepsilon} \in \Sigma\big\}, \tag{4.1.32}$$

where $\Gamma_{\underline{\varepsilon}}$, defined as the limit of (4.1.30) when the finite sequence $\varepsilon_{-m} \ldots, \varepsilon_0 \ldots \varepsilon_n$ converges to $\underline{\varepsilon}$, is a single point. Λ is a Cantor set, obtained by taking the product of two one-dimensional Cantor sets. The map $\phi : \underline{\varepsilon} \mapsto \Gamma_{\underline{\varepsilon}}$ is a bijection from Σ to Λ, which is continuous in the topology defined by (4.1.31). It follows from (4.1.30) that T is conjugated to the shift map

$$\sigma = \phi^{-1} \circ T|_\Lambda \circ \phi : \ldots \varepsilon_{-2}\varepsilon_{-1}, \varepsilon_0\varepsilon_1 \ldots \mapsto \ldots \varepsilon_{-1}\varepsilon_0, \varepsilon_1\varepsilon_2 \ldots \tag{4.1.33}$$

This conjugacy can be used to describe the various orbits of $T|_\Lambda$ in a similar way as we did for the tent map. In particular, periodic itineraries $\underline{\varepsilon}$ correspond to periodic orbits of T, aperiodic itineraries correspond to chaotic orbits. Smale has shown that these qualitative properties are robust under small perturbations of T:

Theorem 4.1.4 (Smale). *The horseshoe map T has an invariant Cantor set Λ such that*

- *Λ contains a countable set of periodic orbits of arbitrarily long periods;*
- *Λ contains an uncountable set of bounded nonperiodic orbits;*
- *Λ contains a dense orbit.*

Moreover, any map $\widetilde{T}$ sufficiently close to T in the $\mathcal{C}^1$ topology has an invariant Cantor set $\widetilde{\Lambda}$ with $\widetilde{T}|_{\widetilde{\Lambda}}$ topologically equivalent to $T|_\Lambda$.

The example of the horseshoe map can be generalised to a class of so-called *axiom A* systems with similar chaotic properties. These systems can be used to show the existence of chaotic orbits in a large class of dynamical systems, including (but not limited to) those with a transverse homoclinic intersection.

Note that the examples of hyperbolic invariant sets that we have encountered have measure zero and are not attracting. Showing the existence of invariant sets containing chaotic orbits and attracting nearby orbits is a much more difficult task.

4.2 Strange Attractors

4.2.1 Attracting Sets and Attractors

We consider in this section a dynamical system on $\mathcal{D} \subset \mathbb{R}^n$, defined by a flow φ_t. One can include the case of iterated maps by restricting t to integer values and setting $\varphi_t = F^t$. Various subsets of $\mathcal{D}$ can be associated with the flow.

Definition 4.2.1.

- *A subset $\mathcal{S} \subset \mathcal{D}$ is called* invariant *if $\varphi_t(\mathcal{S}) = \mathcal{S}$ for all t.*
- *A subset $\mathcal{A} \subset \mathcal{D}$ is called an* attracting set *if there exists a neighbourhood $\mathcal{U}$ of $\mathcal{A}$ such that for all $x \in \mathcal{U}$, $\varphi_t(x) \in \mathcal{U}$ for all $t \geqslant 0$ and $\varphi_t(x) \to \mathcal{A}$ as $t \to \infty$.*
- *A point x is* nonwandering *for φ_t if for every neighbourhood $\mathcal{U}$ of x and every $T > 0$, there exists a $t > T$ such that $\varphi_t(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The set of all nonwandering points is the* nonwandering set Ω.

These sets can be partly determined by looking at the asymptotic behaviour of various orbits of the flow.

Definition 4.2.2. *The ω-limit set of x for φ_t is the set of $y \in \mathcal{D}$ such that there exists a sequence $t_n \to \infty$ with $\varphi_{t_n}(x) \to y$. The α-limit set of x for φ_t is the set of $y \in \mathcal{D}$ such that there exists a sequence $t_n \to -\infty$ with $\varphi_{t_n}(x) \to y$.*

The α- and ω-limit sets of any x are invariant sets, and the ω-limit set is included in the nonwandering set. One can show that asymptotically stable equilibrium points, periodic orbits, and invariant tori are all ω-limit sets of the orbits in their basin of attraction, nonwandering sets and attracting sets. There is, however, a problem with the definition of attracting set, as shows the following example.

Example 4.2.3. Consider the differential equation

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 \\ \dot{x}_2 &= -x_2. \end{aligned} \tag{4.2.1}$$

The ω-limit set of x is $(-1, 0)$ if $x_1 < 0$, $(0, 0)$ if $x_1 = 0$ and $(1, 0)$ if $x_1 > 0$. $(0, 0)$ is also the α-limit set of all points in $(-1, 1) \times \{0\}$. The nonwandering set is composed of the three equilibrium points of the flow, while $(\pm 1, 0)$ are attracting sets. However, the segment $[-1, 1] \times \{0\}$ is also an attracting set.

One would like to exclude attracting sets such as the segment $[-1, 1] \times \{0\}$ in the example, which contains wandering points. This is generally solved in the following way.

Definition 4.2.4. *A closed invariant set Λ is* topologically transitive *if φ_t has an orbit which is dense in Λ. An* attractor *is a topologically transitive attracting set.*

Remark 4.2.5. Sometimes, one uses a weaker notion of indecomposability than topological transitivity, based on the notion of *chain recurrence*. In this case, one requires that for any pair of points x, y and any $\varepsilon > 0$, there exist points $x_0 = x, x_1, \ldots, x_n = y$ and times $t_1, \ldots, t_n$ such that $\|\varphi_{t_j}(x_{j-1}) - x_j\| \leqslant \varepsilon$ for all j.

Remark 4.2.6. A slightly weaker definition of attractor is proposed in [GH83]: An *attractor* is an indecomposable closed invariant set Λ with the property that, given $\varepsilon > 0$, there is a set $\mathcal{U}$ of positive Lebesgue measure in the ε-neighbourhood of Λ such that, if $x \in \mathcal{U}$, the forward orbit of x is contained in $\mathcal{U}$ and the ω-limit set of x is contained in Λ.

4.2.2 Sensitive Dependence on Initial Conditions

A strange attractor is, basically, an attractor on which the dynamics is chaotic. Obviously, this requires that we define what we mean by "chaotic". In [GH83], for instance, a strange attractor is defined as an attractor containing a transversal homoclinic orbit. As we saw in section 4.1.2, the existence of such an orbit implies various chaotic properties (it is not necessary to assume that the system is conservative). Modern definitions are a bit less specific, they require that the dynamics be sensitive to initial conditions:

Definition 4.2.7. *Let Λ be a compact set such that $\varphi_t(\Lambda) \subset \Lambda$ for all $t \geqslant 0$.*

- *The flow φ_t is said to have* sensitive dependence on initial conditions *on Λ if there exists $\varepsilon > 0$ with the following property: For any $x \in \Lambda$ and any neighbourhood $\mathcal{U}$ of x, there exists $y \in \mathcal{U}$ and $t > 0$ such that $\|\varphi_t(x) - \varphi_t(y)\| \geqslant \varepsilon$.*
- *Λ is a* strange attractor *if it is an attractor and φ_t has sensitive dependence on initial conditions on Λ.*

Exercise 4.3. Show that the tent map and Smale's horseshoe map have sensitive dependence on initial conditions.

Sensitive dependence on initial conditions requires that for any point x, one can find an arbitrarily close point y such that the orbits of x and y diverge from each other (the quantity ε should not depend on $\|x - y\|$, though t may depend on it). This is a rather weak property, and one often requires that the divergence occur at an exponential rate.

Definition 4.2.8.

- *Assume φ_t is the flow of a differential equation $\dot{x} = f(x)$. Let $x \in \mathcal{D}$ and let $U(t)$ be the principal solution of the equation linearized around the orbit of x,*

$$\dot{y} = \frac{\partial f}{\partial x}\big(\varphi_t(x)\big)y. \tag{4.2.2}$$

Oseledec [Os68] proved that under quite weak assumptions on f, the limit

$$L = \lim_{t\to\infty} \frac{1}{2t} \log\big(U(t)^T U(t)\big) \tag{4.2.3}$$

exists. The eigenvalues of L are called the Liapunov exponents *of the orbit of x.*

- *If F is an iterated map and $\{x_k\}$ is a given orbit of F, consider the linear equation*

$$y_{k+1} = \frac{\partial F}{\partial x}(x_k)y_k \qquad \Rightarrow \qquad y_k = U_k y_0, \quad U_k = \frac{\partial F}{\partial x}(x_{k-1}) \dots \frac{\partial F}{\partial x}(x_0). \tag{4.2.4}$$

The Liapunov exponents *of the orbit $\{x_k\}$ are the eigenvalues of*

$$L = \lim_{k\to\infty} \frac{1}{2k} \log\big(U_k^T U_k\big). \tag{4.2.5}$$

Note that $U(t)^T U(t)$ is a symmetric positive definite matrix, and hence it is always diagonalizable and has real eigenvalues. This definition implies that the solution $y(t)$ of the equation linearized around the particular solution $\varphi_t(x)$ satisfies

$$\|y(t)\|^2 = \|U(t)y(0)\|^2 = y(0) \cdot U(t)^T U(t) y(0) \simeq y(0) \cdot \mathrm{e}^{2tL}\, y(0). \tag{4.2.6}$$

Let λ_1 be the largest eigenvalue of L. Unless the projection of $y(0)$ on the eigenspace of L associated with λ_1 is zero, $\|y(t)\|$ will grow asymptotically like $\mathrm{e}^{\lambda_1 t}$. We thus say that the flow has *exponentially sensitive dependence on initial conditions* in Λ if the largest Liapunov exponent of all orbits in Λ is positive.

Proposition 4.2.9.

- *If φ_t is conservative, then the sum of all Liapunov exponents of any orbit is zero.*
- *If the flow is dissipative, then this sum is negative.*
- *Assume $\{\varphi_t(x)\}$ is an orbit of the differential equation $\dot{x} = f(x)$, such that $\|f\|$ is bounded below and above by strictly positive constants on $\{\varphi_t(x)\}$. Then this orbit has at least one Liapunov exponent equal to zero.*

PROOF: We saw in the proof of Proposition 2.1.10 that the determinant of $U(t)$ is constant if the system is conservative, and decreasing if the system is dissipative. By definition, $\det U(0) = 1$ and by uniqueness of solutions, $\det U(t) > 0$ for all t. Thus for $t > 0$, the product of all eigenvalues of $U(t)$ is equal to 1 in the conservative case, and belongs to $(0, 1)$ in the dissipative case. The same is true for $\det U(t)^T U(t)$. But for any matrix B, $\det \mathrm{e}^B = \mathrm{e}^{\mathrm{Tr}\, B}$ because the eigenvalues of e^B are exponentials of the eigenvalues of B.

Assume now $x(t) = \varphi_t(x)$ is a solution of $\dot{x} = f(x)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(x)) = \frac{\partial f}{\partial x}(\varphi_t(x))\dot{x}(t),$$

and thus $\dot{x}(t) = U(t)\dot{x}(0)$ by definition of $U(t)$. Hence,

$$\dot{x}(0) \cdot U(t)^T U(t)\dot{x}(0) = \|U(t)\dot{x}(0)\|^2 = \|\dot{x}(t)\|^2 = \|f(\varphi_t(x))\|^2.$$

By the assumption on $\|f\|$, it follows that as $t \to \infty$, the function

$$\dot{x}(0) \cdot \mathrm{e}^{2tL}\, \dot{x}(0)$$

is bounded above and below by strictly positive constants. Being symmetric, L admits an orthonormal set of eigenvectors $u_1, \ldots, u_n$, that is, they satisfy $u_i \cdot u_j = \delta_{ij}$. If $c_j = u_j \cdot \dot{x}(0)$, then $\dot{x}(0) = \sum_j c_j u_j$ and thus

$$\dot{x}(0) \cdot \mathrm{e}^{2tL}\, \dot{x}(0) = \sum_{j=1}^{n} c_j^2\, \mathrm{e}^{2\lambda_j t}\,.$$

Since $\dot{x}(0) \neq 0$, at least one of the coefficients, say c_k, is different from zero, and thus λ_k must be equal to zero. □

It is easy to find systems with positive Liapunov exponents. Consider for instance the linear system $\dot{x} = Ax$. In this case, $U(t) = \mathrm{e}^{At}$ and one shows (using, for instance, the Jordan canonical form of A), that the Liapunov exponents are exactly the real parts of the eigenvalues of A. Thus if A has an eigenvalue with positive real part, all orbits have exponentially sensitive dependence on initial conditions.

More generally, if $x^\star$ is a linearly unstable equilibrium point, its largest Liapunov exponent will be positive. Similarly, if Γ is a periodic orbit, then we have seen in Theorem 2.3.3 that $U(t) = P(t)\, \mathrm{e}^{Bt}$, where $P(t)$ is periodic. The Liapunov exponents are the real parts of the eigenvalues of B, that is, the real parts of the characteristic exponents. e^{Bt} has at least one eigenvalue equal to 1, corresponding to translations along the periodic orbit (seen in Proposition 2.3.5). Thus one of the Liapunov exponents is equal to zero. More generally, orbits on an invariant torus of dimension m will have at least m Liapunov exponents equal to zero. If one or more of the remaining exponents are positive, then the periodic or quasiperiodic orbits on the torus will depend sensitively on initial conditions.

Attractor	Sign of Liapunov exponents	Asymptotic dynamics
Stable equilibrium	$(-,-,-)$	Stationary
Stable periodic orbit	$(0,-,-)$	Periodic
Attracting torus	$(0,0,-)$	Quasiperiodic
Strange attractor	$(+,0,-)$	Chaotic

TABLE 4.1. Examples of attractors of a three-dimensional flow.

However, all the above invariant sets (equilibrium, periodic orbit or torus) must be unstable in order to have sensitive dependence on initial conditions, and thus they have zero measure and are not attractors. Orbits starting near these sets will not necessarily have positive Liapunov exponents.

By contrast, orbits of a strange attractor must attract all nearby orbits that do *not* belong to the attractor, while they repel nearby orbits that *do* belong to it. In order to attract nearby orbits, the flow must be dissipative in a neighbourhood of the attractor, so that by Proposition 4.2.9, the sum of all Liapunov exponents must be negative. Since orbits on the attractor are bounded and not attracted by an equilibrium point, one Liapunov exponent is equal to zero. Hence a two-dimensional flow cannot admit a strange attractor.[2] If a three-dimensional flow has a strange attractor, its Liapunov exponents must satisfy $\lambda_1 > \lambda_2 = 0 > \lambda_3$ and $|\lambda_3| > \lambda_1$ (see Table 4.1). Because of volume contraction, the attractor must have zero volume, but it cannot be a surface and have positive Liapunov exponents. This accounts for the fractal nature of many observed attractors.

For iterated maps, the situation is less restrictive, since their orbits need not have one Liapunov exponent equal to zero. Thus two-dimensional dissipative maps may admit a strange attractor, with Liapunov exponents satisfying $|\lambda_2| > \lambda_1 > 0 > \lambda_2$. For instance, the intersection of the strange attractor of a three-dimensional flow with a Poincaré section is also a strange attractor of the associated two-dimensional Poincaré map. One-dimensional iterated maps may have strange attractors if they are non-invertible.

4.2.3 The Hénon and Lorenz Attractors

The first dynamical system for which the existence of a strange attractor was proved is the Hénon map

$$\begin{aligned} x_{k+1} &= 1 - \lambda x_k^2 + y_k \\ y_{k+1} &= b x_k. \end{aligned} \tag{4.2.7}$$

This map does not describe a physical system. It has been introduced as a two-dimensional generalization of the one-dimensional map $x_{k+1} = 1 - \lambda x_k^2$, which is equivalent to the logistic map. When $b = 0$, (4.2.7) is reduced to this one-dimensional map. If $b \neq 0$, the Hénon map is invertible, and it is dissipative if $|b| < 1$. Numerical simulations indicate that for some parameter values, the Hénon map has indeed a strange attractor with a self-similar structure (Fig. 4.7).

[2] A theorem due to Poincaré and Bendixson states that a non-empty compact ω- or α-limit set of a planar flow is either a periodic orbit, or contains equilibrium points, which also rules out the existence of strange attractors for two-dimensional flows.

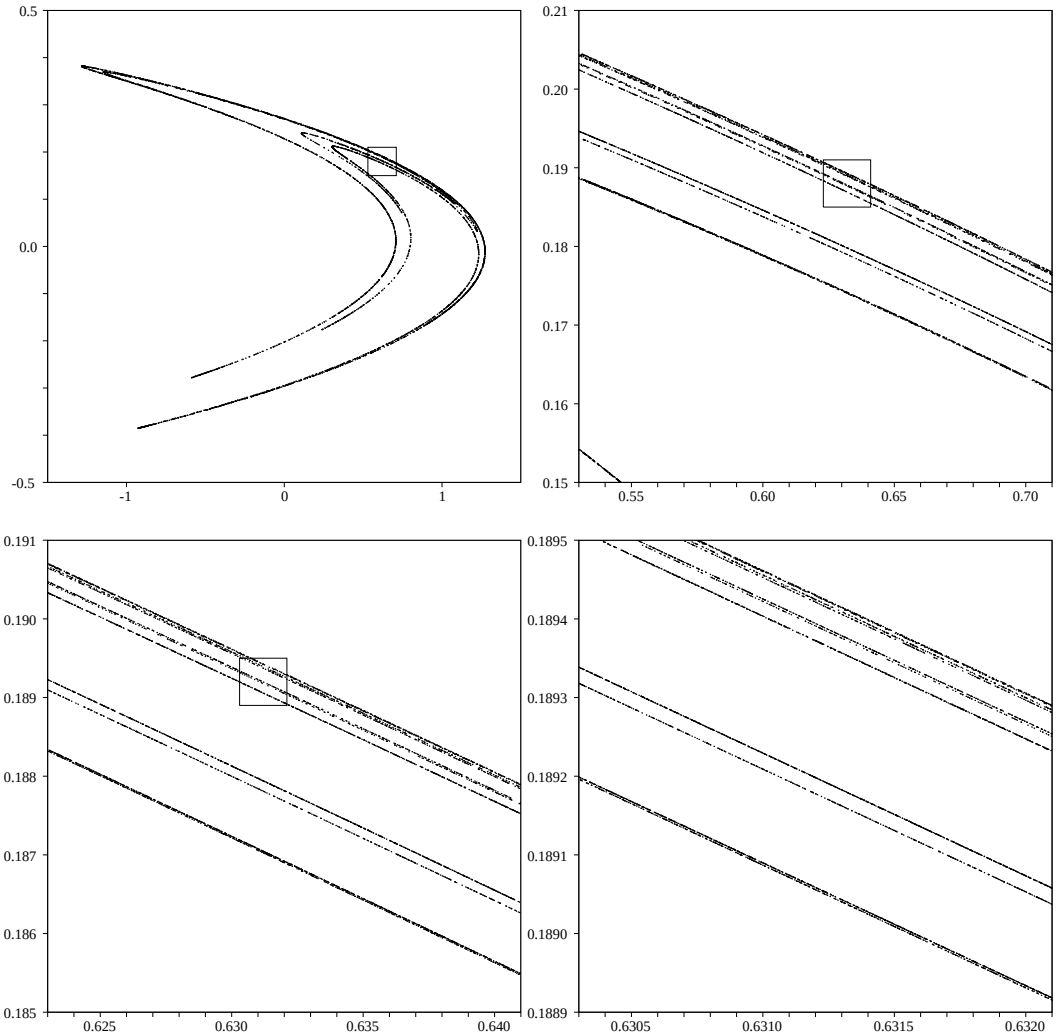

FIGURE 4.7. Hénon attractor for $\lambda = 1.4$ and $b = 0.3$. Successive magnifications of details, by a factor 10, show its self-similar structure.

Theorem 4.2.10 (Benedicks, Carleson [BC91]). *Let $z^\star = (x^\star, y^\star)$ be the fixed point of (4.2.7) with $x^\star$, $y^\star > 0$, and let W^{u} be its unstable manifold. For all $c < \log 2$, there is a set of positive Lebesgue measure of parameters (b, λ) for which*

1. *there is an open set $\mathcal{U} \subset \mathbb{R}^2$, depending on λ and b, such that for all $z \in \mathcal{U}$,*

$$T^k(z) \to \overline{W^{\mathrm{u}}} \qquad as \qquad k \to \infty; \tag{4.2.8}$$

2. *there is a point $z_0 \in W^{\mathrm{u}}$ such that*
   (a) *the positive orbit of z_0 is dense in W^{u};*
   (b) *the largest Liapunov exponent of the orbit of z_0 is larger than c.*

Hence the closure of the unstable manifold W^{u} is a strange attractor. Locally, the strange attractor is smooth in the unstable direction (with positive Liapunov exponent), and has the structure of a Cantor set in the transverse, stable direction (with negative Liapunov exponent).

As a final illustration, we return to the Lorenz equations

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -b x_3 + x_1 x_2. \end{aligned} \tag{4.2.9}$$

The existence of a strange attractor for this system has not been proved to our knowledge, although there is strong numerical evidence that such an attractor exists for certain parameter values, including in particular $\sigma = 10$, $b = 8/3$ and $r = 28$. The qualitative properties of dynamics are nonetheless quite well understood (see for instance [Sp82]). The strange attractor seems to appear after a rather subtle sequence of bifurcations. Let us consider the case $\sigma = 10$, $b = 8/3$, and take r as bifurcation parameter.

1. For $0 \leqslant r \leqslant 1$, the origin is globally asymptotically stable, that is, all orbits converge to the origin (c.f. Exercise 2.3).
2. At $r = 1$, the origin undergoes a pitchfork bifurcation (c.f. Exercise 3.1), and two new stable equilibria $C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ appear. These points become unstable in a Hopf bifurcation at $r = \frac{470}{19} \simeq 24.74$ (c.f. Exercise 2.2). The Hopf bifurcation is subcritical, and thus corresponds to the destruction of an unstable periodic orbit, that must have been created somehow for a smaller value of r.
   The dynamics for $r > 1$ can be described by taking a Poincaré section on the surface $\Sigma : x_3 = r - 1$ (we only take into account intersections with $\dot{x}_3 < 0$). Consider a rectangle containing the segment C_-C_+ and the intersection S of the two-dimensional stable manifold of the origin with Σ (Fig. 4.8). S is attracted by the origin, which has the effect to pinch the rectangle and map it to two pieces of triangular shape. One vertice of each triangle belongs to a piece of the one-dimensional unstable manifold W^{u} of the origin. Due to the dissipation, the angle at this vertice is quite small. In first approximation, the dynamics can thus be described by a one-dimensional map for a coordinate transverse to S, parametrizing the long side of the triangles. For r sufficiently small, this map is increasing, discontinuous at 0, and admits two stable fixed points $c_\pm$ corresponding to $C_\pm$ (Fig. 4.8a).
3. At $r = r_1 \simeq 13.296$, a homoclinic bifurcation occurs (Fig. 4.8b): the unstable manifold W^{u} belongs to the stable manifold W^{s}, and thus the sharpest vertices of both triangles belong to S. The one-dimensional approximation of the Poincaré map is continuous, but still monotonously increasing.
4. For r slightly larger than r_1, each piece of W^{u} hits Σ on the opposite side of S (Fig. 4.8c). The one-dimensional map becomes non-invertible, and has two new unstable equilibria b_- and b_+. The map does not leave the interval $[b_-, b_+]$ invariant, but maps two of its subintervals onto $[b_-, b_+]$. The situation is thus similar to that of the tent map g_3: there exists an invariant Cantor set containing chaotic orbits. The same can be seen to hold for the two-dimensional Poincaré map, which resembles the horseshoe map. Thus there exists a strange nonwandering set Λ, but it is repelling and has zero measure (thus it is difficult to observe numerically).
5. As r increases, the fixed points $b_\pm$ move towards the fixed points $c_\pm$. It appears that for $r \geqslant r_2 \simeq 24.06$, the interval $[b_-, b_+]$ is mapped into itself (Fig. 4.8d). The strange nonwandering set Λ becomes attracting, although it does not yet attract a full neighbourhood, since orbits starting in $b_\pm$ may converge to $c_\pm$.
6. The subcritical Hopf bifurcation at $r = r_3 \simeq 24.74$ makes the fixed points $C_\pm$ unstable, so that finally all orbits can converge to the nonwandering set Λ, which has become

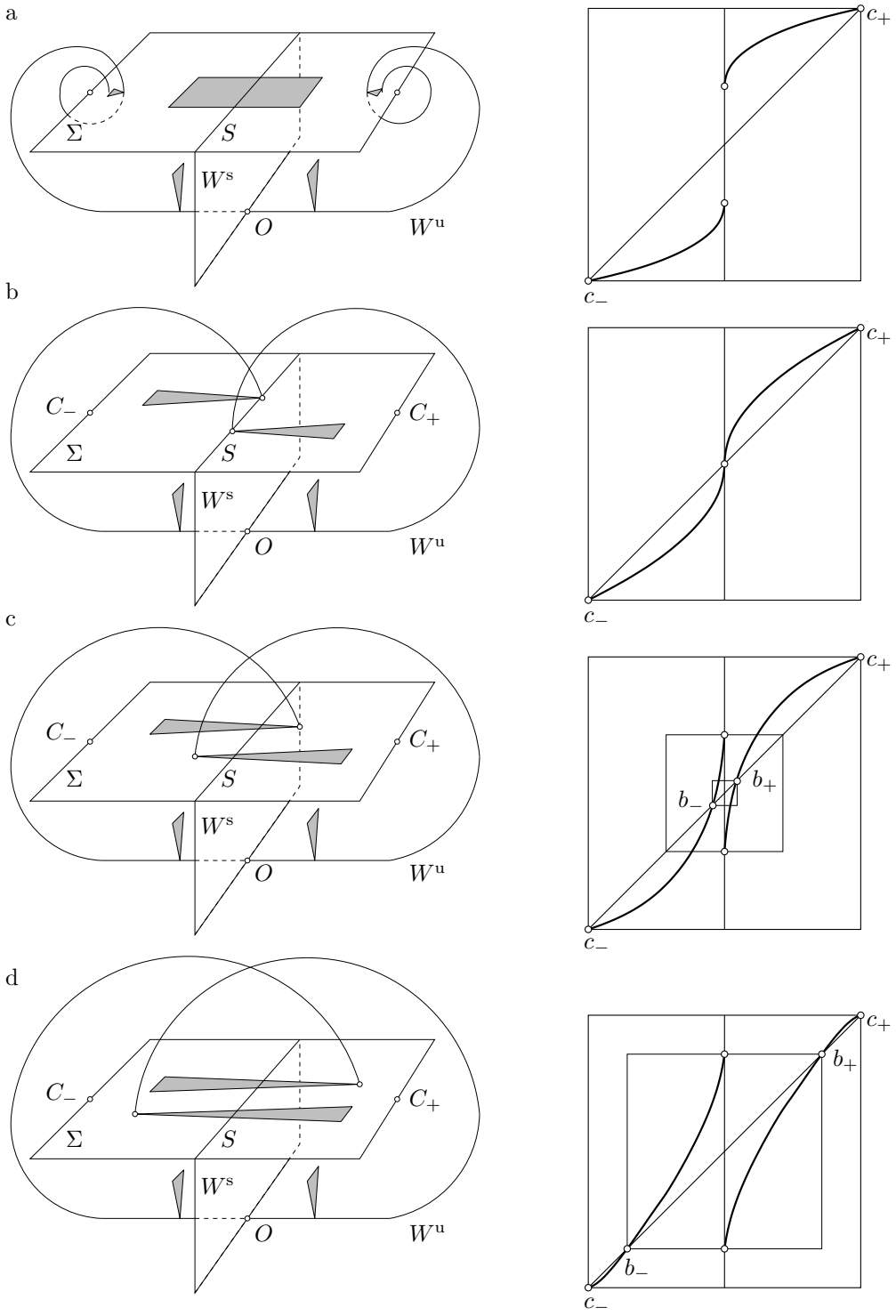

FIGURE 4.8. Schematic representation of the stable and unstable manifolds of the Lorenz equations (left), and one-dimensional approximation of the Poincaré map for the coordinate perpendicular to W^s (right). (a) $1 < r < r_1$, (b) homoclinic bifurcation at $r = r_1$, (c) existence of a nonwandering repeller for $r_1 < r < r_2$, and (d) bifurcation to a strange attractor at $r = r_2$.

an attractor.

The method used here to describe the dynamics near a homoclinic bifurcation by a Poincaré map has been applied to other systems. Transitions to chaotic behaviour are quite frequently associated with such homoclinic bifurcations. The Poincaré map in a vicinity of the unstable manifold is described as the composition of an almost linear map, reflecting the motion near the equilibrium point, and a nonlinear, but usually rather simple map, reflecting the motion near W^{u} away from the equilibrium. The composition of two rather innocent-looking maps contains all the ingredients necessary for the existence of chaotic dynamics.

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