

# Can a local repulsive potential trap an electron?

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We study the classical dynamics of a charged particle in two dimensions, under the influence of a perpendicular magnetic and an in-plane electric field. We prove the surprising fact that there is a finite region in phase space that corresponds to the otherwise drifting particle being trapped by a local *repulsive* potential. Our result is a direct consequence of KAM-theory and, in particular, of Moser's theorem. We illustrate it by numerical phase portraits and by an analytic approximation to invariant curves.

The detailed dynamics of electrons in two dimensions (2D), in electromagnetic fields and with localized scatterers, is important for several aspects of the quantum Hall effect. Classical dynamics is more relevant in this context than one might think, see the recent review by Trugman [1]. In addition, classical kinetic theory for such systems is surprisingly subtle [2], and requires for its foundation an understanding of the underlying dynamics. In this letter we focus on the *classical* dynamics of an “electron” (i.e., a particle with charge  $-e$ ) confined to the  $xy$ -plane. Perpendicular to these two dimensions there is a constant magnetic field  $\vec{B} = B\mathbf{e}_z$ . In addition, there is a constant in-plane electric field  $\vec{E} = E\mathbf{e}_y$ . When no forces act on the electron except those stemming from these fields, the nature of the motion is well known. With  $E = 0$ ,  $B \neq 0$ , the electron with energy  $E = \frac{1}{2}mv^2$  moves in a circular orbit with cyclotron radius  $R = v/\omega_c$  and cyclotron frequency  $\omega_c = eB/m$ . Addition of a finite electric field results in the center of this orbit acquiring a constant drift velocity  $\vec{v}_d = (\vec{E} \times \vec{B})/B^2 = (E/B)\mathbf{e}_x$ . The question we address in this letter is the following: With both  $\vec{E}$ - and  $\vec{B}$ -field present, is there a finite region in phase space which corresponds to the otherwise drifting electron being trapped by a single scatterer, i.e., by a local *repulsive* potential? (For concreteness, we let the repulsive potential be a hard disk of radius  $a$ .) This question arises, for example, when one tries to construct a kinetic theory for the Lorentz gas in 2D, with both electromagnetic fields present.

When  $\vec{E} = 0$ ,  $\vec{B} \neq 0$  all electrons are, in a sense,

trapped by the magnetic field, and the hard disk simply modifies the cyclotron orbits into orbits skipping around the periphery of the disk (see for instance [3]). On the other hand, when  $\vec{E} \neq 0$ ,  $\vec{B} = 0$ , one or more collisions with the disk will only temporarily perturb the dynamics determined by a constant acceleration in the negative electric field direction  $-\mathbf{e}_y$ . With both fields present, the question is whether an electron initially colliding with the disk, will ultimately miss it, due to the drift caused by the electric field.

There are two dimensionless parameters in the problem, the ratio of the cyclotron radius and that of the disk,  $r \equiv R/a$ , and the ratio of the displacement,  $v_d \cdot 2\pi/\omega_c$ , during one cyclotron period and the radius of the disk,  $\varepsilon \equiv (2\pi m/ae) \cdot (E/B^2)$ . Clearly, the answer to our question whether the disk can trap the electron when drift is included, must depend on where in this two-dimensional parameter space it is posed. Our aim in this letter is not to give a complete answer to the question, with parameter regions and corresponding measures precisely delineated. Our more modest aim is to demonstrate the surprising *qualitative* fact that a repulsive potential can, indeed, trap a charged particle in crossed electromagnetic fields in 2D.

We first emphasize that an electron drifting towards the hard disk from “infinity” cannot be trapped. The main elements of the proof are: To the first collision with the disk is associated a finite measure in phase space. Consider the subset of those incoming trajectories (if they exist) that will hit the disk at least  $N$  times. Since classical dynamics is measure preserving, the regions in phase space associated with each of the  $N$  collisions have the same measure. Reversibility (invariance under reversal of time and magnetic field) implies that they do not overlap. Since the total phase space associated with an electron colliding with the disk is finite, it follows that the measure of the trajectories coming from infinity and hitting the scatterer at least  $N$  times has to converge to zero as  $N \rightarrow \infty$ . (It may, of course, vanish for a finite  $N$ .) Trapping is, therefore, only possible for electrons that are initially close to the scatterer.

Naively one might think that conditions are more favorable for trapping when  $r \ll 1$ . If this were so, not

much room would be left for trapping: When  $r \ll 1$ , the perimeter of the disk can, on the scale set by the cyclotron radius, be considered a straight line. An electron skipping upwards along the right side of the disk will, unless the curvature of the disk makes itself felt in time, sooner or later miss the disk and drift away from it.

Surprisingly, it is the opposite regime, when  $r \gg 1$ , which is more favorable for trapping. Here we can, on the scale of the radius of the disk, consider the parts of the electron orbit close to the disk as straight lines. Two subsequent collisions are shown in Fig.1. The direction of the motion just prior to collision no.  $n$  is defined by the angle  $\phi_n$ . To the extent that the orbit locally can be considered to consist of straight lines, the scattering angle of collision no.  $n$  is given by  $\psi_n = \phi_{n+1} - \phi_n$ . Due to the electric field, the incoming line heading for collision no.  $(n+1)$  is shifted by the amount  $\varepsilon a$  in the positive  $x$ -direction with respect to the outgoing line from collision no.  $n$ . In terms of the angles  $\phi_n$  and the dimensionless impact parameters  $\beta_n = b_n/a$ , inspection of Fig.1 gives the map,

$$\begin{aligned}\phi_{n+1} &= \phi_n + \pi - 2 \sin^{-1} \beta_n \\ \beta_{n+1} &= \beta_n - \varepsilon \sin \phi_{n+1}.\end{aligned}\quad (1)$$

In contrast to standard scattering conventions in three dimensions, inherent in the present map is the following choice: The impact parameter  $\beta$  lives on the interval  $(-1, 1)$  and the scattering angle  $\psi$  on the interval  $[0, 2\pi)$ , with  $\beta \rightarrow -1$  corresponding to  $\psi \rightarrow 2\pi$ .

One can show that the map resulting from the full dynamics for arbitrary  $\mathcal{B}$  reduces to the map (1) when the time interval between successive collisions is approximated by the cyclotron period  $2\pi/\omega_c$ . This implies that corrections to (1) are uniformly bounded by a constant times  $a/R = r^{-1}$ . Consequently, the map (1) is reliable when  $r \gg 1$ .

The map (1) has several interesting properties. First of all, since its Jacobian equals unity, it is area-preserving. Thus, our weak  $\mathcal{B}$ -field approximation, basic to (1), has exactly preserved this general property of Hamiltonian maps. Next, for small  $\varepsilon$ , (1) is a small perturbation of the integrable map

$$\begin{aligned}\phi_{n+1} &= \phi_n + \Omega(\beta_n) \\ \beta_{n+1} &= \beta_n,\end{aligned}\quad (2)$$

where  $\Omega(\beta) = \pi - 2 \sin^{-1} \beta$  describes scattering off a hard disk. From (2) we recover the well-known fact that the scattering angle  $\psi_n = \phi_{n+1} - \phi_n$  remains the same in subsequent collisions, when the electric field vanishes [3]. Furthermore,  $\Omega(\beta)$  is a monotonic function of its argument and, as a result, our map (1) belongs to the category of twist maps [4], for which a number of properties have been rigorously established.

However, the question of principal interest from our point of view is the existence of trapped orbits, and the

phase space associated with them, when  $\varepsilon > 0$ . The function  $\Omega(\beta)$  is undefined when  $|\beta| > 1$ . Physically, this corresponds to the electron missing the disk. The regions of first and last collision are represented in phase space by areas of order  $\varepsilon$  near  $\beta = +1$  and  $\beta = -1$ . The nontrivial question is, therefore, whether there is a finite region of phase space associated with orbits that *never* enter the escape region.

For small enough  $\varepsilon$ , Kolmogorov–Arnold–Moser (KAM) theory enables us to give a positive answer to this question. Specifically, in our case of a two-dimensional map, we will use Moser’s theorem [5] which proves the existence of invariant curves in phase space for small perturbations, and for some initial conditions. The existence of two distinct invariant curves winding around phase space, together with area preservation, shows the finite region enclosed between these two curves to be an invariant set, corresponding to trapped orbits. It is straightforward to verify that the map (1) (or any small perturbation of it that remains area preserving) satisfies the hypotheses of Moser’s theorem, provided that we restrict phase space to a smaller strip  $|\beta| < c < 1$ , in order to avoid the escape regions and the singularities of  $\Omega$ . The important point is that this theorem does *not* require the entire strip to be invariant under the map. We have thus proved the existence, for small electric and magnetic fields, of invariant curves in phase space, and hence of trapped orbits with positive measure.

This rigorous result is illustrated numerically by figures 2 and 3, where we show two different phase portraits of the map. When  $\varepsilon = 0.2$ , many invariant curves are present. We can divide them into two classes: those winding around phase space, which we study in this letter, and those forming “elliptic islands” around periodic orbits. When  $\varepsilon = 0.4$ , the structure of phase space has become more complex. Invariant curves winding around phase space are now scarce and, numerically, all of them seem to be destroyed between  $\varepsilon = 0.4$  and  $\varepsilon = 0.5$ . On the other hand, elliptic islands, which are numerous in Fig.3, survive at much stronger electric fields (certainly up to  $\varepsilon \sim 1.1$ ). However, most of the chaotic trajectories, which remained bounded for small  $\varepsilon$ , diffuse until they reach the escape region.

As an analytic illustration of our rigorous result, we have also constructed a perturbation scheme around (2), close in spirit to that used in the proof of Moser’s theorem [5]. We start from the Ansatz [6],

$$\begin{aligned}\phi_n &= \zeta + n\omega + \varepsilon u_1(n; \zeta, \omega) + \varepsilon^2 u_2(n; \zeta, \omega) + \cdots \\ \psi_n &= \phi_{n+1} - \phi_n = \omega + \varepsilon v_1(n; \zeta, \omega) + \varepsilon^2 v_2(n; \zeta, \omega) + \cdots,\end{aligned}\quad (3)$$

with  $v_i(n) = u_i(n+1) - u_i(n)$ . Both the “effective initial condition”  $\zeta$ , and the “angular frequency”  $\omega$  depend on  $\varepsilon$  and on the true initial conditions  $(\phi_0, \psi_0)$ . Setting  $n = 0$  in (3) gives the inverse relations  $\phi_0 = \phi_0(\zeta, \omega; \varepsilon)$ ,  $\psi_0 = \psi_0(\zeta, \omega; \varepsilon)$  via the functions  $u_i(0; \zeta, \omega)$ ,  $v_i(0; \zeta, \omega)$ . We

determine  $u_i(n)$  and  $v_i(n)$  by inserting the Ansatz (3) into the map (1) and doing straightforward perturbation theory, treating  $\zeta$  and  $\omega$  as constants in the process. We fix the “initial values”  $u_i(0)$  and  $v_i(0)$  by *requiring* that the oscillating functions  $u_i(n)$  and  $v_i(n)$  average to zero for large  $n$ . (This is indeed necessary for the Ansatz to be consistent.) Convergence of the scheme is assured [5] if  $\omega/2\pi$  is a Diophantine number (loosely speaking: an irrational number which is hard to approximate by a sequence of rationals).

To zeroth order the result is simple,  $\zeta = \phi_0$ ,  $\omega = \psi_0$ , reproducing the skipping orbits in zero electric field. To first order, perturbation theory gives

$$v_1(n) = \frac{1}{\sin^2(\omega/2)} \{ \cos[\zeta + \omega/2] - \cos[\zeta + (n + \frac{1}{2})\omega] \} + v_1(0). \quad (4)$$

In (4) we determine  $v_1(0)$  by requiring  $v_1(n)$  to average to zero for large  $n$ . We can then calculate  $u_1(n)$  by direct summation, again fixing  $u_1(0)$  by the stipulation that  $u_1(n)$  averages to zero. The results are,

$$v_1(0) = -\frac{\cos(\zeta + \omega/2)}{\sin^2(\omega/2)} \quad ; \quad u_1(0) = -\frac{\sin \zeta}{2 \sin^3(\omega/2)}. \quad (5)$$

In this way one can continue. We are primarily interested in the frequency  $\omega(\varepsilon)$ . To second order, we determine  $v_2(0)$  in the same manner, and construct the inverse relation from (3) to  $\mathcal{O}(\varepsilon^2)$  as

$$\psi_0 = \omega - \varepsilon \frac{\cos(\zeta + \omega/2)}{\sin^2(\omega/2)} + \varepsilon^2 \frac{\cos(2\zeta + \omega)}{8 \sin^3(\omega/2) \cos(\omega/2)}. \quad (6)$$

Inversion of this relation (with appeal to the first order result for  $\zeta$ ) to  $\mathcal{O}(\varepsilon^2)$  finally yields

$$\omega = \psi_0 + \varepsilon \frac{\cos(\phi_0 + \psi_0/2)}{\sin^2(\psi_0/2)} - \varepsilon^2 \left[ \frac{\cos(2\phi_0 + \psi_0)}{8 \sin^3(\psi_0/2) \cos(\psi_0/2)} + \frac{\cos(\psi_0/2) [\cos(2\phi_0 + \psi_0) + 3]}{4 \sin^5(\psi_0/2)} \right] + \mathcal{O}(\varepsilon^3). \quad (7)$$

Note that in the second order term of (6), there appears a “small denominator”,  $\cos(\omega/2)$ , which vanishes at  $\omega = \pi$ , i.e.,  $\beta = 0$  (where an orbit of period two exists). Higher order terms will generate a proliferation of other small denominators. This denominator appearing in second order is the first symptom indicating that convergence of the scheme requires  $\omega/2\pi$  to be a Diophantine number. To see that the solution indeed describes an invariant curve, it is sufficient to note that, by construction,  $v_i(n; \zeta, \omega) = v_i(0; \zeta + n\omega, \omega)$ , with a similar equation for  $u_i$ . Thus, for given  $\varepsilon$  and  $\omega$ ,  $\phi_n$  and  $\psi_n$  depend only on the combination  $\zeta + n\omega$ . Elimination of this parameter and use of (2) yields an explicit equation for the invariant curve,  $\beta_n = \Omega^{-1}(\psi_n) = f(\phi_n; \omega, \varepsilon)$ . As stated above, once two distinct invariant curves have been constructed,

they delimit an invariant region of positive measure corresponding to trapped electrons.

For Diophantine  $\omega/2\pi$ , when the perturbation scheme implied by (3) converges, by taking the average one concludes that  $\langle \psi_n \rangle = \omega$ . In Fig.4 we show a comparison of the scattering angle averaged over 1 000 000 collisions, and our analytic perturbation result for the frequency, with the initial conditions  $(\phi_0 = 0.3, \psi_0 = 2.0)$ . For small  $\varepsilon$  the results are in very good agreement, despite the fact that  $\omega$  takes both rational and irrational values. For larger  $\varepsilon$ , one can observe the breakdown of perturbation theory, due to the phenomenon of “mode locking”: On the interval of roughly  $0.20 < \varepsilon < 0.45$  the electron is trapped in the vicinity of a periodic orbit with period 3 and, as a consequence,  $\langle \psi_n \rangle / 2\pi = 1/3$ , a rational number.

In conclusion, we have shown that for sufficiently small electric and magnetic fields, bound states associated with a hard disk scatterer constitute a set of positive measure. Indeed, both fields add small and smooth perturbations to the integrable map (2), so that KAM theory applies. By investigating the stability of periodic orbits, it is in fact possible to show that trapping occurs also for moderate values of the magnetic field. Note, finally, that generalization from a hard disk to an arbitrary, rotationally invariant repulsive potential of strictly finite range, leaves the scattering function  $\Omega(\beta)$  monotonic. In other words, the corresponding map is still a twist map, and all qualitative conclusions remain unchanged. In short, we have accomplished our goal and, within classical mechanics, answered the question: *Yes*, a local repulsive potential *can* trap an electron! With quantum mechanics, the question remains open.

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- [1] S. A. Trugman, *Physica D* **83**, 271 (1995).
  - [2] A. V. Bobylev, Frank A. Maaø, Alex Hansen and E. H. Hauge, *Phys.Rev.Lett.* **75**, 197 (1995); *J.Stat.Phys.* (in press); J. Piasecki, Alex Hansen and E. H. Hauge, to be published.
  - [3] N. Berglund and H. Kunz, *J. Stat. Phys.* **83**, 81 (1996).
  - [4] J. D. Meiss, *Rev. Mod. Phys.* **64**, 795 (1992).
  - [5] J. Moser, *Nachr. Akad. Wiss., Göttingen, Math. Phys. Kl.*, 1 (1962); *Stable and Random Motions in Dynamical*

*Systems* (Princeton University Press, New Jersey, 1973).

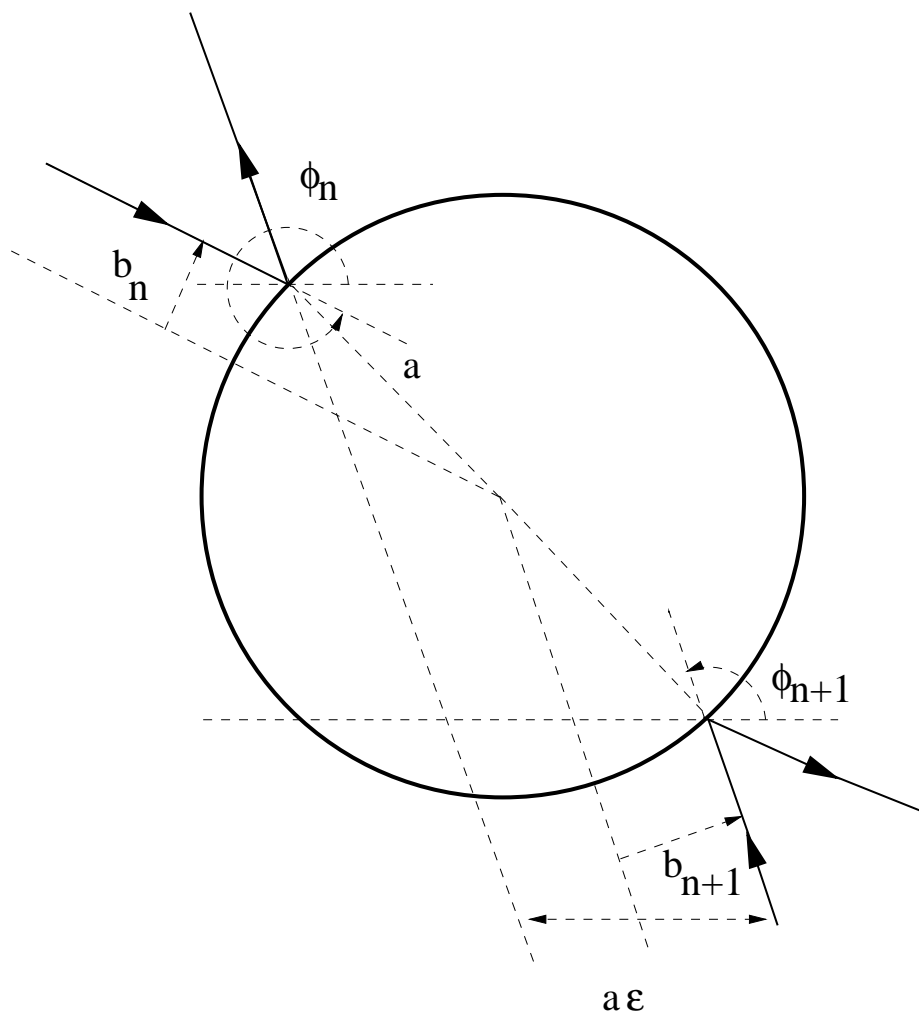
[6] Some readers may find the following motivation for the Ansatz useful: An invariant curve has a parametric equation of the form  $\phi = \xi + \varepsilon U(\xi, \omega, \varepsilon)$ ,  $\psi = \omega + \varepsilon V(\xi, \omega, \varepsilon)$ . The parameter  $\xi$  grows proportionally to time with “angular frequency”  $\omega$ :  $\xi_n = \zeta + n\omega$ .

FIG. 1. Geometry of two successive collisions with the hard disk.

FIG. 2. Phase portrait of the mapping for  $\varepsilon = 0.2$ . The escape regions are recognized as those close to the upper and lower part of the figure. Note that orbits of period 2, 3 and 4 are easily located. Between the periodic orbits are bands of invariant curves, described by the analytic theory of the text.

FIG. 3. Same as Fig.2, but for  $\varepsilon = 0.4$ . The escape regions have grown in size, more periodic orbits can be recognized, and the regions with invariant curves have shrunk considerably. In addition, chaotic regions (which remain trapped!) have become manifest.

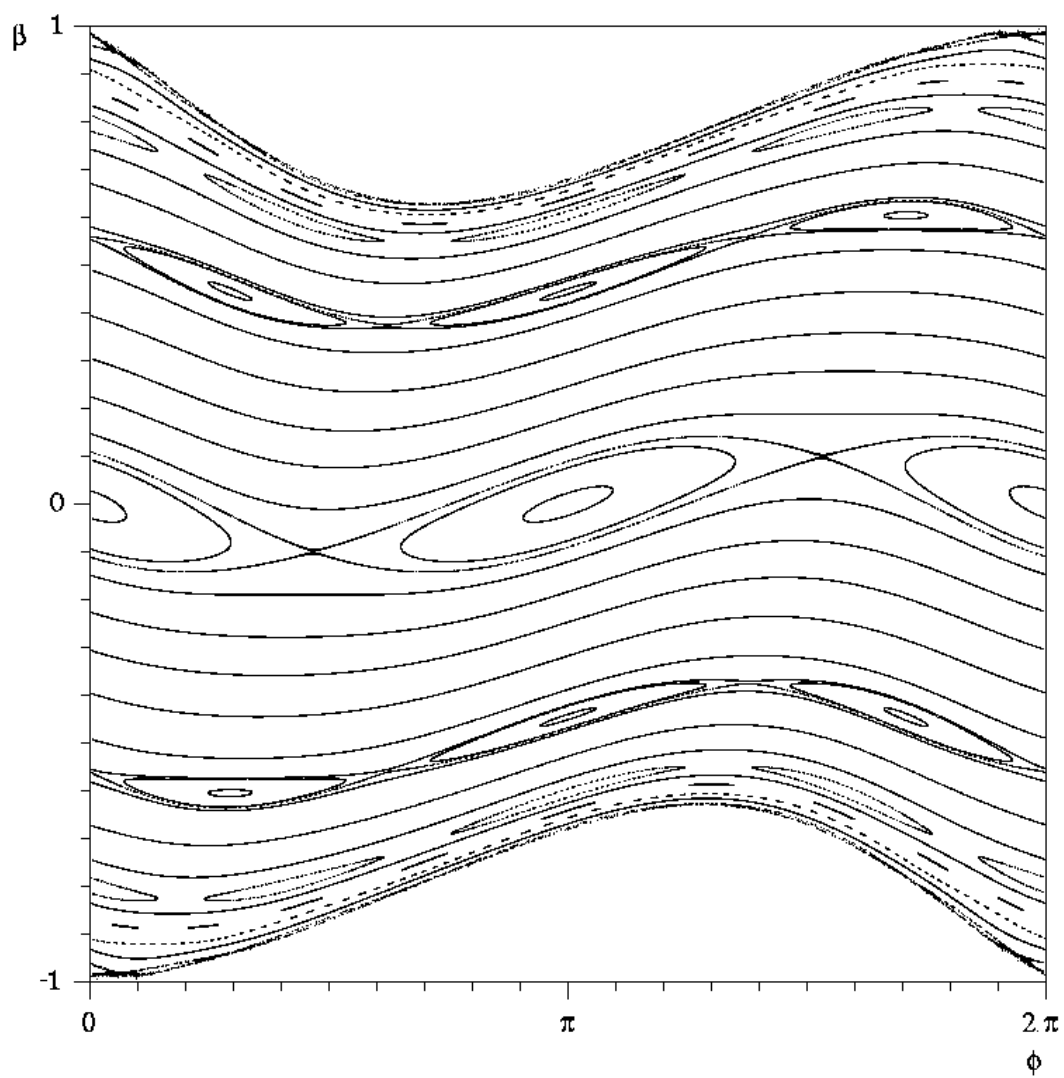
FIG. 4. Comparison between the “frequency”  $\omega/2\pi$  as computed from our analytic approximations to first and second order, and  $\langle\psi_n\rangle/2\pi$  obtained from the simulation, where the average has been taken over 1 000 000 collisions, starting from the initial condition  $\phi_0 = 0.3, \psi_0 = 2.0$ . Note, with reference to Fig.2, that the electron is mode locked to a period 3 orbit,  $\langle\psi_n\rangle/2\pi = 1/3$ , from  $\varepsilon \approx 0.2$  to  $\varepsilon \approx 0.45$ .



**Figure 1**

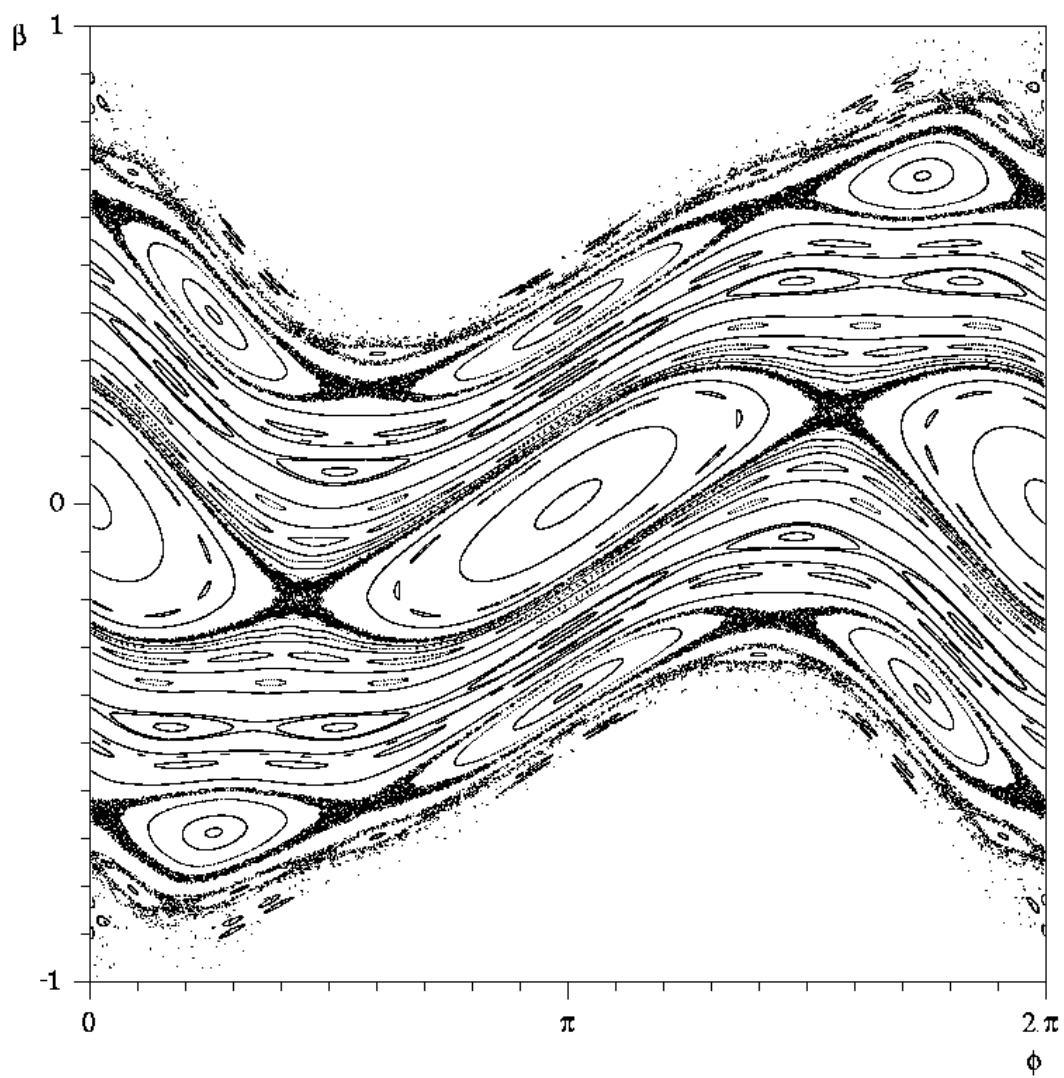
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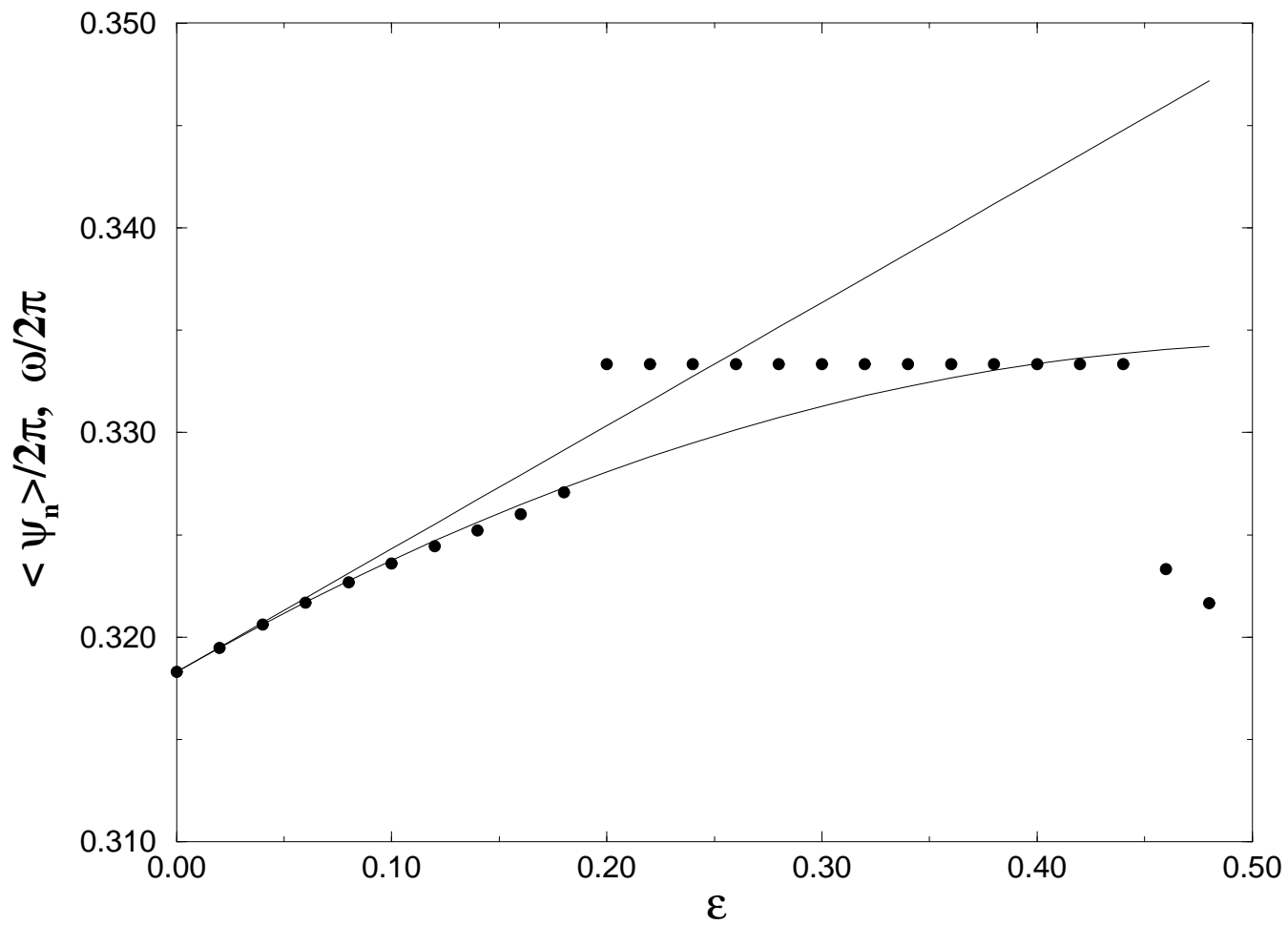
$\varepsilon = 0.2$

Figure 2  
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$\varepsilon = 0.4$

Figure 3  
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**Figure 4**

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