Bifurcations, scaling laws and hysteresis in singularly perturbed systems

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Abstract

Relations between the solutions of singularly perturbed differential equations $\varepsilon\dot{x} = f(x,t)$ and bifurcations in the associated one-parameter families of differential equations $\dot{x} = f(x,\lambda)$ are examined. We discuss bifurcations with single or double zero eigenvalue, as well as the Hopf bifurcation.

We consider singularly perturbed dynamical systems of the form

$$\varepsilon\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), \qquad x \in \mathbb{R}^d,\ t \in \mathbb{R},\ 0 < \varepsilon \ll 1, \tag{1}$$

where f is analytic in an open subset of $\mathbb{R}^d \times \mathbb{R}$. Our aim is to determine the qualitative behaviour of solutions of (1), using properties of the associated family of dynamical systems

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f(x,\lambda), \tag{2}$$

where λ is considered as a fixed parameter. Equation (1) can be viewed as a version of (2) in which the parameter $\lambda = \varepsilon s = t$ is made slowly time–dependent.

The existence of a relation between solutions of (1) and (2) is confirmed by the following result [1, 2, 3, 4]: Assume that $x^\star(\lambda)$ is a family of hyperbolic equilibria of (2). Then (1) admits a particular solution $\bar{x}(t)$ such that $\|\bar{x}(t) - x^\star(t)\| \leqslant c\varepsilon$, uniformly for τ in compact intervals. If $x^\star(\lambda)$ is asymptotically stable, $\bar{x}(t)$ attracts nearby solutions exponentially fast. If $x^\star(\lambda)$ has both stable and unstable manifolds, one can associate time-dependent invariant manifolds with $\bar{x}(t)$ on which the motion is either contracting or expanding.

This naturally raises the question of what effect a bifurcation in (2) has on the dynamics of (1). We will discuss several types of local bifurcations. See [5, 6, 7] for related results.

1 Bifurcations with simple zero eigenvalue

Bifurcations with a simple zero eigenvalue have two main effects: Firstly, when approaching the bifurcation point, solutions tracking an equilibrium are no longer attracted at an exponential rate. This results in the distance between solution and equilibrium scaling as another power of ε than 1. Secondly, there may be several equilibrium branches emerging from the bifurcation point, or no branch at all. The solutions will then choose between tracking one of the outgoing branches, jumping on another attractor, or diverging.

If the function f depends periodically on time, the second mechanism may result in hysteresis phenomena [8, 9], which may be chaotic, even for arbitrarily small ε [10]. In this case, the first mechanism produces non-trivial scaling laws for the area of hysteresis cycles. The scaling exponents can be obtained in a simple way from the Newton polygon associated with the bifurcation

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point. Consider (1) for $d=1$ and

$$f(x,t)=\sum_{n,m} c_{nm}x^n t^m, \qquad c_{00}=c_{10}=0. \tag{3}$$

A well-known result of bifurcation theory states that if $f(x,t)$ vanishes on a curve[2] $x=x^\star(\tau)\approx|\tau|^q$, then $-q$ is necessarily the slope of a segment of Newton's polygon, obtained by taking the convex envelope of the points $(n,m)\in\mathbb{N}^2$ such that $c_{nm}\neq 0$. One can show that generically, the function $a(t)=\partial_x f(x^\star(t),t)$ satisfies

$$|a(t)|\approx|t|^p, \qquad p=\min_{n\geqslant 1, m\geqslant 0}\bigl\{q(n-1)+m \bigm| c_{nm}\neq 0\bigr\}. \tag{4}$$

Graphically, p is the ordinate at 1 of the tangent to Newton's polygon with slope $-q$.

Theorem 1 ([11, 9]). *Assume that for $-T\leqslant t<0$, f admits an equilibrium branch $x^\star(t)\approx|t|^q$, satisfying (4), and such that $f(x^\star(t)+y,t)<0$ for small positive y. Then (1) admits a particular solution satisfying*

$$\bar{x}(t)-x^\star(t)\approx\begin{cases}\varepsilon|t|^{q-p-1} & \text{if } -T\leqslant t\leqslant -\varepsilon^{1/p+1}\\ \varepsilon^{q/p+1} & \text{if } -\varepsilon^{1/p+1}\leqslant t\leqslant 0\end{cases} \tag{5}$$

2 Hopf Bifurcation

Neishtadt showed that the slow passage through a Hopf bifurcation results in the delayed appearance of oscillations. Assume that the linearization around the equilibrium $x^\star(\lambda)$ has two eigenvalues $a(\lambda)\pm\mathrm{i}\,\omega(\lambda)$, where $a(\lambda)$ has the same sign as λ, $\omega(0)\neq 0$, and all other eigenvalues have a strictly negative real part.

Theorem 2 ([12]). *All solutions of (1) starting in some neighbourhood of $x^\star(t_0)$ at $t_0<0$ track the equilibrium $x^\star(t)$ at a distance of order ε for $t_0+\delta(\varepsilon)\leqslant t\leqslant\hat{t}-\delta(\varepsilon)$, where $\delta(\varepsilon)$ goes continuously to zero as $\varepsilon\to 0$, and $\hat{t}=\min\{\Pi(t_0),t_+\}$ is constructed in the following way. For $t\in\mathbb{C}$ define*

$$\Psi(t)=\int_0^t a(s)+\mathrm{i}\,\omega(s)\,\mathrm{d}s. \tag{6}$$

Then $\Pi(t_0)$ is the smallest positive time such that $\operatorname{Re}\Psi(t)=\operatorname{Re}\Psi(t_0)$. The buffer time t_+ is the largest positive time which can be connected to the negative real axis by a path in the complex plane with constant $\operatorname{Re}\Psi$, having certain regularity properties stated in [12].

3 Bifurcations with double zero eigenvalue

The delayed appearance of large amplitude oscillations in the dynamic Hopf bifurcation may have disastrous consequences, if the equation describes a slowly ageing device which reaches a stability boundary, since it results in a sudden jump of some state variable. A way to prevent this delay is to control the system by a state feedback of the form

$$\varepsilon\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,t)+b\,u(x,t), \tag{7}$$

[2] We write $\varphi(\varepsilon,t)\approx\psi(\varepsilon,t)$ if there exist constants $c_\pm>0$ such that $c_-\varphi\leqslant\psi\leqslant c_+\varphi$ for small ε,t.

where $f(x,t)$ is the original vector field undergoing Hopf bifurcation, b is a given vector in $\mathbb{R}^n$, and $u(x,t)$ is the scalar feedback control. Theorem 2 indicates that in order to suppress the bifurcation delay, one may try to decrease the buffer time t_+ by producing a bifurcation with double zero eigenvalue. In [13] we show that the control u can be constructed in such a way that the dynamics of (7) is described by the effective two-dimensional equation

$$\begin{aligned} \varepsilon\dot{\xi} &= \eta \\ \varepsilon\dot{\eta} &= \mu(t)\xi + 2a(t)\eta + \gamma(t)\xi^2 + \delta(t)\xi\eta - \xi^3 - \xi^2\eta + \varepsilon R(\xi,\eta,t,\varepsilon) + \mathcal{O}((\xi^2+\eta^2)^2), \end{aligned} \tag{8}$$

where $a(0) = \mu(0) = \gamma(0) = \delta(0) = 0$, and $R(0,0,t,0)$ is related to the drift $\mathrm{d}_t x^\star(t)$. The associated system is a so-called cubic Liénard equation, which is a codimension four unfolding of the singular vector field $(\eta, -\xi^3 - \xi^2\eta)$ [14].

Theorem 3 ([13]). *Assume that $\mu'(0) > 2a'(0) > 0$ and $R(0,0,0,0) \neq 0$. Then solutions of (8) starting sufficiently close to the origin at $t_0 < 0$ will track, for $t > 0$, one of the equilibrium branches of (8) given by $\eta = 0$ and $\xi = \pm ct^{1/2} + \mathcal{O}(t)$, at a distance at most $\mathcal{O}(\varepsilon^{1/3})$. In other words, the bifurcation delay is suppressed since solutions do not track the original equilibrium represented by $\xi = \eta = 0$.*

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