# Concentration estimates for SPDEs driven by fractional Brownian motion

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**Abstract**

The main goal of this work is to provide sample-path estimates for the solution of slowly time-dependent SPDEs perturbed by a cylindrical fractional Brownian motion. Our strategy is similar to the approach by Berglund and Nader for space-time white noise. However, the setting of fractional Brownian motion does not allow us to use any martingale methods. Using instead optimal estimates for the probability that the supremum of a Gaussian process exceeds a certain level, we derive concentration estimates for the solution of the SPDE, provided that the Hurst index $H$ of the fractional Brownian motion satisfies $H > \frac{1}{4}$. As a by-product, we also obtain concentration estimates for one-dimensional fractional SDEs valid for any $H \in (0,1)$.

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## 1 Introduction

Fractional Brownian motion (fBm) is a famous example of stochastic process used in order to model memory effects or long-range dependencies. An fBm is a centered, stationary Gaussian process parameterized by a so-called Hurst index/parameter $H \in (0,1)$. For $H = \frac{1}{2}$, one recovers the classical Brownian motion. However, for $H \in (\frac{1}{2},1)$ and $H \in (0,\frac{1}{2})$, fBm exhibits a different behaviour than Brownian motion. Its increments are no longer independent, but positively correlated for $H > \frac{1}{2}$, and negatively correlated for $H < \frac{1}{2}$. Fractional Brownian motion has been used to model a wide range of phenomena, extending from mathematical finance [19] to fluid dynamics [22].

Since fractional Brownian motion is neither a Markov process, nor a semi-martingale, it is a challenging task to construct solutions of SDEs/SPDEs driven by such a process and to analyze their dynamical properties [12, 13]. Here we contribute to this topic by deriving concentration estimates for slowly time-dependent SDEs perturbed by an additive fBm for all ranges of the Hurst index $H \in (0,1)$, and for slowly time-dependent semilinear SPDEs perturbed by a cylindrical fBm, provided $H \in (\frac{1}{4},1)$. In this case, the well-posedness of the SPDEs is well-known [8, 9, 10, 21]. We mention that concentration estimates for SDEs driven by fBm with $H \in (\frac{1}{2},1)$ were previously obtained in [11]. One of the main novelties of this work is to extend this finite-dimensional result to $H \in (0,\frac{1}{2})$. The tools used in [11], based on results in [6], break down in this case, since they require the covariance of the fBm to be increasing, and rely on Lyapunov-type equations for the variance of a non-autonomous fractional Ornstein–Uhlenbeck process.

The SDEs we consider in this work have the form

$$\mathrm{d}x_t = f(\varepsilon t, x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^H ,$$

where the drift term $f$ admits a so-called stable uniformly hyperbolic critical manifold, that is, a smooth curve on which $f$ vanishes and has a uniformly negative $x$-derivative. In the deterministic case $\sigma = 0$, it is well known that the equation admits a so-called slow solution, staying $\varepsilon$-close to such a curve, and attracting nearby solutions exponentially fast. In the Brownian case $H = \frac{1}{2}$, it was shown in [1] that for $\sigma > 0$, sample paths are concentrated in a neighbourhood of size of order $\sigma$ of such a slow solution. Our first main result, Theorem 3.6, provides similar concentration estimates for any $H \in (0,1)$, with explicit bounds on the probability of leaving such a neighbourhood.

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$$\mathrm{d}\phi(t,x) = \big[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\big]\,\mathrm{d}t + \sigma\,\mathrm{d}W^H(t,x) ,$$

where $x$ belongs to the one-dimensional torus $\mathbb{T}$, and $W^H(t,x)$ denotes a cylindrical fractional Brownian motion on $\mathbb{T}$. In the Brownian case $H = \frac{1}{2}$, concentration estimates near stable uniformly hyperbolic critical manifolds have been obtained in [2] for the one-dimensional torus, and in [3] for the two-dimensional torus $\mathbb{T}^2$, provided the equation is suitably renormalised. Our second main result, Theorem 4.6, extends the concentration results on $\mathbb{T}$ to all $H \in (\frac{1}{4}, 1)$, for all fractional Sobolev norms of index $s \in (0, 2H - \frac{1}{2})$.

Numerous extensions and applications of these results are imaginable. For instance, one could investigate bifurcations in SDEs/SPDEs with fractional noise. For example [4] analyzes pitchfork bifurcations using finite-time Lyapunov-exponents and approximations with amplitude equations derived in [5], whereas [15] computes early-warning signs for fast-slow systems perturbed by additive fractional noise. In the Brownian case $H = \frac{1}{2}$, the works [2, 3] have obtained similar results for pitchfork and avoided transcritical bifurcations in slowly time-dependent SPDEs. Another exciting direction is given by concentration estimates for SDEs/SPDEs perturbed by multiplicative fractional noise, using tools from rough path theory and slow-fast systems [12].

This manuscript is structured as follows. Section 2 gives a precise definition of fBm, and provides a criterion allowing to estimate the probability that the supremum of a mean-square Hölder continuous Gaussian process exceeds a certain level. Section 3 deals with sample-path estimates for one-dimensional SDEs. The key result in this setting is a suitable upper bound on the variance of a non-autonomous fractional Ornstein–Uhlenbeck process. The concentration inequality is extended to semilinear SPDEs in Section 4, by analyzing the Fourier components of its solution and using Schauder-type estimates.

## 2 Fractional Brownian motion

In this section we collect basic results on fractional Brownian motion and Gaussian processes which will be required later on.

**Definition 2.1.** *A fractional Brownian motion (fBm)* $(W_t^H)_{t\geq 0}$ *with Hurst index* $H \in (0,1)$ *is a centered Gaussian process with covariance*

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) . \tag{2.1}$$

For $H = \frac{1}{2}$ we recover the standard Brownian motion, whereas for $H \neq \frac{1}{2}$ we obtain a process which is neither Markov nor a semi-martingale.

A useful estimate for the probability that the supremum of a Gaussian process exceeds a certain threshold is given by the following theorem [18, Theorem D.4]. It is obtained by comparing the probability of exceeding a certain level with the one of a suitable stationary process, using Slepian's lemma [20]. The mean-square Hölder continuity enables one to define such a process.

**Theorem 2.2.** *Let $(X_t)_t$ be a continuous Gaussian process with zero mean on $[0,T]$ for $T>0$. Assume that $(X_t)_t$ is mean-square Hölder continuous, i.e., there are constants G and $\gamma$ such that*

$$\mathbb{E}\left[(X_t - X_s)^2\right] \leq G|t-s|^\gamma \qquad \text{for all } t,s \in [0,T]\,.$$

*Then there exists a constant $K := K(G,\gamma)$ such that for $c>0$ and $A \subset [0,T]$, one has*

$$\mathbb{P}\left\{\sup_{t\in A} X_t > c\right\} \leq K T c^{2/\gamma} \exp\left\{-\frac{c^2}{2\sigma^2(A)}\right\}\,,$$

*where $\sigma^2(A) := \sup_{t\in A} \operatorname{Var}(X_t)$.*

**Remark 2.3.** By a simple scaling argument, one can infer that

$$K(G,\gamma) = G^{-1/\gamma} K(1,\gamma) =: G^{-1/\gamma} K_0(\gamma)\,.$$

Indeed, the process $\tilde{X}_t = G^{-1/2} X_t$ has Hölder constant 1, maximal variance $G^{-1}\sigma^2(A)$, and satisfies $\sup_{t\in A} \tilde{X}_t = G^{-1/2} \sup_{t\in A} \tilde{X}_t$. ◊

In our case, we use this result for a non-autonomous fractional Ornstein-Uhlenbeck process of Hurst index $H \in (0,1)$, which is known to be mean-square Hölder continuous with exponent $\gamma = 2H$.

# 3 The one-dimensional SDE case

## 3.1 Linear case

We start by considering linear fractional SDEs, driven by an fBm $(W_t^H)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$, given by

$$\mathrm{d}x_t = a(\varepsilon t) x_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^H\,, \tag{3.1}$$

where $\varepsilon, \sigma > 0$ are small parameters, and $a : [0,T] \to \mathbb{R}$ is of class $\mathscr{C}^1$. It will be convenient to scale time by a factor $\varepsilon$, which turns the SDE (3.1) into

$$\mathrm{d}x_t = \frac{1}{\varepsilon} a(t) x_t \,\mathrm{d}t + \frac{\sigma}{\varepsilon^H} \,\mathrm{d}W_t^H\,. \tag{3.2}$$

We will assume that the function $a$ satisfies

$$a(t) \leq -a_0\,, \qquad |a'(t)| \leq a_1 \qquad \forall t \in [0,T] \tag{3.3}$$

for some constants $a_0, a_1 > 0$, and write

$$\alpha(t) = \int_0^t a(s)\,\mathrm{d}s\,, \qquad \alpha(t,u) = \int_u^t a(s)\,\mathrm{d}s\,.$$

In order to apply Theorem 2.2, we will need to control the variance of $x_t$. The following result is an adaptation of [14, Theorem 1.4.3] to the non-autonomous case.

**Lemma 3.1.** *Assume the initial condition $x_0$ in* (3.2) *is deterministic. For any $H \in (0,1)$, the variance of $x_t$ satisfies the upper bound*

$$\mathrm{Var}(x_t) \leqslant \frac{2H\sigma^2}{\varepsilon^{2H}} \int_0^t \Big[ \mathrm{e}^{\alpha(t,s)/\varepsilon}(t-s)^{2H-1} - \mathrm{e}^{\alpha(t)/\varepsilon}(1-\mathrm{e}^{\alpha(t,s)/\varepsilon}) s^{2H-1} \Big] \,\mathrm{d}s\,. \tag{3.4}$$

PROOF: The proof is based on the representation

$$x_t = x_0\,\mathrm{e}^{\alpha(t)/\varepsilon} + \frac{\sigma}{\varepsilon^H} \int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}\, W^H(s)\,\mathrm{d}s + \frac{\sigma}{\varepsilon^H} W^H(t)$$

obtained by integration by parts. Since $x_0\,\mathrm{e}^{\alpha(t)/\varepsilon}$ is deterministic, we obtain

$$\begin{aligned}\mathrm{Var}(x_t) = \frac{\sigma^2}{\varepsilon^{2H}} \Big[ &\mathbb{E}[(W^H(t))^2] + 2\int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathbb{E}[W^H(t)W^H(s)]\,\mathrm{d}s \\ &+ \int_0^t\int_0^t \frac{a(u)}{\varepsilon}\frac{a(v)}{\varepsilon}\,\mathrm{e}^{\alpha(t,u)/\varepsilon}\,\mathrm{e}^{\alpha(t,v)/\varepsilon}\,\mathbb{E}[W^H(u)W^H(v)]\;\mathrm{d}u\;\mathrm{d}v \Big]\,.\end{aligned} \tag{3.5}$$

By the expression (2.1) of the covariance function of the fBm, the first term in square brackets gives $\mathbb{E}[(W^H(t))^2] = t^{2H}$, while the second term becomes

$$2\int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathbb{E}[W^H(t)W^H(s)]\,\mathrm{d}s = \int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}[t^{2H} + s^{2H} - (t-s)^{2H}]\,\mathrm{d}s\,.$$

We split this into three integrals that we compute separately. The first one gives

$$I_1 := t^{2H}\int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}\;\mathrm{d}s = t^{2H}[\mathrm{e}^{\alpha(t)/\varepsilon} - 1]\,,$$

while the other two integrals can be evaluated using integration by parts, yielding

$$\begin{aligned}I_2 &:= \int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}\,s^{2H}\;\mathrm{d}s = -t^{2H} + 2H\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,s^{2H-1}\;\mathrm{d}s\,, \\ I_3 &:= -\int_0^t \frac{a(s)}{\varepsilon}\,\mathrm{e}^{\alpha(t,s)/\varepsilon}(t-s)^{2H}\;\mathrm{d}s = -\mathrm{e}^{\alpha(t)/\varepsilon}\,t^{2H} + 2H\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}(t-s)^{2H-1}\;\mathrm{d}s\,.\end{aligned}$$

We further split the last term in the expression (3.5) for the variance of $x_t$ into three integrals. By symmetry, the first two are equal and add up to

$$\begin{aligned}I_4 &:= \int_0^t\int_0^t \frac{a(u)}{\varepsilon}\frac{a(v)}{\varepsilon}\,\mathrm{e}^{\alpha(t,u)/\varepsilon}\,\mathrm{e}^{\alpha(t,v)/\varepsilon}\,u^{2H}\;\mathrm{d}u\;\mathrm{d}v \\ &= t^{2H} - t^{2H}\,\mathrm{e}^{\alpha(t)/\varepsilon} + 2H(\mathrm{e}^{\alpha(t)/\varepsilon} - 1)\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,s^{2H-1}\;\mathrm{d}s\,.\end{aligned}$$

The last integral, given by

$$-\int_0^t\int_0^t \frac{a(u)}{\varepsilon}\frac{a(v)}{\varepsilon}\,\mathrm{e}^{\alpha(t,u)/\varepsilon}\,\mathrm{e}^{\alpha(t,v)/\varepsilon}\,|u-v|^{2H}\;\mathrm{d}u\;\mathrm{d}v\,,$$

can be dropped since $a$ is negative. The result follows by exploiting cancellations, and writing the factor $t^{2H}$ occurring in one remaining term $-t^{2H}\,\mathrm{e}^{\alpha(t)/\varepsilon}$ as the integral of $2Hs^{2H-1}$. □

Laplace asymptotics and our assumptions (3.3) on $a$ allow us to obtain the following simplified expression for the variance for small $\varepsilon$.

**Lemma 3.2.** *There exists a constant $r_1$, depending only on $a_0$, $a_1$ and $H$, such that for any $H \in (0,1)$ and any $t \geq 0$, one has*

$$\mathrm{Var}(x_t) \leq \frac{\sigma^2 2H\Gamma(2H)}{|a(t)|^{2H}}(1 + r_1\varepsilon) \,. \tag{3.6}$$

PROOF: It is sufficient to bound the first term in the integrand in (3.4), since the second term is negative. The assumptions (3.3) on $a$ imply that whenever $0 \leq s \leq t \leq T$, one has

$$\alpha(t,s) \leq -a_0(t-s) \qquad \text{and} \qquad \alpha(t,s) \leq a(t)(t-s) + \frac{1}{2}a_1(t-s)^2 \,.$$

Using the substitution $y = a(t)(t-s)$, we obtain

$$\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}(t-s)^{2H-1}\,\mathrm{d}s = \frac{1}{|a(t)|^{2H}}\int_0^{|a(t)|t} \mathrm{e}^{\alpha(t,t-\frac{y}{|a(t)|})/\varepsilon}\,y^{2H-1}\,\mathrm{d}y \,.$$

If $|a(t)|t > \frac{a_0}{a_1}$, we split the integral at $y = \frac{a_0}{a_1}$. To bound the integral over the interval $[0, \frac{a_0}{a_1}]$, we use the fact that

$$\alpha\Big(t, t - \frac{y}{|a(t)|}\Big) \leq -y + \frac{a_1}{2a(t)^2}y^2 \leq -y + \frac{a_1}{2a_0}y^2 \,,$$

to obtain

$$\begin{aligned}\int_0^{a_0/a_1} \mathrm{e}^{\alpha(t,s)/\varepsilon}(t-s)^{2H-1}\,\mathrm{d}s &\leq \frac{1}{|a(t)|^{2H}}\int_0^{a_0/a_1} y^{2H-1}\exp\Big\{-\frac{1}{\varepsilon}\Big[y - \frac{a_1}{2a_0}y^2\Big]\Big\}\,\mathrm{d}y \\ &= \frac{\varepsilon^{2H}}{|a(t)|^{2H}}\int_0^{a_0/\varepsilon a_1} z^{2H-1}\exp\Big\{-\Big[z - \varepsilon\frac{a_1}{2a_0}z^2\Big]\Big\}\,\mathrm{d}z \,.\end{aligned}$$

The integral can easily be shown to be bounded above by $\Gamma(2H)[1 + \mathcal{O}(\varepsilon)]$, see for instance [16, Theorem 8.1].

Using the fact that $\alpha(t, t - \frac{y}{|a(t)|}) \leq -y$, one finds that the integral over the remaining interval $[\frac{a_0}{a_1}, |a(t)|t]$ is exponentially small in $\varepsilon$, and therefore negligible with respect to the error of order $\varepsilon$. Finally, if $|a(t)|t < \frac{a_0}{a_1}$, we can use the integral over $[0, \frac{a_0}{a_1}]$ as an upper bound. □

**Remark 3.3.** For $H = \frac{1}{2}$, this result is consistent with the case of Brownian motion investigated by Berglund and Gentz in [1]. In particular, the $H$-dependent constant in (3.6) is given by $2H\Gamma(2H) = 1$ for $H = \frac{1}{2}$. ◊

As a first consequence of the bound (3.6), we obtain a concentration result for the solutions of the linear SDE (3.2). This is based on Theorem 2.2, taking into account the scaling argument in Remark 2.3.

**Proposition 3.4** (Concentration estimate for the linear SDE)**.** *Assume $x_0 = 0$. Then there exists a constant $r_2$, depending only on $a_0$, $a_1$ and $H$, such that*

$$\mathbb{P}\Big\{\sup_{0\leq t\leq T}|x_t||a(t)|^H \geq h\Big\} \leq C\Big(T;\frac{h}{\sigma},a_0\Big)\exp\Big\{-\kappa(\varepsilon)\frac{h^2}{2\sigma^2}\Big\} \,, \tag{3.7}$$

*where the prefactor and exponent are given by*

$$C\Big(T;\frac{h}{\sigma},a_0\Big) = \frac{2K_0(2H)T^2}{a_0}\Big(\frac{h}{\sigma}\Big)^{1/H} \,, \qquad \kappa(\varepsilon) = \frac{1-r_2\varepsilon}{2H\Gamma(2H)} \,. \tag{3.8}$$

PROOF: We introduce a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$ given by $t_k = k\varepsilon$ for $0 \leq k \leq N-1 = \lfloor T/\varepsilon \rfloor$, and write $I_k = [t_k, t_{k+1}]$ for the $k$th interval in the partition. Then the probability on the left-hand side of (3.7) is bounded by

$$\begin{aligned}
&\sum_{k=0}^{N-1} \mathbb{P}\Big\{\sup_{t\in I_k} |x_t||a(t)|^H \geq h\Big\} \\
&\quad\leq \sum_{k=0}^{N-1} \mathbb{P}\Big\{\sup_{t\in I_k} |x_t| \geq h \inf_{t\in I_k} \frac{1}{|a(t)|^H}\Big\} \\
&\quad\leq 2K_0(2H)\varepsilon T\Big(\frac{h}{\sigma}\Big)^{1/H} \sum_{k=0}^{N-1} \inf_{t\in I_k} \frac{1}{|a(t)|} \exp\Big\{-\frac{h^2}{2}\Big(\inf_{t\in I_k} \frac{1}{|a(t)|^{2H}}\Big)\Big(\sup_{t\in I_k} \mathrm{Var}(x_t)\Big)^{-1}\Big\}\,.
\end{aligned}$$

To obtain the last line, we have applied Theorem 2.2 to $x_t$ with $\gamma = 2H$ and $G = \sigma^2/\varepsilon^{2H}$, which is justified for $H > \frac{1}{2}$ by [11, Theorem 3.7]. For $H < \frac{1}{2}$ a computation similar to the one in Lemma 3.1 entails the mean-square Hölder continuity of the non-autonomous fractional Ornstein-Uhlenbeck process with the same coefficients $\gamma = 2H$ and $G = \sigma^2/\varepsilon^{2H}$. Now we observe that, setting

$$\hat{v}(t) = \frac{2H\Gamma(2H)}{|a(t)|^{2H}}\,,$$

(3.6) implies $\mathrm{Var}(x_t) \leq \sigma^2\hat{v}(t)(1 + \mathcal{O}(\varepsilon))$ for all $t \in [0,T]$. The result thus follows from the regularity properties (3.3) of $a$ and the fact that the length of $I_k$ is bounded by $\varepsilon$. □

The proposition shows that as soon as $h \gg \sigma$, the sample paths $(x_t)_{t\in[0,T]}$ are unlikely to leave a strip of width $h|a(t)|^{-H}$ before time $T$. In other words, we obtain a "confidence strip" for these sample paths.

**Remark 3.5.** Instead of using a partition of spacing $\varepsilon$, one could choose a partition given by $t_k = k\delta$ for an arbitrary $\delta \in (0,T]$. This yields an extra factor $\varepsilon/\delta$ in the prefactor $C$, and an additional error term of order $\delta$ in the exponent. Taking $\delta < \varepsilon$ is not of interest, since it increases the prefactor while it does not improve the exponent. Otherwise, the optimal value of $\delta$ has order $\sigma^2/h^2$, and yields a prefactor of order $\varepsilon T^2(h/\sigma)^{2+1/H}$. Because of the condition $\delta \geq \varepsilon$, this is only of interest if $h^2 < \sigma^2/\varepsilon$. ♢

### 3.2 Nonlinear case

This result can now easily be extended to concentration estimates for sample paths near stable slow manifolds of non-linear slowly time-dependent fractional SDEs. Consider the equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(t, x_t)\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W_t^H\,, \tag{3.9}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is of class $\mathscr{C}^2$. Assume that $x^\star : [0,T] \to \mathbb{R}$ is a stable uniformly hyperbolic slow manifold (or stable equilibrium branch), meaning that

- $f(t, x^\star(t)) = 0$ for all $t \in [0,T]$,
- and there exists $a_0 > 0$ such that $a^\star(t) = \partial_x f(t, x^\star(t)) \leq -a_0$ for all $t \in [0,T]$.

Note that by the implicit function theorem, such a function $x^\star$ is of class $\mathscr{C}^2$ as well. Classical results by Tihonov and Fenichel ensure that for sufficiently small $\varepsilon$, the deterministic equation $\varepsilon\dot{x} = f(t,x)$ admits a particular solution $\bar{x}(t)$ satisfying $\bar{x}(t) = x^\star(t) + \mathcal{O}(\varepsilon)$ uniformly in $t \in [0,T]$. We set

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t))$$

and observe that it is bounded above by $-\bar{a}_0 = -a_0 + \mathcal{O}(\varepsilon)$, which is still negative for $\varepsilon$ small enough. We define the set

$$\mathscr{B}(h) = \left\{(t,x) \colon t \in [0,T], |x - \bar{x}(t)||\bar{a}(t)|^H \leq h\right\},$$

which is a strip of width $h|\bar{a}(t)|^{-H}$ around the graph of $\bar{x}$, and write $\tau_{\mathscr{B}}(h)$ for the first-exit time of $(x_t)_t$ from $\mathscr{B}(h)$. Then we have the following concentration result, which is the main result of this section.

**Theorem 3.6** (Concentration estimate for the nonlinear SDE)**.** *There exist $\varepsilon_0, h_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ and $h \leq h_0$, the solution of* (3.9) *with initial condition $x_0 = \bar{x}(0)$ satisfies*

$$\mathbb{P}\left\{\tau_{\mathscr{B}(h)} \leq T\right\} \leq C\Big(T; \frac{h}{\sigma}, \bar{a}_0\Big) \exp\Big\{-\kappa(\varepsilon)\frac{h^2}{2\sigma^2}[1 - \mathcal{O}(h)]\Big\},$$

*where the constants $C(T; \frac{h}{\sigma}, \bar{a}_0)$ and $\kappa(\varepsilon)$ are the same as in Proposition 3.4.*

PROOF: The difference $y_t = x_t - \bar{x}(t)$ satisfies the SDE

$$\mathrm{d}y_t = \frac{1}{\varepsilon}\big[\bar{a}(t) + b(t, y_t)\big]\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W_t^H,$$

where there exist constants $M, d > 0$ such that $|b(t,y)| \leq My^2$ whenever $t \in [0,T]$ and $|y| \leq d$. Its solution can be represented as $y_t = y_t^0 + y_t^1$, where

$$y_t^0 = \frac{\sigma}{\varepsilon^H}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s^H, \qquad y_t^1 = \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\, b(s, y_s)\,\mathrm{d}s.$$

For any decomposition $h = h^0 + h^1$ with $h^0, h^1 > 0$, continuity of sample paths allows us to write

$$\begin{aligned}\mathbb{P}\left\{\tau_{\mathscr{B}(h)} \leq T\right\} &= \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}} |y_t||\bar{a}(t)|^H \geq h\Big\} \\ &\leq \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}} |y_t^0||\bar{a}(t)|^H \geq h^0\Big\} + \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}} |y_t^1||\bar{a}(t)|^H \geq h^1\Big\}. \end{aligned} \tag{3.10}$$

Since

$$|y^1_{t\wedge\tau_{\mathscr{B}}(h)}| \leq \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\frac{Mh^2}{|\bar{a}(s)|^H}\,\mathrm{d}s \leq \frac{Mh^2}{\bar{a}_0^{1+H}},$$

the second term on the right-hand side of (3.10) vanishes for an $h^1$ of order $h^2$. The result then follows from Proposition 3.4, taking $h^0 = h - h^1 = h[1 - \mathcal{O}(h)]$. □

# 4 The SPDE case

## 4.1 Linear case

We now turn to the analysis of linear SPDEs on the one-dimensional torus $\mathbb{T}$, of the form

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}[\Delta + a(t)]\phi(t,x)\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W^H(t,x), \tag{4.1}$$

where $a : [0,T] \to \mathbb{R}$ satisfies again (3.3). The SPDE is driven by a cylindrical fractional Brownian motion $(W^H(t))_{t\geq 0}$ with Hurst parameter $H \in (\frac{1}{4},1)$. This means that the noise is fractional-in-time and white-in-space. The existence of mild solutions for such linear SPDEs was established in [8, Example 3.1] for $H \in (\frac{1}{4},1)$.

The $k$th Fourier component of $\phi$ satisfies

$$\mathrm{d}\phi_k(t) = -\frac{1}{\varepsilon}\lambda_k(t)\phi_k(t)\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W_k^H(t)\,,$$

where $\lambda_k(t)$ is the $k$th eigenvalue of $-\Delta - a(t)$ given by $\lambda_k(t) = (2\pi)^2k^2 - a(t)$. It has the order $\langle k\rangle^2$, where $\langle k\rangle = \sqrt{1+k^2}$, and satisfies $|\lambda_k(t)| \geq (2\pi)^2k^2 + a_0$.

**Remark 4.1.** Rescaling time as $t = \mu_k\tilde{t}$, where $\mu_k = (a_0 + ck^2)^{-1}$ for a constant $c > 0$, we obtain

$$\mathrm{d}\phi_k = -\frac{1}{\varepsilon}\tilde{\lambda}_k(\tilde{t})\phi_k(\tilde{t})\,\mathrm{d}\tilde{t} + \frac{\sigma\mu_k^H}{\varepsilon^H}\,\mathrm{d}W_k^H(\tilde{t}) \tag{4.2}$$

with $\tilde{t} \in [0, T/\mu_k]$, where $\tilde{\lambda}_k(\tilde{t}) = \mu_k\lambda_k(\tilde{t}/\mu_k)$, and consequently $\tilde{\lambda}_k(\tilde{t}) \geq 1$ and $|\tilde{\lambda}'_k(\tilde{t})| \leq a_1$ for all $\tilde{t} \in [0, T/\mu_k]$. This allows us to use Proposition 3.4 with values of $a_0$ and $a_1$ that do not depend on $k$. Therefore, we need not worry about a possible $k$-dependence of the error term $r_2\varepsilon$ in (3.8). $\diamondsuit$

We recall that the (fractional) Sobolev norm on $H^s(\mathbb{T})$ for $s > 0$ is given by

$$\|\phi(t,\cdot)\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|\phi_k(t)|^2\,. \tag{4.3}$$

While one can work with this norm, it turns out that we can obtain slightly sharper bounds using a time-dependent Sobolev norm defined by

$$\|\phi(t,\cdot)\|_{s,t}^2 := \sum_{k\in\mathbb{Z}} a_{k,s}(t)^2|\phi_k|^2\,,$$

where we will choose $a_{k,s}(t) = \lambda_k^H(t)\langle k\rangle^{s-2H}$, so that $a_{k,s}(t) \asymp \langle k\rangle^s$. Note that both norms are equivalent, and are not sensitive to Fourier modes with large $|k|$. However, the time-dependent norm will give a sharper control for Fourier modes with small $|k|$, and especially for $k = 0$.

**Proposition 4.2** (Concentration estimate for the linear SPDE)**.** *Let $H \in (0, \frac{1}{4})$ and $s \in (0, 2H - \frac{1}{2})$. Then there exist constants $c_0, c_1, r_2 > 0$ such that the solution of* (4.1) *with initial condition $\phi(0,\cdot) = 0$ satisfies the concentration inequality*

$$\mathbb{P}\Big\{\sup_{0\leq t\leq T}\|\phi(t,\cdot)\|_{s,t} \geq h\Big\} \leq C\Big(T;\frac{h}{\sigma},s\Big)\exp\Big\{-\kappa(\varepsilon)Q(s)\frac{h^2}{2\sigma^2}\Big\}\,,$$

*where $Q(s) \geq c_0(2H - \frac{1}{2} - s)$, $\kappa(\varepsilon)$ is the same as in* (3.8)*, and*

$$C\Big(T;\frac{h}{\sigma},s\Big) = 2K_0(2H)T^2a_0^2\Big(Q(s)^{1/2}\frac{h}{\sigma}\Big)^{1/H}\big[1 + \mathcal{O}(\mathrm{e}^{-c_1h^2/\sigma^2})\big]\,.$$

PROOF: It is known that the fractional stochastic convolution has continuous trajectories in $H^s$ for $s \in (0, 2H - \frac{1}{2})$, see [8, Corollary 3.1] for $H > \frac{1}{2}$, respectively [9, Lemma 11.10] for $H \in (\frac{1}{4}, \frac{1}{2})$. For any decomposition $h^2 = \sum_{k\in\mathbb{Z}} h_k^2$ with $h_k > 0$ for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\mathbb{P}\Big\{\sup_{0\leq t\leq T}\|\phi(t,\cdot)\|_{s,t} \geq h\Big\} &= \mathbb{P}\Big\{\sup_{0\leq t\leq T}\sum_{k\in\mathbb{Z}} a_{k,s}(t)^2|\phi_k(t)|^2 \geq h^2\Big\}\\ &\leq \sum_{k\in\mathbb{Z}}\mathbb{P}\Big\{\sup_{0\leq t\leq T}|\phi_k(t)|^2\lambda_k(t)^{2H} \geq \frac{h_k^2}{\langle k\rangle^{2s-4H}}\Big\}\end{aligned} \tag{4.4}$$

by the choice of the time-dependent coefficients $a_{k,s}(t) = \lambda_k^H(t)\langle k\rangle^{s-2H}$. According to Proposition 3.4, we have for each component $\phi_k$ solving (4.2) that

$$\mathbb{P}\Big\{\sup_{0\le t\le T}|\phi_k(t)|\lambda_k(t)^H \ge \frac{h_k}{\langle k\rangle^{s-2H}}\Big\} = \mathbb{P}\Big\{\sup_{0\le\tilde t\le T/\mu_k}|\phi_k(\tilde t)|\frac{\tilde\lambda_k(\tilde t)^H}{\mu_k^H} \ge \frac{h_k}{\langle k\rangle^{s-2H}}\Big\}$$
$$\le C_k\exp\Big\{-\kappa(\varepsilon)\frac{h_k^2\mu_k^{2H}}{2\tilde\sigma_k^2\langle k\rangle^{2s-4H}}\Big\},$$

where $\tilde\sigma_k = \mu_k^H\sigma$ due to the scaling in (4.2), and

$$C_k = C\Big(\frac{T}{\mu_k}, \frac{h_k\mu_k^H}{\tilde\sigma_k\langle k\rangle^{s-2H}}, 1\Big) = \frac{2K_0(2H)T^2}{\mu_k^2}\Big(\frac{h_k}{\sigma\langle k\rangle^{s-2H}}\Big)^{1/H}.$$

Plugging this in (4.4) and simplifying the factors $\mu_K^H$ entails

$$\mathbb{P}\Big\{\sup_{0\le t\le T}\|\phi(t,\cdot)\|_{s,t}\ge h\Big\} \le \sum_{k\in\mathbb{Z}} C_k\exp\Big\{-\kappa(\varepsilon)\frac{h_k^2}{2\sigma^2\langle k\rangle^{2s-4H}}\Big\}.$$

We pick $\eta > 0$ and choose $h_k^2 = Q(s)h^2\langle k\rangle^{-(4H-2s-\eta)}$, where the condition $h^2 = \sum_{k\in\mathbb{Z}} h_k^2$ imposes

$$Q(s)^{-1} = \sum_{k\in\mathbb{Z}}\frac{1}{\langle k\rangle^{4H-2s-\eta}} < \infty.$$

This means that we need to have $4H-2s-\eta > 1$, and since both $s$ and $\eta$ must be positive we obtain the restriction $H > \frac14$. The claimed lower bound on $Q(s)$ follows from the behaviour of Riemann's zeta function $\zeta(u)$ as $u \to 1$, choosing for instance $\eta = 2H - s - \frac12$. Based on this choice of the $h_k$, we further obtain

$$\mathbb{P}\Big\{\sup_{0\le t\le T}\|\phi(t,\cdot)\|_{s,t}\ge h\Big\} \le \sum_{k\in\mathbb{Z}} C_k\exp\Big\{-\kappa(\varepsilon)Q(s)\frac{h^2\langle k\rangle^\eta}{2\sigma^2}\Big\}$$
$$\le 2K_0(2H)T^2\Big(Q(s)^{1/2}\frac{h}{\sigma}\Big)^{1/H}\sum_{k\in\mathbb{Z}}\frac{\langle k\rangle^{\eta/(2H)}}{\mu_k^2}\exp\big\{-\beta\langle k\rangle^\eta\big\},$$

where $\beta := \kappa(\varepsilon)Q(s)\frac{h^2}{2\sigma^2}$. We claim that the sum over $k$ is dominated by the term $k = 0$. In fact, by an argument similar to the one in [2, Theorem 2.4], we can bound this sum by

$$f(0) + 2f(1) + \int_1^\infty f(x)\,\mathrm{d}x, \qquad \text{where} \qquad f(x) := (a_0 + cx^2)(1+x^2)^\gamma\,\mathrm{e}^{-\beta(1+x^2)^{\eta/2}},$$

with an exponent $\gamma = \eta/(4H)$. The integral can be shown to be of order $\mathrm{e}^{-c\beta}$ with $c > 1$, which yields the result. □

**Remark 4.3.** The condition $H > \frac14$ is consistent with the solution theory for such SPDEs driven by cylindrical fractional noise [8, 21]. ♢

## 4.2 Nonlinear case

We finally consider the nonlinear SPDE given by

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}[\Delta\phi(t,x) + f(t,\phi(t,x))]\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W^H(t,x).$$

We assume $f(t,\phi) = -\partial_\phi U(t,\phi)$, where the potential $U$ can be decomposed as $U(t,\phi) = P(t,\phi) + g(t,\phi)$. Here $P$ is polynomial of even degree $2p$ with smooth bounded coefficients such that the leading order coefficient $a_{2p}(t) > 0$ for all $t \in [0,T]$, and the function $g \in \mathscr{C}^2([0,T]\times\mathbb{R};\mathbb{R})$ satisfies the boundedness assumptions

$$|g(t,\phi)\phi^{-1}|,\ |\partial_\phi g(t,\phi)|,\ |\partial_{\phi\phi} g(t,\phi)|,\ |\partial_t g(t,\phi)| \leq \widetilde{M}$$

for all $(t,\phi) \in [0,T]\times\mathbb{R}$, for a constant $\widetilde{M} > 0$.

The well-posedness of semilinear SPDEs with drift given by $f(t,\phi) = -\partial_\phi P(t,\phi)$ was established in [10, Section 4]. This remains valid for $U(t,\phi) = P(t,\phi) + g(t,\phi)$, due to the smoothness and boundedness assumptions on $g$. As before, we assume that we are in a stable situation, meaning that the there exists a map $\phi^* : [0,T] \to \mathbb{R}$ such that

- $f(t,\phi^*(t)) = 0$ for all $t \in [0,T]$,
- there exists $a_0 > 0$ such that $a^*(t) = \partial_\phi f(t,\phi^*(t)) \leq -a_0$ for all $t \in [0,T]$.

Similarly to the finite-dimensional case, the deterministic PDE admits, according to [2, Proposition 2.3], a solution $\bar{\phi}$ such that $\bar{\phi}(t) = \phi^*(t) + \mathscr{O}(\varepsilon)$ for all $t \in [0,T]$. We set as before $\bar{a}(t) = \partial_x f(t,\bar{\phi}(t))$ and introduce for $s \in (0, 2H - \frac{1}{2})$

$$\mathscr{B}(h) := \left\{(t,\phi) : t \in [0,T], \|\phi(t,\cdot) - \bar{\phi}(t,\cdot)\|_{s,t} \leq h\right\}.$$

The SPDE for the difference $\psi(t,x) := \phi(t,x) - \bar{\phi}(t,x)$ reads as

$$\mathrm{d}\psi(t,x) = \frac{1}{\varepsilon}[\Delta\psi(t,x) + \bar{a}(t)\psi(t,x) + b(t,\psi(t,x))]\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W^H(t,x)\,, \tag{4.5}$$

where there exists constants $M, d > 0$ such that $|b(t,\psi)| \leq M\psi^2$ and $|\partial_\psi b(t,\psi)| \leq M|\psi|$ for all $t \in [0,T]$ and $\psi \in \mathbb{R}$ with $|\psi| \leq d$. By the variation of constants formula, its solution is given by

$$\begin{aligned}\psi(t,x) &= \frac{\sigma}{\varepsilon^H}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{e}^{[(t-s)/\varepsilon]\Delta}\ \mathrm{d}W^H(s) + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{e}^{[(t-s)/\varepsilon]\Delta}\, b(s,\psi(s,x))\ \mathrm{d}s \\ &:= \psi^0(t,x) + \psi^1(t,x)\,.\end{aligned}$$

As before, $\alpha(t,s) = \int_s^t \bar{a}(r)\,\mathrm{d}r$. In order to analyze the stochastic convolution we rely on Schauder-type estimates.

**Lemma 4.4** (Schauder-type estimates)**.** *Let $f \in H^r$ with $r \in (0, 2H - \frac{1}{2})$. Then for all $q < r+2$, there exists a constant $c(q,r) > 0$ such that*

$$\|\mathrm{e}^{t\Delta} f\|_{H^q} \leq c(q,r)\, t^{-\frac{q-r}{2}}\, \|f\|_{H^r}\,.$$

PROOF: This follows from regularizing properties of analytic semigroups [17, Theorem 6.13], according to which

$$\|\mathrm{e}^{t\Delta} f\|_{D((-\Delta)^{q/2})} \leq c(q,r)\, t^{-\frac{q-r}{2}}\, \|f\|_{D((-\Delta)^{r/2})}$$

for $q - r/2 < 1$, leading to the restriction $q < r+2$. The statement follows, considering that $H^q = D((-\Delta)^{q/2})$. Here we use the equivalence of the time-dependent (fractional) Sobolev norm with (4.3). □

For $\psi(t,\cdot) \in H^s$, one can easily prove that $\beta(t) := b(t,\psi(t,\cdot))$ belongs to $H^s$. A proof of this statement in $H^s$ for $s \in (0,\frac{1}{2})$ relying on Young's inequality is provided in [2, Lemma 3.4]. We now apply the Schauder estimate in order to obtain a bound on $\psi^1(t,\cdot)$ similar to [2, Corollary 3.6].

**Lemma 4.5.** *Assume that there exists $r \in (0,2H-\frac{1}{2})$ such that $\beta(t) \in H^r$ for all $t \in [0,T]$. Then for all $q < r+2$ there exists a constant $c(q,r) > 0$ such that for all $t \in [0,T]$ we have*

$$\|\psi^1(t,\cdot)\|_{H^q} \leq c(q,r)\varepsilon^{\frac{q-r}{2}-1} \sup_{0\leq s\leq t} \|\beta(s)\|_{H^q} .$$

PROOF: We have

$$\begin{aligned}\|\psi^1(t,x)\|_{H^q} &\leq \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-a_0(t-s)/\varepsilon}\,\|\mathrm{e}^{[(t-s)/\varepsilon]\Delta}\beta(s)\|_{H^q}\,\mathrm{d}s \\ &\leq c(q,r)\varepsilon^{\frac{q-r}{2}-1}\sup_{0\leq s\leq t}\|\beta(s)\|_{H^r}\int_0^t (t-s)^{-\frac{q-r}{2}}\,\mathrm{d}s < \infty ,\end{aligned}$$

since $q < r+2$. In the last step we used the uniform negative bound on $\bar{a}$ and the Schauder estimate. □

**Theorem 4.6** (Concentration estimate for the nonlinear SPDE)**.** *For every $s \in (0,2H-\frac{1}{2})$ and any $\nu > 0$, there exists positive constants $\varepsilon_0, h_0$ such that for $\varepsilon \leq \varepsilon_0$ and $h \leq h_0\varepsilon^\nu$, the solution of* (4.5) *with initial condition $\phi(0,\cdot) = \bar{\phi}(0,\cdot)$ satisfies*

$$\mathbb{P}\{\tau_{\mathscr{B}(h)} \leq T\} \leq C\Big(T;\frac{h}{\sigma},s\Big)\exp\Big\{-\kappa(\varepsilon)Q(s)\frac{h^2}{2\sigma^2}\Big[1-\mathscr{O}\Big(\frac{h}{\varepsilon^\nu}\Big)\Big]\Big\} ,$$

*with the same $C\big(T;\frac{h}{\sigma},s\big)$, $Q(s)$ and $\kappa(\varepsilon)$ as in Proposition 4.2.*

PROOF: For any decomposition $h = h_0 + h_1$ we have

$$\begin{aligned}\mathbb{P}\{\tau_{\mathscr{B}(h)} \leq T\} &= \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}}\|\psi(t,\cdot)\|_{s,t} \geq h\Big\} \\ &\leq \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}}\|\psi^0(t,\cdot)\|_{s,t} + \|\psi^1(t,\cdot)\|_{s,t} \geq h\Big\} \\ &\leq \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}}\|\psi^0(t,\cdot)\|_{s,t} \geq h_0\Big\} + \mathbb{P}\Big\{\sup_{0\leq t\leq\tau_{\mathscr{B}(h)}}\|\psi^1(t,\cdot)\|_{s,t} \geq h_1\Big\} . \qquad (4.6)\end{aligned}$$

We bound the first term using Proposition 4.2. For the second one we have, similarly to the proof of Theorem 3.6, that for $t \leq \tau_{\mathscr{B}(h)}$

$$\|\psi^1(t,\cdot)\|_{H^q} \leq c(q,r)\varepsilon^{\frac{q-r}{2}-1}Mh^2 ,$$

since $\|\beta(t)\|_{H^r} \leq M\|\psi^1(t,\cdot)\|_{H^q}^2 \leq Mh^2$. Therefore the second term in (4.6) vanishes provided that $h_1 = c(q,r)\varepsilon^{\frac{q-r}{2}-1}Mh^2$. The statement follows, choosing

$$h_0 = h - h_1 = h - c(q,r)\varepsilon^{\frac{q-r}{2}-1}Mh^2 = h(1-\mathscr{O}(h/\varepsilon^\nu))$$

for $\nu = 1-\frac{q-r}{2}$. □

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