Autour de la troisième vitesse critique en théorie de Gross-Pitaevskii

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Troisième rencontre du GDR dynamique quantique, Orléans
2 Février 2011

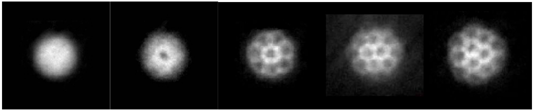
1. **Introduction : the Gross-Pitaevskii theory for rotating Bose-Einstein condensates**
2. Critical speeds in the GP theory with anharmonic confinement
3. Main results on the third critical speed
4. Some tools of the proofs

Bose-Einstein Condensation

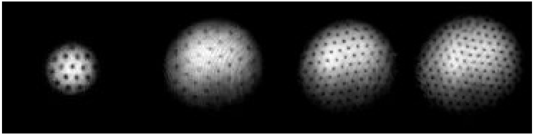
- a Bose-Einstein condensate exhibits quantum properties on a macroscopic scale ($\sim 100\mu m$ typically)
- many atoms (bosons) of a cold gas occupy the same quantum state of lowest energy
- One can describe a BEC with a single complex macroscopic wave function ψ, with $|\psi|^2$ giving the matter density.
- phenomenon first predicted by Bose and Einstein (1924, 1925)
- first experimental observation in 1995 (2001 Nobel prize in physics attributed to Cornell, Wieman and Ketterle)

Superfluidity : vortices in rotating BECs

A BEC can react to rotation by the nucleation of vortices.
The observation of vortices in condensates at equilibrium is a manifestation of frictionless flow, ans thus of the superfluidity of the condensate.



Vortices in rotating BECs, experiments at the LKB, ENS Ulm
[K.W. Madison *et al*, Phys. Rev. Lett. 84 806 (2000)]



Vortex lattices, experiments at the MIT
[J.R. Abo-Shaeer *et al*, Science 292 476 (2001)]

A vortex can be described as a line (or point in 2D models) where $|\psi|^2 = 0$ and around which there is a quantized phase circulation (ie topological degree or winding number),

$$\int_{\mathcal{C}} \partial_\tau \phi = 2\pi d, \quad d \in \mathbb{Z}^*$$

if $\psi = \rho e^{i\phi}$ on $\mathcal{C}$, a closed curve around the vortex with tangent vector τ.

Think of a 2D vortex at x_0 having polar coordinates (r_0, θ_0) as

$$\psi(r,\theta) = f(r - r_0) e^{id(\theta-\theta_0)} = f(r - r_0) \left(\frac{z - z_0}{|z - z_0|} \right)^d$$

with $z_0 = r_0 e^{i\theta_0}$

- f radial, real and ≥ 0
- $f(0) = 0$

Two dimensional Gross-Pitaevskii energy in the rotating frame

Set $x = (x_1, x_2)$ and $x^\perp = (-x_2, x_1)$.

$$\mathcal{E}^{GP}(\psi) = \int_{\mathbb{R}^2} \left(|\nabla\psi|^2 - 2\Omega x^\perp.(i\psi, \nabla\psi) + V(x)|\psi|^2 + G|\psi|^4 \right) dx$$

to be minimized under the mass constraint

$$\int_{\mathbb{R}^2} |\psi|^2 dx = 1.$$

- G : strength of interparticle interactions
- $V(x)$: trapping potential, generally a magnetic trap
- Ω : angular velocity at which the trap is rotated
- $(i\psi, \nabla\psi) = -Im(\psi\nabla\psi^*)$: supercurrent $(= \rho^2\nabla\phi$ if $\psi = \rho e^{i\phi})$

See Aftalion Birkhäuser 2006 for many results about the GP energy, with an emphasis on vortices (also results for the corresponding 3D energy).

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The importance of the confinement

$$\mathcal{E}^{GP}(\psi) = \int_{\mathbb{R}^2} \left(|\nabla\psi - i\Omega x^{\perp}\psi|^2 + \left(V(x) - \Omega^2|x|^2\right)|\psi|^2 + G|\psi|^4 \right) dx,$$

- for most experimental situations, V is a harmonic potential
  $V(x) = a_1 x_1^2 + a_2 x_2^2$
  $\Omega^2 \geq \min(a_1, a_2) \Rightarrow \mathcal{E}^{GP}$ is not bounded below, loss of confinement.
- suggestion by Fetter (PRA 2001): take V growing faster than $|x|^2$ at infinity (anharmonic potential). Then Ω can be arbitrarily large.
- experiments at the Ecole Normale Supérieure in Paris (Bretin-Stock-Seurin-Dalibard PRL 2004) with $V(x) \approx |x|^2 + k|x|^4$

Other type of potentials allowing $\Omega \to +\infty$:

- homogeneous traps $V(x) = |x|^s, \quad s > 2$
- flat trap $V_R(x) = +\infty$ for $|x| > R, \quad V(x) = 0$ for $|x| \leq R$, leads to a problem posed on B_R.

The three critical speeds

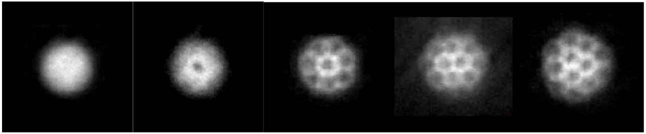
"Conjectures" : the condensate

- contains no vortex if $\Omega \leq \Omega_{c1}$
- contains a vortex lattice if $\Omega_{c1} < \Omega \leq \Omega_{c2}$
- contains a vortex lattice and a central hole if $\Omega_{c2} \leq \Omega \leq \Omega_{c3}$
- contains only a giant vortex if $\Omega_{c3} < \Omega$.

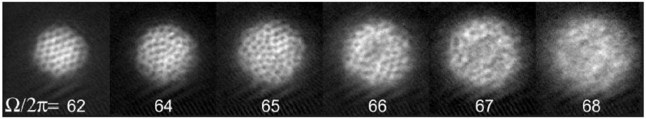
[Fischer-Baym PRL 2003, Kavoulakis-Baym NJP 2003, Fetter-Jackson-Stringari PRA 2005 ...]

Major issue : confirm the conjectures rigorously and provide estimates of Ω_{c1}, Ω_{c2} and Ω_{c3}.

The three phase transitions : experiments

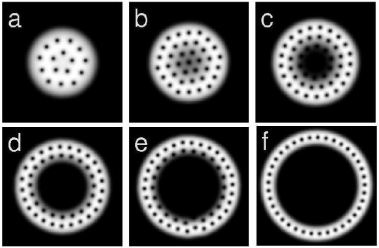


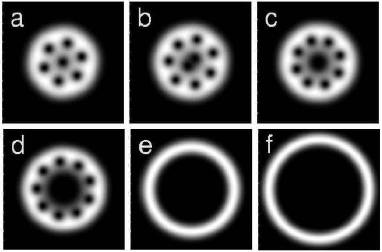
Experiments at LKB, ENS Paris, harmonic trap.
[K.W. Madison *et al* Phys. Rev. Lett. 84, 806 (2000)]



Experiments at LKB, ENS Paris, anharmonic trap.
[V. Bretin *et al*, Phys. Rev. Lett. 92 , 275 (2004)]

The three phase transitions : numerical simulations





[A.L. Fetter, B. Jackson, S. Stringari, Phys. Rev. A 71 (2005)]

Remark : One can design a potential $V(x)$ so that the condensate is annular even at slow rotation speeds. See

- Aftalion-Danaila PRA 2005 for simulations
- Aftalion-Alama-Bronsard ARMA 2005 for theorems.

Setting : strongly interacting regime

We consider the case of a flat trap

$$V(x) = +\infty \text{ for } |x| > 1, \quad V(x) = 0 \text{ for } |x| \leq 1.$$

- Rigorously, this is a model for a superfluid in a solid bucket.
- Formally this is the limit as $s \to +\infty$ of a homogeneous trap.

$$V(x) = |x|^s.$$

This leads to the energy (write $G = \frac{1}{\varepsilon^2}$)

$$\mathcal{E}^{GP}(\psi) = \int_{B_1} \left(|\nabla \psi - i\Omega x^{\perp} \psi|^2 - \Omega^2 |x|^2 |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right) dx,$$

with normalized ground state ψ^{GP} and ground state energy E^{GP} ie

$$E^{GP} = \inf_{\|\psi\|_{L^2(B_1)}=1} \mathcal{E}^{GP}(\psi) = \mathcal{E}^{GP}(\psi^{GP}).$$

We consider the strongly interacting regime $\varepsilon \to 0$.

On the first critical speed

Regime $\Omega \lesssim |\log \varepsilon|$:

- centrifugal forces are negligible
- one can adapt the analysis for harmonic traps by Ignat-Millot (JFA and RMP 2006), see also Rindler-Daller Physica A 2008.
- Evaluation of Ω_{c1}

$$\Omega_{c1} = |\log \varepsilon|(1 + o(1)).$$

- If $\Omega < \Omega_{c1}$: vortex-free condensate, $|\Psi^{\mathrm{GP}}| \sim 1$. Aftalion-Jerrard and Royo-Letelier (2010) prove that if ε is small enough:

$$\Psi^{\mathrm{GP}} = e^{i\alpha} g$$

with g the (real) minimizer of $\mathcal{E}^{\mathrm{GP}}$ with $\Omega = 0$ and α a constant.

- If $\Omega = \Omega_{c_1} + C \log |\log \varepsilon|$ there is a bounded number of vortices. They tend to minimize a renormalized energy depending on their positions (a_i) and degrees (d_i).

The regime $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$

Regime $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-2} |\log \varepsilon|^{-1}$:

- Correggi/Rindler-Daller Yngvason JMP 2007 and Correggi/Yngvason JPA 2008
- If $\Omega \leq \frac{2}{\sqrt{\pi\varepsilon}}$: disc-shaped condensate
- If $\Omega > \frac{2}{\sqrt{\pi\varepsilon}}$: annular condensate, ie Ψ^{GP} exponentially small in the central hole, onset of centrifugal forces
- There is a uniform distribution of vorticity in the bulk of the condensate as long as $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-2} |\log \varepsilon|^{-1}$.
- Transition from the vortex lattice state to the vortex-lattice-plus-hole state at

$$\Omega_{c2} = \frac{2}{\sqrt{\pi\varepsilon}}(1 + o(1)).$$

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The regime $\Omega \propto \frac{1}{\varepsilon^2|\log\varepsilon|}$

Write

$$\Omega = \frac{\Omega_0}{\varepsilon^2|\log\varepsilon|}$$

with a constant Ω_0.

Thomas-Fermi energy functional : for positive densities ρ (playing the role of $|\psi|^2$)

$$\mathcal{E}^{TF}(\rho) = \int_{B_1}\left(-\Omega^2|x|^2\rho + \frac{1}{\varepsilon^2}\rho^2\right)dx,$$

with normalized ground state ρ^{TF} and ground state energy E^{TF} ie

$$E^{TF} = \inf_{\|\rho\|_{L^1(B_1)}=1} \mathcal{E}^{TF}(\rho) = \mathcal{E}^{TF}(\rho^{TF}).$$

$$\rho^{TF} = \frac{\varepsilon^2\Omega^2}{2}\left(r^2 - R_h^2\right)_+$$

with

$$1 - R_h^2 = \frac{2}{\sqrt{\pi}\varepsilon\Omega} \propto \frac{1}{\Omega_0\varepsilon|\log\varepsilon|}.$$

Introduce

$$\mathcal{A}^{bulk} = \left\{R_h + \varepsilon|\log\varepsilon|^{-1} \leq r \leq 1\right\}.$$

Theorem (Correggi, NR, Yngvason, 2010, to appear in CMP)

Suppose

$$\Omega_0 > \frac{2}{3\pi}$$

Then, for any $x \in \mathcal{A}^{bulk}$

$$\left| |\psi^{GP}(x)|^2 - \rho^{TF}(x) \right| \ll \rho^{TF}(x).$$

Moreover

$$\deg(\psi^{GP}, \partial B_1) = \Omega - \frac{2}{3\sqrt{\pi}\varepsilon}(1 - o(1))$$

In particular

- The mass is concentrated in $\mathcal{A}^{bulk}$

$$\int_{\mathcal{A}^{bulk}} |\psi^{GP}(x)|^2 = 1 - o(1)$$

- There can be no vortices in $\mathcal{A}^{bulk}$

$$|\psi^{GP}(x)|^2 \geq \frac{C}{\varepsilon |\log \varepsilon|^{-3}}.$$

Estimating Ω_{c3} : upper bound

If ε is small enough and $\Omega_0 > \frac{2}{3\pi}$ $\Rightarrow$ appearance of a giant vortex state.

Thus, in the limit $\varepsilon \to 0$

$$\Omega_{c3} \leq \frac{2}{3\pi\varepsilon^2|\log\varepsilon|}(1 + o(1)).$$

Is this optimal ?

Consider

$$\Omega = \frac{2}{3\pi\varepsilon^2|\log\varepsilon|} - \frac{\Omega_1}{\varepsilon^2|\log\varepsilon|}$$

with $\Omega_1 > 0$.

Are there any vortices in the condensate for $\Omega_1 \ll 1$?

How to spot vortices ? : extracting the giant vortex phase and density

- For $\omega \in \mathbb{Z}$ and f real-valued, consider the giant-vortex energy

$$\hat{\mathcal{E}}^{\mathrm{GP}}_{\omega}[f] = \mathcal{E}^{\mathrm{GP}}[f(r)e^{i([\Omega]-\omega)\theta}]$$

with ground state energy $\hat{E}^{GP}_{\omega}$.

- Choose ω_0 minimizing $\hat{E}^{GP}_{\omega}$ and denote g the associated ground state.
- Define

$$u = \frac{\psi^{GP}}{g e^{i([\Omega]-\omega_0)}}.$$

How to spot vortices ? : the vorticity measure

Recall that

$$u = \frac{\psi^{GP}}{g e^{i([\Omega]-\omega_0)}}.$$

Define the superfluid current

$$j := (iu, \nabla u)$$

and the vorticity measure

$$\mu := \mathrm{curl}\, j.$$

- One expects $|u| = 1$, $\Rightarrow$ $\mu = 0$ far from the vortices.
- Take some domain $\mathcal{D}$ such that $|u| = 1$ on $\partial\mathcal{D}$: by Stokes' formula

$$\int_{\mathcal{D}} \mu = \int_{\partial\mathcal{D}} j \cdot \tau = 2\pi \deg\{u, \partial\mathcal{D}\}$$

- μ is a measure that counts the number of vortices. One expects (and proves)

$$\mu \approx 2\pi \sum d_i \delta_{a_i}$$

where a_i are the locations of the vortices, and d_i their degrees.

Asymptotics for the vorticity measure (1)

Define a new annulus $\mathcal{A}^{\rm bulk}$

$$\mathcal{A}^{\rm bulk} := \left\{ \vec{r} \mid R^{\rm bulk} \leq r \leq 1 \right\}, \quad R^{\rm bulk} := R_h + \varepsilon |\log \varepsilon| \Omega_1^{1/2},$$

still containing the bulk of the mass

$$\int_{\mathcal{A}^{\rm bulk}} |\psi^{GP}|^2 = 1 - O(\Omega_1^{1/2}).$$

Assume

$$\frac{\log|\log \varepsilon|}{|\log \varepsilon|} \ll \Omega_1 \ll 1.$$

Define for any measure ν in $C_c^1(\mathcal{A}^{\rm bulk})^*$ the norm

$$\|\nu\|_g := \sup_{\phi \in C_c^\infty(\mathcal{A}^{\rm bulk})} \frac{\left| \int_{\mathcal{A}^{\rm bulk}} \nu \phi \right|}{\left(\int_{\mathcal{A}^{\rm bulk}} \frac{1}{g^2} |\nabla \phi|^2 \right)^{1/2} + \varepsilon |\log \varepsilon| \|\nabla \phi\|_{L^\infty(\mathcal{A}^{\rm bulk})}}.$$

Asymptotics for the vorticity measure (2)

Theorem (NR 2010)

There exists a (computable) radius R_* and a (computable) value

$$N_* \propto \frac{\Omega_1}{\varepsilon}$$

such that, denoting δ_* the arc-length measure on the circle of radius R_*, the following holds :

$$\|\mu - N_* \delta_*\|_g \ll \|N_* \delta_*\|_g \,.$$

There are vortices in the condensate, that their number is close to $N_* \propto \frac{\Omega_1}{\varepsilon}$ and that they are evenly distributed on the circle of radius $r = R_*$.

$\Rightarrow$ existence of a circle-of-vortices-plus-hole state when $0 < \Omega_1 \ll 1$
$\Rightarrow$ lower bound to Ω_{c3}, proving that

$$\Omega_{c3} = \frac{2}{3\pi\varepsilon^2 |\log \varepsilon|}(1 + o(1)).$$

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- One expects that, if Ω_0 is large enough

$$\psi^{GP} \sim f(r)e^{in\theta}$$

with f a real-valued function and $n \sim \Omega$.

- Define for any $\omega \in \mathbb{Z}$ and f real-valued the energy functional

$$\hat{\mathcal{E}}^{GP}_{\omega}(f) = \mathcal{E}^{GP}(fe^{i([\Omega]-\omega)\theta})$$

with ground state energy $\hat{E}^{GP}_{\omega}$

- choose ω_0 minimizing $\hat{E}^{GP}_{\omega}$. It turns out that

$$\omega_0 = \frac{2}{3\sqrt{\pi\varepsilon}}(1+o(1)) \ll \Omega$$

- take g the ground state of $\hat{\mathcal{E}}^{GP}_{\omega_0}$ and decouple the original energy into

$$\mathcal{E}^{GP}(\psi) = \hat{\mathcal{E}}^{GP}_{\omega_0}(g) + \int_{B_1} g^2|\nabla v|^2 - 2g^2 B_{\omega_0}\cdot(iv,\nabla v) + \frac{g^4}{\varepsilon^2}\left(1-|v|^2\right)^2$$

where $\psi = gve^{i([\Omega]-\omega_0)\theta}$ and

$$B_{\omega_0} = \left(\Omega r - \frac{[\Omega]-\omega_0}{r}\right)\vec{e}_\theta$$

A reduced problem on an annulus

There is an annulus $\mathcal{A}$, slightly larger than $\mathcal{A}^{TF}$ so that both g and $|\psi^{GP}|$ are $O(\varepsilon^\infty)$ in $B_1 \setminus \mathcal{A}$.
Consider g as a "vortex-free" profile and decompose ψ^{GP}

$$\psi^{GP} = gue^{i([\Omega]-\omega_0)\theta}$$

$\Rightarrow u$ "almost minimizes"

$$\mathcal{E}[v] = \int_{\mathcal{A}} g^2|\nabla v|^2 - 2g^2 B_{\omega_0} \cdot (iv, \nabla v) + \frac{g^4}{\varepsilon^2}\left(1 - |v|^2\right)^2$$

under the mass constraint

$$\int_{\mathcal{A}} g^2|v|^2 = 1.$$

Very reminiscent of the regime $\Omega \propto |\log \varepsilon|$:Ignat-Millot (functional on an disc), Aftalion-Alama-Bronsard (functional on an annulus).
Crucial difference : the problem is posed on a shrinking annulus.

Heuristic comparisons between the regimes $\Omega \propto \frac{1}{\varepsilon^2|\log\varepsilon|}$ and $\Omega \propto |\log\varepsilon|$

Assume g is constant

$$g^2 = \frac{cte}{\varepsilon|\log\varepsilon|}$$

and rescale all lenghts by a factor $\varepsilon|\log\varepsilon|$ (width of $\mathcal{A}$). Consider

$$\tilde{\mathcal{E}}[\tilde{v}] = \int_{\tilde{\mathcal{A}}} |\nabla\tilde{v}|^2 - \tilde{B}_{\omega_0}\cdot(i\tilde{v},\nabla\tilde{v}) + \frac{1}{\tilde{\varepsilon}^2}\left(1-|\tilde{v}|^2\right)^2.$$

- $\tilde{\mathcal{A}}$ is an annulus of fixed width and large diameter ($\propto \tilde{\varepsilon}^{-2/3}|\log\tilde{\varepsilon}|^{-1}$).
- $\left|\tilde{B}_{\omega_0}\right| \propto \frac{1}{\Omega_0}|\log\tilde{\varepsilon}|$

$\Rightarrow$ One expects the same kind of transition as in the regime $\Omega \propto |\log\varepsilon|$, but backwards...

Major new difficulty : the large diameter of $\tilde{\mathcal{A}}$.

- Main tools : vortex ball methods (Béthuel, Brézis, Hélein, Sandier, Jerrard, Serfaty, Soner ...)
- Yields lower bounds of the form

$$\int_{\mathcal{A}} g^2|\nabla u|^2 - g^2 B_{\omega_0} \cdot (iu, \nabla u) \geq \sum_j 2\pi |d_j| \left(\frac{1}{2} g^2(a_j) |\log \varepsilon| + F(a_j) \right)$$

  where (a_j) and (d_j) are the locations and degrees of "approximate vortices"
- For Ω_0 large enough, the cost function

$$H(r) = \frac{1}{2} g^2(r) |\log \varepsilon| + F(r)$$

  is positive.
- If Ω_0 is large enough, vortices are not favorable energetically :

$$0 = \mathcal{E}[1] \geq \mathcal{E}[u] \geq \sum_j 2\pi |d_j| H(a_j)$$

- Key point for the proof of the giant vortex theorem

- Starting point : cover the set $\left\{|1-|u||>|\log\varepsilon|^{-1}\right\}$ with vortex balls $B_j(a_j,r_j)$, with $\sum_j |B_j| \ll |\mathcal{A}|$.
- Requires an upper bound on

$$\mathcal{F}[u] = \int_{\mathcal{A}} g^2|\nabla u|^2 + \frac{g^4}{\varepsilon^2}\left(1-|u|^2\right)^2$$

- Only natural bound (trial function $v_{trial} \equiv 1$)

$$\mathcal{E}[u] \leq 0$$

yields

$$\mathcal{F}[u] \leq \int_{\mathcal{A}} |B_{\omega_0}|^2 g^2 |u|^2 \leq C\varepsilon^{-2}$$

- Problem: only ensures $\left|\left\{|1-|u||>|\log\varepsilon|^{-1}\right\}\right| \leq C\varepsilon|\log\varepsilon| \sim |\mathcal{A}|$.

- Solution : Slice the annulus into $N \propto \varepsilon^{-1}|\log\varepsilon|^{-1}$ cells $\mathcal{A}_1, \dots, \mathcal{A}_N$ of side-length $\varepsilon|\log\varepsilon|$
- Assume that one can localize the estimate :

$$\int_{\mathcal{A}_i} g^2|\nabla u|^2 + \frac{g^4}{\varepsilon^2}\left(1-|u|^2\right)^2 \leq C\varepsilon^{-2}\frac{1}{N} \propto \frac{|\log\varepsilon|}{\varepsilon}$$

- Then

$$\left|\left\{|1-|u|| > |\log\varepsilon|^{-1}\right\} \cap \mathcal{A}_i\right| \leq C\varepsilon^3|\log\varepsilon| \ll |\mathcal{A}_i| \propto \varepsilon^2|\log\varepsilon|^2.$$

Method :

- Distinguish between good and bad cells
- Construct vortex balls in good cells
- Use the lower bounds to $\mathcal{E}[u]$ inside good cells to iteratively reduce the number of bad cells.

The vortex circle

- For $\Omega_1 \ll 1$ the cost function H has a negative minimum at $r = R_*$

$$H(R_*) \propto -\frac{\Omega_1}{\varepsilon}$$

- Vortices are favorable close to the circle $\mathcal{C}_{R_*}$ of radius R_*
- Evaluate the inter-vortex repulsion using an "electric potential" associated to any measure ν

$$\begin{cases} -\nabla\left(\frac{1}{g^2}\nabla h_\nu\right) = \nu \text{ in } \mathcal{A} \\ h_\nu = 0 \text{ on } \partial\mathcal{A}. \end{cases}$$

- One obtains

$$E^{\mathrm{GP}} = \hat{E}^{\mathrm{GP}}_{\omega_0} + \inf_{\mathrm{supp}(\nu)\subset\mathcal{C}_{R_*}} \left(\int_{\mathcal{A}^{\mathrm{bulk}}} \frac{1}{g^2}|\nabla h_\nu|^2 + H(R_*)\int \nu\right)(1+o(1))$$

- μ "minimizes" the above energy $\Rightarrow \mu \propto \delta_*$.

Conclusion/Prospects

- Proof that the giant vortex state (vanishing mass inside the hole, no vortices where the density is sufficiently large) appears above some threshold.
- Estimate of the critical speed in the small ε limit $\Omega_{c3} \sim 2\left(3\pi\varepsilon^2|\log\varepsilon|\right)^{-1}$
- Sligthly below the threshold there is a circle of vortices in the annulus.
- What happens when Ω_1 is no longer small ?
- What about more realistic trapping potentials ?
- What about anisotropic traps (non-radial confinements) ?