Control of Dynamic Bifurcations

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Abstract

We consider differential equations $\dot{x} = f(x, \lambda)$ where the parameter $\lambda = \varepsilon t$ moves slowly through a bifurcation point of f. Such a dynamic bifurcation is often accompanied by a possibly dangerous jump transition. We construct smooth scalar feedback controls which avoid these jumps. For transcritical and pitchfork bifurcations, a small constant additive control is usually sufficient. For Hopf bifurcations, we have to construct a more elaborate control creating a suitable bifurcation with double zero eigenvalue.

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1 Introduction

Consider the nonlinear control system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, u, \lambda), \tag{1}$$

with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^k$, which depends on some parameter $\lambda \in \mathbb{R}^p$. Assume that the uncontrolled system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, 0, \lambda) \equiv f_0(x, \lambda) \tag{2}$$

changes its qualitative behavior when λ passes λ_0, i.e., $\lambda = \lambda_0$ is a bifurcation point for (2). We are interested in bifurcations involving an exchange of stabilities between a family $x^\star(\lambda)$ of "nominal" equilibria of (2) and another family of attractors. These attractors are either other equilibria or periodic orbits (Poincaré–Andronov–Hopf bifurcation).

The motivation to study control systems whose state is close to a bifurcation point comes from the well-known fact that the performance of a control system can be improved if it is maintained to operate at high loading levels, that is, near a stability boundary (see for example [Ab1, VSZ]).

The existence of a bifurcation in the uncontrolled system (2) raises the following questions:

1. How does this bifurcation influence the controllability of (1)?
2. How can we control an exchange of stabilities?

The first problem has been investigated by the means of control sets, see e.g. [CK] for one-dimensional systems, [CHK] for Hopf bifurcations and [HS] for a Takens–Bogdanov singularity (i.e., when a Hopf and a saddle–node bifurcation curve intersect).

The second question is related to the problem of controlling the direction of the bifurcation. In order to avoid escaping trajectories, one usually tries to render the bifurcation supercritical, that is, a stable equilibrium or limit cycle should exist for $\lambda > \lambda_0$, which attracts the orbits departing from the nominal equilibrium $x^\star(\lambda)$. To do this, one has to find a control stabilizing the critical steady state of (2) for $\lambda = \lambda_0$. This problem has been solved by using a smooth state feedback, see [Ae, AF, Ab2] for the continuous-time case and [MS] for the discrete-time case.

In what follows, we are concerned with the problem of *dynamic* exchange of stabilities. In contrast with static bifurcation theory, the theory of dynamic bifurcations considers a process in which the parameter λ depends on time, where one usually assumes that this dependence is slow [Ben]. Such a situation occurs for instance if the device modelled by the equation is ageing, so that its characteristics are slowly modified.

Instead of (2), we thus consider an uncontrolled system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_0(x, \varepsilon t), \qquad 0 < \varepsilon \ll 1. \tag{3}$$

Basically, an exchange of stability in the static system (2) may result in two types of behaviour for (3): immediate exchange or delayed exchange. In the first case, the solution of (3) tracks the stable branch emerging from the bifurcation point immediately after the bifurcation [LS1, LS2, NS2, B1]. In the second case, the solution tracks the *unstable* branch for some time before jumping on the stable equilibrium (see [Sh, Ne1, Ne2, BER, Ben, HE] for the Hopf bifurcation and [Hab, EM, BK1, BK2, NS1] for pitchfork and transcritical bifurcations). Since a jump of a state variable may have catastrophic consequences for the device, our goal is to construct a control ensuring an immediate exchange of stabilities. Note that this feature may be used to detect the bifurcation point. We restrict our analysis to affine scalar feedback controls of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_0(x, \varepsilon t) + b\,u(x, \varepsilon t), \tag{4}$$

where b is a fixed vector in $\mathbb{R}^n$, and u is a scalar function.

This paper is organized as follows. In Section 2, we present a few elements of the theory of dynamic bifurcations, and show how the center manifold theorem can be used to reduce the dimension of the system. In Section 3, we consider one-dimensional cases such as the transcritical and pitchfork bifurcation, which are relatively easy to control. Section 4 is devoted to two-dimensional bifurcations. We first discuss the Hopf bifurcation, which displays a delay which is more robust than for one-dimensional bifurcations. To suppress this delay, we have to shift the eigenvalues' imaginary parts in order to produce a double zero eigenvalue, for which we present a result on immediate exchange of stability.

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2 Dynamic Bifurcations

Consider a one-parameter family of dynamical systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda), \qquad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}. \tag{5}$$

In the theory of dynamic bifurcations, one is concerned with the slowly time-dependent system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \varepsilon t), \qquad 0 < \varepsilon \ll 1, \tag{6}$$

that one wants to study on the time scale ε^{-1}. It is convenient to introduce the slow time $\tau = \varepsilon t$, in order to transform (6) into the singularly perturbed system

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau). \tag{7}$$

The basic idea is to use information on the bifurcation diagram of (5) in order to analyse solutions of (7).

Assume first that for $\lambda \in [a, b]$, (5) admits a family of asymptotically stable equilibria $x^\star(\lambda)$. That is, we require that $f(x^\star(\lambda), \lambda) = 0$ and that all eigenvalues of the Jacobian matrix $A(\lambda) = \partial_x f(x^\star(\lambda), \lambda)$ have real parts smaller than some $K < 0$, uniformly for $\lambda \in [a, b]$. It is known [PR, Fe, VBK, B2] that all solutions of (7) starting at $\tau = a$ in a sufficiently small neighbourhood of $x^\star(a)$ will reach an $\mathcal{O}(\varepsilon)$–neighbourhood of $x^\star(\tau)$ after a slow time of order $\varepsilon|\ln \varepsilon|$ and remain there until $\tau = b$. Thus, a sufficiently slow drift of the parameter λ will cause the system to track the nominal equilibrium $x^\star(\lambda)$ as closely as desired.

A new situation arises when $x^\star(\lambda)$ undergoes a bifurcation. Assume that at $\lambda = 0$, the Jacobian matrix $A(0)$ has m eigenvalues with zero real parts and $n - m$ eigenvalues with negative real parts. We can introduce coordinates $(y, z) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ such that (6) can be written in the form

$$\begin{aligned} \mathrm{d}y/\mathrm{d}t &= A_- y + g_-(y, z, \tau) \\ \mathrm{d}z/\mathrm{d}t &= A_0 z + g_0(y, z, \tau) \\ \mathrm{d}\tau/\mathrm{d}t &= \varepsilon \\ \mathrm{d}\varepsilon/\mathrm{d}t &= 0, \end{aligned} \tag{8}$$

where all eigenvalues of A_- have negative real parts, and all eigenvalues of A_0 have zero real parts. The functions g_- and g_0 vanish at $\tau = 0$ together with their derivatives with respect to y and z. Thus, at the bifurcation point z can be considered as a slow variable as well as τ. By the center manifold theorem [Ca] there exists a locally invariant manifold $y = h(z, \tau, \varepsilon)$, on which the dynamics is governed by the m-dimensional equation

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}\tau} = A_0 z + g_0(h(z, \tau, \varepsilon), z, \tau). \tag{9}$$

Moreover, trajectories starting close to this manifold are locally attracted by it with an exponential rate (see Lemma 1, p. 20 in [Ca]).

This observation allows us to restrict the analysis of (7) near the bifurcation point to the analysis of the lower-dimensional equation (9). Note, however, that we have to pay attention to the following points:

1. If we add a control to (7), we will modify the shape of the center manifold.
2. The center manifold is not analytic in general.

To simplify the discussion, we will only consider the low-dimensional systems on the center manifold. The above remarks imply that some additional verifications are necessary before conclusions about the reduced equations can be carried over to the general ones.

3 One-Dimensional Center Manifold

We consider the scalar equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = f_0(x, \tau) + u(x, \tau), \qquad x \in \mathbb{R}. \tag{10}$$

3.1 Transcritical Bifurcation

Assume that the uncontrolled vector field $f_0(x, \tau)$ has two families of equilibria $x = \varphi_1(\tau)$ and $x = \varphi_2(\tau)$ intersecting at $\tau = 0$. The family $\varphi_1(\tau)$ is stable for $\tau < 0$, while the family $\varphi_2(\tau)$ is stable for $\tau > 0$. This kind of bifurcation is referred to as transcritical bifurcation. We introduce the so-called singular stable solution

$$\varphi(\tau) = \begin{cases} \varphi_1(\tau) & \text{if } \tau < 0, \\ \varphi_2(\tau) & \text{if } \tau > 0. \end{cases} \tag{11}$$

Our goal is to find a control u such that the solution of (10) starting at $\tau_0 < 0$ in the basin of attraction of $\varphi_1(\tau)$ always stays in a small neighborhood of the singular stable solution $\varphi(\tau)$ for $\tau > \tau_0$. It turns out that the dynamics depends essentially on the values of $\varphi_1'(0)$ and $\varphi_2'(0)$. We discuss three representative cases.

Example 3.1 (Immediate exchange of stability).
Assume that the uncontrolled system has the form

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (x+\tau)(\tau - x). \tag{12}$$

Then we have $\varphi_1(\tau) = -\tau$, $\varphi_2(\tau) = \tau$ and $\varphi(\tau) = |\tau|$. It is shown in [LS1] that the solutions starting above $\varphi_2(\tau_0)$ at $\tau_0 < 0$ will track the singular stable solution $\varphi(\tau)$, so that no control is necessary. More precisely, it follows from [B1] that (12) admits a particular solution $x(\tau)$ satisfying

$$|x(\tau) - \varphi(\tau)| \leqslant \begin{cases} M\varepsilon|\tau|^{-1} & \text{if } \varepsilon^{1/2} \leqslant |\tau| \leqslant T, \\ M\varepsilon^{1/2} & \text{if } |\tau| \leqslant \varepsilon^{1/2}, \end{cases} \tag{13}$$

for some positive M and T, which attracts nearby solutions exponentially fast (Fig. 1a). Moreover this exchange of stability is robust in the following sense: it is shown in [BK2] that there exists a constant $c > 0$ such that solutions still track the stable equilibrium curve if we add a constant term $u_0 > -c\varepsilon$ to (12) (Fig. 1b). The same is true for more general bifurcations, for which $\varphi_1'(0) < 0$ and $\varphi_1'(0) < \varphi_2'(0)$.

Example 3.2 (Delayed exchange of stability).
The uncontrolled system

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(\tau - x) \tag{14}$$

has the equilibria $\varphi_1(\tau) = 0$ and $\varphi_2(\tau) = \tau$. This happens to be an explicitly solvable Bernoulli equation. The important fact is that the solution starting at $\tau_0 < 0$ at some $x_0 > 0$ remains close to the origin for $\tau_0 < \tau < -\tau_0$ as $\varepsilon \to 0$, and jumps to the branch $\varphi_2(\tau)$ near $\tau = -\tau_0$ (Fig. 1c).

Is is relatively easy to find a control which guarantees that the solution remains close to $\varphi(\tau)$. This is due to the fact that the vector field $-x^2$ is a codimension two singularity with unfolding

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\lambda - x) + \mu. \tag{15}$$

If μ is a positive constant, this equation has two families of equilibria which do not intersect. The family located in the half plane $x > 0$ is asymptotically stable and lies at a distance of order $\mu^{1/2}$ from $x = \varphi(\tau)$. This implies that the solution of the initial value problem

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(\tau - x) + u_0, \qquad x(\tau_0) > \tau_0,\ \tau_0 < 0,\ u_0 > 0 \tag{16}$$

stays near $\varphi(\tau)$ provided ε is sufficiently small (Fig. 1d). More precisely, there exists a continuous function $\delta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that the solution will track the upper equilibrium if $u_0 > \delta(\varepsilon)$. Thus, the smaller the drift velocity ε, the weaker the control has to be. It is known [EM] that $\delta(\varepsilon)$ goes to zero faster than any power law.

Example 3.3 (Diverging solutions).
For the equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (x - 2\tau)(\tau - x), \tag{17}$$

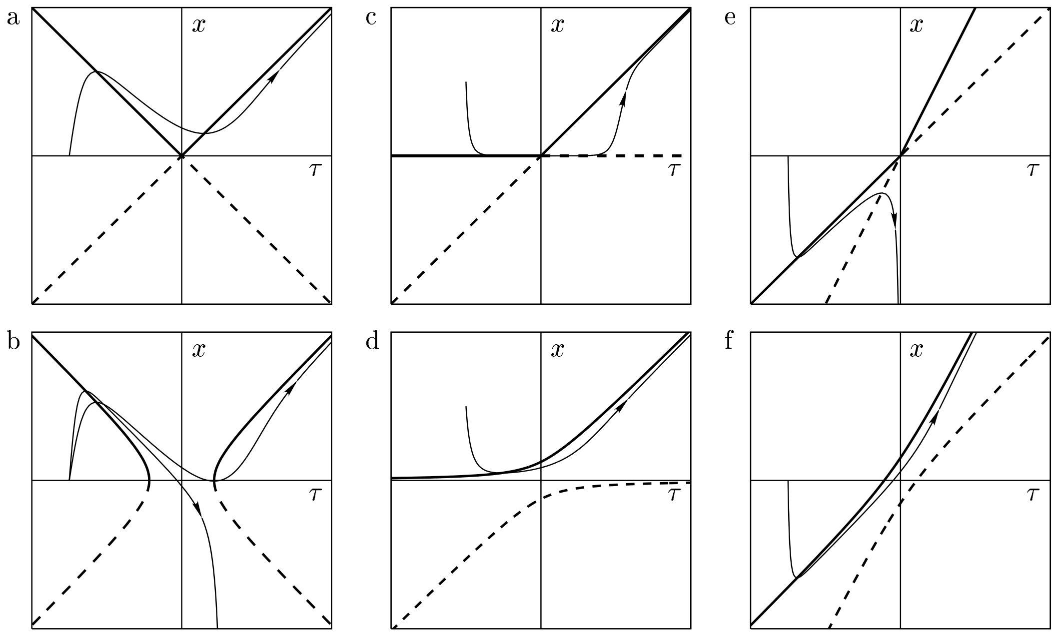

FIGURE 1. Exchange of stability for dynamic transcritical bifurcations. Light curves represent solutions of the time-dependent equation, heavy curves represent stable (full) and unstable (broken) equilibria of the static system. (a) Solutions of (12) track the singular stable solution $\varphi = |\tau|$. (b) This behaviour subsists if we add a negative constant to (12), provided ε is large enough. If ε is too small, the solution slips through the gap. (c) The solution of (14) with initial condition $x_0 > 0$ at $\tau_0 < 0$ exhibits a jump at $\tau = -\tau_0$. (d) This jump is suppressed if we add a small positive control. (e) The uncontrolled system (17) has diverging solutions. (f) A sufficiently large additive control suppresses this divergence.

we have $\varphi_1(\tau) = \tau$ and $\varphi_2(\tau) = 2\tau$. Solutions of this equation diverge for some $\tau \leqslant 0$ (Fig. 1e). This can be avoided by adding a control

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (x - 2\tau)(\tau - x) + u_0, \tag{18}$$

which splits the equilibrium branches if $u_0 > 0$ (Fig. 1f). In this case, we must have $u_0 > \delta(\varepsilon) = \varepsilon$. Note that if $u_0 = \varepsilon$, the change of variables $y = x - \tau$ transforms (18) into (14). The same qualitative features hold if $\varphi_2'(0) > \varphi_1'(0) > 0$.

3.2 Pitchfork Bifurcation

Similar results hold for the pitchfork bifurcation. Assume that the uncontrolled vector field $f_0(x, \tau)$ has a family of equilibria $x = \varphi_0(\tau)$ which is stable for $\tau < 0$ and unstable for $\tau > 0$. For positive τ, there exist two additional stable equilibria $\varphi_\pm(\tau) = \pm c\sqrt{\tau} + \mathcal{O}(\tau)$.

Example 3.4 (Immediate exchange of stability).
Assume that the uncontrolled system has the form

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (x - \tau)(\tau - x^2). \tag{19}$$

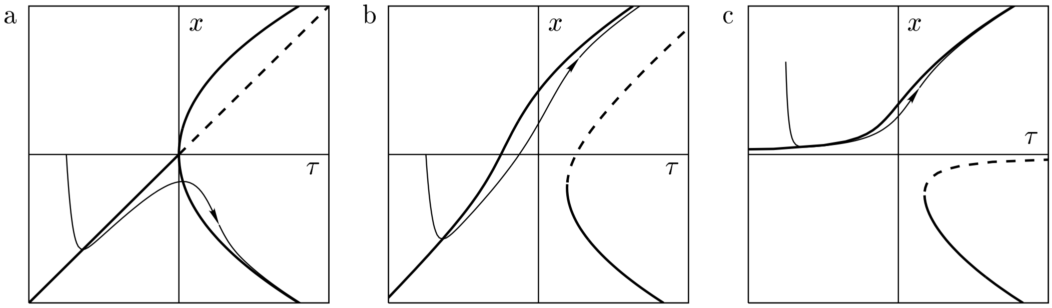

FIGURE 2. Exchange Of stability for pitchfork bifurcations. (a) Solutions of (19) track the lower stable equilibrium. (b) A sufficiently large positive control makes the system follow the upper equilibrium. (c) The same occurs for system (23).

It is shown in [LS2] that the solutions will track the lower branch $\varphi_-(\tau)$ after the bifurcation (Fig. 2a). More precisely, let

$$\varphi(\tau) = \begin{cases} \varphi_0(\tau) & \text{if } \tau < 0, \\ \varphi_-(\tau) & \text{if } \tau > 0. \end{cases} \tag{20}$$

In [B1] we obtained the existence of an attracting particular solution $x(\tau)$ satisfying

$$|x(\tau) - \varphi(\tau)| \leqslant \begin{cases} M\varepsilon|\tau|^{-1} & \text{if } -T \leqslant \tau \leqslant -\varepsilon^{1/2}, \\ M\varepsilon^{1/2} & \text{if } -\varepsilon^{1/2} \leqslant \tau \leqslant \varepsilon, \\ M\tau^{1/2} & \text{if } \varepsilon \leqslant \tau \leqslant \varepsilon^{1/2}, \\ M\varepsilon|\tau|^{-3/2} & \text{if } \varepsilon^{1/2} \leqslant \tau \leqslant T, \end{cases} \tag{21}$$

for some positive M and T.

One may wish to make the solution track the *upper* equilibrium $\varphi_+(\tau)$ after the bifurcation. This can be achieved by adding a constant control of the form

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (x-\tau)(\tau - x^2) + u_0, \tag{22}$$

with $u_0 > \delta(\varepsilon) = \varepsilon$ (Fig. 2b).

Example 3.5 (Delayed exchange of stability).
The uncontrolled system

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = \tau x - x^3 \tag{23}$$

displays a bifurcation delay similar to Example 3.2. One can provoke an immediate exchange of stability by adding a constant control u_0; there exists a function $\delta(\varepsilon)$ such that solutions track the upper equilibrium $\tau^{1/2}$ if $u_0 > \delta(\varepsilon)$ (Fig. 2c) and the lower equilibrium $-\tau^{1/2}$ if $u_0 < -\delta(\varepsilon)$. The function $\delta(\varepsilon)$ goes to zero faster than any power law [EM].

3.3 Bifurcations with Identically Zero Equilibrium

One can encounter systems of the form (10) for which $f(0,\tau) = 0$ for all τ. This happens, for instance, when f is symmetric under the transformation $x \to -x$. In such a case, we can write the uncontrolled system in the form

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = a(\tau)x + g(x,\tau), \tag{24}$$

where $|g(x,\tau)| \leqslant Mx^2$ for $|x| \leqslant d$. Assume that we start in the basin of attraction of the origin at a time τ_0 at which $a(\tau_0) < 0$. Then one can show [B1, NS1] that

$$x(\tau) = \mathcal{O}(\varepsilon) \quad \text{for } \tau_0 + \mathcal{O}(\varepsilon|\ln\varepsilon|) \leqslant \tau \leqslant \Pi(\tau_0) + \mathcal{O}(\varepsilon|\ln\varepsilon|), \tag{25}$$

where $\Pi(\tau_0) > \tau_0$ is the first time such that

$$\int_{\tau_0}^{\Pi(\tau_0)} a(\tau)\,\mathrm{d}\tau = 0. \tag{26}$$

If, for instance, $a(\tau)$ is negative for $\tau < 0$ and positive for $\tau > 0$, the delay time $\Pi(\tau_0)$ is obtained by making equal the areas delimited by the τ-axis, the curve $a(\tau)$ and the times τ_0, 0 and $\Pi(\tau_0)$, see Fig. 3a.

The delay can be suppressed by adding a constant control as in Examples 3.2 and 3.5. If, however, one does not wish to destroy the equilibrium $x = 0$, it is possible to influence the delay by a linear control $u(x) = cx$. The behaviour will strongly depend on the initial condition.

Example 3.6 (Pitchfork bifurcation).
Consider again the equation of Example 3.5, but with a linear control

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = \tau x - x^3 + cx. \tag{27}$$

If we start at some distance of $x = 0$ at $\tau_0 < 0$, the bifurcation is translated to the time $-c$, while the delay time is given by $\Pi(\tau_0) = -2c - \tau_0$. The effect of the control is thus twice as large as expected from the static theory.

Example 3.7 (Relaxation oscillations).
If $f_0(x,\tau)$ depends periodically on time, the system may exhibit *relaxation oscillations*, which are periodic solutions with alternating slow and fast motions [MR]. This happens for instance for the equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = (A + \sin\tau)x - x^3, \tag{28}$$

if $0 < A < 1$. The oscillations may be suppressed by adding a linear control $u(x) = -Ax$, although from the static theory one would expect that a control $u(x) = -(A+1)x$ were necessary.

4 Two-Dimensional Center Manifold

We consider now the two-dimensional version of (4). By an appropriate choice of variables, it can be written as

$$\begin{aligned} \varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} &= f_1(x,\tau) \\ \varepsilon \frac{\mathrm{d}y}{\mathrm{d}\tau} &= f_2(x,\tau) + u(x,\tau). \end{aligned} \tag{29}$$

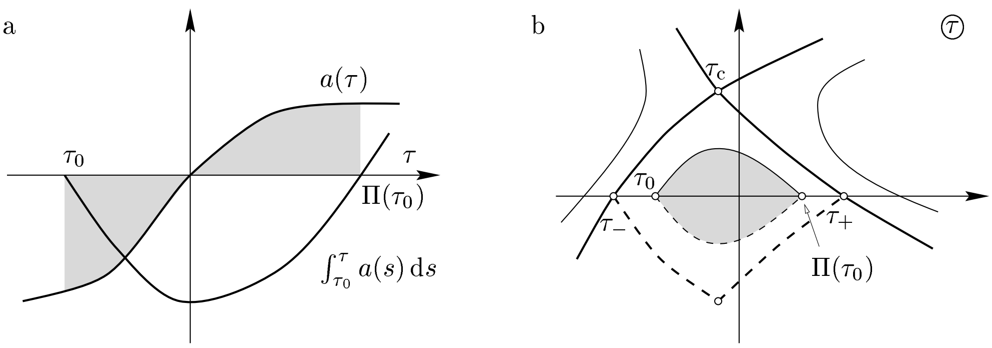

FIGURE 3. Determination of the bifurcation delay time. (a) In the case of the one-dimensional equation (24), the delay is simply obtained by making two areas the same. (b) In the case of a Hopf bifurcation, there is a maximal delay τ_+ given by the largest real time which can be connected to the negative real axis by a path with constant $\operatorname{Re}\Psi$. This path must have certain properties described in [Ne3], in particular the equation should be analytic in the shaded region.

4.1 Hopf Bifurcation

We assume that the static uncontrolled system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f_1(x,\lambda) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f_2(x,\lambda)\end{aligned} \tag{30}$$

admits a family of equilibria such that the eigenvalues of the linearization are of the form $a(\lambda) \pm \mathrm{i}\,\omega(\lambda)$, with $a(0) = 0$, $a'(0) > 0$ and $\omega(0) \neq 0$. It is known that the system can in general be controlled by a smooth feedback in such a way that the bifurcation is supercritical, that is, a stable periodic orbit exists for positive λ [Ae].

When $\lambda = \varepsilon t$ is made slowly time-dependent and the right-hand side of (30) is analytic, the bifurcation is delayed, as has been proved by Neishtadt [Ne1, Ne2] (Fig. 5a). Unlike in the case of pitchfork bifurcations, this delay also exists when the equilibrium depends on λ, and is stable with respect to analytic deterministic perturbations.

Example 4.1 (Hopf bifurcation).
Consider the system

$$\begin{aligned}\varepsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \tau(x-\tau) + \omega_0 y - \big[(x-\tau)^2 + y^2\big](x-\tau) \\ \varepsilon\frac{\mathrm{d}y}{\mathrm{d}\tau} &= -\omega_0(x-\tau) + \tau y - \big[(x-\tau)^2 + y^2\big]y,\end{aligned} \tag{31}$$

which admits the family of equilibria $(\tau, 0)$. The linearization around them has eigenvalues $\tau \pm \mathrm{i}\,\omega_0$. The complex variable $\zeta = x - \tau + \mathrm{i}\,y$ satisfies the equation

$$\varepsilon\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = (\tau - \mathrm{i}\,\omega_0)\zeta - |\zeta|^2\zeta - \varepsilon. \tag{32}$$

The delay phenomenon can be understood by considering the linearization of (32), which admits the solution

$$
\begin{aligned}
\zeta(\tau) &= \mathrm{e}^{[\Psi(\tau)-\Psi(\tau_0)]/\varepsilon}\,\zeta(\tau_0) - \int_{\tau_0}^{\tau} \mathrm{e}^{[\Psi(\tau)-\Psi(s)]/\varepsilon}\,\mathrm{d}s, \\
\Psi(\tau) &= \int_0^{\tau} (s - \mathrm{i}\,\omega_0)\,\mathrm{d}s = \frac{1}{2}\tau^2 - \mathrm{i}\,\omega_0 \tau.
\end{aligned}
\tag{33}
$$

The first term is small for $\tau_0 < \tau < -\tau_0$, as in the case of the pitchfork bifurcation. A crucial role is played by the second term, which is due to the τ-dependence of the equilibria. It can be evaluated using a deformation of the integration path into the complex plane. The function $\Psi(\tau)$ can be extended to complex τ and we have

$$
\operatorname{Re}\Psi(\tau) = \frac{1}{2}\big[(\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau - \omega_0)^2 + \omega_0^2\big]. \tag{34}
$$

The level lines of this function are hyperbolas centered at $\tau = \mathrm{i}\,\omega_0$. The integral in (33) is small if we manage to connect τ_0 and τ by a path on which $\operatorname{Re}\Psi(s) \geqslant \operatorname{Re}\Psi(\tau)$, i.e., if we never go uphill in the landscape of $\operatorname{Re}\Psi(s)$. This is possible if

$$
\tau \leqslant \hat\tau = \min\big\{-\tau_0, \omega_0\big\}. \tag{35}
$$

The existence of the maximal delay $\tau = \omega_0$ is a nonperturbative effect, entirely determined by the linearization around the equilibria.

The computation of the delay in the general case is discussed in [Ne3]. It is given by the formula

$$
\hat\tau = \min\big\{\Pi(\tau_0), \tau_+\big\}, \tag{36}
$$

where $\Pi(\tau_0)$ is defined by (26), and the maximal delay τ_+ can be determined by the level lines of $\operatorname{Re}\Psi(\tau)$ (Fig. 3b).

This shows in particular that the delay is robust, and the jump transition occurring at the delay time $\hat\tau$ cannot be avoided by adding a small constant control, as in the case of the pitchfork bifurcation. We may, of course, use a linear control which shifts the real part of the eigenvalues of the linearization, in order to increase the delay as in Section 3.3. This, however, will only postpone the problem to some later time, if the real part of the linearization is monotonically increasing.

Here we propose a different strategy to avoid a jump. We would like to provoke an immediate exchange of stability in order to detect the bifurcation point before it is too late. Expression (36) for the delay time shows that this can only be done by decreasing the buffer time, which might be achieved by shifting the *imaginary* part of the eigenvalues. This will create a bifurcation with double zero eigenvalue, which we study below.

4.2 Double Zero Eigenvalue

We start by analysing the autonomous control system

$$
\frac{\mathrm{d}z}{\mathrm{d}t} = f(z,\lambda) + b\,u(z,\lambda), \qquad z \in \mathbb{R}^2, \tag{37}
$$

where $f(z,\lambda)$ admits an equilibrium branch $z^\star(\lambda)$ such that the linearization $\partial_z f(z^\star(\lambda),\lambda)$ has eigenvalues $a(\lambda) \pm \mathrm{i}\,\omega(\lambda)$, with $a(0) = 0$, $a'(0) > 0$ and $\omega(0) = 1$ (this value of $\omega(0)$ may be achieved by a rescaling of time). Let $F(z,\lambda) = f(z,\lambda) + b\,u(z,\lambda)$. The scalar feedback $u(z,\lambda)$ is determined by two requirements:

1. The matrix $\partial_z F(0,0)$ should have a double zero eigenvalue.
2. In analogy with works on stabilization of bifurcations [Ae], the origin should be a stable equilibrium of (37) when $\lambda = 0$.

After a suitable affine transformation, we can write (37) as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \quad a(\lambda)x + \omega(\lambda)y + g_1(x,y,\lambda) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\omega(\lambda)x + a(\lambda)y + g_2(x,y,\lambda) + \tilde{u}(x,y,\lambda),\end{aligned} \tag{38}$$

where g_1 and g_2 are of order x^2+y^2. We have used the fact that the linear part is rotation invariant, so that we may take $b = \binom{0}{1}$.

For $\lambda = 0$, we propose the control

$$\tilde{u}(x,y,0) = x + v_1 x^2 + v_2 xy + v_3 y^2 + v_4 x^3, \tag{39}$$

where the coefficients $v_1, \dots, v_4$ have yet to be determined. The system (38) takes the form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y + c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^3 + \cdots \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (d_1 + v_1)x^2 + (d_2 + v_2)xy + (d_3 + v_3)y^2 + (d_4 + v_4)x^3 + \cdots,\end{aligned} \tag{40}$$

where c_i and d_i are the Taylor coefficients of g_1 and g_2 at the origin, respectively. A normal form of (40) is

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \gamma x^2 + \delta xy + \alpha x^2 y + \beta x^3 + \mathcal{O}(\|z\|^4),\end{aligned} \tag{41}$$

where the coefficients α, β, γ and δ are algebraic functions of c_i, d_i and v_i. This system has already been studied by Takens [Ta, GH]. If $\gamma \neq 0$ or $\delta \neq 0$, the origin is an unstable Bogdanov–Takens singularity. We thus require that $\gamma = \delta = 0$ (which amounts to imposing that $v_1 = -d_1$, $v_2 = -d_2 - 2c_1$). For the origin to be stable, we require moreover that $\alpha, \beta < 0$, which imposes some inequalities on v_3 and v_4.

For general values of λ, we choose a control of the form

$$\tilde{u}(x,y,\lambda) = (1 + C\lambda)\tilde{u}(x,y,0). \tag{42}$$

After inserting this into (38), carrying out a linear transformation and computing the normal form, we get the system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu(\lambda)x + 2a(\lambda)y + \gamma(\lambda)x^2 + \delta(\lambda)xy - x^2 y - x^3 + \mathcal{O}(\|z\|^4),\end{aligned} \tag{43}$$

where $\gamma(0) = \delta(0) = 0$, and

$$\mu(\lambda) = [C - \omega'(0)]\lambda + \mathcal{O}(\lambda^2) \tag{44}$$

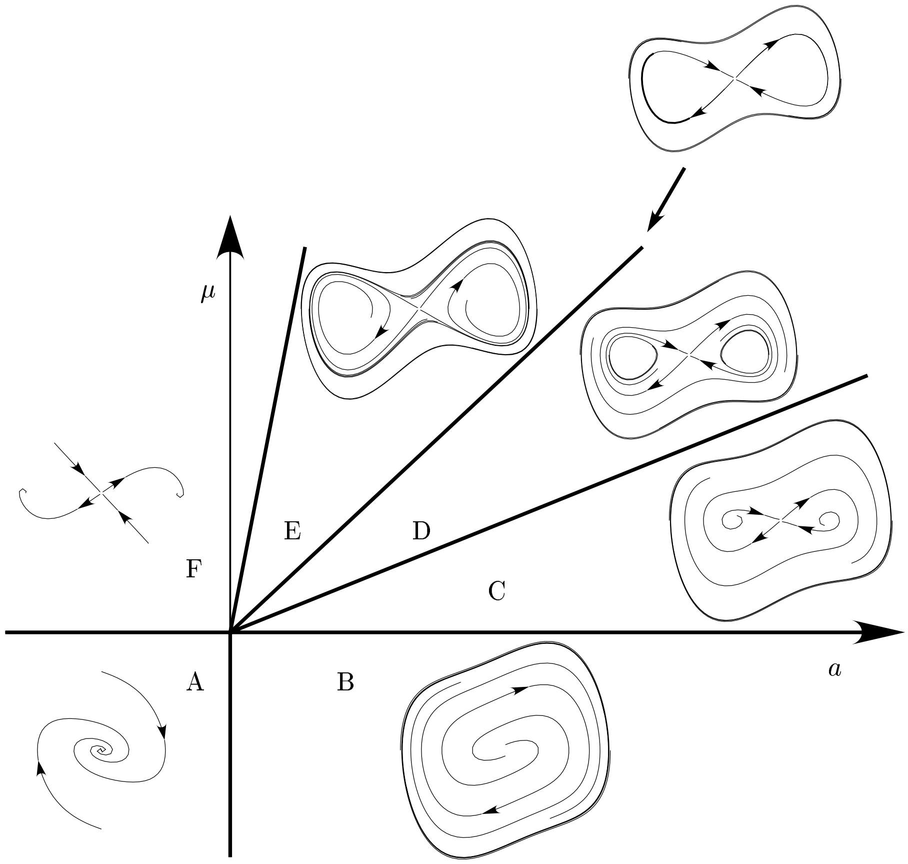

FIGURE 4. Schematic bifurcation diagram of equation (43) in the plane $\gamma = \delta = 0$. The transition A-B is the original Hopf bifurcation. By moving the eigenvalues' imaginary parts to 0, we change the function $\mu(\lambda)$ in such a way that $\mu(0) = 0$. This produces new bifurcation lines. The transition A-F is a supercritical saddle-node bifurcation, the transition C-B a subcritical one. D-C is a subcritical Hopf bifurcation, D-E a homoclinic bifurcation and E-F a saddle-node bifurcation of periodic orbits.

can be influenced by the choice of C. This equation happens to be a codimension-four unfolding of the singular vector field $(y, -x^2y - x^3)$ which has been studied in detail, see [KKR, VT] and references therein. The bifurcation diagram in the section $\gamma = \delta = 0$ has already been studied in [Ta], it is shown in Fig. 4.

We now consider the time–dependent version of (37),

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}\tau} = f(z,\tau) + b\,u(z,\tau). \tag{45}$$

It can be shown that similar transformations as above yield the equation

$$\begin{aligned} \varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} &= y \\ \varepsilon \frac{\mathrm{d}y}{\mathrm{d}\tau} &= \mu(\tau)x + 2a(\tau)y + \gamma(\tau)x^2 + \delta(\tau)xy - x^2y - x^3 + \mathcal{O}(\|z\|^4) + \varepsilon R(x,y,\tau,\varepsilon), \end{aligned} \tag{46}$$

where $R(0,0,\tau,0)$ is directly related to the drift $\frac{\mathrm{d}}{\mathrm{d}\tau} z^\star(\tau)$ of the nominal equilibrium.

The dynamics of (46) depends essentially on the path $((a(\tau),\mu(\tau))$ through the bifurcation diagram of Fig. 4, the effect of $\gamma(\tau)$ and $\delta(\tau)$ is small in a neighbourhood of the

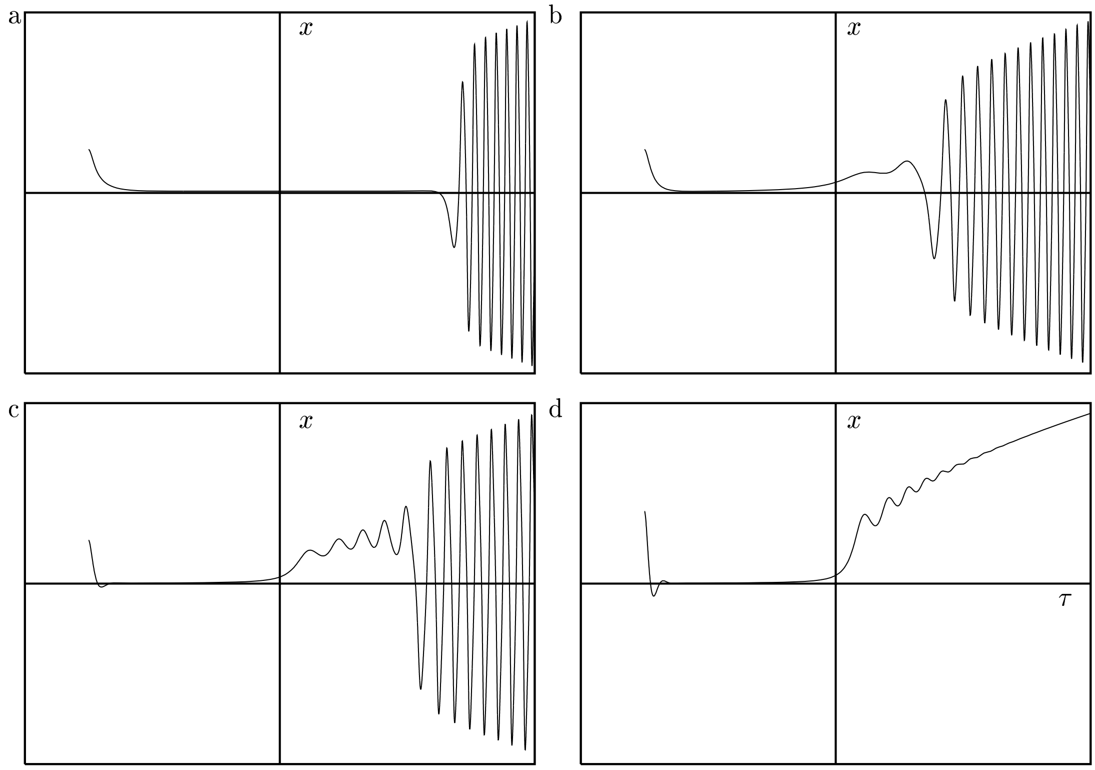

FIGURE 5. Solutions of equation (46) in the case $\gamma = \delta \equiv 0$, $R \equiv 1$, $a(\tau) = 2\tau$ and different functions $\mu(\tau)$. (a) $\mu(\tau) = -0.2$: We traverse the bifurcation diagram of Fig. 4 from region A to region B. The system undergoes a Hopf bifurcation, which results in the delayed appearance of large amplitude oscillations. (b) $\mu(\tau) = 0$: The delay is suppressed, but we still have oscillations. (c) $\mu(\tau) = a(\tau)$: We cross the bifurcation diagram from region A to region C. The trajectory starts by following the unstable focus, before being attracted by the limit cycle. (d) $\mu(\tau) = 2.5a(\tau)$: Theorem 4.2 applies, there is immediate exchange of stabilities between the nominal equilibrium and a stable focus.

bifurcation point. Various typical solutions are shown in Fig. 5. If we go from region A to region B, the Hopf bifurcation induces the usual delayed appearance of oscillations (Fig. 5a). If we go into region C, the delay is suppressed, but we still have oscillations (Fig. 5b,c). If, however, $\mathrm{d}\mu/\mathrm{d}a(0)$ is large enough to reach one of the regions D, E or F, there is an immediate exchange of stabilities with a stable focus (Fig. 5d). In [B3] we prove the following result on exchange of stabilities:

Theorem 4.2. *Assume that $\mu'(0) > 0$. There exist positive constants d, T, M, κ and a neighbourhood $\mathcal{M}$ of the origin in $\mathbb{R}^2$ with the following property. For every $\tau_0 \in [-T, 0)$, there is a constant $c_1 > 0$ such that for sufficiently small ε, any solution of* (46) *with initial condition $(x,y)(\tau_0) \in \mathcal{M}$ satisfies*

$$|x(\tau)| \leqslant M\frac{\varepsilon}{|\tau|}, \qquad |y(\tau)| \leqslant M\frac{\varepsilon}{|\tau|^{1/2}}, \qquad \text{for } \tau_1(\varepsilon) \leqslant \tau \leqslant -\Big(\frac{\varepsilon}{d}\Big)^{2/3}, \tag{47}$$

$$|x(\tau)| \leqslant M\varepsilon^{1/3}, \qquad |y(\tau)| \leqslant M\varepsilon^{2/3}, \qquad \text{for } -\Big(\frac{\varepsilon}{d}\Big)^{2/3} \leqslant \tau \leqslant \Big(\frac{\varepsilon}{d}\Big)^{2/3}, \tag{48}$$

where $\tau_1(\varepsilon) = \tau_0 + c_1\varepsilon|\ln\varepsilon|$*. If, moreover, the relations*

$$\frac{a'(0)}{\mu'(0)} < \frac{1}{2}, \qquad R(0,0,0,0) \neq 0 \tag{49}$$

hold, then for $(\varepsilon/d)^{2/3} \leqslant \tau \leqslant T$ *we have*

$$\begin{aligned} |x(\tau) - x_+(\tau)| &\leqslant M\Big[\frac{\varepsilon}{\tau} + \frac{\varepsilon^{1/2}}{\tau^{1/4}}\,\mathrm{e}^{-\kappa\tau^2/\varepsilon}\Big], \\ |y(\tau)| &\leqslant M\Big[\frac{\varepsilon}{\tau^{1/2}} + \varepsilon^{1/2}\tau^{1/4}\,\mathrm{e}^{-\kappa\tau^2/\varepsilon}\Big], \end{aligned} \tag{50}$$

where

$$x_+(\tau) = \begin{cases} \sqrt{\mu} + \mathcal{O}(\tau), & \textit{if } R(0,0,0,0) > 0, \\ -\sqrt{\mu} + \mathcal{O}(\tau), & \textit{if } R(0,0,0,0) < 0 \end{cases} \tag{51}$$

are equilibria of (43)*, i.e., the right-hand side of* (43) *vanishes when* $x = x_+$ *and* $y = 0$*.*

The control that we have constructed is robust in the following sense. If the coefficients in the feedback (39) are not perfectly adjusted, the functions $\mu(\tau)$, $\gamma(\tau)$ and $\delta(\tau)$ will not vanish exactly at the same time. This means that the bifurcation diagram of Fig. 4 will be traversed on a line which misses the origin. Depending on whether the path passes above or below the origin, solutions will either track a stable branch emerging from a pitchfork bifurcation, or start oscillating, but with a relatively small amplitude. This can be considered as an almost immediate transfer of stability from the nominal equilibrium to the limit cycle.

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