Metastability for the Ginzburg–Landau equation with space-time noise

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Ginzburg–Landau equation

$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3$

$x \in [0, L]$, $u(x,t) \in \mathbb{R}$ represents e.g. magnetisation

- Periodic b.c.
- Neumann b.c. $\partial_x u(0,t) = \partial_x u(L,t) = 0$

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Noise: weak, white in space and time

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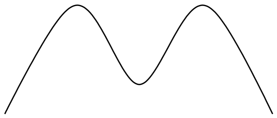
Deterministic system is gradient

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 = -\frac{\partial V}{\partial \delta_x}[u(\cdot,t)]$$

$$V[u] = \int_0^L \left[\frac{1}{2} u'(x)^2 - \frac{1}{2} u(x)^2 + \frac{1}{4} u(x)^4\right] \mathrm{d}x$$

Stationary solutions

$u''(x) = -u(x) + u(x)^3 = -\frac{\mathsf{d}}{\mathsf{d}x}[$  $]$

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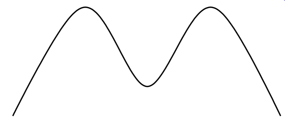
$$\Bigg]$$

- $u_{\pm}(x) \equiv \pm 1$: global minima of V, stable
- $u_0(x) \equiv 0$: unstable
- Periodic b.c: for $k = 1, 2, \ldots$, if $L > 2\pi k$,
  $u_{k,\varphi}(x) = \sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \varphi, m\Big)$ $\qquad 4k\sqrt{m+1}\,\mathrm{K}(m) = L$
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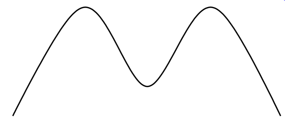
$$u''(x) = -u(x) + u(x)^3 = -\frac{\mathsf{d}}{\mathsf{d}x}\Big[$$  $$\Big]$$

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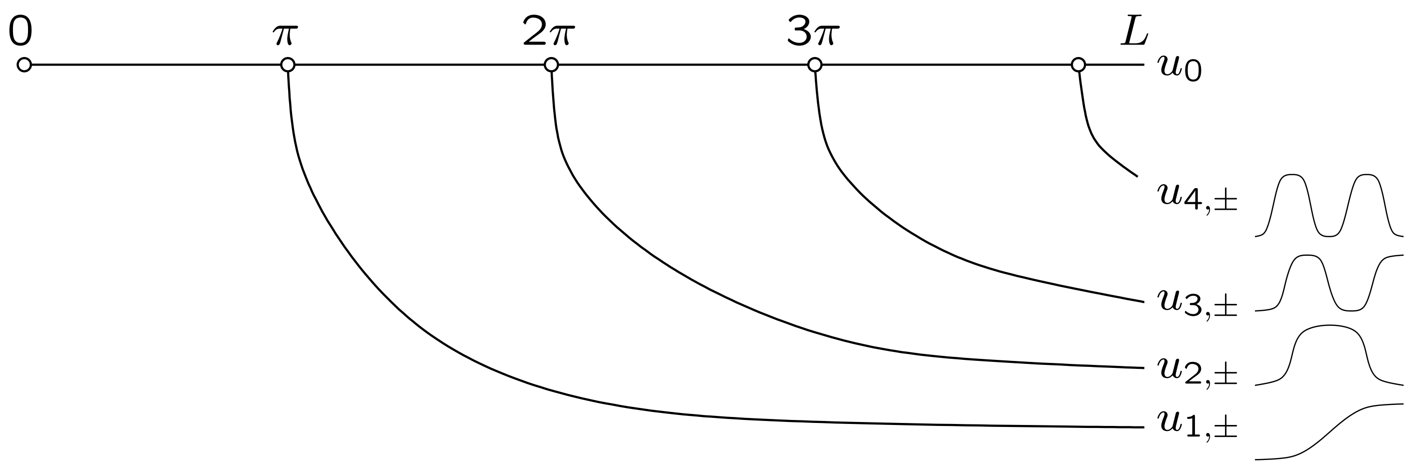

Stability of stationary solutions

Linearisation at $u(x)$: $\partial_t \varphi = A[u]\varphi$, $A[u] = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1 - 3u(x)^2$

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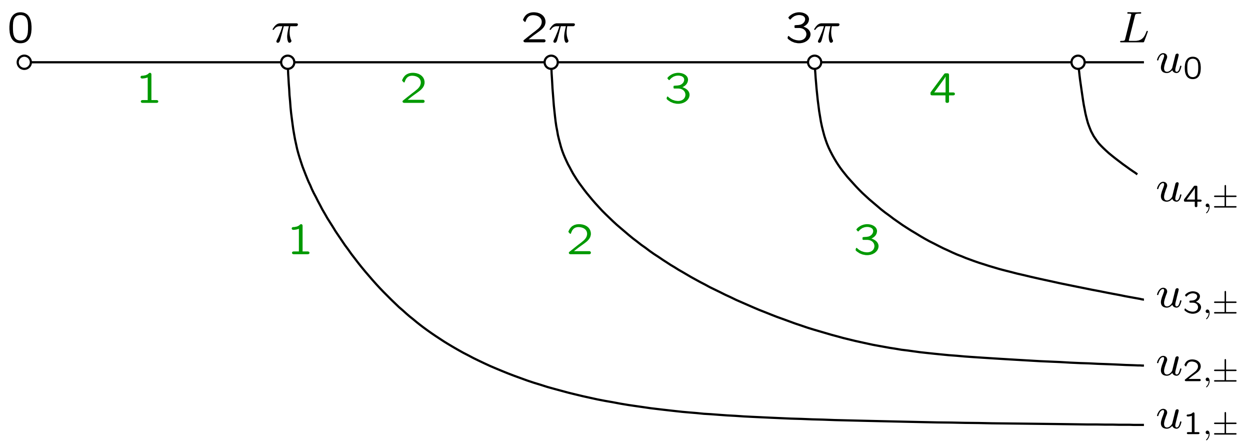
Number of positive eigenvalues:



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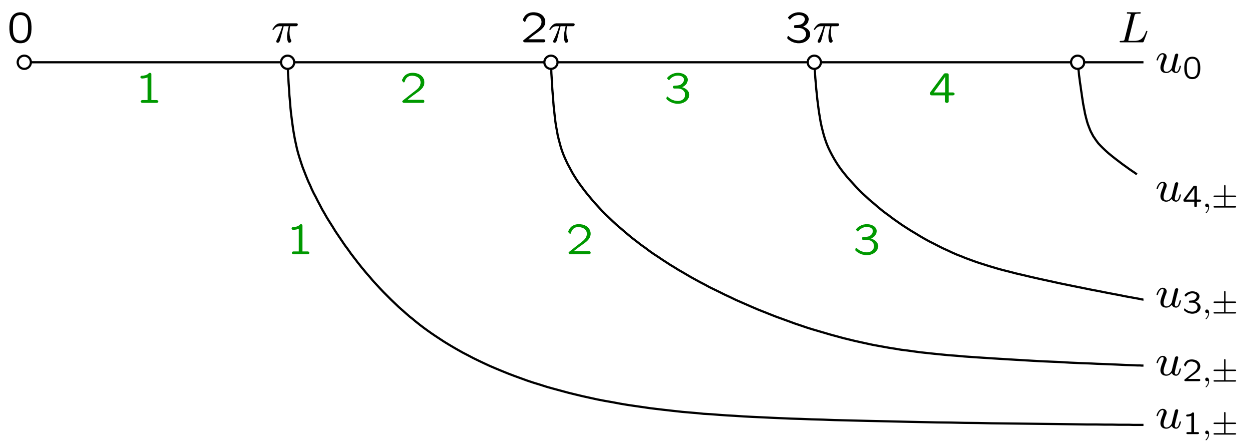

Periodic b.c:

- eigenvalues at u_0 are doubly degenerate
- one eigenvalue at $u_{k,\varphi}$ is always zero

Ginzburg–Landau equation with noise

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 \quad + \sqrt{2\varepsilon}\,\xi(x,t)$$

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Definition of space–time white noise $\xi(x,t)$:

1. By taking derivative of Brownian sheet
2. By adding independent Brownian motions to each Fourier mode

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$$\dot{y}_k = -\lambda_k y_k - \frac{1}{L} \sum_{k_1+k_2+k_3=k} y_{k_1} y_{k_2} y_{k_3} \; + \sqrt{2\varepsilon} \dot{W}_t^{(k)}$$

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Energy functional:

$$V[\{y_k\}] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \lambda_k |y_k|^2 + \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} y_{k_1} y_{k_2} y_{k_3} y_{k_4}$$

The question

How long does the system take to get from $u_-(x) \equiv -1$ to (a neighbourhood of) $u_+(x) \equiv 1$?

Metastability: Time of order $\mathrm{e}^{\mathit{const}/\varepsilon}$
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We seek constants ΔW (activation energy), Γ_0 and α such that the random transition time τ satisfies

$$\mathbb{E}[\tau] = \left[\Gamma_0^{-1} + \mathcal{O}(\varepsilon^\alpha)\right] \mathrm{e}^{\Delta W/\varepsilon}$$

Reversible diffusion

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- $\triangleright$ $V : \mathbb{R}^d \to \mathbb{R}$: potential, growing at infinity
- $\triangleright$ W_t: d-dim Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$

Reversible w.r.t.
invariant measure:

$$\mu_\varepsilon(\mathrm{d}x) = \frac{\mathrm{e}^{-V(x)/\varepsilon}}{Z_\varepsilon}\,\mathrm{d}x$$

(detailed balance)

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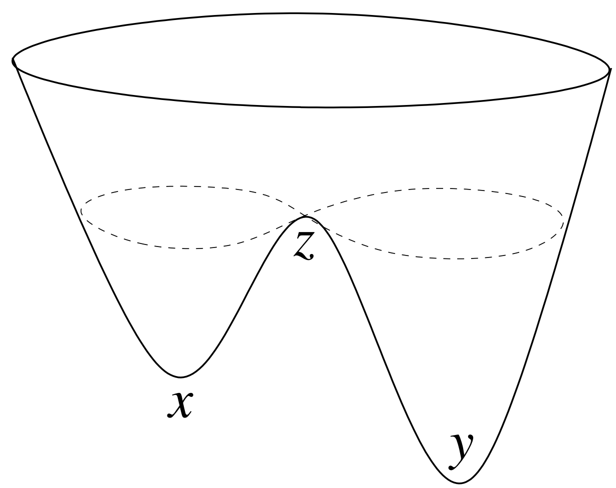

τ_y^x: first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, starting in x
"Eyring–Kramers law" (Eyring 1935, Kramers 1940)

- Dim 1: $\mathbb{E}[\tau_y^x] \simeq \dfrac{2\pi}{\sqrt{V''(x)|V''(z)|}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}$
- Dim$\geqslant$ 2: $\mathbb{E}[\tau_y^x] \simeq \dfrac{2\pi}{|\lambda_1(z)|}\sqrt{\dfrac{|\det(\nabla^2 V(z))|}{\det(\nabla^2 V(x))}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}$

Towards a proof of Kramers' law

- Large deviations (Wentzell & Freidlin 1969):

$$\lim_{\varepsilon\to 0} \varepsilon \log(\mathbb{E}[\tau_y^x]) = V(z) - V(x) \qquad = \Delta W$$

- Analytic (Helffer, Sjöstrand 85, Miclo 95, Mathieu 95, Kolokoltsov 96,...): low-lying spectrum of generator
- Potential theory/variational (Bovier, Eckhoff, Gayrard, Klein 2004):

$$\mathbb{E}[\tau_y^x] = \frac{2\pi}{|\lambda_1(z)|}\sqrt{\frac{|\det(\nabla^2 V(z))|}{\det(\nabla^2 V(x))}}\, e^{[V(z)-V(x)]/\varepsilon}\Big[1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{1/2})\Big]$$

  and similar asymptotics for eigenvalues of generator
- Witten complex (Helffer, Klein, Nier 2004): full asymptotic expansion of prefactor
- Distribution of τ_y^x (Day 1983, Bovier *et al* 2005):

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big\{\tau_y^x > t\mathbb{E}[\tau_y^x]\Big\} = e^{-t}$$

Formal computation for Ginzburg–Landau (R.S. Maier, D. Stein, 01)

Take e.g. Neumann b.c, $L < \pi$

Saddle state: $u_0 \equiv 0$, $V[u_0] = 0$

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Eigenvalues at $V[u_0] \equiv 0$: $\lambda_k = -1 + (\pi k/L)^2$

Thus formally $\Gamma_0 \simeq \dfrac{|\lambda_0|}{2\pi} \sqrt{\prod_{k=0}^{\infty} \dfrac{\mu_k}{|\lambda_k|}} = \dfrac{1}{2^{3/4}\pi} \sqrt{\dfrac{\sinh(\sqrt{2}L)}{\sin L}}$

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Problems:

1. What happens when $L \to \pi_-$? (bifurcation)
2. Is the formal computation correct in infinite dimension?

Potential theory

Consider first Brownian motion $W_t^x = x + W_t$

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$G_{A^c}(x,y)$ Green's function $\Rightarrow w_A(x) = \int_{A^c} G_{A^c}(x,y)\,\mathrm{d}y$

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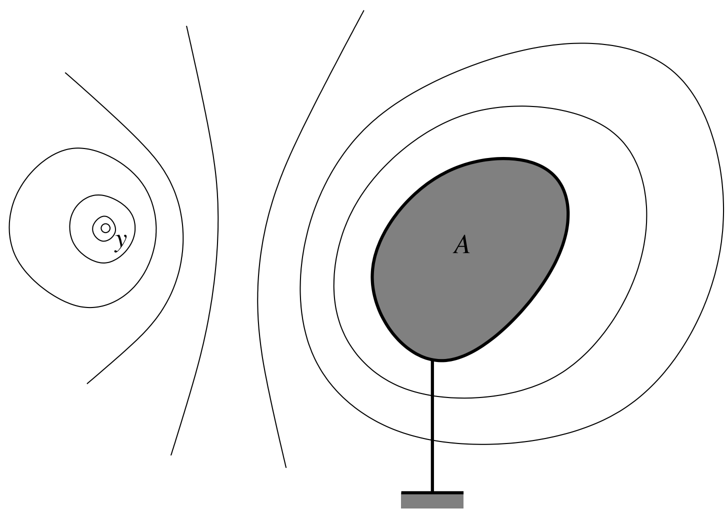

Potential theory

Fact 2: $h_{A,B}(x) = \mathbb{P}[\tau_A^x < \tau_B^x]$ satisfies

$$\begin{aligned} \Delta h_{A,B}(x) &= 0 \qquad x \in (A \cup B)^c \\ h_{A,B}(x) &= 1 \qquad x \in A \\ h_{A,B}(x) &= 0 \qquad x \in B \end{aligned}$$

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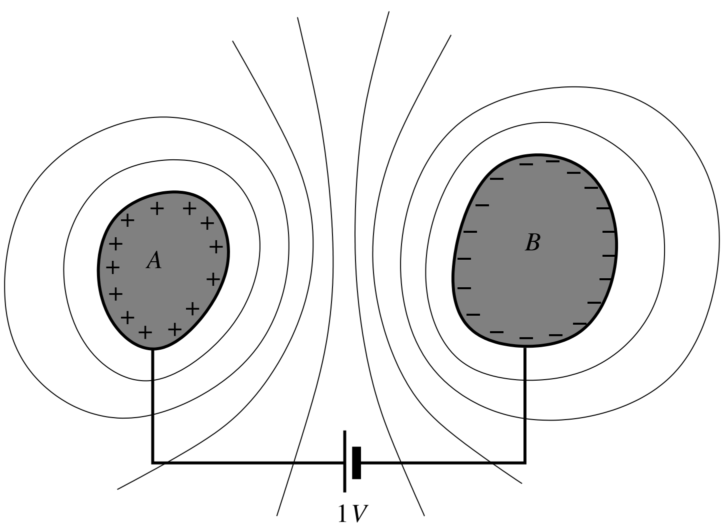
$\rho_{A,B}$: "surface charge density" on ∂A



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Key observation: let $C = \mathcal{B}_\varepsilon(x)$, then (using $G(y,z) = G(z,y)$)

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$$\Rightarrow \qquad \mathbb{E}[\tau_A^x] = w_A(x) \simeq \frac{\int_{A^c} h_{\mathcal{B}_\varepsilon(x),A}(y)\ \mathrm{d}y}{\text{cap}_{\mathcal{B}_\varepsilon(x)}(A)}$$

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Variational representation: Dirichlet form

$$\text{cap}_A(B) = \int_{(A\cup B)^c} \|\nabla h_{A,B}(x)\|^2\,\mathrm{d}x = \inf_{h\in\mathcal{H}_{A,B}} \int_{(A\cup B)^c} \|\nabla h(x)\|^2\,\mathrm{d}x$$

($\mathcal{H}_{A,B}$: set of sufficiently smooth functions satisfying b.c.)

Potential theory

General case: $\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

Generator: $\Delta \mapsto \varepsilon\Delta - \nabla V \cdot \nabla$

Then

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Rough a priori bounds on h show that if x potential minimum,

$$\int_{A^c} h_{\mathcal{B}_\varepsilon(x),A}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \frac{(2\pi\varepsilon)^{d/2}\,\mathrm{e}^{-V(x)/\varepsilon}}{\sqrt{\det(\nabla^2 V(x))}}$$

Estimation of capacity

Neumann b.c.

Truncated energy functional: retain only modes with $k \leqslant d$

$$V[\{y_k\}] = -\frac{1}{2}y_0^2 + u_1(y_1) + \frac{1}{2}\sum_{k=2}^{d} \lambda_k |y_k|^2 + \dots$$

$u_1(y_1) = \frac{1}{2}\lambda_1 y_1^2 + \frac{3}{8L} y_1^4$

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Theorem: For all $L < \pi$,

$$\operatorname{cap}_{\mathcal{B}_\varepsilon(u_-)}(\mathcal{B}_\varepsilon(u_+)) = \varepsilon \frac{\int_{-\infty}^{\infty} \mathrm{e}^{-u_1(y_1)/\varepsilon}\,\mathrm{d}y_1}{\sqrt{2\pi\varepsilon}} \prod_{j=2}^{d} \sqrt{\frac{2\pi\varepsilon}{\lambda_j}} \big[1 + R(\varepsilon)\big]$$

where $R(\varepsilon) = \mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})$ is *uniform in* d.

Sketch of proof

Upper bound:

$$\text{cap} = \inf_h \Phi(h) \leqslant \Phi(h_+) \qquad \Phi(h) = \varepsilon \int \|\nabla h(y)\|^2 \, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y$$

Let $\delta = \sqrt{c\varepsilon|\log \varepsilon|}$, choose

$$h_+(y) = \begin{cases} 1 & \text{for } y_0 < -\delta \\ f(y_0) & \text{for } -\delta < y_0 < \delta \\ 0 & \text{for } y_0 > \delta \end{cases}$$

where $\varepsilon f''(y_0) + \partial_{y_0} V(y_0, 0) f'(y_0) = 0$ with b.c. $f(\pm\delta) = 0, 1$

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Lower bound:

Bound Dirichlet Φ form below by restricting domain, taking only 1st component of gradient and use for b.c. a priori bound on h

Applications

Neumann b.c., $L < \pi$:

$$\Gamma_0 = \frac{1}{2^{3/4}\pi}\sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}$$

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$$\Gamma_0 = \frac{1}{2^{3/4}\pi}\sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}\sqrt{\frac{\lambda_1}{\lambda_1+\sqrt{3\varepsilon/4L}}}\Psi_+\left(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}}\right)$$

where $\lambda_1 = -1 + (\pi/L)^2$ and

$$\Psi_+(\alpha) = \sqrt{\frac{\alpha(1+\alpha)}{8\pi}}\,\mathrm{e}^{\alpha^2/16}\,K_{1/4}\left(\frac{\alpha^2}{16}\right)$$

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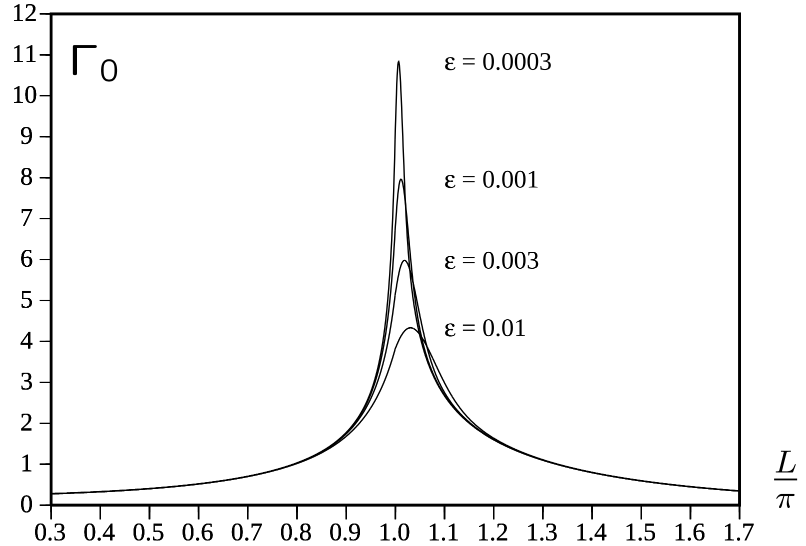
Similar expression for $L > \pi$

(product of eigenvalues computed using path-integral techniques, cf. Maier and Stein)



Applications

Periodic b.c., $L < 2\pi$:

$$\Gamma_0 = \frac{1}{2\pi}\frac{\sinh(L/\sqrt{2})}{\sin(L/2)}\frac{\lambda_1}{\lambda_1+\sqrt{3\varepsilon/4L}}\widetilde{\Psi}_+\left(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}}\right)$$

where $\lambda_1 = -1 + (2\pi/L)^2$, some bounded function $\widetilde{\Psi}_+$

Similar expression for $L > 2\pi$

$$\lim_{L\to 2\pi_-} \Gamma_0 = \frac{\sinh(\sqrt{2}\pi)}{\sqrt{3}\pi}\varepsilon^{-1/2}$$

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