

# Hysteresis in Adiabatic Dynamical Systems: an Introduction

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## Abstract

We give a nontechnical description of the behaviour of dynamical systems governed by two distinct time scales. We discuss in particular memory effects, such as bifurcation delay and hysteresis, and comment the scaling behaviour of hysteresis cycles. These properties are illustrated on a few simple examples.

**Key words:** adiabatic theory, slow-fast systems, bifurcation theory, dynamic bifurcations, bifurcation delay, hysteresis, scaling laws

**Note:** This text is the introduction to, and summary of the author's Ph.D. dissertation. Some references are to chapters of the dissertation. Postscript files are available at <http://dpwww.epfl.ch/instituts/ipt/berglund/these.html>  
Please contact the author for further information.



*“Try not to have a good time ... This is supposed to be educational.”*

*Charles Schulz*

# 1 Introduction

## 1.1 Dynamic Variables and Parameters

Since the discovery of Newton’s equation and its application to the study of the Solar System, it has become apparent that an important number of physical problems could be modeled, more or less accurately, by ordinary differential equations (ODEs). Sometimes, these equations are direct consequences of the fundamental laws of Physics, like Newton’s equation (for classical mechanical systems) or Maxwell’s equations (for electromagnetic problems). Macroscopic systems, for which we cannot neglect the fact that they are composed of a very large number of atoms or molecules, may sometimes be modeled by somewhat more phenomenological laws, taking into account the interaction of a small number of effective degrees of freedom. This applies to the equations of thermodynamics (applicable for instance to kinetics of chemical reactions), master equations (lasers) or mean field equations (phase transitions). There also exist a number of systems, which are not directly related to Physics, but are nevertheless modeled, on a very phenomenological level, by ODEs: this is the case, for instance, for population dynamics in ecology.

When we consider some specific examples, like those given in Table 1, we realize that such differential equations will depend on two kinds of variables: **dynamic variables** and **parameters**. As far as the mathematical model is concerned, the distinction between these two types of variables is clear:

- dynamic variables define the state of the system; their role is twofold: on one hand, they evolve in time, specifying the state of the system at each instant; on the other hand, they determine the future evolution of the system;
- parameters also influence the future evolution, but their value remains fixed; in fact, a different dynamical system is obtained for each value of the parameters.

System	Dynamic variables	Parameters
Mechanical system	Positions and momenta	External driving force
Electric device	Charges and currents	Power supply, tunable resistance
Chemical reaction	Concentration of reacting substances	Supply flux, temperature
Laser	Level population, internal field	External field
Magnet	Order parameter (magnetization)	Magnetic field, temperature
Population dynamics	Number of individuals of each species	Climate, reproduction rate

TABLE 1. Examples of systems which can be modeled by ODEs, with associated dynamic variables and parameters.

Are the “parameters” of Table 1 really always fixed? Let us examine more closely different kinds of parameters which may appear in a physical experiment. We may distinguish the following types:

- parameters which are related to physical constants or technical specifications of the experimental set-up, and are, therefore, fixed during the experiment; this applies to masses and coupling constants of particles, and dimensions of a cavity or reactor;
- **control parameters**, which can be accurately tuned, say by turning a knob of the experimental device; this may be the case for the supply voltage of an electric device, an applied external field, or the temperature difference between two sides of a cavity;
- parameters that one would like to maintain fixed, but which are not so easy to control in a real experiment, like a supply flux of chemicals, or the temperature in a reactor.

One usually characterizes a dynamical system by its **bifurcation diagram**, representing the asymptotic state (which may be stationary, periodic or more complicated) against the control parameter (Fig. 1). What do we mean when we say that the bifurcation diagram is determined experimentally by varying the control parameter?

According to the mathematical modeling, the bifurcation diagram should be determined as follows. Fix the control parameter and choose an initial state for the system. Let the system evolve until it has reached an asymptotic state. Repeat this procedure for different initial conditions, in order to find other possible asymptotic states. Then increase the control parameter, reset the initial state, and repeat the whole experiment. Apply this procedure for the desired set of parameter values, and plot the asymptotic state(s) against the control parameter.

In practice, it is not always possible to carry out this rather elaborate program. We may not have the time to wait for the system relaxing to equilibrium for each parameter value, or we may not be able to reset the initial condition. In fact, it is very tempting<sup>1</sup> to turn slowly the knob controlling the parameter during the experiment, in the hope that if this parameter variation is sufficiently slow, it will not affect the bifurcation diagram very much.

Is this hope justified? The answer to this question is not immediate at all. It requires a precise understanding of the relation that exists between, on one hand, a one-parameter family of autonomous Dynamical Systems and, on the other hand, the system with slowly time-dependent parameter. This relation is by no means trivial in all cases, since memory effects, in particular hysteresis, may show up in such systems. An understanding of this relation would allow us, for example, to solve the following problems:

- If the control parameter is swept slowly in time, do we obtain a trustworthy representation of the bifurcation diagram?
- How do parameters, which cannot be controlled completely, but are subject to slow fluctuations, affect our modeling of the system?
- Consider a system subject to a slowly time-dependent driving force. Can we use the static bifurcation diagram (which is analytically more tractable) to gain some information on the time-dependent system?

To deal with this kind of questions, we should begin by understanding the role of time scales in Physics.

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<sup>1</sup>Every person who has ever seen an experimental device with a knob for the control parameter knows that it is indeed *very* difficult not to turn this knob during the experiment.

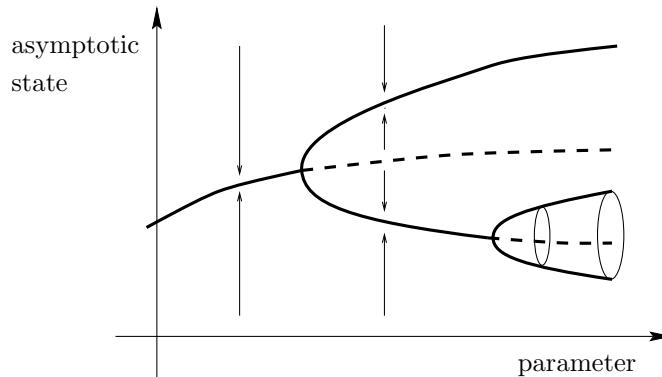


FIGURE 1. Example of a bifurcation diagram. For each value of the parameter, one plots the asymptotic state of the system. In this example, there is a unique stable equilibrium state for small parameter (thick full line). At some parameter value, this equilibrium becomes unstable (dotted line), while two new stable equilibria are formed. One of them is then replaced by a limit cycle, i.e., a stable periodic orbit. To determine this diagram experimentally, one should fix a value of the parameter and an initial condition, and wait for the system to relax to equilibrium (vertical arrows). This procedure should be repeated for several parameter values.

## 1.2 Slow–Fast Systems and Hysteresis

Physical systems are often characterized by one or several time scales. A characteristic time might be the period of a typical periodic solution, or the relaxation time to equilibrium. Let us consider a dynamical system with characteristic time  $T_1$ , called the **fast system**, and couple it to another system with much larger characteristic time  $T_2 \gg T_1$ , called the **slow system**.

Two particular situations are of interest:

1. The evolution of the slow system is imposed from outside, and acts on the fast system as a slowly time–dependent parameter. For this purpose, it need not be governed by a differential equation. We call this coupled system an **adiabatic system**.
2. The slow system is also a dynamical system, which is coupled to and influenced by the fast one. In this situation, we speak of a **slow–fast system**.

As an illustration, let us imagine the following population model. In some relatively small ecosystem, predators and prey reproduce, say, a couple of times a year. Their populations have attained a cyclic regime, with a period of a few years. Now the climate begins to change slowly, due for instance to human impact, modifying the reproduction rate of the predator. This would be an example of an adiabatic system, since the climate change is imposed from outside. Another situation appears when, due to continual food consumption by the prey, vegetation and *micro-climate* are slowly modified, changing the reproduction rates in turn. This would be an example of a slow–fast system.

In this work, we are mainly interested in adiabatic systems. We believe, however, that most results can be transposed to slow–fast systems (see Chapter 4).

What do we expect from the behaviour of an adiabatic system? To fix the ideas, we can keep in mind the example of the motion of a damped particle, in a slowly time–dependent potential. Let us first examine the case when the static system (obtained by freezing the potential) admits a stable stationary state (a potential minimum), depending smoothly on

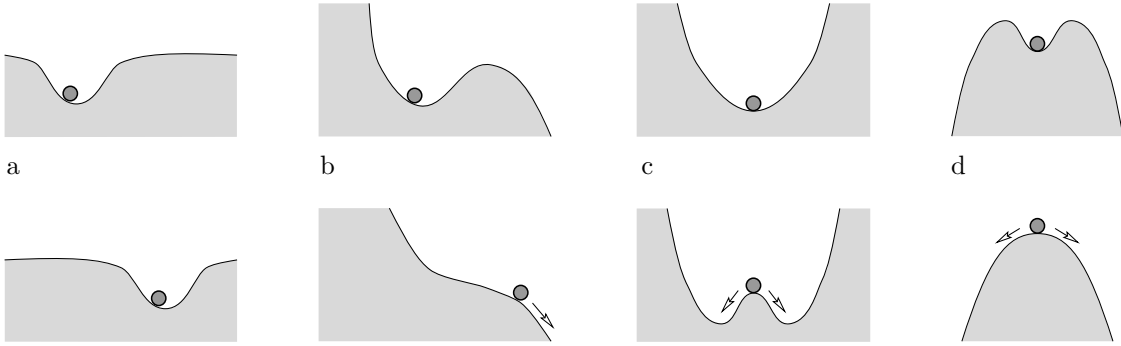


FIGURE 2. The damped motion of a particle in a slowly varying potential provides a simple example of adiabatic system. If the potential admits an isolated, slowly moving minimum, the particle will follow this well adiabatically (a). Bifurcations correspond to situations where this minimum interacts with other equilibrium points. For instance, the minimum may annihilate with a maximum (saddle-node bifurcation), and the particle leaves the vicinity of the bifurcation point (b). We may also have creation of two new equilibria (direct pitchfork bifurcation), so that the particle has to choose between its current, unstable position, and two potential wells (c). Or the minimum may disappear in favor of a maximum (indirect pitchfork bifurcation) (d).

the parameter. When the parameter is fixed, orbits starting in its neighborhood will relax to this equilibrium. When the parameter is swept slowly in time, it is generally believed that the orbit will follow the equilibrium curve adiabatically, i.e., the particle will remain close (in a sense to be made precise later) to the potential minimum.

This behaviour has the following physical interpretation: in the adiabatic limit, the asymptotic state will be identical with the static equilibrium curve. In other words, the fast system is enslaved by the slow one, its state being entirely determined by the value of the slow variables (i.e., the parameters).<sup>2</sup>

New phenomena arise when the equilibrium loses stability, a situation known as **bifurcation**. Different scenarios are possible (Fig. 2): the equilibrium may simply disappear, or it may become unstable after interacting with one or several other equilibria. The particle's motion depends a lot on the local structure of the bifurcation. In some cases, it leaves the vicinity of the bifurcation point, until reaching some other equilibrium or limit cycle. It may also follow a new equilibrium branch created in the bifurcation, or even remain close to an *unstable* equilibrium for some time, a phenomenon known as **bifurcation delay**, which can be interpreted as metastability. These problems belong to the field of **dynamic bifurcations**, which has received much attention in recent years.

These local features of dynamics have a strong influence on global properties. Let us focus on the situation when the parameter is varied periodically in time. Without bifurcations, the solution will merely follow the periodic motion of a stable equilibrium, independently of whether the parameter is increasing or decreasing. The situation changes in presence of bifurcations. It may happen, for instance, that the fast system follows a different equilibrium branch for increasing or decreasing parameter. This phenomenon is known as **hysteresis**: the asymptotic state depends not only on the present value of the parameter, but also on its history (Fig. 3).

<sup>2</sup>To avoid a confusion due to terminology, we point out that in thermodynamics, such a motion will be called **quasistatic** rather than adiabatic.

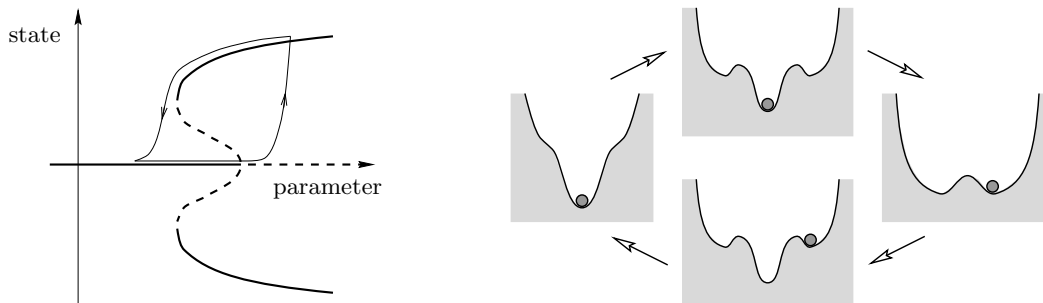


FIGURE 3. Example of a bifurcation diagram leading to hysteresis (a similar diagram is found in [Wi]). It can be seen as a combination of the bifurcations in Fig. 2b and d. For increasing parameter, the solution follows the stable origin at least until the bifurcation (in fact, we will see that it may even follow the unstable origin for some time). When it finally reaches the new stable branch, and the parameter is decreased again, it stays close to this branch down to a smaller value of the parameter, hence describing a hysteresis cycle.

Hysteresis can be interpreted as the non-commutation of two limits, the asymptotic and the adiabatic one. Mathematically, it is easier to take the adiabatic limit first, which amounts to freezing the slow system. The motion of the fast system is then governed by an autonomous (closed) equation, and taking the asymptotic limit merely corresponds to analysing its equilibria (or other attractors).

This is, however, not the physically interesting information. We would like instead to fix some small, but positive frequency of the parameter variation, and study the asymptotic motion of the time-dependent fast system (whatever this motion may be). *Then* we would like to determine how this asymptotic motion behaves in the adiabatic limit, i.e., when the frequency of parameter variation goes to zero.

Without bifurcation, the two limits can be taken in either order: the asymptotic motion will approach a simple function of the parameter in the adiabatic limit (see Example 1 below). In presence of bifurcations, hysteresis may occur. This is the central topic of this work. To understand how hysteresis arises in adiabatic Dynamical Systems, we first need to develop methods which enable us to determine solutions for small, but positive parameter sweeping rates. In particular, we have to understand if, for a periodically varied parameter, these solutions tend asymptotically to periodic ones, or if more complicated dynamics are possible. Then we will be able to study their behaviour in the adiabatic limit.

### 1.3 Historical Account

We have no intention of giving here an exhaustive historical account of the theory of Dynamical Systems with multiple time scales. Besides the fact that such an exposition would take many pages, we do not feel sufficiently well acquainted with the multiple aspects of this large domain to be able to cite correctly the numerous researchers who contributed to one (or several) of its facets. Instead, we would like to mention at this place the major sources of inspiration of this work.

Research on adiabatic systems, hysteresis and related subjects appears to have been pursued almost independently by mathematicians and physicists. The former have been

mostly interested in slow-fast systems, adiabatic invariants and, more recently, in dynamic bifurcations and bifurcation delay. The latter have rediscovered several times during this century the importance of adiabatic systems. Recently, there has been renewed interest in hysteresis appearing in lasers and magnets. Different models have been considered, and studied mainly by numerical methods.

### Mathematics: slow-fast systems and bifurcation delay

Slow-fast systems have been studied almost since the beginning of differential equations theory itself. They appear naturally in perturbed integrable systems, where angle variables define the fast system, and action variables the slow one. For instance, in the Solar System, fast variables describe the motion of planets in their orbits, while slow variables describe the spatial orientation of these orbits.<sup>3</sup> When the interaction between planets is neglected, these orbits are frozen in space, whereas they begin to deform slowly in time when their interaction is taken into account.

Research on these systems has mainly focused on the dynamics of *slow* variables. The method of averaging, for instance, aims at replacing the dynamics of the slow variables by an effective equation, where the fast variables have been averaged out [Ar2]. One often tries to construct **adiabatic invariants**, which are functions on phase space remaining (almost) constant in time. A highlight of this line of research is the celebrated Kolmogorov–Arnol’d–Moser (KAM) theorem, which proves the existence of exact adiabatic invariants for some initial conditions.

Adiabatic dynamics have been, for a long time, mainly studied in relation with quantum mechanics [Berry]. The **quantum adiabatic theorem** states that solutions of the slowly time-dependent Schrödinger equation will adiabatically follow the eigenspaces of the instantaneous Hamiltonian. Although this problem is relatively old, rigorous proofs have been given only very recently [JKP]. Classical adiabatic systems (mostly linear ones) have been studied in some detail by Wasow [Wa].

An early result on nonlinear slow-fast systems is due to Pontryagin and Rodygin [PR] in 1960. They showed that orbits of the fast system, which start sufficiently close to a stable equilibrium or limit cycle, will follow this attractor adiabatically. Problems involving bifurcations seem to have been studied for the first time by Lebovitz and Schaar [LS] in 1977. They considered problems where two equilibrium branches exchange stability, and showed that under some generic conditions, the orbit will follow a stable branch after the bifurcation.

In 1979, Haberman [Hab] considered a class of one and two-dimensional problems. He introduced the notion of slowly varying states, computed as series in the adiabatic parameter, and studied in particular jump phenomena (also known as catastrophes) occurring near saddle-node bifurcations.

The topic which would soon be given the name of **dynamic bifurcations** developed rapidly in the second half of the eighties. The importance of the **bifurcation delay** phenomenon in various physical situations (lasers, neurons) was emphasized by Mandel, Erneux and co-workers [ME1, ME2, BER], who derived an approximate formula for the delay time using slowly varying states. This phenomenon, and the related problem of **ducks** (also called **canards**) were then studied by several mathematicians, using non-standard analysis (see [Ben] for a summary of these works and a more detailed history).

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<sup>3</sup>See for instance Laskar’s article in [DD] for a non-technical discussion.



A common feature of most of these works, including Wasow's, is that the authors try to construct particular solutions as series in the adiabatic parameter. The problem is, however, that these series are in general *not* convergent. A naive treatment of such equations may therefore yield, in some cases, incorrect results. In order to obtain the right answers with these methods (for instance the fact that there exists a maximal value for the bifurcation delay), one has to use rather elaborate techniques, as resummation of divergent series (see [Ben], in particular the articles by Diener and Diener, and by Canalis–Durand).

An entirely new direction to treat these problems was initiated by Neishtadt [Ne1, Ne2]. Returning to the old technique of successive changes of variables, but combined with estimations inspired by Nekhoroshev, he was able to prove rigorously the existence of a bifurcation delay. Moreover, with the help of a technique involving deformation of an integration path into the complex plane, he could give an explicit lower bound to the delay time. Diener and Diener [Ben] have examined under which generic conditions this formula gives an upper bound as well.

Recently, these results have been generalized to the case of a periodic orbit undergoing Hopf bifurcation [NST].

### Physics: hysteresis and scaling laws

Research on hysteresis has been pursued by physicists, almost independently of mathematicians, and mostly with numerical methods. For a long time, the standard model for hysteretic phenomena has been the Preisach model [May, MNZ]. This model, however, is artificial and provides no derivation of hysteresis from microscopic principles.

Interest in microscopic models of magnetic hysteresis was renewed in 1990, by an important article by Rao, Krishnamurthy and Pandit [RKP]. They analyse numerically two models, an Ising model with Monte–Carlo dynamics, and a continuous model with  $O(N)$  symmetry in the large  $N$  limit. They proposed in particular that the area  $\mathcal{A}$  enclosed by the hysteresis cycle should scale with the amplitude  $H_0$  of the magnetic field and its frequency  $\Omega$  according to the power law  $\mathcal{A} \sim H_0^\alpha \Omega^\beta$ , where  $\alpha \sim 0.66$  and  $\beta \sim 0.33$  for small frequency and amplitude.

This work inspired a large number of articles trying to exhibit scaling laws for hysteresis cycles. In the case of a laser system [JGRM], analytical arguments showed that a one-dimensional model equation admits a hysteresis cycle with area  $\mathcal{A}(\Omega) \sim \mathcal{A}(0) + \Omega^{2/3}$ . The discrepancy between this result and the one in [RKP] lead in following years to some controversy [Ra].

Still in the year 1990, a numerical study of a mean field approximation of the Ising model introduced the concept of a **dynamic phase transition** [TO]: regions with zero and non-zero average magnetization by cycle are separated by a transition line in the temperature–magnetic-field-amplitude plane.

These papers were followed by various numerical simulations (on lattice models and continuous ones) and experiments, which proposed new sets of exponents. We show some of them in Table 2. The trouble is that even for one and the same model, these exponents differ widely from one experiment to the other.

There have been several attempts to derive these exponents analytically. Relatively simple systems, like lasers, seem to be described satisfactorily by one-dimensional equations, as shown in [HL&, GBS] which extend results in [JGRM]. However, for magnetic

	System	Scaling of $\mathcal{A}$
Experiments	Iron [Ste] Fe/Au film [HW] Co/Cu film [JYW] Fe/W film [SE]	$H_0^{1.6}$ $H_0^{0.59}\Omega^{0.31}$ $\mathcal{A}_0 + (H_0^2 - H_c^2)^{0.34}\Omega^{0.66}$ $H_0^{0.25}\Omega^{0.03}$
Numerical simulations	$(\Phi^2)^2$ model [RKP] $(\Phi^2)^2$ model [ZZS] $(\Phi^2)^3$ model [ZZS] Ising 2D Monte-Carlo [LP] Ising 2D Monte-Carlo [ZZL] Ising 2D Monte-Carlo [AC1] Ising 3D Monte-Carlo [AC1] Cell-dynamical system [ZZL] Mean field [LZ]	$H_0^{0.66}\Omega^{0.33}$ $\Omega^{0.5}$ $\mathcal{A}_0 + \Omega^{0.7}$ $H_0^{0.46}\Omega^{0.36}$ $\mathcal{A}_0 + \Omega^{0.36}$ $H_0^{0.7}\Omega^{0.36}$ $H_0^{0.67}\Omega^{0.45}$ $\mathcal{A}_0 + \Omega^{0.66}$ $\mathcal{A}_0 + H_0^{0.66}\Omega^{0.66}$
Analytical arguments	Mean field [JGRM] $(\Phi^2)^2$ model [DT] $(\Phi^2)^2$ model [SD] $(\Phi^2)^2$ model [ZZ2] Ising dD [SRN]	$\mathcal{A}_0 + (H_0^2 - H_c^2)^{1/3}\Omega^{2/3}$ $H_0^{1/2}\Omega^{1/2}$ $H_0^{1/2}\Omega^{1/2}$ $\Omega^{1/2}$ $ \ln \Omega ^{-1/(d-1)}$

TABLE 2. Some results on the scaling behaviour of the area  $\mathcal{A}$  enclosed by a hysteresis cycle, as a function of magnetic field amplitude  $H_0$  and frequency  $\Omega$ . Recent experiments were made with ultrathin films. Numerical Monte-Carlo simulations have been carried out on the two-dimensional (2D) and three-dimensional (3D) Ising model with Glauber dynamics. Other numerical experiments concern the Langevin equation in a Ginzburg-Landau  $(\Phi^2)^2$  or  $(\Phi^2)^3$  potential with  $O(N)$  symmetry. In the large  $N$  limit, the noise can be eliminated from the equation, and one obtains deterministic ODE. The proposed exponents differ a lot from one experiment to another. In particular, it is not clear whether the area should go to zero, or to a finite limit  $\mathcal{A}_0$  when  $\Omega \rightarrow 0$ . It is amusing to note that results of one experiment [JYW] could be fitted on the mean field result [JGRM], while another one [HW] was fitted on results of the  $(\Phi^2)^2$  model studied in [RKP]. Although the mean-field studies in [JGRM] and [LZ] predict the same  $\Omega$ -dependence, they do not agree on the  $H_0$ -dependence. In fact, we will show that both laws are incorrect.

systems, no satisfactory explanation has been obtained. Some analytical arguments, using rescaling [SD] or renormalization [ZZ2] seem to indicate that the area should scale as  $\mathcal{A} \sim H_0^{1/2} \Omega^{1/2}$ . Various explanations have been proposed for these discrepancies, for instance logarithmic corrections [DT].

In fact, it is not clear at all whether the area should really follow a power law [SRN]. It depends probably in a crucial way on the detailed dynamics of droplets during magnetization reversal. At any rate, understanding how these scaling laws may appear in the model equations would be a good criterion to test their adequacy against real physical systems. Recently, several authors have introduced other models, including quantum effects [BDS]; they have also become interested in other indicators, like pulse susceptibility [AC2].

## 2 Mathematical Formulation

### 2.1 Adiabatic Systems and Slow–Fast Systems

We will consider dynamical systems described by ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, \lambda), \quad (1)$$

where  $x \in \mathbb{R}^n$  is the vector of dynamic variables, and  $\lambda \in \mathbb{R}^p$  is a set of parameters. We shall assume that  $f$  is a function of class  $\mathcal{C}^2$  at least.

The slow variation of parameters is described by a function  $G(\varepsilon t)$ , where  $0 < \varepsilon \ll 1$  is the **adiabatic parameter**:

$$\frac{dx}{dt} = f(x, G(\varepsilon t)). \quad (2)$$

This formulation should be interpreted as follows:  $f(x, \lambda)$  and  $G(\tau)$  are given functions, fixed once and for all,<sup>4</sup> and we would like to understand the behaviour of (2) in the **adiabatic limit**  $\varepsilon \rightarrow 0$ . For instance,  $G(\varepsilon t) = \sin(\varepsilon t)$  would describe a periodic variation of the parameter, with small frequency  $\varepsilon$ .

The adiabatic limit should be taken with some care. If we naively replace  $\varepsilon$  by 0 in (2), we obtain the autonomous system  $\frac{dx}{dt} = f(x, G(0))$ . This is due to the fact that with respect to the slow time scale, we have zoomed on a particular instant. This is not what we are interested in: it is more natural, for our purpose, to study the system on the slow time scale of parameter variation. We do that by introducing a **slow time**  $\tau = \varepsilon t$ , so that (2) can be rewritten

$$\varepsilon \frac{dx}{d\tau} =: \varepsilon \dot{x} = f(x, G(\tau)). \quad (3)$$

We call this equation an **adiabatic system**. In the adiabatic limit, it reduces to the algebraic equation  $f(x, G(\tau)) = 0$ . We will see that although this limit is singular, it is less problematic to analyse than for (2).

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<sup>4</sup>One may, in fact, allow for an  $\varepsilon$ -dependence of  $f$ , provided  $f$  behaves smoothly (in some sense) in the limit  $\varepsilon \rightarrow 0$ , see Section 4.1.

By contrast, a **slow-fast system** is described by a set of coupled ODE of the form

$$\begin{aligned}\varepsilon \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y).\end{aligned}\tag{4}$$

In some circumstances, adiabatic and slow-fast systems are equivalent and may be transformed into one another. For instance, if  $G(\tau)$  is the solution of a differential equation  $\dot{y} = g(y)$ , the adiabatic system (3) can be transformed into a slow-fast system. If  $\lambda \in \mathbb{R}$ , this transformation is only possible for monotonous  $G(\tau)$ . There are other ways to write (3) as a vector field, for instance by considering the slow time  $\tau$  as a dynamic variable (see next subsection). In some particular cases, it may be helpful to introduce additional variables, for instance  $G(\tau) = \sin \tau$  is a solution of  $\dot{y} = z$ ,  $\dot{z} = -y$ .

On the other hand, if  $g(x, y)$  depends only on  $y$ , the slow-fast system (4) is equivalent to the adiabatic system (3), with  $G(\tau)$  given by the solution of  $\dot{y} = g(y)$ . If  $g$  depends on  $x$  as well, this reduction is not possible, but one can sometimes construct a solution in the following way: in first approximation,  $x$  is related to  $y$  by the algebraic equation  $f(x, y) = 0$ . If this equation admits a unique solution  $x = x^*(y)$ ,  $y(\tau)$  may be approximated by a solution of the equation  $\dot{y} = g(x^*(y), y)$ , which can be used in turn to estimate corrections to the solution  $x(\tau) \simeq x^*(y(\tau))$ .

## 2.2 Adiabatic Systems and Vector Fields

We can exploit the similarities with slow-fast systems to obtain valuable informations on the solutions of the adiabatic system (3) without any analytical calculation. This is done by using geometric properties of vector fields. For simplicity, we consider the case of a scalar parameter  $\lambda \in \mathbb{R}$ .

It is always possible to write (3) as a vector field by considering the slow time  $\tau$  as a dynamic variable:

$$\begin{aligned}\frac{dx}{d\tau} &= f(x, G(\tau)) \\ \frac{d\tau}{dt} &= \varepsilon.\end{aligned}\tag{5}$$

A major drawback is that this vector field has no singular points. One can however deduce some general properties of the flow. When  $f(x, G(\tau)) \neq 0$ , orbits have a large slope of order  $\varepsilon^{-1}$ , due to the short characteristic time of the fast variable  $x$ . On the other hand, when  $f(x, G(\tau)) = 0$ , the vector field is parallel to the  $\tau$ -axis. In fact, in a neighborhood of order  $\varepsilon$  of an equilibrium branch, the motion of the fast variable becomes slow: it is the region where **adiabatic solutions** (also known as slowly varying states) may exist.

In the case  $x \in \mathbb{R}$  ( $n = 1$ ), the form of the vector field (5) imposes strong constraints on the solutions. It is possible to show, using only geometric arguments, that some solutions will remain in the neighborhood of equilibrium branches of  $f$  (Fig. 4). We will see that this property can be generalized to the  $n$ -dimensional case.

There are two particular classes of functions  $G(\tau)$  for which it is possible to say more:

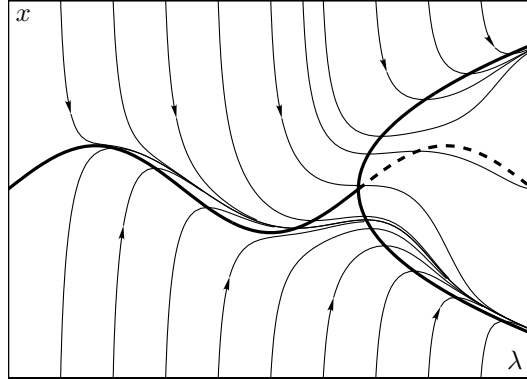


FIGURE 4. Solutions of the equation  $\varepsilon \dot{x} = f(x, \lambda)$ , for  $\lambda(\tau) = \tau$  [here, the function  $f$  is given by  $f(x, \lambda) = (x - \sin(\pi\lambda)/2)(\lambda - x^2)$ ]. The curves on which  $f(x, \lambda) = 0$  (thick lines) delimit regions where the vector field has positive or negative slope. This imposes geometrical constraints on the solutions. In the left half of the picture, there exists a stable equilibrium branch. Solutions lying above this branch are decreasing, while those lying below are increasing. From this construction, one can already deduce existence of adiabatic solutions, remaining close to the equilibrium. For a special parameter value, there is a bifurcation: the equilibrium becomes unstable, and new stable branches are created. In this case, adiabatic solutions coming from the left follow the lower branch.

### Monotonous case

If  $G(\tau)$  is strictly monotonous, it admits an inverse function  $G^{-1}$ . We may thus use  $\lambda = G(\tau)$  as a dynamic variable, giving

$$\begin{aligned} \varepsilon \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= g(\lambda) := G'(G^{-1}(\lambda)). \end{aligned} \tag{6}$$

If  $G'(\tau)$  goes to zero in some limit, fixed points may appear in the vector field.

Consider for instance the case  $G(\tau) = \tanh \tau$ . Then  $g(\lambda) = 1 - \lambda^2$  vanishes at  $\lambda = \pm 1$ . As  $\tau$  goes to infinity ( $\lambda \rightarrow 1$ ), trajectories will be attracted by stable fixed points of  $f(x, 1)$ . We conclude that if  $\lambda$  moves infinitely smoothly from an initial to a final value, we can construct a smooth transformation which compactifies phase space, and in this way the asymptotic limit  $\tau \rightarrow \infty$  can be properly defined (Fig. 5a).

### Periodic case

Assume  $G(\tau)$  is periodic, say  $G(\tau) = \sin \tau$ . We can write the adiabatic system in the form (5), with the particularity that  $\tau$  can be considered as a periodic variable (i.e., the phase space has the topology of a cylinder). Since the flow is transverse to every plane  $\tau = \text{constant}$ , dynamics can be characterized by the **Poincaré section** at  $\tau = 0$  (say), and its **Poincaré map**  $T : x(0) \mapsto x(1)$ . In particular, periodic orbits correspond to fixed points of  $T$ . In the one-dimensional case, this fact can be used to prove that every orbit is either periodic, or attracted by a periodic orbit.

Of course, to study hysteresis properties, we would like to go back to  $(\lambda, x)$ -variables, which is done by “wrapping” the  $(\tau, x)$ -space (Fig. 5b). Some information can also be gained by using a representation of the form (5) on each interval in which  $G(\tau)$  is

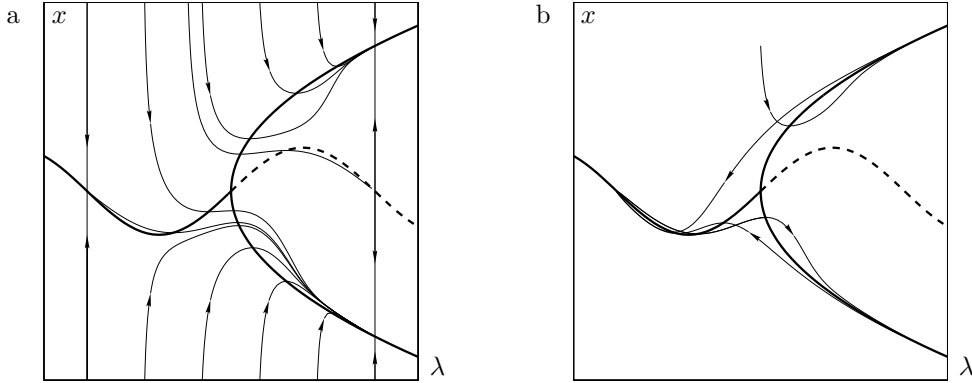


FIGURE 5. Same equation as in Fig. 4, but with (a)  $\lambda(\tau) = \text{th}(\tau)$  and (b)  $\lambda(\tau) = \sin(\tau)$ . In (a), the system admits hyperbolic fixed points at  $(\pm 1, 0)$ , and stable nodes at  $(1, \pm 1)$ , which define the asymptotic states. The stable manifold of  $(1, 0)$  delimits the basins of attraction. In this case, all trajectories reach the lower equilibrium. In (b), during the first cycle the solution follows the upper branch, which is still a transient motion. From the next cycle on, it is attracted by a periodic orbit following the lower branches.

monotonous. One should, however, pay attention to the fact that this transformation introduces artificial singularities in the vector field at those points where  $G'(\tau)$  vanishes.

### 2.3 Some Simple Examples

Let us return to the example of the damped motion of a particle in a potential, which is described by an equation of the form

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{\partial \Phi}{\partial x}(x, \lambda) = 0. \quad (7)$$

We will show in Chapter 3 that for sufficiently large friction, this system is governed by the one-dimensional equation

$$\frac{dx}{dt} = f(x, \lambda) \simeq -\frac{\partial \Phi}{\partial x}(x, \lambda). \quad (8)$$

Thus, any one-dimensional system of the form

$$\varepsilon \dot{x} = f(x, \lambda(\tau)) \quad (9)$$

may be interpreted as describing the overdamped motion of a particle in a slowly varying potential  $\Phi(x, \lambda(\tau))$ , with  $f \simeq -\partial_x \Phi$ . Let us examine some particular cases, in order to illustrate the previously discussed concepts.

**Example 1.** Consider the equation

$$\varepsilon \dot{x} = -x + \lambda(\tau), \quad \lambda(\tau) = \sin \tau, \quad (10)$$

which describes the motion of an overdamped, adiabatically forced harmonic oscillator. It can be solved explicitly, with the result

$$x(\tau) = \left[ x(0) + \frac{\varepsilon}{1 + \varepsilon^2} \right] e^{-\tau/\varepsilon} + \frac{1}{1 + \varepsilon^2} (\sin \tau - \varepsilon \cos \tau). \quad (11)$$

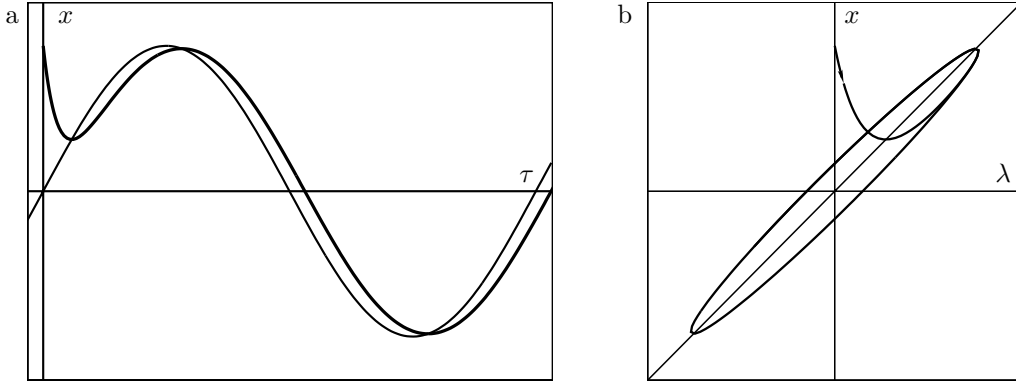


FIGURE 6. Solutions of the equation  $\varepsilon \dot{x} = -x + \sin \tau$  of Example 1. (a) After a short transient, the solution  $x(\tau)$  (thick line) follows adiabatically the forcing  $\sin \tau$  (thin line), with a phase shift of order  $\varepsilon$ . (b) The “Lissajous plot” of this solution in the  $(\lambda, x)$ -plane is attracted by an ellipse, at a distance of order  $\varepsilon$  from the line  $x = \lambda$ . This cycle encloses an area of order  $\varepsilon$  which vanishes in the adiabatic limit, thus we do not consider it as a hysteresis cycle.

The second term is a periodic particular solution of (10). It follows the forcing  $\lambda(\tau)$  with a phase shift of order  $\varepsilon$  (Fig. 6a). This is precisely what we call an **adiabatic solution**, since it remains in a neighborhood of order  $\varepsilon$  of the static equilibrium  $x = \lambda(\tau)$ . In the  $(\lambda, x)$ -plane, it is represented by an ellipse with width of order  $\varepsilon$  (which can be interpreted as a Lissajous plot of the solution), see Fig. 6b.

The first term in (11) is a transient one, which decreases exponentially fast. In fact, it is of order  $\varepsilon$  as soon as  $\tau \geq \tau_1(\varepsilon) = \varepsilon |\ln \varepsilon|$ . Since  $\lim_{\varepsilon \rightarrow 0} \tau_1(\varepsilon) = 0$ , we may write

$$\lim_{\varepsilon \rightarrow 0} x(\tau; \varepsilon) = \lambda(\tau) \quad \text{for } \tau > 0. \quad (12)$$

The state of the system is thus determined entirely by the slow variable  $\lambda$ . According to the discussion of Subsection 1.2, we are in a situation without hysteresis, since the adiabatic and asymptotic limit commute. Indeed, the physically meaningful procedure is to take the asymptotic limit first: we find that trajectories converge to the periodic solution  $\bar{x}(\tau, \varepsilon) = (\sin \tau - \varepsilon \cos \tau)/(1 + \varepsilon^2)$ . Then we see that  $\bar{x}(\tau, \varepsilon)$  tends to  $\lambda(\tau)$  in the adiabatic limit  $\varepsilon \rightarrow 0$ . On the other hand, taking the adiabatic limit directly in (10) yields the correct result  $x = \lambda(\tau)$ .

The fact that all orbits are attracted by a periodic one can also be seen on the Poincaré map (taken at  $\tau = 0 \equiv 2\pi$ ), which reads

$$T : x \mapsto \left[ x + \frac{\varepsilon}{1 + \varepsilon^2} \right] e^{-2\pi/\varepsilon} - \frac{\varepsilon}{1 + \varepsilon^2}, \quad (13)$$

and admits a stable fixed point at  $x^* = -\varepsilon/(1 + \varepsilon^2)$ . Let us finally point out that the fact that the periodic solution  $\bar{x}(\tau; \varepsilon)$  admits a convergent series in  $\varepsilon$  is rather exceptional, in general we will only be able to obtain asymptotic series.

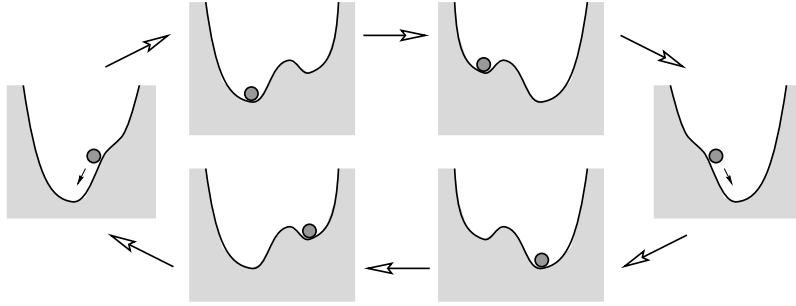


FIGURE 7. Equation (14) can be interpreted as describing the overdamped motion of a particle in a slowly varying potential of the form shown here. When  $\lambda < -\lambda_c$ , the particle joins the equilibrium  $x_-^*(\lambda)$ . For  $-\lambda_c < \lambda < \lambda_c$ , a new minimum  $x_+^*(\lambda)$  has appeared, but the particle still remains in the left well. Only at  $\lambda = \lambda_c$ , when the left equilibrium disappears, will the particle join  $x_+^*(\lambda)$ , which it follows as long as  $\lambda > -\lambda_c$ . For intermediate values of  $\lambda$ , the position of the particle depends not only on  $\lambda(\tau)$ , but also on  $\dot{\lambda}(\tau)$ : the system displays hysteresis.

**Example 2.** The equation

$$\varepsilon \dot{x} = x - x^3 + \lambda(\tau), \quad \lambda(\tau) = \sin \tau, \quad (14)$$

describes the overdamped motion of a particle in a Ginzburg–Landau type double-well potential  $\Phi(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ , with an external field  $\lambda$  (Fig. 7). This is the most common example for hysteresis in ODE found in textbooks [MR, MK&].

Taking the adiabatic limit  $\varepsilon \rightarrow 0$  in (14), we obtain the algebraic equation of a cubic  $\lambda = -x + x^3$ , admitting stationary points  $\pm(x_c, -\lambda_c)$ , where  $x_c = 1/\sqrt{3}$  and  $\lambda_c = 2/3\sqrt{3}$ . When  $|\lambda| > \lambda_c$ , there is a single solution  $x^*(\lambda)$ , which corresponds to a stable equilibrium of the static system. But when  $|\lambda| < \lambda_c$ , there are *three* equilibrium curves (two stable and one unstable), and it is not clear from this analysis which one the trajectory will follow. Let us denote by  $x_+^*(\lambda)$  the upper stable equilibrium, and  $x_-^*(\lambda)$  the lower one.

Despite its simplicity, equation (14) admits no exact solution. But the qualitative behaviour of orbits can be easily understood by drawing the vector field in the  $(\tau, x)$ -plane (Fig. 8a). Starting for instance at  $(\tau, x) = (0, 0)$ , the orbit will be attracted by the upper branch  $x_+^*$ , and follow it until it disappears when  $\lambda$  becomes smaller than  $-\lambda_c$ . If  $\varepsilon$  is small enough, the trajectory will quickly reach the lower branch  $x_-^*$ , and follow it until  $\lambda$  becomes larger than  $\lambda_c$  again. This behaviour will repeat itself periodically, and it can be checked (using only geometric properties of the Poincaré map) that the trajectory is attracted by a periodic solution  $\bar{x}(\tau; \varepsilon)$ . We thus obtain an asymptotic cycle, characterized by alternating phases with slow and fast motion. Such a solution is called a **relaxation oscillation**.

When wrapping this solution to the  $(\lambda, x)$ -plane, we obtain that

$$\lim_{\varepsilon \rightarrow 0} \bar{x}(\tau; \varepsilon) = \begin{cases} x_+^*(\lambda) & \text{if } \lambda > \lambda_c \text{ or } \lambda > -\lambda_c \text{ and } \dot{\lambda} < 0, \\ x_-^*(\lambda) & \text{if } \lambda < -\lambda_c \text{ or } \lambda < \lambda_c \text{ and } \dot{\lambda} > 0. \end{cases} \quad (15)$$

This solution displays the most familiar type of hysteresis. When  $|\lambda| < \lambda_c$ , the asymptotic state (in the adiabatic limit) depends not only on  $\lambda$ , but also on its derivative.<sup>5</sup>

<sup>5</sup>If  $\lambda(\tau)$  is a more complicated function than  $\sin \tau$ , admitting several different maxima and minima, it



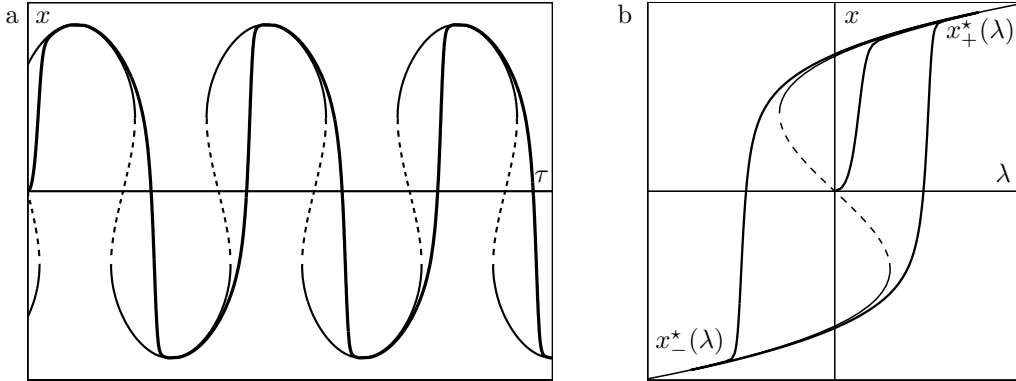


FIGURE 8. Solutions of equation (14), (a) in the  $(\tau, x)$ -plane and (b) in the  $(\lambda, x)$ -plane. Thin full lines indicate stable equilibria of the static system, dashed lines indicate unstable equilibria. These curves are solutions of the equation  $x - x^3 + \lambda = x - x^3 + \sin \tau = 0$ . The first representation (a) is useful to draw the vector field. One easily understands that the solution follows stable branches until the next saddle-node bifurcation, and then moves rapidly to the other branch. We obtain a periodic solution with alternating slow and fast motions, called a **relaxation oscillation**. When this solution is wrapped to the  $(\lambda, x)$ -plane, we obtain a familiar-looking hysteresis cycle. In the limit  $\varepsilon \rightarrow 0$ , this cycle approaches a curve delimited by the equilibrium branches  $x_{\pm}^*(\lambda)$  and two verticals.

The limiting hysteresis cycle (Fig. 8b) has a well-defined area, given by the geometric formula

$$\mathcal{A}(0) = 2 \int_{-\lambda_c}^{\lambda_c} x_+^*(\lambda) d\lambda. \quad (16)$$

It is clear from the vector field analysis that  $\mathcal{A}(\varepsilon)$  increases with  $\varepsilon$ . In fact, it has been shown in [JGRM] that

$$\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{2/3}. \quad (17)$$

We will show in Chapter 4 that this exponent  $2/3$  can be computed in a very simple way, using only local properties of the bifurcation points. We point out that in this example, we have assumed the amplitude of  $\lambda(\tau)$  to be larger than  $\lambda_c$ , so that  $x(\tau)$  necessarily changes sign. We will examine in Chapter 7 what happens when the amplitude approaches  $\lambda_c$ .

**Example 3.** The equation

$$\varepsilon \dot{x} = \lambda(\tau)x - x^3, \quad \lambda(\tau) = \sin \tau, \quad (18)$$

describes the overdamped motion in a potential  $\Phi(x, \lambda) = -\frac{1}{2}\lambda x^2 + \frac{1}{4}x^4$ . In the Ginzburg-Landau analogy, the parameter  $\lambda$  controls the temperature. The potential has a single well at the origin if  $\lambda < 0$  ( $T > T_c$ ,  $T_c$  being the critical temperature of a phase transition), and a double well if  $\lambda > 0$  ( $T < T_c$ ). In the language of dynamical systems, we have a pitchfork bifurcation at  $\lambda = 0$ . For positive  $\lambda$ , the adiabatic system has to choose between two stable equilibria  $\pm\sqrt{\lambda}$  and the unstable origin.

may require more information than  $\lambda(\tau)$  and  $\dot{\lambda}(\tau)$  to compute the asymptotic state at time  $\tau$ . In fact, this state will depend on the velocity of the last passage of  $\lambda(\tau)$  through  $\pm\lambda_c$ .

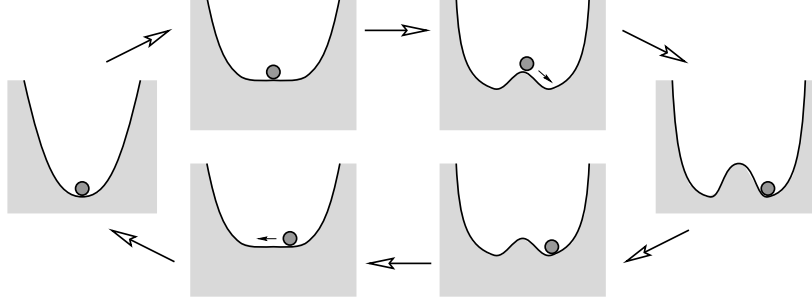


FIGURE 9. The potential corresponding to equation (18). For negative  $\lambda$ , the particle joins the single well at the origin. When this equilibrium becomes unstable, and two new wells are formed, the particle does not react immediately to the bifurcation: it remains for some time in unstable equilibrium near the origin. This situation is called **delayed bifurcation**. Finally, the particle chooses a potential minimum, and follows it until the minima merge to form a single well again. This system displays hysteresis.

Equation (18) admits the explicit solution

$$x(\tau) = \frac{x(\tau_0) e^{\alpha(\tau)/\varepsilon}}{\sqrt{1 + \frac{2}{\varepsilon} x(\tau_0)^2 \int_{\tau_0}^{\tau} e^{2\alpha(s)/\varepsilon} ds}}, \quad \alpha(\tau) := \int_{\tau_0}^{\tau} \lambda(s) ds = \cos \tau_0 - \cos \tau. \quad (19)$$

It is not straightforward to analyse this solution analytically. Let us consider the special case  $\tau_0 = -\pi/2$ ,  $\xi(\tau_0) = 1$ . Then

$$x(\tau) = \frac{e^{-\cos \tau/\varepsilon}}{\sqrt{1 + \frac{2}{\varepsilon} \int_{-\pi/2}^{\tau} e^{-2 \cos s/\varepsilon} ds}}. \quad (20)$$

For  $-\pi/2 < \tau < \pi/2$ ,  $-\cos \tau$  is negative, and the behaviour is governed by the numerator  $e^{-\cos \tau/\varepsilon}$ , which is exponentially small. Thus, the solution remains exponentially close to the origin until  $\tau = \pi/2$ . For negative  $\tau$ , this is not surprising, since the origin is stable. Although the origin becomes unstable at  $\tau = 0$ , the trajectory still remains close to it until  $\tau = \pi/2$ . This is a simple example of **bifurcation delay**: the effective bifurcation takes place at  $\tau = \pi/2$  rather than at  $\tau = 0$ . In the Ginzburg–Landau analogy, this phenomenon may be interpreted as **metastability**.

When  $\tau > \pi/2$ , the solution leaves the origin, and, in fact, settles near the equilibrium position at  $x = \sqrt{\sin \tau}$  until  $\tau = \pi$ , when this branch merges with the origin again. One can show (using for instance the saddle point method to estimate the integral in (20)) that  $x(\pi) = \mathcal{O}(\varepsilon^{1/4})$ . If we plot this solution in the  $(\lambda, x)$ -plane, we find that the bifurcation delay leads to **hysteresis**, since the trajectory always follows a stable branch for decreasing  $\lambda$ , but sometimes follows an unstable one for increasing  $\lambda$ .

In fact, the solution analysed here is still a transient one. During the next cycle of  $\lambda$ , the bifurcation delay is so large that the trajectory ends up by always following the origin. But it is sufficient to add an offset to  $\lambda(\tau)$ , of the form  $\lambda(\tau) = \sin \tau + c$ , to obtain an asymptotic hysteresis cycle as in Fig. 10b. We will show that its area scales as

$$\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{3/4}. \quad (21)$$

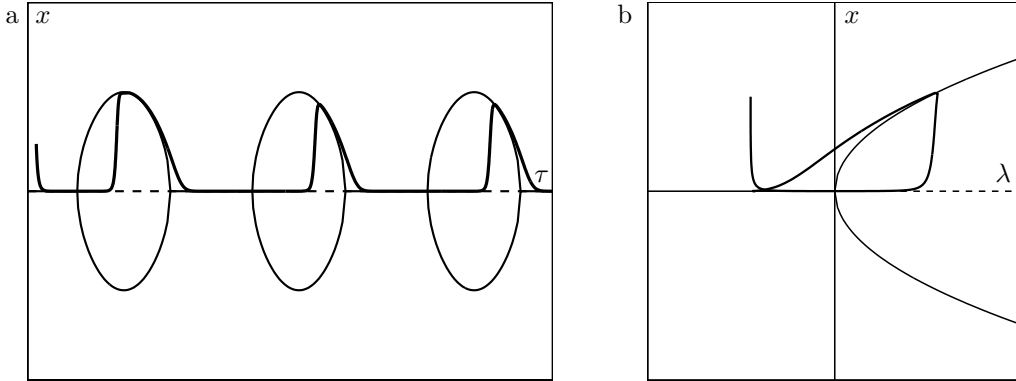


FIGURE 10. Solutions of equation (18), (a) in the  $(\tau, x)$ -plane and (b) in the  $(\lambda, x)$ -plane, but for the function  $\lambda(\tau) = \sin \tau + 0.1$ . Thin full lines indicate stable equilibria of the static system, dashed lines indicate unstable equilibria. These curves are solutions of the equation  $\lambda x - x^3 = 0$ . When the origin is stable, solutions reach it after a short time. They follow the origin for some macroscopic time after it has become unstable, a phenomenon known as **bifurcation delay**. If the solution finally jumps on another equilibrium, we obtain a hysteresis cycle.

Considering the one-dimensional equations studied in these three examples, we observe that

- solutions of a periodically forced system are always attracted by periodic ones;
- without bifurcations, the periodic solution encloses an area of order  $\varepsilon$ , and does not display hysteresis in the adiabatic limit;
- when bifurcations are present, the periodic solution displays hysteresis, and encloses an area which follows a scaling relation of the form  $\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^\mu$ , where  $\mu$  is a nontrivial, fractional exponent.

One of the goals of this work will be to find out if these properties remain valid for more general equations. We will see that asymptotic solutions are not necessarily periodic. However, if such a periodic solution exists, its area will usually follow a scaling law of the above mentioned form, with an exponent  $\mu$  which can be computed in a relatively simple way.

### 3 About this Thesis

#### 3.1 Objectives

We pursue two major objectives in this work:

1. Establish a coherent mathematical framework in order to deal with adiabatic systems of the form (3). In particular, we would like to understand the relation between an adiabatic system, and the corresponding family of autonomous equations. We are also interested in developing some practical tools allowing to establish existence of periodic orbits and hysteresis cycles, and to determine their scaling behaviour as a function of the adiabatic parameter.
2. Apply these methods to some concrete examples. This should allow to check their efficiency to deal with a given equation, and to detect aspects of the theory which need further development. Since many authors, after spending much effort to derive

equations describing magnetic hysteresis, analyse them by numerical simulations, we would like to show how the theory of Dynamical Systems can be used to obtain valuable information on such equations with relatively small effort.

As we discussed in Subsection 1.3, much work has already been done on adiabatic systems, in particular on bifurcation delay. We feel, however, that this work is worth extending in two directions. Firstly, results obtained by mathematicians are often formulated in a rather abstract language, which is not easily accessible to the average (even theoretical) physicist. Thus it is certainly useful to translate them into a language facilitating their application to concrete problems. Secondly, several aspects of the fundamental theory still need to be clarified. For instance, hysteresis itself, and the associated scaling behaviour, have almost not been studied by mathematicians. We also discovered, when analysing particular examples, that several basic concepts still needed to be developed, for instance adiabatic manifolds.

We have chosen two types of applications. The first one, which we call nonlinear oscillators, concerns various situations where a damped particle is placed in a slowly varying force field. Such low-dimensional Dynamical Systems are interesting for several reasons: we have some physical intuition for their behaviour; they are sufficiently simple to be analysed in great detail, so that we have a better chance to understand fundamental mechanisms of hysteresis; still, some of these systems are known to exhibit chaotic motion when forced periodically, and it is important to understand what happens when this forcing becomes adiabatic.

As a second application, we will consider a few models of magnetic hysteresis. This program appears to be much more ambitious, since magnets are so complicated systems that it is not clear at all whether they may be modeled by finite dimensional equations. We think, however, that such an attempt is justified by the mere fact that it will reveal both the power and limits of such a kind of modeling. It may give some hints as to what characteristics a realistic model should include, and in what directions the theory should be extended in order to give more reliable predictions.

### 3.2 Philosophy

In this work, we adopt the point of view of Mathematical Physics. This implies that physicists may regard it as an unnecessarily pedantic way of establishing evidences, while mathematicians may consider it as a pedestrian approach to a problem, which might be described much more nicely using non-standard analysis and Borel series.

To the former, we would like to point out that there exist numerous examples of problems, for which it was considered as evident that their solutions behave in some special way, until this evidence was proved wrong by a serious analysis. The precise mathematical understanding of a problem is always desirable when it reveals the power and limits of an empirical approach. For the latter, we would like to underline that our work aims at providing a method of practical use, allowing the physicist to obtain useful information on a concrete adiabatic system, with a minimum of technical tools.

There are different approaches to Mathematical Physics. One of them relies on exact solutions. We believe that this approach is useful as far as it provides very precise information on a particular model equation, which is assumed to be generic. There are, however, two major drawbacks: Firstly, differential equation which can be solved exactly are very scarce, so that only very few model equations are likely to be analysable in this

way. Secondly, even when a system has been solved exactly, the interesting features are not immediately apparent, and it may require a lot of hard analysis to derive them. This approach does not favour the physical intuition, and often yields incorrect interpretations.<sup>6</sup>

A good illustration of these difficulties is provided by Example 3: this system is still relatively simple to solve, if one knows about Bernoulli's equation. But it turns out that the important phenomenon, namely bifurcation delay, can also be obtained in a much simpler way, by studying the linearized equation  $\varepsilon \dot{x} = \lambda(\tau)x$ . The behaviour of solutions far from the origin can be analysed by different methods, that do not depend on the detailed form of the nonlinear term, which is necessary for the equation to be exactly solvable. We will show that even the scaling law (21) can be obtained using only a local analysis around the bifurcation point.

We will thus prefer those methods which favour the physical intuition. To analyse some complicated equation, one has to understand first which terms are important, and which terms have a negligible influence. Then one starts by solving the simplified equation containing only the important terms. Perturbative methods are often well adapted so such a procedure.

But one has to be careful not to confuse perturbation and approximation. It is very tempting (and often done) to assume that a solution can be written as power series of some small parameter, to insert this series into the equation, and to solve it for the first few orders. This procedure is often dangerous, since these power series do usually *not* converge.<sup>7</sup> In fact, it is better to apply perturbation theory to the equation than to its solution.

We will often proceed in two steps. Firstly, we will derive an iterative scheme that allows to decrease the order of some remainder in the equation, which prevents us from solving it. Secondly, we have to prove in an independent way that the influence of this small remainder on the solution can be *bounded*. Thus, if we write that a solution contains a remainder  $R = \mathcal{O}(\varepsilon)$ , we mean something very precise: namely there exist positive constants  $c$  and  $\varepsilon_0$  such that  $|R| \leq c\varepsilon$  for  $0 < \varepsilon \leq \varepsilon_0$ . These constants are independent of  $\varepsilon$  and could be computed explicitly (although this computation may turn out to be quite cumbersome).

Once such a bound on the remainder is known, when applying the theory to a concrete example one can forget about the proof of the second part, and use the iterative scheme to determine the behaviour of the solution at leading order in the small parameter. It appears that the bounds  $c$  and  $\varepsilon_0$  are often far too pessimistic, and that, at least for finite dimensional systems, the asymptotic theory provides rather accurate information for fairly “large” values of the “small” parameter.

### 3.3 Reader's Guide

There are different ways to write a Ph.D. dissertation. One can choose to present a summary of the major results, or one can give a more detailed account with background information on the subject and complete proofs. We preferred the second possibility. We believe that such a detailed presentation is justified for the kind of subject we have

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<sup>6</sup>A startling example of such a misunderstanding is found in [AC1], who analyse an equation linearized around an *unstable* equilibrium, which is never reached by the solution.

<sup>7</sup>For instance, the perturbative analysis of a Hopf bifurcation in [ME1, BER] fails to reveal the phenomenon of maximal delay.

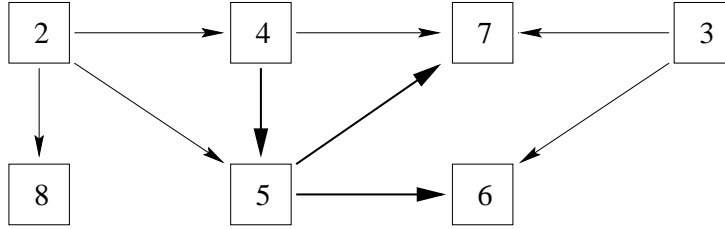


FIGURE 11. Logical organization of the chapters. Arrows indicate that a chapter relies on the contents of another one, in an essential way if the arrow is thick.

been working on, which lies on the boundary of Mathematics and Physics, provided the structure of the text is sufficiently apparent.

Although the chapters of this dissertation are not self-contained, we tried to write them as far as possible in an independent way. Thus it is certainly not necessary, in order to understand the contents of a given chapter, to have read all the preceding ones.

Roughly speaking, Chapters 2 and 3 present some aspects of mathematical and physical theory which are already well known. Chapters 4 and 5 are dedicated to the abstract, mathematical theory which we have developed to deal with adiabatic Dynamical Systems. Chapters 6 and 7 provide applications of this theory to some concrete examples. Chapter 8 contains extensions of some results to iterated maps.

## Chapter 2: Mathematical Tools

We present some important notions from the theory of Dynamical Systems (as equilibrium points, stability and Lyapunov functions, invariant manifolds, bifurcations and normal forms), and some elements of analysis which are used in the proofs (Banach spaces, Fréchet derivatives, asymptotic series and differential equations). The notations we use do in general not differ from standard ones. This chapter should be considered as a reference chapter, and the reader who is already well acquainted with the theory, as well as the reader who is not interested in detailed mathematics, can safely skip it.

## Chapter 3: Physical Models

We describe some physical concepts on which rely the examples discussed in Chapters 6 and 7. The damped motion of a particle in a potential serves as paradigm for a wide class of Dynamical Systems. We discuss the most common models for ferromagnets at equilibrium and out of equilibrium, and show how to derive a deterministic evolution equation from the governing master equation. Finally, we briefly present some existing phenomenological models of hysteresis.

## Chapter 4: One-Dimensional Systems

We present a detailed mathematical framework to deal with one-dimensional adiabatic equations of the form  $\varepsilon \dot{x} = f(x, \tau)$ . We first discuss the properties of adiabatic solutions, which are particular solutions remaining close to non-bifurcating equilibria, and admitting asymptotic series in the adiabatic parameter  $\varepsilon$ . We then analyse in detail the dynamics near bifurcation points, in particular the way how they scale with  $\varepsilon$ . We provide a simple geometrical method, based on the Newton polygon, to determine the scaling exponents.

Finally, we discuss some global aspects of the flow, in particular how to determine periodic orbits, and the  $\varepsilon$ -dependence of hysteresis cycles.

## **Chapter 5: $n$ -Dimensional Systems**

We extend results of the previous chapter to  $n$ -dimensional equations. The discussion of adiabatic solutions is quite similar to the 1D case. We then examine the linear equation  $\varepsilon \dot{y} = A(\tau)y$ , which describes the linearized motion around an adiabatic solution. This is a rather lengthy task, but we show that the problem of diagonalizing such an equation can be transformed into the problem of finding adiabatic solutions of an auxiliary equation. This transformation allows to treat eigenvalue crossings and bifurcations in a unified way. Next we develop some tools to deal with nonlinear terms, in particular adiabatic manifolds and dynamic normal forms. Finally, we examine some global properties of the flow.

## **Chapter 6: Nonlinear Oscillators**

We consider different examples involving the damped motion of a particle in a slowly varying potential. The most important one is equivalent to a simple physical system, namely a pendulum on a table rotating with slowly modulated angular frequency. This system displays two important phenomena: a bifurcation delay leading to hysteresis, and the possibility of a chaotic motion, even for arbitrarily small adiabatic parameter. We use the methods developed in Chapters 4 and 5 to compute an asymptotic expression of the Poincaré map, which allows to delineate precisely the parameter regions where hysteresis and chaos occur. The other two examples discussed in this chapter illustrate the effect of eigenvalue crossings.

## **Chapter 7: Magnetic Hysteresis**

We discuss a few simple models of ferromagnets in a slowly varying magnetic field. When the interaction between spins has infinite range (i.e., in a Curie–Weiss type model), the dynamics can be described in the thermodynamic limit by a low-dimensional differential equation, of Ginzburg–Landau type. We examine the phenomenon of dynamic phase transition for 1D spins, and the effect of anisotropy on the dynamics of 2D spins. Finally, we explain why models with short range interaction are much harder to analyse, and present a few simple approximations.

## **Chapter 8: Iterated Maps**

We extend some of the previous results to adiabatic iterated maps. We start by showing that some basic properties of adiabatic ODEs, such as existence of adiabatic solutions and the behaviour of linear systems, can be extended to maps depending on a slowly varying parameter. We conclude by presenting some results on existence of adiabatic invariants for slow–fast maps, and illustrate them on a few billiard problems.

## Acknowledgments

This work would not have been possible without the help of many people, to whom I would like to express my gratitude at this place.

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