Chaotic hysteresis in an adiabatically oscillating double well

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(October 3, 1996)

We consider the motion of a damped particle in a potential oscillating slowly between a simple and a double well. The system displays hysteresis effects which can be of periodic or chaotic type. We explain this behaviour by computing an analytic expression of a Poincaré map.

Although hysteresis is a quite familiar and ubiquitous phenomenon, no general theory achieves to describe its many facets. In condensed matter, hysteresis often accompanies a phase transition, which by nature results from the cooperative effect of a large number of degrees of freedom. This has recently led some workers to analyze it by means of Langevin type [1] or master equations [2] with infinitely many degrees of freedom, and to propose various scaling laws for the area of the hysteresis loop. A mean field treatment of this problem [3] reduces it to an ordinary differential equation for the order parameter, with some slowly time dependent external parameter like the magnetic field. Noise can be incorporated to the problem. Similar equations appear naturally to describe mechanical or electrical systems, as well as lasers [4]. Hysteresis effects may appear if the equilibria of the dynamical system with static parameter undergo a bifurcation, and scaling laws have been found also in this case [5].

A paradigmatical example is that of a damped particle in a slowly varying potential. If the symmetric potential depends on a parameter smoothly interpolating between a simple and a double well, the static bifurcation diagramm looks like the inset of Fig. 2. Imagine now that the parameter is oscillating slowly between these extreme values. The particle starting close to the initially stable origin does not react immediately to the bifurcation. Rather, it remains for some time close to the now unstable origin, before falling into one of the newly formed minima, which it follows until it merges again with the origin. Thus this *bifurcation delay*, which has been observed in various physical systems [4], and was rigorously analyzed for the first time by Neishtadt [6], is responsible for metastability leading to hysteresis.

In this letter we address the following question: for a periodic parameter variation, which symmetric potential well will the particle fall into during each cycle? In the overdamped case, it always chooses the same minimum (Fig. 1a). But at low friction, we found that depending on the frequency of the parameter variation, the particle may also fall alternatively into the left and right equilibrium (Fig. 1b), or even go from one minimum to the other in a random way (Fig. 1c). We observed the same phenomenon in an experimental realization of the system, a pendulum on a rotating table. In this work, we show that this surprising behaviour can be understood by means of a Poincaré map, that we compute explicitly to lowest order of the parameter variation. We only outline the derivation of this fairly complicated, although essentially one–dimensional map, postponing rigorous proofs to a further publication [7].

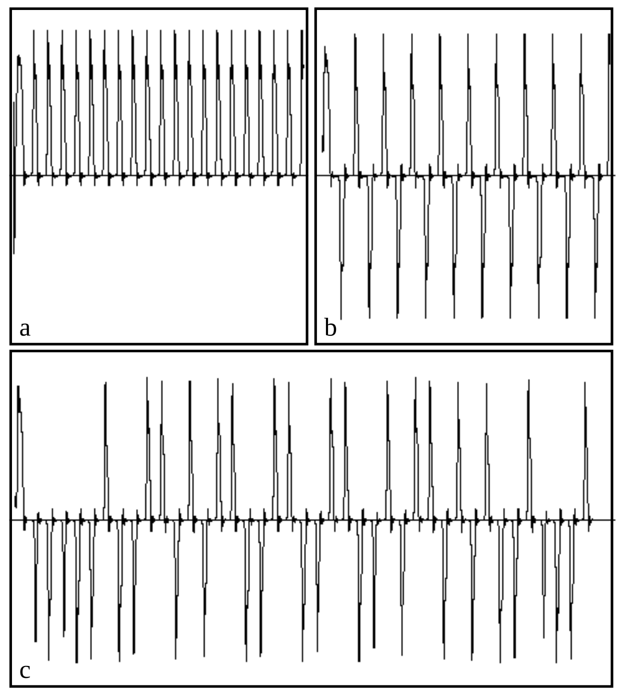

FIG. 1. Position of the particle (or the pendulum) as a function of time, each peak corresponding to the particle falling into one of the wells. The sequence of visited wells may be periodic, biperiodic or chaotic.

The equation of motion of a damped particle in a slowly varying potential can be written

$$\ddot{q} + 2\gamma\dot{q} + \Phi'(q, \lambda(\varepsilon t)) = 0, \tag{1}$$

where the prime denotes derivation with respect to q and $0 < \varepsilon \ll 1$ is the adiabatic parameter.

Introducing the variables $x = (x_1, x_2) = (q, \dot{q})$, this equation can be transformed into the non–autonomous first order system

$$\begin{aligned} \varepsilon\dot{x}_1 &= x_2 \\ \varepsilon\dot{x}_2 &= -2\gamma x_2 - \Phi'(x_1, \lambda(\tau)) \end{aligned} \tag{2}$$

where the dots indicate now derivation with respect to the slow time $\tau = \varepsilon t$.

For fixed $\lambda(\tau) \equiv \lambda_0$, the dynamics of (2) are well-known. Around the equilibria $x_1 = q^*(\lambda_0)$, $x_2 = 0$, where $\Phi'(q^*(\lambda_0), \lambda_0) = 0$, the linearization of (2) has eigenvalues $a_\pm = -\gamma \pm \sqrt{\gamma^2 - \phi''}$, with $\phi'' = \Phi''(q^*(\lambda_0), \lambda_0)$. Hence the fixed point is a saddle if $\phi'' < 0$, a stable node if $0 < \phi'' < \gamma^2$ and a stable focus if $\phi'' > \gamma^2$. With these informations, the phase portrait is easily drawn.

We will consider potentials of the following type: $\Phi(q, \lambda)$ is an even analytic function of q such that the origin O is hyperbolic when $\lambda > 0$, a node for $\lambda_-(\gamma) < \lambda < 0$ and a focus for $\lambda < \lambda_-$. For positive λ, Φ has two minima $\pm q^*(\lambda)$ which are nodes when $\lambda < \lambda_+(\gamma)$ and focuses when $\lambda > \lambda_+$.

The simplest example is the *Ginzburg–Landau potential* $\Phi(q, \lambda) = -\frac{1}{2}\lambda q^2 + \frac{1}{4}q^4$ (here $\lambda_- = -\gamma^2$, $\lambda_+ = \gamma^2/2$), q being the order parameter and λ the difference between the temperature and its critical value. But there is also a simple mechanical system which can be described by an equation of this type, namely a plane pendulum on a rotating table. In the frame rotating with frequency Ω, it experiences a torque $-LMg\sin q$ due to its weight and a centrifugal torque $I\Omega^2 \sin q \cos q$, where L is the distance between suspension point P and center of mass G, I the moment of inertia with respect to P, and q the angle between PG and the vertical. Taking as a time unit $\Omega_{\rm cr}^{-1} = (LMg/I)^{1/2}$, we obtain the equation of motion (1) with $\Phi'(q, \lambda) = \sin q[1 - (\lambda + 1)\cos q]$, $\lambda = \Omega^2 - 1$.

Finally, the function $\lambda(\tau)$ we consider is periodic with period 1, and has exactly one minimum and one maximum. In order to analyze the behaviour of the dynamical system, we want to compute the Poincaré map in the (x_1, x_2)–plane during one period to dominant order in ε. This proceeds essentially by following the motion of x along its static equilibria. Let us denote by $a^o_\pm(\tau)$ and $a^*_\pm(\tau)$ the eigenvalues of the linearization around O and $q^*(\lambda(\tau))$ respectively, and use the notation

$$\alpha^{o,*}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \operatorname{Re} a^{o,*}_+(\tau)\, d\tau,$$
$$\phi^{o,*}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \operatorname{Im} a^{o,*}_+(\tau)\, d\tau,$$
$$\delta^{o,*}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \operatorname{Re}(a^{o,*}_+ - a^{o,*}_-)(\tau)\, d\tau.$$

We first study (2) in a neighborhood of the origin. The linearized system $\varepsilon\dot{x} = A(\tau)x$ can be solved by constructing a linear change of variables $x = S(\tau, \varepsilon)y$ transforming the equation into $\varepsilon\dot{y} = B(\tau, \varepsilon)y$, where B is as simple as possible. If $A(\tau)$ has distinct eigenvalues, one can find S regular in a neighborhood of $\varepsilon = 0$ such that B has exponentially small off–diagonal terms. Complete diagonalization is possible if we only require S to admit an asymptotic expansion in ε [8]. The full system (2) thus becomes

$$\begin{aligned} \varepsilon\dot{y}_1 &= a^o_+(\tau, \varepsilon)y_1 + b_+(y_1, y_2; \tau, \varepsilon) \\ \varepsilon\dot{y}_2 &= a^o_-(\tau, \varepsilon)y_2 + b_-(y_1, y_2; \tau, \varepsilon), \end{aligned} \qquad (3)$$

with $a^o_\pm(\tau, \varepsilon) = a^o_\pm(\tau) + \mathcal{O}(\varepsilon)$ and $b_\pm = \mathcal{O}(|y|^3)$. We assume that $a^o_+(\tau_0) > 0$ and define an *adiabatic unstable manifold* $y_2 = u(y_1; \tau, \varepsilon)$ by the equation

$$\varepsilon\dot{u} = a^o_- u + b_-(y_1, u) - \partial_y u[a^0_+ y_1 + b_+(y_1, u)],$$

where the initial condition coincides with the instantaneous unstable manifold at $\tau = \tau_0$. The solution, whose asymptotic expansion can be computed, may be continued to all times such that the origin is a saddle or a node. We then carry out the change of variables $y_2 = u(y_1) + \eta$ and define in a similar way an *adiabatic stable manifold* $y_1 = v(\eta; \tau, \varepsilon)$. With $y_1 = v(\eta) + \xi$, (3) becomes

$$\begin{aligned} \varepsilon\dot{\xi} &= [a^o_+(\tau, \varepsilon) + \beta_+(\xi, \eta; \tau, \varepsilon)]\xi \\ \varepsilon\dot{\eta} &= [a^o_-(\tau, \varepsilon) + \beta_-(\xi, \eta; \tau, \varepsilon)]\eta \end{aligned} \qquad (4)$$

with $\beta_\pm = \mathcal{O}(\xi^2 + \eta^2)$.

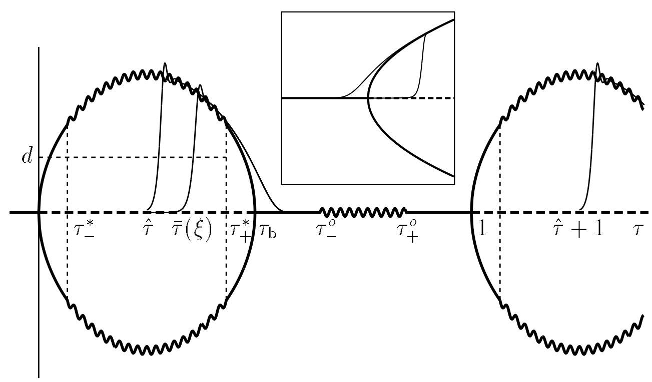

FIG. 2. Fixed points in the (τ, x_1)–plane. Full lines represent nodes, dashed lines hyperbolic points, and wavy lines focuses. The thin lines are orbits of the system. The inset shows the static bifurcation diagramm in the (λ, x_1) plane (thick lines) with an asymptotic hysteresis cycle (thin lines).

In the overdamped case, when $\lambda(\tau)$ always remains in the interval (λ_-, λ_+), the problem is easy to analyze because it can be reduced to a one-dimensional one. Assume that $\lambda(\tau) > 0$ for $0 < \tau < \tau_{\rm b}$ and $\lambda(\tau) < 0$ for $\tau_{\rm b} < \tau < 1$. One shows that it is always possible to transform (2) into (4), with $a^o_- + \beta_- < 0$. Thus, we conclude from the second equation that $\eta(\tau)$ will go to zero exponentially fast, and it is sufficient to consider the reduced equation on the adiabatic manifold, $\varepsilon\dot{\xi} = [a^o_+ + \beta_+(\xi, 0)]\xi \equiv g(\xi; \tau, \varepsilon)$, which undergoes a direct pitchfork bifurcation at $\tau = \mathcal{O}(\varepsilon)$, and an inverse pitchfork bifurcation at $\tau = \tau_{\rm b} + \mathcal{O}(\varepsilon)$.

The most important effect of the adiabaticity of the parameter variation is that the bifurcation is *delayed*. The orbit starting with a positive ξ at $\tau_0 \in (\tau_{\rm b} - 1, 0)$ reaches the $\mathcal{O}(\varepsilon)$–neighborhood of the origin at $\tau_1 =$

$\tau_0 + \mathcal{O}(\varepsilon|\ln\varepsilon|)$. As long as $\xi = \mathcal{O}(\varepsilon)$, we have $\varepsilon\dot{\xi} = [a^o_+(\tau) + \mathcal{O}(\varepsilon)]\xi$ so that

$$\xi(\tau) = \xi(\tau_1)\,\exp\frac{1}{\varepsilon}[\alpha^o(\tau_1,\tau) + \mathcal{O}(\varepsilon)].$$

The smallness of ξ is thus guaranteed as long as $\alpha^o < 0$. For $\tau_1 < \tau < 0$, α^o is decreasing; it has increased again to zero only at $\tau = \Psi(\tau_1) > 0$, which is by definition the *delayed bifurcation time*. Afterwards, the orbit quickly jumps on the upper branch q^* and follows it adiabatically until the inverse bifurcation at $\tau_{\rm b}$, where we can prove that $\xi(\tau_{\rm b}) = C\varepsilon^{1/4}$. The next delayed bifurcation occurs at $\hat{\tau} = \Psi(\tau_{\rm b})$.

It follows that the Poincaré map, defined by $\xi(\hat{\tau}+1) = T(\xi(\hat{\tau}))$, is a monotonic odd function of ξ, such that for $\xi > \exp -\frac{1}{\varepsilon}\alpha^o(\hat{\tau},\hat{\tau}+1)$, one has $0 < c_1\varepsilon^{1/4} < T(\xi) < c_2\varepsilon^{1/4}$. This implies that the map has three fixed points, the unstable origin and two symmetric stable points. The positive solution corresponds to the attractive periodic orbit of Fig. 1a, and, if plotted in the (λ,ξ)–plane, to an asymptotic hysteresis cycle (see inset of Fig. 2), whose area we are able to prove scales like $\mathcal{A}(\varepsilon) \simeq \mathcal{A}(0) + c\varepsilon^{3/4}$.

We now turn to the most complicated case, when the amplitude of $\lambda(\tau)$ is large enough for the origin to be a focus for $\tau^o_- < \tau < \tau^o_+$, and for q^* to be a focus for $\tau^*_- < \tau < \tau^*_+$. We make a Poincaré section at the delay time $\hat{\tau} = \Psi(\tau_{\rm b}-1)$, which we assume to be in the interval (τ^*_-,τ^*_+) (Fig. 2).

The variables $\zeta = (\xi,\eta)$ are defined for $\tau \notin [\tau^o_-,\tau^o_+]$ and for $|\xi| < d$ where d is a constant such that x_1 is closer to the origin than the focus at q^*. We want to compute $\zeta(\hat{\tau}+1) = (\xi_1,\eta_1)$ as a function of $\zeta(\hat{\tau}) = (\xi_0,\eta_0)$, in the case where $\xi_0 \in (0,d)$ and $\eta_0 = \mathcal{O}(\varepsilon)$. From (4), we deduce that $\xi(\tau) = d$ at $\tau = \bar{\tau}(\xi_0) + \mathcal{O}(\varepsilon)$, where $\bar{\tau}(\xi_0)$ is the solution of

$$\alpha^o(\hat{\tau},\bar{\tau}) = -\varepsilon\ln(\xi_0/d). \qquad (5)$$

For most trajectories, $\bar{\tau}$ is very close to $\hat{\tau}$, but orbits which are exponentially close to the origin at $\hat{\tau}$ can be appreciably delayed. $\bar{\tau}(\xi_0)$ is a decreasing function of ξ_0 for $\xi_{\rm c} \leq \xi_0 \leq d$, with $\bar{\tau}(d) = \hat{\tau}$ and $\bar{\tau}(\xi_{\rm c}) = \tau^*_+$, where $\xi_{\rm c} = d\exp -\frac{1}{\varepsilon}\alpha^o(\hat{\tau},\tau^*_+)$.

Next, we construct a particular solution of (2) $\bar{x}(\tau)$, wich remains in a neighborhood of the upper branch $q^*(\tau)$ and such that $\bar{\xi}(\tau_{\rm b}) = C\varepsilon^{1/4}$, $\bar{\eta}(\tau_{\rm b}) = 0$. Writing $x = \bar{x}(\tau) + y$, we obtain the equation $\varepsilon\dot{y} = \bar{A}(\tau;\varepsilon)y + b(y,\tau)$, where $\bar{A}$ has eigenvalues $a^*_\pm(\tau) + \mathcal{O}(\varepsilon^{1/2})$. The nonlinear term $b(y,\tau)$ may be decreased to order ε by the change of variables $y = z + \chi(z)$, which puts the instantaneous system into normal form. When $\tau \neq \tau^*_+$, the linear part of the equation can be diagonalized. Then we have to take care of the turning point τ^*_+, where the eigenvalues of $\bar{A}(\tau)$ cross and $S(\tau)$ diverges. The problem is solved by carrying out, in the interval $[\tau^*_+ - \varepsilon^{2/3}, \tau^*_+ + \varepsilon^{2/3}]$, a change of variables which puts B into Jordan form, so that one obtains Airy's equation. Putting together these steps, we finally obtain

$$\begin{aligned}\xi(\tau_{\rm b}) &= \bar{\xi}(\tau_{\rm b}) + {\rm e}^{\alpha_*/\varepsilon}\sin(\phi_*/\varepsilon)\\ \eta(\tau_{\rm b}) &= {\rm e}^{(\alpha_*-\delta^*)/\varepsilon}\sin(\phi_*/\varepsilon + \theta^*),\end{aligned} \qquad (6)$$

where $\delta^* = \delta^*(\tau^*_+,1) + \mathcal{O}(\varepsilon^{1/2})$ is a constant and $\alpha_* = \alpha_*(\xi_0) + \mathcal{O}(\varepsilon^{1/2})$, $\phi_* = \phi_*(\xi_0) + \mathcal{O}(\varepsilon)$, with

$$\begin{aligned}\alpha_*(\xi_0) &= \alpha^*(\bar{\tau}(\xi_0),1)\\ \phi_*(\xi_0) &= \phi^*(\bar{\tau}(\xi_0),\tau^*_+).\end{aligned} \qquad (7)$$

Of course $\zeta(\tau_{\rm b})$ depends also on η_0 via $y(\bar{\tau})$, but only at next–to–leading order. For $\xi_0 \in [\xi_{\rm c}, d]$, (6) is the parametric equation of an exponentially squeezed spiral.

Proceeding in a similar way around the origin, we find $\zeta(\hat{\tau}+1) = U\zeta(\tau_{\rm b})$, with

$$U = \begin{pmatrix} \sin\left(\frac{\phi^o}{\varepsilon}\right) & {\rm e}^{-\delta^o_2/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon}+\theta^o_2\right) \\ {\rm e}^{-\delta^o_1/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon}+\theta^o_1\right) & {\rm e}^{-\delta^o_3/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon}+\theta^o_3\right)\end{pmatrix}, \qquad (8)$$

where $\phi^o = \phi^o(\tau^o_-,\tau^o_+) + \mathcal{O}(\varepsilon^{1/2})$ is the dynamic phase, $\delta^o_1 = \delta^o(0,\tau^o_-) + \mathcal{O}(\varepsilon^{1/2})$, $\delta^o_2 = \delta^o(\tau^o_+,\hat{\tau}) + \mathcal{O}(\varepsilon^{1/2})$, and $\delta^o_3 = \delta^o_1 + \delta^o_2$. The geometric phase shifts θ_i can be expressed in terms of Airy functions (plus corrections of order $\varepsilon^{1/3}$). This transformation acts like a rotation of angle ϕ^o/ε and an exponential contraction along the stable manifold.

Combining (6) and (8), we finally obtain the expression of the Poincaré map $\xi_1 = T_1(\xi_0,\eta_0;\varepsilon)$, $\eta_1 = T_2(\xi_0,\eta_0;\varepsilon)$ with

$$\begin{aligned}T_1 = \sin\left(\frac{\phi^o}{\varepsilon}\right)&\left[C\varepsilon^{1/4} + {\rm e}^{\alpha_*/\varepsilon}\sin\left(\frac{\phi_*}{\varepsilon}\right)\right]\\ &+ {\rm e}^{(\alpha_*-\delta^*-\delta^o_2)/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon}+\theta^o\right)\sin\left(\frac{\phi_*}{\varepsilon}+\theta^*\right)\end{aligned} \qquad (9)$$

and $T_2 = \mathcal{O}({\rm e}^{-\delta^o_1/\varepsilon})$.

The expression (9) of T_1 is valid for $\xi_0 > \xi_{\rm c}$. By symmetry, T_1 is an odd function of ξ_0, and for $|\xi_0| < \xi_{\rm c}$, it is monotonic as in the overdamped case. All variables appearing in (9) are independent of ξ_0, η_0 at lowest order in ε, except α_* and ϕ_* which are given by (7).

Since η is exponentially contracted at each iteration, we may replace the two–dimensional invertible Poincaré map by the non–invertible one–dimensional map $\xi_0 \mapsto T_1(\xi_0,0)$. Its graph is oscillating around $C\varepsilon^{1/4}\sin(\phi^o/\varepsilon)$ with increasing amplitude and frequency as $\xi \searrow \xi_{\rm c}$. One can prove that there is a positive constant μ such that when $\sin(\phi^o/\varepsilon) > {\rm e}^{-\mu/\varepsilon}$, T_1 admits only one positive fixed point at $\xi \sim \varepsilon^{1/4}\sin(\phi^o/\varepsilon)$, which is stable. In this case, we obtain a hysteresis cycle of period 1. Similarly, when $\sin(\phi^o/\varepsilon) < -{\rm e}^{-\mu/\varepsilon}$, the map has a stable orbit of period 2, corresponding to the particle falling alternatively into the left and right well.

The most interesting situation occurs when $|\sin(\phi^o/\varepsilon)| < \mathrm{e}^{-\mu/\varepsilon}$. In this case, we observed numerically a great variety of behaviours, including period doubling cascades and chaotic orbits. These "chaotic" zones occur at integer values of $k = \phi^o/\pi\varepsilon$, and the width and height of the nth zone are proportional to $\mathrm{e}^{-(\mu\pi/\phi^o)n}$. This prediction has been confirmed numerically.

The map (9) certainly deserves further study. But the most important fact to us is that it explains accurately the alternance of periodic, biperiodic and chaotic hysteresis, in accordance with numerical simulations (Fig. 3) as well as with the laboratory experiment of the rotating pendulum.

We thank P. Braissant and B. Egger for carrying out the laboratory experiment of the rotating pendulum. This work is supported by the Fonds National Suisse de la Recherche Scientifique.

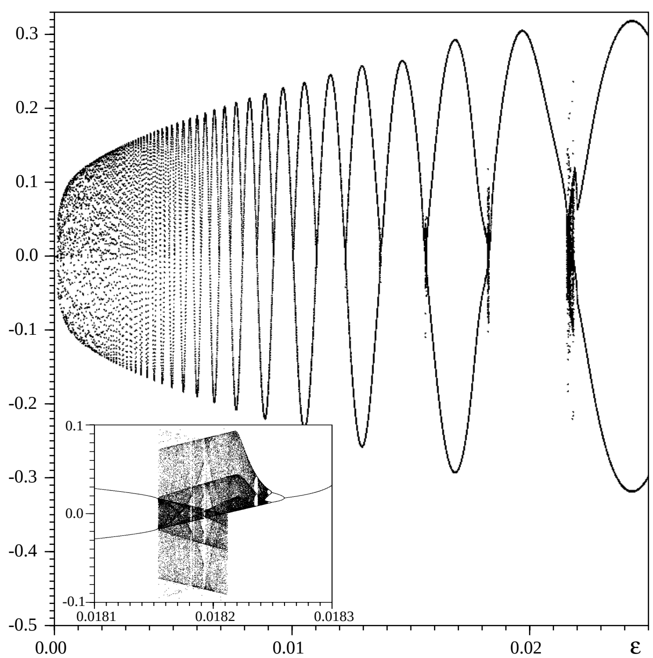

FIG. 3. Numerically computed bifurcation diagramm of the Poincaré map for the rotating pendulum. For each value of ε on the abscissa, we plot the asymptotic behaviour of $\xi(\hat{\tau} + n)$, $n \in \mathbb{N}$, for *one* initial condition. Regions with a period–1 and period–2 cycle are separated by small chaotic zones, the inset showing an enlargement of the second zone from the right.

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