

An Eyring–Kramers law for slowly oscillating bistable diffusions

Nils Berglund

Abstract

We consider two-dimensional stochastic differential equations, describing the motion of a slowly and periodically forced overdamped particle in a double-well potential, subjected to weak additive noise. We give sharp asymptotics of Eyring–Kramers type for the expected transition time from one potential well to the other one. Our results cover a range of forcing frequencies that are large with respect to the maximal transition rate between potential wells of the unforced system. The main difficulty of the analysis is that the forced system is non-reversible, so that standard methods from potential theory used to obtain Eyring–Kramers laws for reversible diffusions do not apply. Instead, we use results by Landim, Mariani and Seo that extend the potential-theoretic approach to non-reversible systems.

Date. July 16, 2020.

2020 Mathematical Subject Classification. 60H10, 34F05 (primary), 60J45, 81Q20 (secondary)

Keywords and phrases. Stochastic exit problem, diffusion exit, first-exit time, limit cycle, large deviations, potential theory, semiclassical analysis, cycling, Gumbel distribution.

1 Introduction

This work is concerned with time-periodic perturbations of the stochastic differential equation (SDE)

$$dx_t = -V'_0(x_t) dt + \sigma dW_t, \quad (1.1)$$

describing the overdamped motion of a Brownian particle in a double-well potential $V_0 : \mathbb{R} \rightarrow \mathbb{R}$, which is bounded below and grows at least quadratically at infinity.

Let us start by recalling some well-known properties of the unperturbed system (1.1). It's unique invariant measure has density $Z^{-1} e^{-2V_0(x)/\sigma^2}$ with respect to Lebesgue measure, where Z is the normalisation. Furthermore, the dynamics is reversible with respect to this measure. Denote the local minima of V_0 by x_{\pm}^* , and its local maximum by x_0^* , with $x_-^* < x_0^* < x_+^*$. Let $\tau_+ = \inf\{t > 0 : x_t = x_+^*\}$ be the first-hitting time of x_+^* . Then one has the explicit expression

$$\mathbb{E}_x[\tau_+] = \frac{2}{\sigma^2} \int_x^{x_+^*} \int_{-\infty}^{x_2} e^{2[V_0(x_2) - V_0(x_1)]/\sigma^2} dx_1 dx_2$$

for the expectation of τ_+ when starting at any $x < x_+^*$. This result is obtained by solving an ordinary differential equation (ODE) satisfied by the function $x \mapsto \mathbb{E}_x[\tau_+]$, owing to Dynkin's formula. In particular, the Laplace method shows that when starting in x_-^* , this expectation satisfies the so-called Eyring–Kramers law [23, 29]

$$\mathbb{E}_{x_-^*}[\tau_+] = \frac{2\pi}{\sqrt{|V_0''(x_0^*)|V_0''(x_-^*)}} e^{2[V_0(x_0^*) - V_0(x_-^*)]/\sigma^2} [1 + \mathcal{O}(\sigma^2)]. \quad (1.2)$$

Furthermore, in [17], Day has shown that the law of τ_+ is asymptotically exponential, in the sense that

$$\lim_{\sigma \rightarrow 0} \mathbb{P}\{\tau_+ > s \mathbb{E}_{x_-^*}[\tau_+]\} = e^{-s} \quad (1.3)$$

holds for all $s > 0$.

While the expected transition time from x_-^* to x_+^* is exponentially long, the actual successful transition, also known as the reactive or transition path, takes much less time. In [16], Cérou, Guyader, Lelièvre and Malrieu have shown that for any fixed $a < x_0 < x_0^* < b$ in (x_-^*, x_+^*) , one has the convergence in law

$$\lim_{\sigma \rightarrow 0} \text{Law}(|V_0''(x_0^*)|\tau_b - 2 \log(\sigma^{-1}) \mid \tau_b < \tau_a) = \text{Law}\left(Z + T(x_0, b)\right), \quad (1.4)$$

where $T(x_0, b)$ is an explicit deterministic quantity independent of σ , and Z is a standard Gumbel variable, that is, $\mathbb{P}\{Z \leq t\} = \exp\{-e^{-t}\}$ holds for all $t \in \mathbb{R}$. Therefore, the duration of a transition is of order $\log(\sigma^{-1})$. See also Bakhtin's works [1, 2] for insights on the relation of this result to extreme-value theory.

Several of these results have been generalised to multidimensional diffusions of the form

$$dx_t = -\nabla V_0(x_t) dt + \sigma dW_t,$$

where now $V_0 : \mathbb{R}^d \rightarrow \mathbb{R}$. These are still reversible with respect to the invariant measure $Z^{-1} e^{-2V_0/\sigma^2}$. A weaker form of the Eyring–Kramers law (that is, without a sharp control of the prefactor of the exponential in (1.2)), known as Arrhenius law, follows from the theory of large deviations developed for diffusions by Freidlin and Wentzell [24]. In [14, 15], Bovier, Eckhoff, Gayard and Klein used potential theory to prove a generalisation of (1.2) to the multidimensional gradient case, as well as the asymptotically exponential character (1.3) of the law of transition times. Similar results have been obtained by Helffer, Klein and Nier in [26] using methods from semiclassical analysis. See also [31, 32, 34, 37] for generalisations to diffusions on manifolds with or without boundary. The potential-theoretic approach has also been successfully applied to obtain Eyring–Kramers laws for stochastic PDEs [10, 3, 7]. See also [4, 6] and references therein, as well as [13] for a comprehensive account of the potential-theoretic approach.

The situation is much less understood for non-gradient diffusions, whose invariant measure is not explicitly known in general, and which are not reversible. While the theory of large deviations in [24] allows to derive Arrhenius laws for these systems as well, determining precise asymptotics on transition times of Eyring–Kramers type is much harder than in the reversible case. Some partial results in this direction have nevertheless been obtained. In [12], Bouchet and Reygner proposed an Eyring–Kramers law for non-reversible diffusions in a bistable situation, based on formal asymptotic computations. In [30], Landim, Mariani and Seo obtained a generalisation of the potential-theoretic approach of [14, 15] to non-reversible systems. This allowed them in particular to justify the formal result of Bouchet and Reygner for a particular class of systems whose invariant measure is known explicitly. See also the work [33] by Le Peutrec and Michel for semiclassical results on non-reversible diffusions with known invariant measure. In a different direction, a reactive path theory for multidimensional, non-reversible diffusions was developed by Lu and Nolen in [35], based on ideas by E and Vanden-Eijnden [22].

In this work, we are concerned with extensions of (1.2) to systems of another type, namely to periodically perturbed versions of (1.1) of the form

$$\begin{aligned} dx_t &= -\partial_x V_0(x_t, y_t) dt + \sigma dW_t^x, \\ dy_t &= \varepsilon dt + \sigma \sqrt{\varepsilon} \varrho dW_t^y, \end{aligned} \quad (1.5)$$

where $\{W_t^x\}_t$ and $\{W_t^y\}_t$ are independent standard Wiener processes. The parameter ϱ has to be strictly positive for technical reasons (we need the diffusion to be elliptic), but our results do not depend on ϱ to leading order. This system is a particular case of systems studied by the author and Barbara Gentz in [11]. The main result in that work gives a rather sharp description of the density of τ_0 , the first-passage time at the saddle $x_0^*(y)$ of $x \mapsto V_0(x, y)$ (or, more precisely, at the deterministic periodic solution tracking the saddle). A slightly less precise, but more transparent way of formulating this result is that

$$\lim_{\sigma \rightarrow 0} \text{Law} \left(\theta(y_{\tau_0}) - \log(\sigma^{-1}) - \frac{\lambda_+}{\varepsilon} Y^\sigma \right) = \text{Law} \left(\frac{Z}{2} - \frac{\log 2}{2} \right), \quad (1.6)$$

where

- $\theta(y)$ is a convenient and explicit parametrisation of the periodic orbit tracking $x_0^*(y)$;
- λ_+ is the Lyapunov exponent of this orbit;
- Z follows again a standard Gumbel law;
- and Y^σ is asymptotically geometric, meaning that it has positive integer values and satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}\{Y^\sigma = n + 1 | Y^\sigma > n\} = p(\sigma),$$

for a constant $p(\sigma)$ that is exponentially small in σ^2 .

(In fact, we have slightly simplified the precise result, which is given in [5, Theorem 4.2].) The most striking feature of (1.6) is that the law of $\theta(y_{\tau_0})$ is shifted by an amount $\log(\sigma^{-1})$ as σ decreases, and thus does not admit a limit as $\sigma \rightarrow 0$. This is the phenomenon of cycling discovered by Day [18, 19, 20, 21]. In fact, this shift by $\log(\sigma^{-1})$ is also present in (1.4). As for Y^σ , its interpretation is as follows: under a non-degeneracy assumption, the system has a “window of opportunity” during each period to make a transition, which is defined by the minimisers of its large-deviation rate function. The integer variable Y^σ simply gives the period during which the actual transition takes place.

The expectation $\mathbb{E}[\tau_0]$ can be deduced from (1.6), and is close to the inverse of the parameter $p(\sigma)$ (see [8]). Since transitions from the saddle to the local minima $x_\pm^*(y)$ take a time of order $\log(\sigma^{-1})$ (see [5, Theorem 6.2]), the expectation of the first-hitting time τ_+ of $x_+^*(y)$ has the same sharp asymptotics as $\mathbb{E}[\tau_0]$. In [11], we did not attempt to obtain sharp asymptotics for $p(\sigma)$, but only showed that it is close, in the sense of logarithmic equivalence, to e^{-I/σ^2} where I is the Freidlin–Wentzell quasipotential, which can be expressed as the solution of a variational principle.

The aim of the present work is to obtain sharp asymptotics of Eyring–Kramers type for $\mathbb{E}[\tau_+]$, which is equivalent to getting precise asymptotics for $p(\sigma)$. Our main result, Theorem 2.4, states that for any starting point on $x_-^*(y)$,

$$\mathbb{E}[\tau_+] = \frac{2\pi[1 + R_1(\varepsilon, \sigma)]}{\int_0^1 \sqrt{|\partial_{xx} V_0(x_0^*(y), y)| \partial_{xx} V_0(x_-^*(y), y)} e^{-2[V_0(x_0^*(y), y) - V_0(x_-^*(y), y)]/\sigma^2} dy}, \quad (1.7)$$

where $R_1(\varepsilon, \sigma)$ is some (complicated) error term. The result applies to values of ε which are large with respect to the integrand in this expression, but still have to be exponentially small in σ^2 owing to technical reasons. Note that (1.7) is indeed a generalisation of the static Eyring–Kramers law (1.2).

The remainder of this paper is organised as follows. In Section 2, we define precisely the considered equations, and state all main results. These are proved in Sections 3 to 7, see Section 2.7 for a more precise outline of the structure of the proofs. Finally, the appendix contains some of the more technical proofs.

Notations

The system studied in this work depends on two small parameters ε and σ . We write $X \lesssim Y$ to indicate that $X \leq cY$ for a constant c independent of ε and σ , as long as ε and σ are small enough. The notation $X \asymp Y$ indicates that one has both $X \lesssim Y$ and $Y \lesssim X$, while Landau's notation $X = \mathcal{O}(Y)$ means that $|X| \lesssim Y$. The canonical basis of \mathbb{R}^2 is denoted $(\mathbf{e}_x, \mathbf{e}_y)$. If $a, b \in \mathbb{R}$, $a \wedge b$ denotes the minimum of a and b , and $a \vee b$ denotes the maximum of a and b . Finally, we write $1_{\mathcal{D}}(x)$ for the indicator function of a set or event \mathcal{D} .

Acknowledgments

This work is supported by the ANR project PERISTOCH, ANR-19-CE40-0023. To a large extent, it was written during the Spring 2020 lockdown due to the Covid-19 epidemics, which probably goes a long way toward explaining why it contains a lot of detailed computations. The author wishes to thank all members of the probability community who helped keeping the community's spirits intact by organising and giving many interesting online seminars.

2 Results

2.1 Set-up

We will consider a version of (1.5) in which time has been scaled by a factor ε , given by

$$\begin{aligned} dx_t &= \frac{1}{\varepsilon} b(x_t, y_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t^x, \\ dy_t &= dt + \sigma \varrho dW_t^y, \end{aligned} \quad (2.1)$$

where $\{W_t^x\}_{t \geq 0}$ and $\{W_t^y\}_{t \geq 0}$ are independent Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, ε , σ and ϱ are strictly positive parameters, and the drift term b satisfies the following assumptions:

- $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class \mathcal{C}^4 and is periodic, of period 1, in its second argument.
- For any $y \in [0, 1]$, the map $x \mapsto b(x, y)$ vanishes at exactly 3 points $x_-^*(y) < x_0^*(y) < x_+^*(y)$, and the derivative $\partial_x b(x, y)$ is nonzero for these 3 values of x .
- There are constants $M, L > 0$ such that $xb(x, y) \leq -Mx^2$ whenever $|x| \geq L$.

The above conditions guarantee existence of a pathwise unique strong solution $(x_t, y_t)_{t \geq 0}$ for any initial condition (x_0, y_0) . We denote by $\mathbb{P}_{x,y}\{\cdot\}$ the law of the process starting in (x, y) , and by $\mathbb{E}_{x,y}[\cdot]$ expectations with respect to $\mathbb{P}_{x,y}\{\cdot\}$.

We define the potential

$$V_0(x, y) = - \int_{x_0^*(y)}^x b(\bar{x}, y) d\bar{x}.$$

The assumptions on b imply that for any y , $x \mapsto V_0(x, y)$ has local minima at $x_{\pm}^*(y)$, a local maximum at $x_0^*(y)$, and grows at least quadratically for large $|x|$. We say that V_0 is a double-well potential (Figure 1). We denote the well depths by

$$h_{\pm}(y) = V_0(x_0^*(y), y) - V_0(x_{\pm}^*(y), y) = -V_0(x_{\pm}^*(y), y),$$

and measure the curvatures at stationary points by

$$\begin{aligned} \omega_{\pm}(y) &= \sqrt{\partial_{xx} V_0(x_{\pm}^*(y), y)} = \sqrt{-\partial_x b(x_{\pm}^*(y), y)}, \\ \omega_0(y) &= \sqrt{|\partial_{xx} V_0(x_0^*(y), y)|} = \sqrt{\partial_x b(x_0^*(y), y)}. \end{aligned}$$

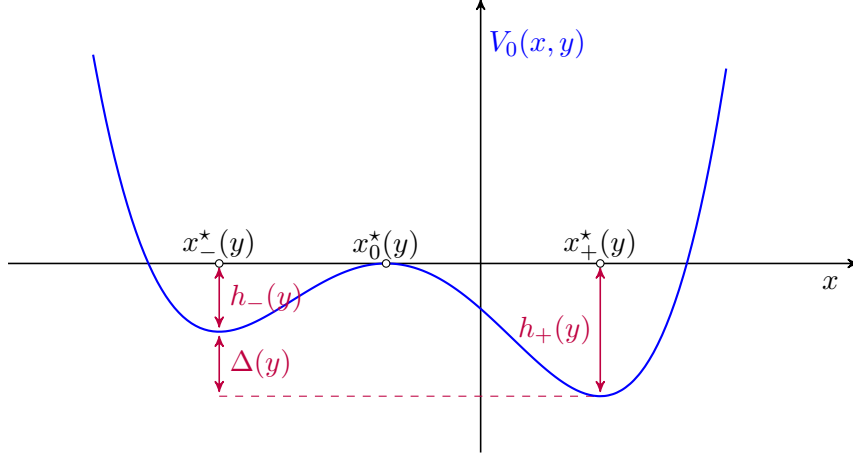


FIGURE 1. For each y , the map $x \mapsto V_0(x, y)$ is a double-well potential.

The assumptions on b imply that all these quantities are finite and bounded away from zero, uniformly in y . We further write $\Delta(y) = h_+(y) - h_-(y)$ for the difference of the two potential well depths.

2.2 Static system

We recall some well-known properties of the static system

$$dx_t = b(x_t, y) dt + \sigma dW_t^x \quad (2.2)$$

in which y is kept constant. Its direct and adjoint infinitesimal generators are the differential operators

$$\begin{aligned} \mathcal{L}_x f &= \frac{\sigma^2}{2} \partial_{xx} f + b \partial_x f, & \mathcal{L}_x^\dagger \mu &= \frac{\sigma^2}{2} \partial_{xx} \mu - \partial_x [b \mu], \\ &= \frac{\sigma^2}{2} e^{2V_0/\sigma^2} \partial_x (e^{-2V_0/\sigma^2} \partial_x f), & &= \frac{\sigma^2}{2} \partial_x (e^{-2V_0/\sigma^2} \partial_x (e^{2V_0/\sigma^2} \mu)). \end{aligned} \quad (2.3)$$

In particular, the kernel of \mathcal{L}_x is spanned by constant functions, while the kernel of \mathcal{L}_x^\dagger is spanned by the density $\pi_0(x|y)$ of the invariant measure of (2.2), which is given by

$$\pi_0(x|y) = \frac{1}{Z_0(y)} e^{-2V_0(x,y)/\sigma^2}, \quad Z_0(y) = \int_{-\infty}^{\infty} e^{-2V_0(x,y)/\sigma^2} dx$$

(Figure 2). We denote the eigenvalues of \mathcal{L}_x and \mathcal{L}_x^\dagger by

$$0 = -\lambda_0(y) > -\lambda_1(y) > -\lambda_2(y) \geq \dots,$$

and the corresponding L^2 -normalised eigenfunctions by $\phi_n(\cdot|y)$ and $\pi_n(\cdot|y)$. These are related by

$$\pi_n(x|y) = \pi_0(x|y) \phi_n(x|y).$$

There is a spectral gap of order 1, separating $\lambda_1(y)$ from $\lambda_2(y)$ and all subsequent eigenvalues, which is why an important role will be played by $\lambda_1(y)$ and the associated eigenfunctions. The eigenvalue satisfies

$$\lambda_1(y) = [r_+(y) + r_-(y)] [1 + \mathcal{O}(\sigma^2)], \quad (2.4)$$

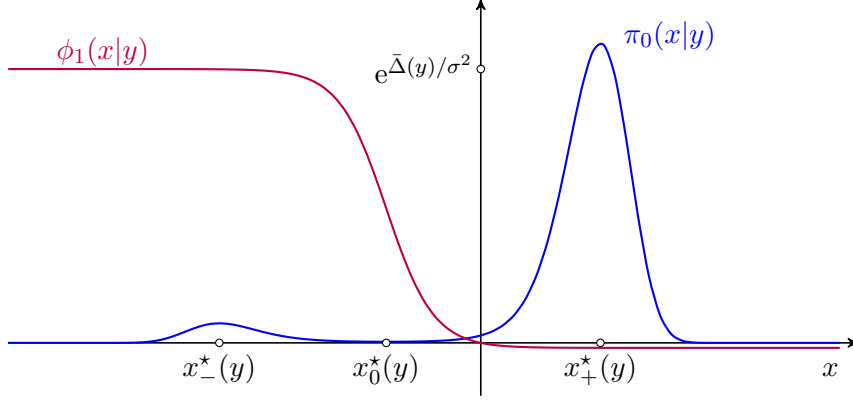


FIGURE 2. Sketch of the static eigenfunctions $\pi_0(x|y)$ and $\phi_1(x|y)$ for the potential of Figure 1.

where

$$r_{\pm}(y) = \frac{\omega_{\pm}(y)\omega_0(y)}{2\pi} e^{-2h_{\pm}(y)/\sigma^2}.$$

The corresponding eigenfunction can be approximated in terms of the committor

$$h_0(x|y) = \mathbb{P}_x\{\tau_{x_-^*}(y) < \tau_{x_+^*}(y)\}, \quad \tau_{\bar{x}} = \inf\{t > 0: x_t = \bar{x}\},$$

which satisfies $\mathcal{L}_x h_0 = 0$ with boundary conditions $h_0(x_-^*(y)|y) = 1$ and $h_0(x_+^*(y)|y) = 0$. Solving this equation, one obtains that for all $x \in [x_-^*(y), x_+^*(y)]$,

$$h_0(x|y) = \frac{1}{N(y)} \int_x^{x_+^*(y)} e^{2V_0(\bar{x},y)/\sigma^2} d\bar{x}, \quad N(y) = \int_{x_-^*(y)}^{x_+^*(y)} e^{2V_0(\bar{x},y)/\sigma^2} d\bar{x}. \quad (2.5)$$

The first eigenfunction of \mathcal{L}_x is related to $h_0(x|y)$ by

$$\phi_1(x|y) = \left[e^{\bar{\Delta}(y)/\sigma^2} h_0(x|y) - e^{-\bar{\Delta}(y)/\sigma^2} (1 - h_0(x|y)) \right] [1 + \mathcal{O}(\lambda_1(y) \log(\sigma^{-1}))], \quad (2.6)$$

where $\bar{\Delta}(y)$ is defined by

$$e^{2\bar{\Delta}(y)/\sigma^2} = \frac{r_-(y)}{r_+(y)} \quad \Rightarrow \quad \bar{\Delta}(y) = \Delta(y) + \frac{\sigma^2}{2} \log\left(\frac{\omega_-(y)}{\omega_+(y)}\right). \quad (2.7)$$

The function $x \mapsto \phi_1(x|y)$ is almost constant except near $x_0^*(y)$, with a value close to $e^{\bar{\Delta}(y)/\sigma^2}$ for $x < x_0^*(y)$ and close to $-e^{-\bar{\Delta}(y)/\sigma^2}$ for $x > x_0^*(y)$ (Figure 2). We give a precise statement of (2.6), including bounds on derivatives of ϕ_1 , in Section 4.1.

2.3 Two-state jump process

The spectral-gap property implies that for small σ , the dynamics of the static system (2.2) is well-approximated by a two-state Markovian jump process with rates $r_{\pm}(y)$. It is thus natural to expect that the dynamics of the fast-slow system (2.1) is well-approximated by a time-dependent two-state process, in which y plays the role of time (Figure 3). Its law $(p_-(y), p_+(y))$ satisfies the system

$$\begin{aligned} \varepsilon p'_-(y) &= r_+(y)p_+(y) - r_-(y)p_-(y) \\ \varepsilon p'_+(y) &= -r_+(y)p_+(y) + r_-(y)p_-(y). \end{aligned} \quad (2.8)$$

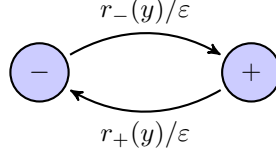


FIGURE 3. Time-dependent two-state markovian jump process.

Let

$$A(y) = \frac{r_-(y) - r_+(y)}{r_-(y) + r_+(y)} = \tanh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right), \quad (2.9)$$

and let $\delta(y)$ be the 1-periodic solution of

$$\varepsilon \delta'(y) = -\lambda_1(y) [\delta(y) - A(y)]. \quad (2.10)$$

Then it is straightforward to check that the solution of System (2.8) with initial condition $(p_+(y_0), p_-(y_0))$ satisfying $p_+(y_0) + p_-(y_0) = 1$ is given by

$$p_{\pm}(y) = \frac{1}{2} [1 \pm \delta(y)] \pm \frac{1}{2} [p_+(y_0) - p_-(y_0) - \delta(y_0)] e^{-\Lambda(y, y_0)/\varepsilon}, \quad (2.11)$$

where

$$\Lambda(y, y_0) = \int_{y_0}^y \lambda_1(\bar{y}) d\bar{y}.$$

Note that $\delta(y)$ admits the explicit integral representation

$$\delta(y) = \frac{1}{\varepsilon(e^{\Lambda(1,0)/\varepsilon} - 1)} \int_y^{y+1} \lambda_1(\bar{y}) A(\bar{y}) e^{\Lambda(\bar{y}, y)/\varepsilon} d\bar{y}.$$

Two regimes are of particular interest:

- In the *fast forcing regime* $\varepsilon \gg \max_{y \in [0,1]} \lambda_1(y)$, the dynamics is averaged, and $\delta(y)$ satisfies

$$\delta(y) = \frac{1}{\Lambda(1,0)} \int_0^1 \lambda_1(\bar{y}) A(\bar{y}) d\bar{y} \left[1 + \mathcal{O}\left(\frac{\max_{y \in [0,1]} \lambda_1(y)}{\varepsilon}\right) \right].$$

In this case, $\delta(y)$ and $p_{\pm}(y)$ are asymptotically almost constant.

- In the *super-adiabatic regime* $\varepsilon \ll \min_{y \in [0,1]} \lambda_1(y)$, integration by parts shows that

$$\delta(y) = A(y) \left[1 + \mathcal{O}\left(\frac{\varepsilon}{\min_{y \in [0,1]} \lambda_1(y)}\right) \right].$$

Thus $\delta(y)$ tracks $A(y)$, which is close to the sign of $\bar{\Delta}(y)$, meaning that with high probability, the jump process is found in the currently deepest potential well.

It is also possible to compute explicitly the expectation of the transition time τ_+^{jump} from the $-$ state to the $+$ state. We give the simple proof of the following result in Appendix A.

Proposition 2.1. *For any $y_0 \in [0, 1]$, one has*

$$\mathbb{E}_{-, y_0} [\tau_+^{\text{jump}}] = \frac{1}{1 - e^{-R_-(1,0)/\varepsilon}} \int_0^1 e^{-R_-(y_0+y, y_0)/\varepsilon} dy,$$

where

$$R_-(y_1, y_0) = \int_{y_0}^{y_1} r_-(\bar{y}) d\bar{y}.$$

A similar expression holds for the transition time τ_-^{jump} from the $+$ state to the $-$ state.

The same distinction between regimes as above can be made here:

- If $\varepsilon \gg \max_{y \in [0,1]} r_-(y)$, then the expected jump time does not depend on y_0 to leading order, and is given by the average

$$\mathbb{E}_{-,y_0} [\tau_+^{\text{jump}}] = \frac{\varepsilon}{R_-(0,1)} \left[1 + \mathcal{O}\left(\frac{\max_{y \in [0,1]} r_-(y)}{\varepsilon}\right) \right].$$

- If $\varepsilon \ll \min_{y \in [0,1]} r_-(y)$, then the expected jump time is much shorter than the oscillation period, and thus given by the instantaneous value

$$\mathbb{E}_{-,y_0} [\tau_+^{\text{jump}}] = \frac{\varepsilon}{r_-(y_0)} \left[1 + \mathcal{O}\left(\frac{\varepsilon}{\min_{y \in [0,1]} r_-(y)}\right) \right].$$

2.4 Invariant measure

We now return to the fast-slow SDE (2.1). In order to be able to apply the potential-theoretic approach of [30], it is necessary to control the invariant measure of the system. The main result of this section is the following theorem, which will be proved in Section 4.

Theorem 2.2 (Invariant measure). *For sufficiently small σ and ε , the invariant measure of the system (2.1) has the density*

$$\pi(x, y) = \pi_0(x|y) [1 + \alpha_1(y)\phi_1(x|y) + \Phi_\perp(x, y)], \quad (2.12)$$

where

$$\alpha_1(y) = \sinh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right) - \delta_1(y) \cosh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right).$$

Here $\bar{\Delta}(y)$ is given by (2.7), and $\delta_1(y)$ is the unique periodic solution of the linear second-order equation

$$\frac{\varrho^2}{2} \varepsilon \sigma^2 \delta_1'' - \varepsilon q_1(y) \delta_1' - \lambda_1(y) q_2(y) \left[\delta_1 - \tanh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right) \right] + q_3(y) = 0, \quad (2.13)$$

where

$$\begin{aligned} q_1(y) &= 1 + \mathcal{O}(\lambda_1(y) \log(\sigma^{-1})^2), \\ q_2(y) &= 1 + \mathcal{O}\left(\frac{\varepsilon}{\sigma^2} \log(\sigma^{-1})^3\right), \\ q_3(y) &= \mathcal{O}\left(\frac{\varepsilon}{\sigma^2} \lambda_1(y) \log(\sigma^{-1})^3\right) + \mathcal{O}\left(\frac{\varepsilon^3}{\sigma^6} \sqrt{\lambda_1(y) \log(\sigma^{-1})^3}\right). \end{aligned} \quad (2.14)$$

Furthermore, the error term $\Phi_\perp(x, y)$ in (2.12) is orthogonal to the span of ϕ_0 and ϕ_1 , and satisfies

$$\langle \pi_0, \Phi_\perp^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2} \cosh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right), \quad (2.15)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product for $L^2(\mathbb{R}, dx)$.

As we will see in Section 4, the periodic solution of (2.13) is in fact close to the periodic solution of the first-order equation

$$\varepsilon \delta_1' = -\lambda_1(y) \frac{q_2(y)}{q_1(y)} \left[\delta_1 - A(y) \right] + \frac{q_3(y)}{q_1(y)},$$

which is similar to (2.10). The function $\delta_1(y)$ also has a similar interpretation as $\delta(y)$ in (2.11). Indeed, one has (by a similar argument as in the proof of Corollary 5.2)

$$p_-(y) := \int_{-\infty}^{x_0^*(y)} \pi(x, y) dx = \frac{1}{2} [1 - \delta_1(y) + \mathcal{O}(\sigma^2)] .$$

By analogy with (2.11), $p_-(y)$ can be interpreted as the “instantaneous” probability to be in the left-hand potential well at equilibrium.

Given a function $f : [0, 1] \rightarrow \mathbb{R}$, we introduce the notation

$$\langle f \rangle = \int_0^1 f(y) dy . \quad (2.16)$$

We will mainly be concerned with the fast-forcing regime $\varepsilon \gg \langle \lambda_1 \rangle$. Then $\delta_1(y)$ is actually nearly constant, in the sense that

$$\delta_1(y) = \bar{\delta}_1 \left[1 + \mathcal{O}\left(\frac{\langle \lambda_1 \rangle}{\varepsilon}\right) \right] , \quad (2.17)$$

where

$$\bar{\delta}_1 = \frac{1}{\langle \lambda_1 \rangle} \left[\langle \lambda_1 A \rangle + \mathcal{O}\left(\frac{\varepsilon}{\sigma^2} \log(\sigma^{-1})^3 \langle \lambda_1 \rangle\right) + \mathcal{O}\left(\frac{\varepsilon^3}{\sigma^6} \log(\sigma^{-1})^{3/2} \langle \sqrt{\lambda_1} \rangle\right) \right] . \quad (2.18)$$

One should note that the main limitation of Theorem 2.2 lies in the error term proportional to $\sqrt{\lambda_1(y)}$ in (2.14), which causes the error term in $\langle \sqrt{\lambda_1} \rangle$ in (2.18). This is due to technical difficulties in controlling Φ_\perp , and will limit the applicability of our results to the regime

$$\varepsilon \ll \langle \lambda_1 \rangle^{1/4} .$$

In fact, there is already a substantial amount of work involved in getting an error term proportional to $(\varepsilon/\sigma^2)^3 \langle \sqrt{\lambda_1} \rangle$, rather than $(\varepsilon/\sigma^2)^2 \langle \sqrt{\lambda_1} \rangle$. This improvement is due to the fact that we are able to prove that

$$\Phi_\perp(x, y) = \Phi_\perp^*(x, y) + \Phi_\perp^1(x, y) ,$$

where Φ_\perp^* is explicit, and has a contribution of order $\lambda_1(y)$ to $q_3(y)$, while Φ_\perp^1 satisfies a bound of the form (2.15), but with a larger power of ε . See Corollary 4.13 for details.

2.5 Main results: expected transition time

In order to formulate our main result, we introduce two functions

$$\begin{aligned} a(y) &= x_-^*(y) + \rho , \\ b(y) &= x_+^*(y) - \rho , \end{aligned} \quad (2.19)$$

where $\rho > 0$ is a parameter of order 1 that will be taken sufficiently small. We then define two set

$$\begin{aligned} \mathcal{A} &= \{(x, y) \in \mathbb{R} \times [0, 1] : x \leq a(y)\} , \\ \mathcal{B} &= \{(x, y) \in \mathbb{R} \times [0, 1] : x \geq b(y)\} \end{aligned}$$

see Figure 4.

Our first main result gives a general expression for the expected first-hitting time of \mathcal{B} , when starting with a specific distribution on $\partial\mathcal{A}$, the so-called equilibrium measure.

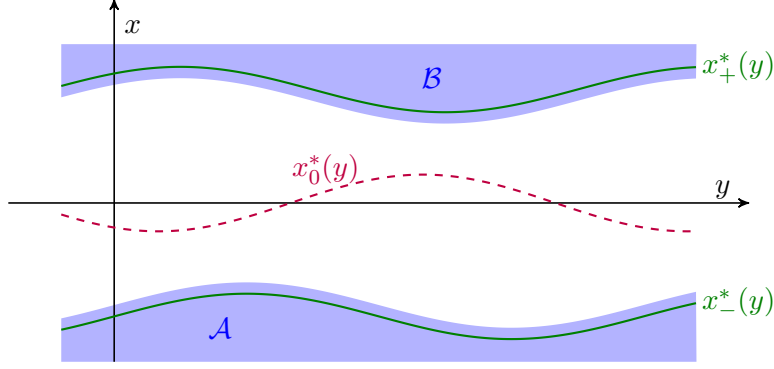


FIGURE 4. Definition of the sets \mathcal{A} and \mathcal{B} , in relation with the extrema $x_{\pm}^*(y)$ and $x_0^*(y)$ of the map $x \mapsto V_0(x, y)$.

Theorem 2.3 (Main result, general case). *There exists a probability measure $\nu_{\mathcal{AB}}$, supported on $\partial\mathcal{A}$, such that*

$$\int_{\partial\mathcal{A}} \mathbb{E}_{(x,y)}[\tau_{\mathcal{B}}] d\nu_{\mathcal{AB}} = \frac{2\varepsilon[1 - \langle\delta_1\rangle]}{\langle\lambda_1[1 - A\delta_1]\rangle} [1 + R_0(\varepsilon, \sigma)] , \quad (2.20)$$

where $R_0(\varepsilon, \sigma)$ is an error term satisfying

$$|R_0(\varepsilon, \sigma)| \lesssim \sigma^2 + \frac{\varepsilon \log(\sigma^{-1}) \langle\sqrt{\lambda_1}\rangle}{\sigma^2[1 - \langle\delta_1\rangle]} + \frac{\varepsilon \log(\sigma^{-1})^2 \langle\lambda_1\rangle}{\sigma^2 \langle\lambda_1[1 - A\delta_1]\rangle} + \frac{\varepsilon^2 \log(\sigma^{-1}) \langle\sqrt{\lambda_1}\rangle}{\sigma^{7/2} \langle\lambda_1[1 - A\delta_1]\rangle} . \quad (2.21)$$

As such, this result has two main limitations. First, it is not immediately apparent for which values of ε and σ the remainder $R_0(\varepsilon, \sigma)$ is actually small. And second, we do not know the equilibrium measure $\nu_{\mathcal{AB}}$.

We will address both issues in the fast-forcing regime $\varepsilon \gg \langle\lambda_1\rangle$. In fact, the discussion of the two-state jump process in Section 2.3 suggests that if $\varepsilon < \langle\lambda_1\rangle$, the expected first-hitting time depends strongly on the starting point, whereas it is almost constant if $\varepsilon \gg \langle\lambda_1\rangle$.

Recalling the expressions (2.4) for $\lambda_1(y)$ and (2.9) for $A(y)$, we obtain

$$\begin{aligned} \langle\lambda_1\rangle &= [\langle r_- \rangle + \langle r_+ \rangle] [1 + \mathcal{O}(\sigma^2)] , \\ \langle\lambda_1 A\rangle &= [\langle r_- \rangle - \langle r_+ \rangle] [1 + \mathcal{O}(\sigma^2)] . \end{aligned} \quad (2.22)$$

Furthermore, if $\varepsilon \gg \langle\lambda_1\rangle$, (2.17) and (2.18) imply that $\delta_1(y)$ is close to

$$\frac{\langle\lambda_1 A\rangle}{\langle\lambda_1\rangle} = \frac{\langle r_- \rangle - \langle r_+ \rangle}{\langle r_- \rangle + \langle r_+ \rangle} [1 + \mathcal{O}(\sigma^2)] .$$

It follows that the leading term in (2.20) is given by

$$\begin{aligned} 2\varepsilon \frac{\langle\lambda_1\rangle - \langle\lambda_1 A\rangle}{\langle\lambda_1\rangle^2 - \langle\lambda_1 A\rangle^2} &= \frac{\varepsilon}{\langle r_- \rangle} [1 + \mathcal{O}(\sigma^2)] \\ &= \frac{2\pi\varepsilon}{\int_0^1 \omega_-(y) \omega_0(y) e^{-2h_-(y)/\sigma^2} dy} [1 + \mathcal{O}(\sigma^2)] , \end{aligned}$$

which agrees with (1.7) (recall that we have scaled time by a factor ε). In order to quantify error terms, we introduce minimal barrier heights

$$h_{\pm}^{\min} = \min_{0 \leq y \leq 1} h_{\pm}(y) , \quad (2.23)$$

and asymmetry factors

$$H = |h_-^{\min} - h_+^{\min}|, \quad H_- = [h_-^{\min} - h_+^{\min}]_+, \quad (2.24)$$

where $[\cdot]_+$ denotes the positive part.

Theorem 2.4 (Main result, fast-forcing regime). *Assume $\varepsilon \gg \langle \lambda_1 \rangle$. Then for any initial condition $(x, y) \in \partial \mathcal{A}$, we have*

$$\mathbb{E}_{(x,y)}[\tau_{\mathcal{B}}] = \frac{\varepsilon}{\langle r_- \rangle} [1 + R_1(\varepsilon, \sigma)], \quad (2.25)$$

where

$$|R_1(\varepsilon, \sigma)| \lesssim \sigma^2 + \left(\frac{\varepsilon \log(\sigma^{-1})^3}{\sigma^2} + \frac{\varepsilon^2 \log(\sigma^{-1})}{\sigma^{7/2} \langle \lambda_1 \rangle^{1/2}} + \frac{\langle \lambda_1 \rangle^2}{\varepsilon} \right) e^{2H/\sigma^2} + \frac{\langle \lambda_1 \rangle}{\varepsilon} (1 + e^{2H_-/\sigma^2}). \quad (2.26)$$

In the symmetric case $h_-^{\min} = h_+^{\min}$, we have $H = H_- = 0$, and the error term takes the simpler form

$$|R_1(\varepsilon, \sigma)| \lesssim \sigma^2 + \frac{\varepsilon \log(\sigma^{-1})^3}{\sigma^2} + \frac{\varepsilon^2 \log(\sigma^{-1})}{\sigma^{7/2} \langle \lambda_1 \rangle^{1/2}} + \frac{\langle \lambda_1 \rangle}{\varepsilon}.$$

Disregarding powers of σ with respect to exponential terms, we see that Theorem 2.4 is applicable when

$$\langle \lambda_1 \rangle \ll \varepsilon \ll \langle \lambda_1 \rangle^{1/4}. \quad (2.27)$$

In the asymmetric case $h_-^{\min} \neq h_+^{\min}$, the error term is larger, and results in stronger conditions on ε . One can however check (see Section 7) that there exists a non-empty interval of values of ε for which Theorem 2.4 is still meaningful as long as

$$\frac{1}{2} h_-^{\min} < h_+^{\min} < 2 h_-^{\min}, \quad (2.28)$$

that is, as long as the asymmetry between the potential wells is not too large.

2.6 Discussion

Theorem 2.4 provides a generalisation of the static Eyring–Kramers law (1.2) to slowly oscillating double-well potentials, when the forcing frequency ε lies in an interval given by (2.27) if the oscillation is symmetric, in the sense that $h_-^{\min} = h_+^{\min}$. We now provide some comments on what we expect to happen outside this domain of validity. This will also serve as a “reality check” of our main results.

If $\varepsilon \leq \langle \lambda_1 \rangle$, Theorem 2.3 is still valid, but perhaps not as useful. The main limitation of the result in that case is that $\mathbb{E}_{(x,y)}[\tau_{\mathcal{B}}]$ is no longer expected to be almost constant, so that the equilibrium measure $\nu_{\mathcal{AB}}$ matters. In fact, we do have an explicit expression for $\nu_{\mathcal{AB}}$, which is given (cf. (3.10)) by

$$d\nu_{\mathcal{AB}} = \frac{\sigma^2}{2\varepsilon \operatorname{cap}(\mathcal{A}, \mathcal{B})} (D\nabla h_{\mathcal{AB}}^* \cdot \mathbf{n}) \pi d\lambda,$$

where all notations are defined in Section 3. In particular, \mathbf{n} is the unit normal vector to $\partial \mathcal{A}$, and $d\lambda$ is the Lebesgue measure on $\partial \mathcal{A}$, so that

$$\mathbf{n} d\lambda = (\mathbf{e}_x - a'(y)\mathbf{e}_y) dy.$$

Using the estimates on the capacity $\text{cap}(\mathcal{A}, \mathcal{B})$ given in Theorem 6.1, the expression (3.11) of the diffusion matrix D , the estimate on the adjoint committor $h_{\mathcal{A}\mathcal{B}}^*$ obtained in Proposition 5.1, and Theorem 2.2 on the invariant measure π , we obtain that to leading order,

$$d\nu_{\mathcal{A}\mathcal{B}} \simeq \frac{\sigma^2}{2\varepsilon} \frac{1 + \alpha_1(y)\phi_1(a(y)|y)}{\tilde{N}(y)Z_0(y)\text{cap}(\mathcal{A}, \mathcal{B})} dy \simeq \frac{\lambda_1(y)[1 + A(y)][1 - \delta_1(y)]}{\langle \lambda_1[1 - A\delta_1] \rangle} dy.$$

Substituting this in the result (2.20) of Theorem 2.3, and using the fact that $\lambda_1(y)[1 + A(y)] = 2r_-(y)$, we obtain that to leading order,

$$\langle \mathbb{E}[\tau_{\mathcal{B}}]r_-(1 - \delta_1) \rangle \simeq \varepsilon \langle 1 - \delta_1 \rangle. \quad (2.29)$$

- In the fast-forcing regime $\varepsilon \gg \langle \lambda_1 \rangle$, the expectation $\mathbb{E}_{(a(y_0), y_0)}[\tau_{\mathcal{B}}]$ being nearly constant, we recover indeed (2.25).
- In the superadiabatic regime $\varepsilon \ll \min_y \lambda_1(y)$, the discussion in Section 2.3 on the two-state jump process suggests that we have

$$\mathbb{E}_{(a(y_0), y_0)}[\tau_{\mathcal{B}}] \simeq \frac{\varepsilon}{r_-(y_0)}, \quad (2.30)$$

which is indeed consistent with (2.29).

- In the intermediate regime $\min_y \lambda_1(y) \leq \varepsilon \leq \max_y \lambda_1(y)$, the situation is more complicated owing to the phenomenon of stochastic resonance (see for instance the discussion in [9, Section 4.1.2]). What we expect then is the following. If the process starts at a point $(a(y_0), y_0) \in \partial\mathcal{A}$ such that $\varepsilon < r_-(y_0)$, the mean hitting time of \mathcal{B} will still satisfy (2.30). Otherwise, the transition to \mathcal{B} will occur near the smallest $y > y_0$ such that $r_-(y) = \varepsilon$, and thus the expectation of $\tau_{\mathcal{B}}$ is dominated by $y - y_0$. This picture is also consistent with large-deviation results obtained in [25].

The other regime not covered by our results is when $\varepsilon \geq \langle \lambda_1 \rangle^{1/4}$. As noted above, this is mainly due to technical difficulties in controlling the part of the invariant measure π which is orthogonal to the span of the first two eigenfunctions π_0 and π_1 of \mathcal{L}_x^\dagger . In fact, it seems quite plausible that the expression (2.25) for the mean transition time still holds as long as $\varepsilon \ll \sigma^2$ (for larger ε , the slow-fast structure of the equation for π changes). To establish such a result, however, new ideas are needed to achieve a better control of the invariant measure.

2.7 Outline of the proof

As already mentioned, the main ingredient of our proof is the potential-theoretic approach to metastability, which was developed in [14, 15] for reversible diffusions, and extended in [30] to general diffusions. We give a quick overview of this approach in Section 3. Its key result relates the expected first-hitting time of a set \mathcal{B} , when starting with in the equilibrium measure $\nu_{\mathcal{A}\mathcal{B}}$ on the boundary of another set \mathcal{A} , with the invariant measure of the diffusion and the so-called *capacity* $\text{cap}(\mathcal{A}, \mathcal{B})$. See Proposition 3.4 below.

The main difficulty in our case is to determine the invariant measure π of the system. While this measure is explicitly known for reversible systems, this is no longer the case here. As shown in [30], π is related to the solution of a Hamilton–Jacobi equation, see Lemma 3.1. However, obtaining an approximate solution of this equation with good enough control of error terms turns out to be difficult. Therefore we adopt another approach, which consists in expanding π on a basis of eigenfunctions of the generator of the system with frozen y , and analysing the resulting system of ODEs. This is done in Section 4, which contains in particular the proof of Theorem 2.2.

In Section 5, we investigate the adjoint system that enters the expression for the mean first-hitting time in Proposition 3.4. In particular, we obtain approximate expressions for the committors $\mathbb{P}\{\tau_A < \tau_B\}$ of the original and adjoint system in Proposition 5.1, using a perturbation theory argument around the committors of the frozen systems. The necessary estimate for Proposition 3.4 is then obtained in Corollary 5.2.

The other quantity that needs to be determined for the potential-theoretic approach to work is the capacity $\text{cap}(A, B)$. This is comparatively easy once the invariant measure is known, since the capacity obeys variational principles (the Dirichlet and Thomson principle) that give upper and lower bounds once one makes a sufficiently good guess of test functions to feed into them. It turns out that the system with frozen y provides such sufficiently good guesses, the only difficulty being to account for the fact that these guesses are not strictly divergence-free. The main result is Theorem 6.1, which provides upper and lower bounds on the capacity.

Section 7 contains the last steps of the proof of Theorems 2.3 and 2.4. While Theorem 2.3 follows directly from the obtained bounds on the invariant measure, committors and capacity, Theorem 2.4 requires a little more work, which consists in simplifying the expressions for the dominant term and error terms, and getting rid of the equilibrium measure ν_{AB} .

In order to increase readability, we have relegated some of the more technical proofs to the appendix. Appendix A contains the proof of Proposition 2.1 on the two-state jump process, Appendix B contains the proofs of the potential-theoretic results in Section 3, Appendix C contains the estimates on static eigenfunctions required for determining the invariant measure, and Appendix D gathers a few auxiliary results involving Laplace asymptotics.

3 Non-reversible potential theory

In this section, we give a short overview of the potential-theoretic results contained in [30, Section 4], slightly adapted to our situation. All proofs are given in Appendix B.

The infinitesimal generator of the system (2.1) is given by

$$\mathcal{L} = \frac{\sigma^2}{2\varepsilon} (\partial_{xx} + \varrho^2 \varepsilon \partial_{yy}) + \frac{1}{\varepsilon} b \partial_x + \partial_y. \quad (3.1)$$

A key idea in [30] is to decompose \mathcal{L} into a symmetric and an antisymmetric part. This allows to define an adjoint stochastic process, and both the direct and adjoint process play a role in the expressions for mean first-passage times.

3.1 Invariant density

Lemma 3.1. *The system (2.1) has an invariant measure with density $\pi(x, y) = Z^{-1} e^{-2V(x, y)/\sigma^2}$, where V solves the Hamilton–Jacobi equation*

$$(\partial_x V)^2 + b \partial_x V + \varepsilon \varrho^2 (\partial_y V)^2 + \varepsilon \partial_y V = \frac{\sigma^2}{2} [\partial_{xx} V + \partial_x b + \varepsilon \varrho^2 \partial_{yy} V]. \quad (3.2)$$

Lemma 3.2. *The infinitesimal generator (3.1) can be written as*

$$\begin{aligned} \mathcal{L}f &= \frac{\sigma^2}{2\varepsilon} e^{2V/\sigma^2} \left\{ \partial_x [e^{-2V/\sigma^2} \partial_x f] + \varepsilon \varrho^2 \partial_y [e^{-2V/\sigma^2} \partial_y f] \right\} + c \cdot \nabla f \\ &=: \frac{\sigma^2}{2\varepsilon} e^{2V/\sigma^2} \nabla \cdot [D e^{-2V/\sigma^2} \nabla f] + c \cdot \nabla f \end{aligned} \quad (3.3)$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \varrho^2 \end{pmatrix}$$

is a diffusion matrix, and

$$c = \frac{1}{\varepsilon} (b + \partial_x V) \mathbf{e}_x + (1 + \varrho^2 \partial_y V) \mathbf{e}_y \quad (3.4)$$

satisfies the vanishing divergence condition

$$\nabla \cdot (e^{-2V/\sigma^2} c) = 0 . \quad (3.5)$$

3.2 Adjoint process

We decompose \mathcal{L} into a symmetric and an antisymmetric part by writing $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$, where

$$\mathcal{L}_s f = \frac{\sigma^2}{2\varepsilon} e^{2V/\sigma^2} \nabla \cdot [D e^{-2V/\sigma^2} \nabla f] , \quad \mathcal{L}_a f = c \cdot \nabla f .$$

We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ for the circle, and endow $L^2(\mathbb{R} \times \mathbb{T})$ with the inner product

$$\langle f, g \rangle_\pi = \int_{\mathbb{R} \times \mathbb{T}} f(x, y) g(x, y) d\pi ,$$

where $d\pi = \pi(x, y) dx dy$. Then one checks that \mathcal{L}_s is self-adjoint with respect to this inner product, while (3.5) and the divergence theorem imply

$$\int_{\mathbb{R} \times \mathbb{T}} f c \cdot \nabla g d\pi = - \int_{\mathbb{R} \times \mathbb{T}} g c \cdot \nabla f d\pi , \quad (3.6)$$

showing that \mathcal{L}_a is anti-self-adjoint (skew-symmetric), that is $\mathcal{L}_a^\dagger = -\mathcal{L}_a$. By definition, the adjoint process has the generator

$$\mathcal{L}^* f = \mathcal{L}_s f - \mathcal{L}_a f = \frac{\sigma^2}{2\varepsilon} e^{2V/\sigma^2} \nabla \cdot [D e^{-2V/\sigma^2} \nabla f] - c \cdot \nabla f .$$

The corresponding SDE is given by

$$\begin{aligned} dx_t &= \frac{1}{\varepsilon} b^*(x_t, y_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t^x , \\ dy_t &= -[1 + 2\varrho^2 \partial_y V(x, y)] dt + \sigma \varrho dW_t^y , \end{aligned} \quad (3.7)$$

where

$$b^* = -\partial_x V - \varepsilon c_x = -2\partial_x V - b .$$

We denote by $\mathbb{P}_{x,y}^* \{\cdot\}$ the law of the adjoint process starting in (x, y) , and by $\mathbb{E}_{x,y}^* [\cdot]$ the corresponding expectations.

3.3 Committed and capacity

Consider two sets $\mathcal{A} = \{(x, y) : x \leq a(y)\}$ and $\mathcal{B} = \{(x, y) : x \geq b(y)\}$, where $a(y) < b(y)$ are smooth periodic functions. The committed $h_{\mathcal{AB}}(x, y) = \mathbb{P}_{x,y}\{\tau_{\mathcal{A}} < \tau_{\mathcal{B}}\}$ and $h_{\mathcal{AB}}^*(x, y) = \mathbb{P}_{x,y}^*\{\tau_{\mathcal{A}} < \tau_{\mathcal{B}}\}$ satisfy the Dirichlet problems

$$\begin{cases} (\mathcal{L}h)(x, y) = 0 & (x, y) \in (\mathcal{A} \cup \mathcal{B})^c, \\ h(x, y) = 1 & (x, y) \in \mathcal{A}, \\ h(x, y) = 0 & (x, y) \in \mathcal{B}, \end{cases} \quad \begin{cases} (\mathcal{L}^*h^*)(x, y) = 0 & (x, y) \in (\mathcal{A} \cup \mathcal{B})^c, \\ h^*(x, y) = 1 & (x, y) \in \mathcal{A}, \\ h^*(x, y) = 0 & (x, y) \in \mathcal{B}. \end{cases}$$

The capacities of the direct and adjoint process are defined via the Dirichlet form associated with \mathcal{L}_s , that is

$$\begin{aligned} \text{cap}(\mathcal{A}, \mathcal{B}) &= \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla h_{\mathcal{AB}} \cdot (D\nabla h_{\mathcal{AB}}) d\pi, \\ \text{cap}^*(\mathcal{A}, \mathcal{B}) &= \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla h_{\mathcal{AB}}^* \cdot (D\nabla h_{\mathcal{AB}}^*) d\pi. \end{aligned}$$

Lemma 3.3. *We have $\text{cap}(\mathcal{A}, \mathcal{B}) = \text{cap}(\mathcal{B}, \mathcal{A})$, and*

$$\text{cap}(\mathcal{A}, \mathcal{B}) = \frac{\sigma^2}{2\varepsilon} \int_{\partial\mathcal{A}} (D\nabla h_{\mathcal{AB}} \cdot \mathbf{n}) \pi d\lambda = \int_{\partial\mathcal{A}} \left(\frac{\sigma^2}{2\varepsilon} D\nabla h_{\mathcal{AB}} + h_{\mathcal{AB}} c \right) \cdot \mathbf{n} \pi d\lambda, \quad (3.8)$$

where \mathbf{n} is the inward-pointing unit normal vector to $\partial\mathcal{A}$, and $d\lambda$ is the arclength on $\partial\mathcal{A}$. An analogous relation, with $h_{\mathcal{AB}}$ replaced by $h_{\mathcal{AB}}^*$, holds for $\text{cap}^*(\mathcal{A}, \mathcal{B})$. Furthermore,

$$\begin{aligned} \text{cap}(\mathcal{A}, \mathcal{B}) &= \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} [\nabla h_{\mathcal{AB}}^* \cdot (D\nabla h_{\mathcal{AB}}) - \varepsilon h_{\mathcal{AB}}^* (c \cdot \nabla h_{\mathcal{AB}})] d\pi \\ &= \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} [\nabla h_{\mathcal{AB}} \cdot (D\nabla h_{\mathcal{AB}}^*) + \varepsilon h_{\mathcal{AB}} (c \cdot \nabla h_{\mathcal{AB}}^*)] d\pi = \text{cap}^*(\mathcal{A}, \mathcal{B}). \end{aligned} \quad (3.9)$$

3.4 Equilibrium measure and mean hitting time

The \mathcal{AB} -equilibrium measure $\nu_{\mathcal{AB}}$ is the probability measure supported on $\partial\mathcal{A}$ defined by

$$d\nu_{\mathcal{AB}} = \frac{\sigma^2}{2\varepsilon \text{cap}(\mathcal{A}, \mathcal{B})} (D\nabla h_{\mathcal{AB}}^* \cdot \mathbf{n}) \pi d\lambda. \quad (3.10)$$

Then we have the following fundamental relation.

Proposition 3.4. *Let $\tau_{\mathcal{B}} = \inf\{t > 0 : (x_t, y_t) \in \mathcal{B}\}$ denote the first-hitting time of \mathcal{B} . Then*

$$\mathbb{E}_{\nu_{\mathcal{AB}}}[\tau_{\mathcal{B}}] := \int_{\partial\mathcal{A}} \mathbb{E}_x[\tau_{\mathcal{B}}] d\nu_{\mathcal{AB}} = \frac{1}{\text{cap}(\mathcal{A}, \mathcal{B})} \int_{\mathcal{B}^c} h_{\mathcal{AB}}^* d\pi.$$

3.5 Variational principles

For φ, ψ two vector fields on $(\mathcal{A} \cup \mathcal{B})^c$, we define the bilinear form

$$\mathcal{D}(\varphi, \psi) := \frac{2\varepsilon}{\sigma^2} \int_{(\mathcal{A} \cup \mathcal{B})^c} \varphi(x, y) \cdot (D^{-1}\psi(x, y)) \frac{dx dy}{\pi(x, y)}, \quad (3.11)$$

and we denote $\mathcal{D}(\varphi, \varphi)$ by $\mathcal{D}(\varphi)$. For $\gamma \in \mathbb{R}$, we write \mathcal{F}_{AB}^γ for the closure with respect to the norm $\mathcal{D}(\cdot)$ of the set of flows φ which are divergence-free, i.e.

$$\nabla \cdot \varphi = 0 \quad \text{in } (\mathcal{A} \cup \mathcal{B})^c ,$$

and such that

$$\int_{\partial \mathcal{A}} (\varphi \cdot \mathbf{n}) \, d\lambda = -\gamma . \quad (3.12)$$

We further denote by $\mathcal{H}_{AB}^{\alpha, \beta}$ the set of functions $f \in L^2(d\pi)$ which have constant values α in \mathcal{A} , and β in \mathcal{B} . For such an f , we use the notations

$$\Phi_f = \frac{\sigma^2}{2\varepsilon} \pi D \nabla f - \pi f c , \quad \Psi_f = \frac{\sigma^2}{2\varepsilon} \pi D \nabla f .$$

Note in particular that

$$\mathcal{D}(\Psi_{h_{AB}}) = \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla h_{AB} \cdot D \nabla h_{AB} \, d\pi = \text{cap}(\mathcal{A}, \mathcal{B}) .$$

$-\Psi_{h_{AB}}$ is called the *harmonic flow* from \mathcal{A} to \mathcal{B} .

Lemma 3.5. *For all $f \in \mathcal{H}_{AB}^{\alpha, 0}$ and $\varphi \in \mathcal{F}_{AB}^\gamma$, we have*

$$\mathcal{D}(\Phi_f - \varphi, \Psi_{h_{AB}}) = \alpha \text{cap}(\mathcal{A}, \mathcal{B}) + \gamma . \quad (3.13)$$

Lemma 3.5 is all we need to prove the Dirichlet and Thomson principles.

Proposition 3.6 (Dirichlet principle). *We have*

$$\text{cap}(\mathcal{A}, \mathcal{B}) = \inf_{f \in \mathcal{H}_{AB}^{1, 0}} \inf_{\varphi \in \mathcal{F}_{AB}^0} \mathcal{D}(\Phi_f - \varphi) ,$$

where the infimum is reached for $f = \bar{f} := \frac{1}{2}(h_{AB} + h_{AB}^*)$ and $\varphi = \bar{\varphi} := \Phi_{\bar{f}} - \Psi_{h_{AB}}$. Furthermore, the bound

$$\text{cap}(\mathcal{A}, \mathcal{B}) \leq \inf_{f \in \mathcal{H}_{AB}^{1, 0}} \mathcal{D}(\Phi_f - \varphi) - 2 \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{AB} \, dx \, dy , \quad (3.14)$$

holds for any φ satisfying (3.12) with $\gamma = 0$.

Proposition 3.7 (Thomson principle). *We have*

$$\text{cap}(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathcal{H}_{AB}^{0, 0}} \sup_{\varphi \in \mathcal{F}_{AB}^1} \frac{1}{\mathcal{D}(\Phi_f - \varphi)} ,$$

where the supremum is reached for $f = \bar{f} := (h_{AB} - h_{AB}^*)/(2 \text{cap}(\mathcal{A}, \mathcal{B}))$ and $\varphi = \bar{\varphi} := \Phi_{\bar{f}} - \Psi_{h_{AB}}/\text{cap}(\mathcal{A}, \mathcal{B})$. Furthermore, the bound

$$\text{cap}(\mathcal{A}, \mathcal{B}) \geq \sup_{f \in \mathcal{H}_{AB}^{0, 0}} \frac{1}{\mathcal{D}(\Phi_f - \varphi)} \left(1 + \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{AB} \, dx \, dy \right)^2 . \quad (3.15)$$

holds for any φ satisfying (3.12) with $\gamma = 1$.

4 The invariant measure

This section is devoted to the proof of Theorem 2.2. Lemma 3.1 shows that the invariant density $\pi(x, y)$ can be obtained by solving the Hamilton–Jacobi equation (3.2). However, it turns out to be difficult to obtain a good control of error terms when trying to do so. We thus use another approach instead, which consists in expanding $\pi(x, y)$ on the basis of eigenfunctions of \mathcal{L}_x^\dagger , and analysing the resulting system of infinitely many coupled ODEs. In order to do so, we will need a number of bounds involving these eigenfunction, which we will derive in Section 4.1. The actual proof of Theorem 2.2 will then be given in Section 4.2.

4.1 Eigenfunctions of the static system

The aim of this section is to obtain estimates on the eigenfunction ϕ_1 , and on the inner products

$$\begin{aligned} f_{nm}(y) &= \sigma^2 \langle \partial_y \pi_m, \phi_n \rangle \\ g_{nm}(y) &= \sigma^4 \langle \partial_{yy} \pi_m, \phi_n \rangle . \end{aligned}$$

Note that taking derivatives of the orthonormality relations $\langle \pi_m, \phi_n \rangle = \delta_{nm}$ yields

$$\begin{aligned} f_{nm}(y) &= -\sigma^2 \langle \pi_m, \partial_y \phi_n \rangle \\ g_{nm}(y) &= -\sigma^4 \langle \pi_m, \partial_{yy} \phi_n \rangle - 2\sigma^4 \langle \partial_y \pi_m, \partial_y \phi_n \rangle =: -\ell_{nm}(y) - 2k_{nm}(y) . \end{aligned} \quad (4.1)$$

In particular, since ϕ_0 is constant, $f_{0m}(y) = 0$ and $g_{0m}(y) = 0$ for all $m \in \mathbb{N}$. Using standard Laplace asymptotics (cf. Appendix D), it is rather easy to obtain estimates on integrals against π_0 up to multiplicative errors of the form $1 + \mathcal{O}(\sigma^2)$. In particular, the normalisation of $\pi_0(x|y)$ satisfies

$$Z_0(y) = \sqrt{\pi}\sigma \left[\frac{1}{\omega_-(y)} e^{2h_-(y)/\sigma^2} + \frac{1}{\omega_+(y)} e^{2h_+(y)/\sigma^2} \right] [1 + \mathcal{O}(\sigma^2)] . \quad (4.2)$$

Similarly, the normalisation of the committor (2.5) satisfies

$$N(y) = \frac{\sqrt{\pi}\sigma}{\omega_0(y)} [1 + \mathcal{O}(\sigma^2)] , \quad (4.3)$$

and bounds of the same type can be obtained for f_{1i} and g_{1i} for $i \in \{0, 1\}$. We will, however, need much sharper estimates with exponentially small errors of order $\lambda_1(y)$, which requires more work.

4.1.1 Eigenfunction ϕ_1

We start by providing sharp estimates on the first eigenfunction ϕ_1 of \mathcal{L}_x and its derivatives.

Let $\tau_\pm = \tau_{x_\pm^*}(y)$ be the first-hitting times of $x_\pm^*(y)$ for the static SDE (2.2), and let $\tau = \tau_- \wedge \tau_+$. The Feynman–Kac formula allows us to write

$$\begin{aligned} \phi_1(x|y) &= \mathbb{E}_x [e^{\lambda_1(y)\tau} \phi_1(x_\tau)] \\ &= \phi_-(y) \mathbb{E}_x [e^{\lambda_1(y)\tau} \mathbf{1}_{\{\tau_- < \tau_+\}}] + \phi_+(y) \mathbb{E}_x [e^{\lambda_1(y)\tau} \mathbf{1}_{\{\tau_+ < \tau_-\}}] \\ &= \phi_-(y) [h_0(x|y) + h_1(x|y)] + \phi_+(y) [1 - h_0(x|y) + \bar{h}_1(x|y)] , \end{aligned} \quad (4.4)$$

where we use the shorthands $\phi_\pm(y) = \phi_1(x_\pm^*(y), y)$, while $h_0(x|y) = \mathbb{P}_x\{\tau_- < \tau_+\}$ is the committor and

$$h_1(x|y) = \mathbb{E}_x [(e^{\lambda_1(y)\tau} - 1) \mathbf{1}_{\{\tau_- < \tau_+\}}] , \quad \bar{h}_1(x|y) = \mathbb{E}_x [(e^{\lambda_1(y)\tau} - 1) \mathbf{1}_{\{\tau_+ < \tau_-\}}] .$$

Recall that $h_0(x|y)$ is given by (2.5) for $x \in (x_-^*(y), x_+^*(y))$. Furthermore, h_0 is constant equal to 1 for $x < x_-^*(y)$, and constant equal to 0 for $x > x_+^*(y)$.

It will be convenient to *define* $\bar{\Delta}(y)$ by the relations

$$\langle \pi_0, h_0 \rangle = \frac{e^{-\bar{\Delta}(y)/\sigma^2}}{e^{-\bar{\Delta}(y)/\sigma^2} + e^{\bar{\Delta}(y)/\sigma^2}} , \quad \langle \pi_0, 1 - h_0 \rangle = \frac{e^{\bar{\Delta}(y)/\sigma^2}}{e^{-\bar{\Delta}(y)/\sigma^2} + e^{\bar{\Delta}(y)/\sigma^2}} . \quad (4.5)$$

Indeed, standard Laplace asymptotics (see Lemma D.1) show that this definition is compatible to leading order with (2.7). We further introduce

$$A(y) = \tanh\left(\frac{\bar{\Delta}(y)}{\sigma^2}\right) , \quad B(y) = \frac{1}{\cosh(\bar{\Delta}(y)/\sigma^2)} .$$

Note carefully that $B(y) \in (0, 1]$ may be exponentially small, and that we have the relations

$$\begin{aligned} A(y)^2 + B(y)^2 &= 1 , & \sigma^2 A'(y) &= \bar{\Delta}'(y) B(y)^2 , \\ \sigma^2 B'(y) &= -\bar{\Delta}'(y) A(y) B(y) . \end{aligned}$$

Combining (4.3) and (4.2) with the expression (2.4) of $\lambda_1(y)$, we obtain the very useful relation

$$Z_0(y) N(y) \lambda_1(y) = \frac{2\sigma^2}{B(y)^2} [1 + \mathcal{O}(\sigma^2)] . \quad (4.6)$$

Finally, to lighten notations, we set

$$\ell(\sigma) = \log(\sigma^{-1}) ,$$

and we will sometimes omit the argument y .

The following results establish some properties of h_0 , h_1 and ϕ_1 . Their proofs are postponed to Appendix C.1.

Proposition 4.1 (Properties of h_0). *We have*

$$|\partial_y h_0(x|y)| \lesssim \frac{1}{\sigma^2} h_0(x|y) (1 - h_0(x|y)) , \quad (4.7a)$$

$$|\partial_{yy} h_0(x|y)| \lesssim \frac{1}{\sigma^4} h_0(x|y) (1 - h_0(x|y)) . \quad (4.7b)$$

Furthermore, the inner product $\eta(y) = \langle \pi_0, h_0(1 - h_0) \rangle$ satisfies

$$0 \leq \eta(y) \lesssim \lambda_1(y) \ell(\sigma) B(y)^2 . \quad (4.8)$$

Proposition 4.2 (Bounds on h_1). *The remainder $h_1(x|h)$ satisfies the bounds*

$$\begin{aligned} |h_1(x|y)| &\lesssim \lambda_1(y) \ell(\sigma) h_0(x|y) \\ |\partial_y h_1(x|y)| &\lesssim \frac{1}{\sigma^2} \lambda_1(y) \ell(\sigma)^2 h_0(x|y) \\ |\partial_{yy} h_1(x|y)| &\lesssim \frac{1}{\sigma^4} \lambda_1(y) \ell(\sigma)^3 h_0(x|y) . \end{aligned}$$

Similar bounds hold for $\bar{h}_1(x|y)$, with h_0 replaced by $1 - h_0$.

Proposition 4.3 (First eigenfunction). *The coefficients $\phi_{\pm}(y)$ of $\phi_1(x|y)$ satisfy*

$$\phi_{\pm}(y) = \mp e^{\mp \bar{\Delta}(y)/\sigma^2} [1 + \mathcal{O}(\lambda_1(y)\ell(\sigma))] \quad (4.9a)$$

$$\phi'_{\pm}(y) = \mp \frac{1}{\sigma^2} \phi_{\pm}(y) [\bar{\Delta}'(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^2)] \quad (4.9b)$$

$$\phi''_{\pm}(y) = \mp \frac{1}{\sigma^4} \phi_{\pm}(y) [\bar{\Delta}'(y)^2 \mp \sigma^2 \bar{\Delta}''(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^3)] . \quad (4.9c)$$

Combining the last two propositions with (4.4), we obtain the following representations of ϕ_1 and its derivatives:

$$\phi_1 = \phi_- h_0 [1 + \mathcal{O}(\lambda_1 \ell)] + \phi_+ (1 - h_0) [1 + \mathcal{O}(\lambda_1 \ell)] , \quad (4.10a)$$

$$\partial_y \phi_1 = \frac{\phi_-}{\sigma^2} h_0 [\bar{\Delta}' + \mathcal{O}(\lambda_1 \ell^2)] - \frac{\phi_+}{\sigma^2} (1 - h_0) [\bar{\Delta}' + \mathcal{O}(\lambda_1 \ell^2)] + (\phi_- - \phi_+) \partial_y h_0 , \quad (4.10b)$$

$$\begin{aligned} \partial_{yy} \phi_1 &= \frac{\phi_-}{\sigma^4} h_0 [(\bar{\Delta}')^2 + \sigma^2 \bar{\Delta}'' + \mathcal{O}(\lambda_1 \ell^3)] - \frac{\phi_+}{\sigma^4} (1 - h_0) [(\bar{\Delta}')^2 - \sigma^2 \bar{\Delta}'' + \mathcal{O}(\lambda_1 \ell^3)] \\ &\quad + 2(\phi'_- - \phi'_+) \partial_y h_0 + (\phi_- - \phi_+) \partial_{yy} h_0 . \end{aligned} \quad (4.10c)$$

It is then straightforward to obtain the following expressions for inner products involving derivatives of the first two eigenfunctions.

Proposition 4.4 (Matrix elements involving ϕ_1). *We have*

$$\begin{aligned} f_{10}(y) &= -B(y) [\bar{\Delta}'(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^2)] , \\ f_{11}(y) &= -A(y) \bar{\Delta}'(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^2) , \\ g_{10}(y) &= B(y) [2A(y) \bar{\Delta}'(y)^2 - \sigma^2 \bar{\Delta}''(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^3)] , \\ g_{11}(y) &= (2A(y)^2 - 1) \bar{\Delta}'(y)^2 - \sigma^2 A(y) \bar{\Delta}''(y) + \mathcal{O}(\lambda_1(y)\ell(\sigma)^3) . \end{aligned}$$

A consequence of these estimates is that we have, for instance,

$$\sigma^2 \partial_y \phi_1 = (A(y) \phi_1 + B(y)) \bar{\Delta}'(y) + R_1(x) , \quad (4.11)$$

where R_1 is a remainder, dominated by the term in $\partial_y h_0$ in (4.10b). One checks that it satisfies

$$\langle \pi_0, R_1 \rangle = \mathcal{O}(\lambda_1 \ell^2 B) , \quad \langle \pi_0, R_1^2 \rangle = \mathcal{O}(\lambda_1 \ell^2) . \quad (4.12)$$

In other words, $\partial_y \phi_1$ lies almost in the space spanned by ϕ_0 and ϕ_1 .

Remark 4.5. A useful observation is that the expression (4.10a) for ϕ_1 implies

$$\begin{aligned} \langle \pi_0, |\phi_1| \rangle &= [\phi_-(y) \langle \pi_0, h_0 \rangle + |\phi_+(y)| \langle \pi_0, 1 - h_0 \rangle] [1 + \mathcal{O}(\lambda_1 \ell)] \\ &\leq B(y) [1 + \mathcal{O}(\lambda_1 \ell)] , \end{aligned}$$

where we have used the definition (4.5) of $\bar{\Delta}$. This is often better than the bound $\langle \pi_0, |\phi_1| \rangle \leq 1$ provided by the Cauchy–Schwarz inequality. \diamond

4.1.2 Bounds involving other eigenfunctions

When analysing the system of ODEs giving the invariant density, we will also need a number of bounds involving other eigenfunctions than ϕ_0 and ϕ_1 . All proofs of these bounds are postponed to Appendix C.2. We start with some simple ℓ^2 estimates.

Proposition 4.6. *There exists a constant M_0 , uniform in σ and $n \geq 1$, such that*

$$\sum_{n \geq 0} f_{ni}(y)^2 \leq M_0, \quad \sum_{n \geq 0} g_{ni}(y)^2 \leq M_0 \quad \forall i \in \{0, 1\}, \quad (4.13)$$

$$\sum_{m \geq 0} f_{nm}(y)^2 \leq M_0, \quad \sum_{m \geq 0} g_{nm}(y)^2 \leq M_0 \quad \forall n \geq 1. \quad (4.14)$$

The following result shows that h_0 is almost orthogonal to the span of π_0 and π_1 .

Proposition 4.7. *We have*

$$\sum_{n \geq 2} \langle \pi_n, h_0 \rangle^2 \lesssim \lambda_1(y) \ell(\sigma) B(y)^2.$$

We can also get exponentially small bounds for a number of sums involving f_{nm} and g_{nm} .

Proposition 4.8. *The following sums are all of order $\lambda_1(y) \ell(\sigma)^a$ for some $a \leq 3$, where $i \in \{0, 1\}$:*

$$\begin{aligned} & \sum_{m \geq 2} f_{1m}(y)^2, \quad \sum_{m \geq 2} f_{1m}(y) f_{mi}(y), \quad \sum_{m \geq 2} g_{1m}(y)^2, \\ & \sum_{m \geq 2} f_{1m}(y) g_{mi}(y), \quad \sum_{m \geq 2} g_{1m}(y) f_{mi}(y), \quad \sum_{m \geq 2} g_{1m}(y) g_{mi}(y). \end{aligned}$$

Finally, the following result provides exponentially small bounds on similar sums, but with all terms divided by λ_n . These bounds are *not* consequences of the previous ones, since the terms of these sums do not have the same sign, so that their smallness is due to cancellations between terms.

Proposition 4.9. *The following sums are all of order $\lambda_1(y) \ell(\sigma)^a$ for some $a \leq 3$, where $i \in \{0, 1\}$:*

$$\begin{aligned} & \sum_{m \geq 2} \frac{1}{\lambda_m(y)} f_{1m}(y) f_{mi}(y), \quad \sum_{m \geq 2} \frac{1}{\lambda_m(y)} f_{1m}(y) g_{mi}(y), \\ & \sum_{m \geq 2} \frac{1}{\lambda_m(y)} g_{1m}(y) f_{mi}(y), \quad \sum_{m \geq 2} \frac{1}{\lambda_m(y)} g_{1m}(y) g_{mi}(y). \end{aligned}$$

4.2 Proof of Theorem 2.2

Since for each y , the eigenfunctions $\pi_n(\cdot|y)$ form a complete orthonormal basis of $L^2(\mathbb{R}, \pi_0 dx)$, we can decompose the density π of the invariant measure as

$$\pi(x, y) = \sum_{n \geq 0} \alpha_n(y) \pi_n(x|y) = \pi_0(x|y) \sum_{n \geq 0} \alpha_n(y) \phi_n(x|y). \quad (4.15)$$

We write the adjoint generator as $\mathcal{L}^\dagger = \frac{1}{\varepsilon} \mathcal{L}_x^\dagger + \mathcal{L}_y^\dagger$, where \mathcal{L}_x^\dagger has been defined in (2.3), and

$$\mathcal{L}_y^\dagger \mu = -\partial_y \mu + \frac{\varrho^2 \sigma^2}{2} \partial_{yy} \mu.$$

The stationarity condition $\mathcal{L}^\dagger \pi = 0$ becomes

$$\sum_{n \geq 1} \lambda_n(y) \alpha_n(y) \pi_n(x|y) = \varepsilon \sum_{n \geq 0} \mathcal{L}_y^\dagger (\alpha_n(y) \pi_n(x|y)), \quad (4.16)$$

where the right-hand side can be evaluated using

$$\mathcal{L}_y^\dagger(\alpha_n \pi_n) = \left(-\alpha'_n + \frac{\varrho^2 \sigma^2}{2} \alpha''_n\right) \pi_n + \left(-\alpha_n + \varrho^2 \sigma^2 \alpha'_n\right) \partial_y \pi_n + \frac{\varrho^2 \sigma^2}{2} \alpha_n \partial_{yy} \pi_n .$$

We now project (4.16) on each eigenfunction ϕ_n . Since $\langle \partial_y \pi_n, \phi_0 \rangle = \partial_y \langle \pi_n, \phi_0 \rangle = 0$, and similarly for the second derivative, the projection on ϕ_0 yields

$$-\alpha'_0(y) + \frac{\varrho^2 \sigma^2}{2} \alpha''_0(y) = 0 .$$

Using periodicity in y and the fact that π is normalised, one easily gets

$$\alpha_0(y) = 1 .$$

The projections on the remaining ϕ_n result in the following statement, whose proof is a simple computation.

Lemma 4.10. *The stationary distribution π is given by (4.15) with $\alpha_0(y) = 1$ and $\{\alpha_n(y)\}_{n \in \mathbb{N}}$ given by the first component of the unique periodic solution of*

$$\begin{aligned} \varrho^2 \sigma^2 \alpha'_n &= 2\alpha_n - 2\beta_n \\ \sigma^2 \beta'_n &= -\frac{\sigma^2}{\varepsilon} \lambda_n(y) \alpha_n + \sum_{m \geq 1} [c_{nm}(y) \alpha_m + d_{nm}(y) \beta_m] + c_{n0}(y) , \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} c_{n0}(y) &= -f_{n0}(y) + \frac{\varrho^2}{2} g_{n0}(y) , \\ c_{nm}(y) &= f_{nm}(y) + \frac{\varrho^2}{2} g_{nm}(y) , \\ d_{nm}(y) &= -2f_{nm}(y) . \end{aligned} \quad m \geq 1 ,$$

4.2.1 The first-order case

It is instructive to consider first the case $\varrho^2 = 0$. Then $\beta_n(y) = \alpha_n(y)$, and $\alpha_n(y)$ satisfies the linear inhomogeneous system

$$\varepsilon \alpha'_n = -\lambda_n(y) \alpha_n - \frac{\varepsilon}{\sigma^2} f_{n0}(y) - \frac{\varepsilon}{\sigma^2} \sum_{m \geq 1} f_{nm}(y) \alpha_m . \quad (4.18)$$

Note that for $n \geq 2$, $\alpha_n(y)$ is a fast variable, which, by the general theory of singularly perturbed ordinary differential equations, is expected to remain ε -close to a value $\alpha_n^*(y)$ such that the right-hand side of the system vanishes.

The case $n = 1$, however, is special since $\lambda_1(y)$ is exponentially small. This makes the system hard to study in the form (4.18), because $\alpha_1(y)$ can become exponentially large. The solution is to observe that, disregarding for a moment the terms α_n with $n \geq 2$ we have by Lemma D.1

$$\begin{aligned} p_-(y) &:= \mathbb{P}\{x(y) < x_0^*(y)\} \simeq \int_{-\infty}^{x_0^*(y)} \pi_0(x|y) [1 + \alpha_1(y) \phi_1(x|y)] dx \\ &= \frac{1}{2} B(y) (e^{-\bar{\Delta}/\sigma^2} + \alpha_1(y)) [1 + \mathcal{O}(\sigma^2)] . \end{aligned} \quad (4.19)$$

This suggests setting

$$\alpha_1(y) = \frac{A(y) - \delta_1(y)}{B(y)}, \quad (4.20)$$

so that $p_-(y) \simeq \frac{1}{2}(1 - \delta_1(y))$, and therefore $\delta_1(y)$ remains of order 1. Then a computation shows that

$$\varepsilon \delta_1' = \left[-\lambda_1(y) + \frac{\varepsilon}{\sigma^2} p_1(y) \right] (\delta_1 - A(y)) + \frac{\varepsilon}{\sigma^2} w_1(y) + \frac{\varepsilon}{\sigma^2} B(y) \sum_{m \geq 2} f_{1m}(y) \alpha_m, \quad (4.21)$$

where

$$\begin{aligned} p_1(y) &= -f_{11}(y) - \bar{\Delta}'(y) A(y) = \mathcal{O}(\lambda_1(y) \ell(\sigma)^2), \\ w_1(y) &= \bar{\Delta}'(y) B(y)^2 + B(y) f_{10}(y) = \mathcal{O}(\lambda_1(y) \ell(\sigma)^2 B(y)^2). \end{aligned} \quad (4.22)$$

The unique periodic solution of this equation is given by

$$\delta_1(y) = \frac{1}{\varepsilon(1 - e^{-\bar{\Lambda}_1(1,0)/\varepsilon})} \int_y^{y+1} e^{-\bar{\Lambda}_1(y+1, \bar{y})/\varepsilon} \left[\bar{\lambda}_1(\bar{y}) A(\bar{y}) + \frac{\varepsilon}{\sigma^2} w_1(\bar{y}) + \frac{\varepsilon}{\sigma^2} \tilde{w}_1(\bar{y}) \right] d\bar{y},$$

where we have set $\bar{\lambda}_1(y) = \lambda_1(y) + \frac{\varepsilon}{\sigma^2} p_1(y)$ and

$$\tilde{w}_1(y) = B(y) \sum_{m \geq 2} f_{1m}(y) \alpha_m(y), \quad (4.23)$$

$$\bar{\Lambda}_1(y_2, y_1) = \int_{y_1}^{y_2} \bar{\lambda}_1(y) dy.$$

In particular, for $\varepsilon \gg \bar{\Lambda}_1(1, 0) = \langle \bar{\lambda}_1 \rangle$, $\delta_1(y)$ is almost constant, that is, we have

$$\begin{aligned} \delta_1(y) &= \bar{\delta}_1 \left[1 + \mathcal{O}\left(\frac{\langle \bar{\lambda}_1 \rangle}{\varepsilon}\right) \right], \\ \bar{\delta}_1 &= \frac{1}{\langle \bar{\lambda}_1 \rangle} \int_0^1 \left[\bar{\lambda}_1(y) A(y) + \frac{\varepsilon}{\sigma^2} w_1(y) + \frac{\varepsilon}{\sigma^2} \tilde{w}_1(y) \right] dy. \end{aligned}$$

To analyse the dynamics of the remaining coefficients $\alpha_n(y)$ with $n \geq 2$, we introduce a vector $\alpha_\perp^*(y)$ with components

$$\alpha_n^*(y) = -\frac{\varepsilon}{\sigma^2} \frac{1}{\lambda_n(y)} \left[f_{n0}(y) + \frac{A(y) - \delta_1(y)}{B(y)} f_{n1}(y) \right], \quad (4.24)$$

and examine in particular the behaviour of $\alpha_\perp^1(y) = \alpha_\perp(y) - \alpha_\perp^*(y)$.

Proposition 4.11. *The unique periodic solution of the system (4.18) satisfies*

$$\alpha_n(y) = \alpha_n^*(y) + \alpha_n^1(y), \quad (4.25)$$

where

$$\sup_{n \geq 2} \lambda_n(y) |\alpha_n^*(y)| \lesssim \frac{\varepsilon}{\sigma^2 B(y)} \quad \forall y \in [0, 1], \quad (4.26)$$

$$\sup_{n \geq 2} \lambda_n(y) |\alpha_n^1(y)| \lesssim \frac{\varepsilon^2}{\sigma^4 B(y)} \quad \forall y \in [0, 1]. \quad (4.27)$$

PROOF: The bound (4.26) is a direct consequence of the bound (4.13) on the sum of f_{ni}^2 . In order to establish (4.27), we first note that the α_n^1 satisfy the equation

$$\varepsilon(\alpha_n^1)' = -\lambda_n(y)\alpha_n^1 - \frac{\varepsilon}{\sigma^2} \sum_{m \geq 2} f_{nm}(y)(\alpha_m^*(y) + \alpha_m^1) - \varepsilon\alpha_n^*(y)' . \quad (4.28)$$

We will show that the set

$$H = \left\{ (\alpha_\perp^1, y) : |\alpha_m^1| \leq \frac{\varepsilon^2 C_0}{\sigma^4 B(y) \lambda_m(y)} \quad \forall m \geq 2 \right\}$$

is invariant under the flow of (4.28) for sufficiently large C_0 . Assume α_\perp^1 belongs to ∂H , and pick n such that $\alpha_n^1 = \pm(\varepsilon^2 C_0)/(\sigma^4 \lambda_n B)$. The Cauchy–Schwarz inequality yields

$$\left(\sum_{m \geq 2} f_{nm}(\alpha_m^* + \alpha_m^1) \right)^2 \leq \sum_{m \geq 2} f_{nm}^2 \sum_{m \geq 2} (\alpha_m^* + \alpha_m^1)^2 \leq \frac{\varepsilon^2 C_1}{B^2 \sigma^4} \left(1 + \frac{\varepsilon^2 C_0^2}{\sigma^4} \right) ,$$

for a constant C_1 , where we have used (4.14) to bound the first sum, and (4.26) and the definition of H to bound the second one. The derivative of $\alpha_n^*(y)$ can be bounded using the relations

$$\sigma^2 f_{ni}'(y) = g_{ni}(y) + k_{ni}(y) , \quad \sigma^2 \left(\frac{A - \delta_1}{B} \right)' = \frac{\bar{\Delta}'(1 - A\delta_1) - \sigma^2 \delta_1'}{B} = \mathcal{O}(B^{-1})$$

and the Hellmann–Feynman theorem (cf. (C.16)), which shows that $\sigma^2 \lambda_n'(y)$ has order 1. The result is that

$$|(\alpha_n^*)'(y)| \leq \frac{\varepsilon C_2}{\sigma^4 B(y) \lambda_n(y)}$$

for a constant C_2 . Plugging these bounds into (4.28) shows that for C_0 large enough, the sign of $\varepsilon(\alpha_n^1)'$ is the opposite of the sign of α_n^1 . This shows the invariance of H , and therefore the bound (4.27). \square

Corollary 4.12. *The error term $\tilde{w}_1(y)$ introduced in (4.23) satisfies*

$$|\tilde{w}_1(y)| \lesssim \frac{\varepsilon}{\sigma^2} \lambda_1(y) \ell(\sigma)^3 + \frac{\varepsilon^2}{\sigma^4} \sqrt{\lambda_1(y) \ell(\sigma)^3} \quad (4.29)$$

uniformly in $y \in [0, 1]$.

PROOF: This follows directly from the decomposition (4.25). Indeed, the contribution of the α_m^* can be bounded via Proposition 4.9, and yields the first term on the right-hand side. The contribution of the α_m^1 can be bounded via the Cauchy–Schwarz inequality, using Proposition 4.8 and (4.27). \square

One consequence of this result that will be useful when estimating capacities is the following. Integrating the ODE (4.21) satisfied by $\delta_1(y)$ over one period, and using (4.22) and (4.29), we obtain

$$|\langle \lambda_1[\delta_1 - A] \rangle| \lesssim \frac{\varepsilon}{\sigma^2} \langle \lambda_1 \rangle \ell^2 + \frac{\varepsilon^2}{\sigma^4} \langle \lambda_1 \rangle \ell^3 + \frac{\varepsilon^3}{\sigma^6} \langle \sqrt{\lambda_1} \rangle \ell^{3/2} . \quad (4.30)$$

Proposition 4.11 also allows us to control the remainder Φ_\perp of the invariant measure. For $\sharp \in \{ , *, 1 \}$, let us write

$$\Phi_\perp^\sharp(x, y) = \sum_{n \geq 2} \alpha_n^\sharp(y) \phi_n(x|y) .$$

Corollary 4.13. *Let $\mathcal{D} = (x_-^*(y), x_+^*(y))$. The L^2 -bounds*

$$\langle \pi_0, (\Phi_\perp^*)^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2 B(y)}, \quad \langle \pi_0, (\Phi_\perp^1)^2 \rangle^{1/2} \lesssim \frac{\varepsilon^2}{\sigma^4 B(y)}, \quad (4.31a)$$

$$\langle \pi_0, (\partial_x \Phi_\perp)^2 \mathbf{1}_{\mathcal{D}} \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^4 B(y)}, \quad \langle \pi_0, (\partial_x \Phi_\perp^1)^2 \mathbf{1}_{\mathcal{D}} \rangle^{1/2} \lesssim \frac{\varepsilon^2}{\sigma^6 B(y)} \quad (4.31b)$$

hold for all $y \in [0, 1]$. Furthermore, the bounds

$$|\Phi_\perp^*(x, y)| \lesssim \frac{\varepsilon e^{V_0(x, y)/\sigma^2}}{\sigma^{3/2} B(y)^2 \sqrt{\lambda_1(y)}}, \quad |\Phi_\perp^1(x, y)| \lesssim \frac{\varepsilon^2 e^{V_0(x, y)/\sigma^2}}{\sigma^{7/2} B(y)^2 \sqrt{\lambda_1(y)}} \quad (4.32)$$

hold for all $x \in \mathbb{R}$ and all $y \in [0, 1]$.

PROOF: The first two L^2 -bounds follow directly from the fact that

$$\langle \pi_0, (\Phi_\perp^*)^2 \rangle = \|\alpha_\perp^*\|_{\ell^2}^2 = \sum_{n \geq 2} (\alpha_n^*)^2,$$

while the L^2 -bound on the derivative is a consequence of Lemma C.5. As for the L^∞ -bounds (4.32), they follow from the fact that \mathcal{L}_x is conjugated to a Schrödinger operator (cf. (C.12) in Appendix C.2), whose eigenfunctions ψ_n are bounded by a constant of order 1, so that

$$|\Phi_\perp^*(x, y)| = \frac{1}{\sqrt{\pi_0(x|y)}} \left| \sum_{n \geq 2} \alpha_n^*(y) \psi_n(x|y) \right| \lesssim \frac{1}{\sqrt{\pi_0(x|y)}} \|\alpha_\perp^*\|_{\ell^1}.$$

The ℓ^1 -norm of α_\perp^* can be bounded using the previous proposition. \square

Part of the importance of Φ_\perp^* lies in the following estimate, which shows that functions bounded by $h_0(1 - h_0)$ are almost orthogonal to Φ_\perp^* , and thus allows to improve a certain number of error bounds when estimating the capacity. Its proof is close in spirit to the proof of Proposition 4.6, so we also give it in Appendix C.2.

Proposition 4.14. *Let f be supported on $\mathcal{D} = (x_-^*(y), x_+^*(y))$, and satisfy either one of the bounds*

$$|f(x)| \leq M h_0(x|y) (1 - h_0(x|y)) \quad \text{or} \quad |f(x)| \leq M e^{2V_0(x, y)/\sigma^2}$$

for all $x \in \mathcal{D}$, and for some constant $M > 0$. Then

$$|\langle \pi_0, \Phi_\perp^* f \rangle| \lesssim \frac{\varepsilon}{\sigma^2} \lambda_1(y) \ell(\sigma) M.$$

4.2.2 The second-order case

Consider now the case $\varrho^2 > 0$. We again carry out the change of variables (4.20), and in addition set

$$\beta_1(y) = \frac{A(y) - \delta_1(y) - \hat{\gamma}_1(y)}{B(y)}, \quad \hat{\gamma}_1(y) = \frac{\sigma}{\sqrt{\varepsilon}} \gamma_1(y) + \frac{\varrho^2}{2} \bar{\Delta}'(y) [1 - A(y) \delta_1(y)].$$

The resulting system for (δ_1, γ_1) is given by

$$\sqrt{\varepsilon} \sigma \delta_1' = -\frac{2}{\varrho^2} \gamma_1 \quad (4.33)$$

$$\sqrt{\varepsilon} \sigma \gamma_1' = \left[-\lambda_1(y) + \frac{\varepsilon}{\sigma^2} p_1(y) \right] (\delta_1 - A(y)) + \frac{\varrho^2}{2} \frac{\sqrt{\varepsilon}}{\sigma} q_1(y) \gamma_1 + \frac{\varepsilon}{\sigma^2} [w_1(y) + \tilde{w}_1(y)],$$

where

$$\begin{aligned}
p_1(y) &= -f_{11}(y) + \frac{\varrho^2}{2}g_{11}(y) - \bar{\Delta}'(y)A(y) \\
&\quad + \frac{\varrho^2}{2}\left[\bar{\Delta}'(y)^2 + 2\bar{\Delta}'(y)A(y)f_{11}(y) + \sigma^2\bar{\Delta}''(y)A(y)\right], \\
q_1(y) &= 1 - \varrho^2\left[f_{11}(y) + \bar{\Delta}'(y)A(y)\right], \\
w_1(y) &= B(y)^2\left[\bar{\Delta}'(y)(1 - \varrho^2f_{11}(y)) - \frac{\varrho^2\sigma^2}{2}\bar{\Delta}''(y)\right] + B(y)\left[f_{10}(y) - \frac{\varrho^2}{2}g_{10}(y)\right],
\end{aligned}$$

and the contribution of the other variables is contained in the term

$$\tilde{w}_1(y) = B(y) \sum_{m \geq 2} \left[f_{1m}(y)(2\beta_m(y) - \alpha_m(y)) - \frac{\varrho^2}{2}g_{1m}(y)\alpha_m(y) \right].$$

It follows from Proposition 4.4 that

$$\begin{aligned}
p_1(y) &= \mathcal{O}(\lambda_1(y)\ell(\sigma)^3), \\
q_1(y) &= 1 + \mathcal{O}(\lambda_1(y)\ell(\sigma)^2), \\
w_1(y) &= \mathcal{O}(\lambda_1(y)\ell(\sigma)^3B(y)^2).
\end{aligned}$$

It is straightforward to check that the system (4.33) is equivalent to the second-order equation

$$\frac{\varrho^2}{2}\varepsilon\sigma^2\delta_1'' - \varepsilon q_1(y)\delta_1' + \left[-\lambda_1(y) + \frac{\varepsilon}{\sigma^2}p_1\right](\delta_1 - A(y)) + \frac{\varepsilon}{\sigma^2}[w_1(y) + \tilde{w}_1(y)] = 0.$$

By a standard argument of singular perturbation theory (see for instance [9, Example 2.1.3]), the solutions of this equation are close, up to multiplicative errors $1 + \mathcal{O}(\varrho^4\sigma^4)$, to those of the first-order equation (4.21).

In order to analyse the behaviour of the remaining coefficients $\alpha_n(y)$ and $\beta_n(y)$ with $n \geq 2$, we introduce, analogously to (4.24),

$$\alpha_n^*(y) = \frac{\varepsilon}{\sigma^2} \frac{1}{\lambda_n(y)} \left[c_{n0}(y) + c_{n1}(y)\alpha_1(y) + d_{n1}(y)\beta_1(y) \right].$$

Proposition 4.15. *The unique periodic solution of the system (4.17) satisfies*

$$\begin{aligned}
\alpha_n(y) &= \alpha_n^*(y) + \alpha_n^1(y), \\
\beta_n(y) &= \alpha_n^*(y) + \beta_n^1(y),
\end{aligned}$$

where

$$\sup_{n \geq 2} \lambda_n(y) |\alpha_n^*(y)| \lesssim \frac{\varepsilon}{\sigma^2 B(y)} \quad \forall y \in [0, 1], \quad (4.34a)$$

$$\sup_{n \geq 2} \lambda_n(y) |\alpha_n^1(y)| \lesssim \frac{\varepsilon^2}{\sigma^4 B(y)} \quad \forall y \in [0, 1], \quad (4.34b)$$

$$\sup_{n \geq 2} \lambda_n(y) |\beta_n^1(y)| \lesssim \frac{\varepsilon}{\sigma^2 B(y)} \quad \forall y \in [0, 1]. \quad (4.34c)$$

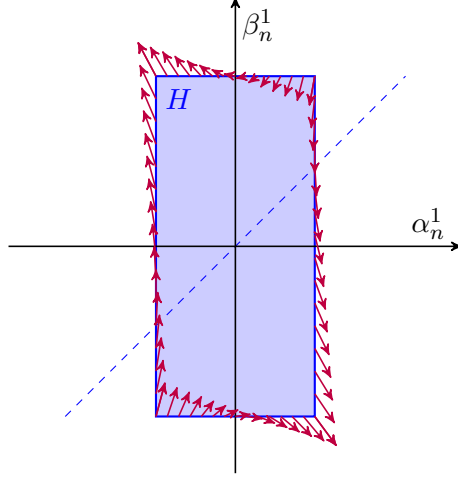


FIGURE 5. Vector field (4.35) on the boundary of the set H , shown for a fixed component n and fixed y . The broken line shows the approximate location of the points where $(\alpha_n^1)'$ changes sign.

PROOF: The bound (4.34a) is again a direct consequence of (4.13), noting that

$$\alpha_n^*(y) = -\frac{\varepsilon}{\sigma^2} \frac{1}{\lambda_n} \left[f_{n0} - \frac{\varrho^2}{2} g_{n0} + \frac{A - \delta_1}{B} \left(f_{n1} - \frac{\varrho^2}{2} g_{n1} \right) - \frac{2}{B} f_{n1} \hat{\gamma}_1 \right].$$

To show (4.34b) and (4.34c), we will use the fact that the pairs (α_n^1, β_n^1) satisfy the system

$$\begin{aligned} \varepsilon(\alpha_n^1)' &= \frac{2\varepsilon}{\varrho^2 \sigma^2} [\alpha_n^1 - \beta_n^1 - \sigma^2(\alpha_n^*)'] \\ \varepsilon(\beta_n^1)' &= -\lambda_n \alpha_n^1 + \frac{\varepsilon}{\sigma^2} \sum_{m \geq 2} [c_{nm}(\alpha_m^* + \alpha_m^1) + d_{nm}(\alpha_m^* + \beta_m^1)] - \varepsilon(\alpha_n^*)'. \end{aligned} \quad (4.35)$$

We will argue that the unique periodic solution of this equation has to be entirely contained in the set

$$H = \left\{ (\alpha_\perp^1, \beta_\perp^1, y) : |\alpha_m^1| \leq \frac{\varepsilon^2 C_0}{\sigma^4 B(y) \lambda_m(y)}, |\beta_m^1| \leq \frac{\varepsilon C_0}{\sigma^2 B(y) \lambda_m(y)} \quad \forall m \geq 2 \right\},$$

provided C_0 is a sufficiently large constant of order 1. Indeed, similar estimates as in the proof of Proposition 4.11 show that whenever $(\alpha_n^1, \beta_n^1, y)$ lies in H , one has

$$\begin{aligned} \varepsilon(\alpha_n^1)' &= \frac{2\varepsilon}{\varrho^2 \sigma^2} \left[\alpha_n^1 - \beta_n^1 + \mathcal{O}\left(\frac{\varepsilon}{\sigma^2 B(y) \lambda_n(y)}\right) \right] \\ \varepsilon(\beta_n^1)' &= -\lambda_n \alpha_n^1 + \mathcal{O}\left(\frac{\varepsilon^2(1 + C_0)}{\sigma^4 B(y)}\right). \end{aligned}$$

For sufficiently large C_0 , this vector field has the following properties on the boundary ∂H (Figure 5):

- on the upper boundary of H , it points to the left, and changes from pointing outward H to pointing inward as α_n^1 increases;
- on the right boundary of H , it points downward, and changes from pointing outward H to pointing inward as β_n^1 increases;
- the situation is reversed on the lower and left boundaries of H .

Combined with the fact that the equation is linear, these properties imply that a solution leaving H cannot enter it again. Therefore, the unique periodic solution has to lie within H . \square

As α_\perp^* and α_\perp^1 satisfy the same bounds as for $\varrho^2 = 0$, it is straightforward to check that Corollary 4.12, Corollary 4.13 and Proposition 4.14 still hold in the present case.

5 Adjoint process and committors

Recall from Lemma 3.1 that the invariant density can be written as $\pi(x, y) = Z^{-1} e^{-2V(x, y)/\sigma^2}$. Since we also have

$$\pi(x, y) = \pi_0(x|y)\Phi(x, y), \quad \Phi(x, y) = 1 + \sum_{n \geq 1} \alpha_n(y)\phi_n(x|y), \quad (5.1)$$

solving for V gives the expression

$$V(x, y) = V_0(x, y) - \frac{\sigma^2}{2} \log \Phi(x, y) + \frac{\sigma^2}{2} \log \frac{Z_0(y)}{Z}. \quad (5.2)$$

One can get a better idea of the difference between V and V_0 by writing

$$\Phi(x, y) = \Phi_0(x, y) + \Phi_\perp(x, y),$$

where

$$\begin{aligned} \Phi_0(x, y) &= 1 + \alpha_1(y)\phi_1(x|y) \\ &= 1 + \frac{A(y) - \delta_1(y)}{B(y)} \left[\phi_+(y) + (\phi_-(y) - \phi_+(y))h_0(x|y) \right] [1 + \mathcal{O}(\lambda_1 \ell)]. \end{aligned}$$

Note that by (4.9a) we have

$$B(y)^2 \Phi_0(x, y) = [1 - A(y)\delta_1(y) + (2h_0(x|y) - 1)(A(y) - \delta_1(y))] [1 + \mathcal{O}(\lambda_1 \ell)]. \quad (5.3)$$

The x -component of the vector field c , defined in (3.4), is thus given by

$$c_x = \frac{1}{\varepsilon} (\partial_x V - \partial_x V_0) = -\frac{\sigma^2}{2\varepsilon} \frac{\partial_x \Phi(x, y)}{\Phi(x, y)}, \quad (5.4)$$

and the adjoint SDE has the form (3.7) with

$$\begin{aligned} b^*(x, y) &= -\partial_x V_0(x, y) + \sigma^2 \frac{\partial_x \Phi(x, y)}{\Phi(x, y)} \\ &= -\partial_x V_0^*(x, y), \end{aligned}$$

where we have defined the adjoint potential by

$$V_0^*(x, y) = V_0(x, y) - \sigma^2 \log \Phi(x, y). \quad (5.5)$$

Using the expression (4.10a) for ϕ_1 , and approximating $\Phi(x, y)$ by $\Phi_0(x, y)$, one can deduce from (5.2) and (5.5) that

$$\begin{aligned} V(x_0^*(y), y) - V(x_\pm^*(y), y) &\simeq \frac{h_-(y) + h_+(y)}{2}, \\ V_0^*(x_0^*(y), y) - V_0^*(x_\pm^*(y), y) &\simeq h_\mp(y). \end{aligned}$$

In other words, the potential well depths are symmetrised for V , and inverted for the adjoint potential V_0^* with respect to the initial potential V_0 (see also Figure 1).

We will need some a priori estimates on the committors h_{AB} and h_{AB}^* . We expect h_{AB} to be close to the static committor \tilde{h}_0 given for $a(y) < x < b(y)$ by

$$\tilde{h}_0(x|y) = \frac{1}{\tilde{N}(y)} \int_x^{b(y)} e^{2V_0(\bar{x},y)/\sigma^2} d\bar{x}, \quad \tilde{N}(y) = \int_{a(y)}^{b(y)} e^{2V_0(x,y)/\sigma^2} dx. \quad (5.6)$$

Note that \tilde{h}_0 only slightly differs from h_0 owing to the different boundary conditions. The difference is however exponentially small in σ^2 , with an exponent that can be made large by taking ρ in (2.19) small. Similarly, h_{AB}^* should be close to

$$\begin{aligned} \tilde{h}_0^*(x|y) &= \frac{1}{\tilde{N}^*(y)} \int_x^{b(y)} e^{2V_0^*(\bar{x},y)/\sigma^2} d\bar{x} & \tilde{N}^*(y) &= \int_{a(y)}^{b(y)} e^{2V_0^*(x,y)/\sigma^2} dx \\ &= \frac{1}{\tilde{N}^*(y)} \int_x^{b(y)} \frac{e^{2V_0(\bar{x},y)/\sigma^2}}{\Phi(\bar{x},y)^2} d\bar{x}, & &= \int_{a(y)}^{b(y)} \frac{e^{2V_0(x,y)/\sigma^2}}{\Phi(x,y)^2} dx. \end{aligned}$$

Proposition 5.1. *We have*

$$h_{AB}(x, y) = \tilde{h}_0(x|y) + g(x, y), \quad h_{AB}^*(x, y) = \tilde{h}_0^*(x|y) + g^*(x, y),$$

where

$$\langle \pi_0, g^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^{3/2} \sqrt{Z_0(y)}} \lesssim \frac{\varepsilon \sqrt{\lambda_1(y)} B(y)}{\sigma^2}, \quad (5.7)$$

and similarly for $\langle \pi_0, (g^*)^2 \rangle^{1/2}$.

PROOF: Since $\mathcal{L}_x \tilde{h}_0 = 0$, the function g satisfies the equation

$$\varepsilon \mathcal{L}_y g = -\mathcal{L}_x g - \varepsilon \mathcal{L}_y \tilde{h}_0$$

with Dirichlet boundary conditions. Consider first the case $\varrho^2 = 0$, in which $\mathcal{L}_y = \partial_y$, and define the Lyapunov function

$$\mathcal{V}(g) = \frac{1}{2} \langle \pi_0, g^2 \rangle.$$

Changing y into $-y$, we obtain

$$\begin{aligned} \varepsilon \partial_y \mathcal{V} &= \langle \pi_0 g, \varepsilon \partial_y g \rangle + \frac{1}{2} \varepsilon \langle \partial_y \pi_0, g^2 \rangle \\ &= \langle \pi_0 g, \mathcal{L}_x g \rangle + \varepsilon \langle \pi_0 g, \partial_y \tilde{h}_0 \rangle + \frac{\varepsilon}{\sigma^2} \langle \pi_0 g, W g \rangle \\ &\leq -c_1 \mathcal{V} + \varepsilon \sqrt{\mathcal{V}} \langle \pi_0, (\partial_y \tilde{h}_0)^2 \rangle^{1/2} + \frac{\varepsilon}{\sigma^2} c_2 \mathcal{V}, \end{aligned}$$

where $c_1 > 0$ is a constant of order 1 related to the spectral gap of \mathcal{L}_x with Dirichlet boundary conditions on $\tilde{D} = (a(y), b(y))$. Using the bounds (C.1) and (C.2) on $\partial_y \tilde{h}_0$ obtained in the proof of Proposition 4.1, we get

$$\langle \pi_0, (\partial_y \tilde{h}_0)^2 \rangle \lesssim \frac{1}{\sigma^2 Z_0(y)} \int_{a(y)}^{b(y)} \frac{e^{2V_0(x,y)/\sigma^2}}{(\sigma + |x - x_0^*(y)|)^2} dx \lesssim \frac{1}{\sigma^3 Z_0(y)}.$$

It follows that

$$\varepsilon \partial_y \mathcal{V} \leq -\bar{c}_1 \mathcal{V} + \frac{\varepsilon \bar{c}_2}{\sigma^{3/2} \sqrt{Z_0(y)}} \sqrt{\mathcal{V}}$$

for some constant $\bar{c}_1, \bar{c}_2 > 0$. The result then follows by applying Gronwall's inequality to $\mathcal{W} = \sqrt{\mathcal{V}}$.

It thus remains to deal with the case $\varrho^2 > 0$. To this end, we introduce

$$k(x, y) = \frac{\sqrt{\varepsilon}}{\sigma} \left(g(x, y) + \frac{\varrho^2 \sigma^2}{2} \partial_y g(x, y) \right).$$

Then the pair (g, k) satisfies the system of hyperbolic type

$$\begin{aligned} \frac{\varrho^2}{2} \sqrt{\varepsilon} \sigma \partial_y g &= k - \frac{\sqrt{\varepsilon}}{\sigma} g \\ \sqrt{\varepsilon} \sigma \partial_y k &= -\mathcal{L}_x g - \varepsilon \partial_y \tilde{h}_0. \end{aligned}$$

This system can be made isotropic via a shearing transformation

$$k(x, y) = \frac{\varrho}{\sqrt{2}} (-\mathcal{L}_x)^{1/2} \bar{k}(x, y),$$

where for any $\gamma \in \mathbb{R}$, we set, in terms of eigenvalues $\bar{\lambda}_n$ and eigenfunctions $\bar{\pi}_n$ and $\bar{\phi}_n$ of \mathcal{L}_x with Dirichlet boundary conditions on $\tilde{\mathcal{D}}$,

$$(-\mathcal{L}_x)^\gamma f = \sum_{n \geq 1} (-\bar{\lambda}_n)^\gamma \langle \bar{\pi}_n, f \rangle \bar{\phi}_n.$$

This results in the system

$$\begin{aligned} \frac{\sqrt{\varepsilon} \varrho \sigma}{\sqrt{2}} \partial_y g &= (-\mathcal{L}_x)^{1/2} \bar{k} - \frac{\sqrt{2\varepsilon}}{\varrho \sigma} g \\ \frac{\sqrt{\varepsilon} \varrho \sigma}{\sqrt{2}} \partial_y \bar{k} &= (-\mathcal{L}_x)^{1/2} g - \varepsilon (-\mathcal{L}_x)^{1/2} \partial_y \tilde{h}_0 - \frac{\sqrt{\varepsilon} \varrho \sigma}{\sqrt{2}} (-\mathcal{L}_x)^{-1/2} \partial_y (-\mathcal{L}_x)^{1/2} \bar{k}. \end{aligned}$$

The result then follows in a similar way as above, by working with the Lyapunov functions

$$\mathcal{V}_\pm(g, \bar{k}) = \frac{1}{2} \langle \pi_0, (g \pm \bar{k})^2 \rangle,$$

and showing that for a periodic solution of the system, both \mathcal{V}_+ and \mathcal{V}_- have to remain small. \square

Corollary 5.2. *We have*

$$\int_{B^c} h_{\mathcal{AB}}^* d\pi = \frac{1}{2} \int_0^1 \left[1 - \delta_1(y) + \mathcal{O}\left(\frac{\varepsilon}{\sigma^2} \ell(\sigma) \sqrt{\lambda_1(y)}\right) \right] dy [1 + \mathcal{O}(\sigma^2)].$$

PROOF: The definition (5.1) of Φ implies

$$\int_{B^c} h_{\mathcal{AB}}^* d\pi = \int_0^1 \langle \pi_0, \Phi h_{\mathcal{AB}}^* \rangle dy.$$

We decompose

$$\langle \pi_0, \Phi h_{\mathcal{AB}}^* \rangle = \langle \pi_0, \Phi_0 \tilde{h}_0^* \rangle + \langle \pi_0, \Phi_0 g^* \rangle + \langle \pi_0, \Phi_\perp \tilde{h}_0^* \rangle + \langle \pi_0, \Phi_\perp g^* \rangle,$$

and estimate the contribution of each term separately. A similar argument as in (4.19) shows that

$$\langle \pi_0, \Phi_0 \tilde{h}_0^* \rangle = \frac{1}{2} [1 - \delta_1(y)] [1 + \mathcal{O}(\sigma^2)].$$

The other terms can be bounded via the Cauchy–Schwarz inequality. Namely, we obtain

$$\begin{aligned} |\langle \pi_0, \Phi_0 g^* \rangle| &\leq \langle \pi_0, \Phi_0^2 \rangle^{1/2} \langle \pi_0, (g^*)^2 \rangle^{1/2} \lesssim \frac{1}{B(y)} \frac{\varepsilon}{\sigma^2} \sqrt{\lambda_1(y) B(y)} , \\ |\langle \pi_0, \Phi_\perp \tilde{h}_0^* \rangle| &\leq \langle \pi_0, \Phi_\perp^2 \rangle^{1/2} \langle \pi_0, (\tilde{h}_0^*)^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2 B(y)} \sqrt{\lambda_1(y) \ell(\sigma) B(y)^2} , \\ |\langle \pi_0, \Phi_\perp g^* \rangle| &\leq \langle \pi_0, \Phi_\perp^2 \rangle^{1/2} \langle \pi_0, (g^*)^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2 B(y)} \frac{\varepsilon}{\sigma^2} \sqrt{\lambda_1(y) B(y)} , \end{aligned}$$

where we have used Proposition 4.7 (which applies to \tilde{h}_0 as well), Corollary 4.13 and Proposition 5.1. \square

6 Estimating the capacity

Recall from (2.16) the notation

$$\langle f \rangle = \int_0^1 f(y) \, dy ,$$

and define

$$C_0 = \frac{1}{4\varepsilon} \langle \lambda_1 [1 - A\delta_1] \rangle .$$

The purpose of this section is to establish the following estimates on the capacity $\text{cap}(\mathcal{A}, \mathcal{B})$.

Theorem 6.1 (Estimate of the capacity). *There exist constants M_\pm such that*

$$\begin{aligned} \frac{\text{cap}(\mathcal{A}, \mathcal{B})}{C_0} &\leq 1 + M_+ \left[\sigma^2 + \frac{\varepsilon \ell(\sigma) \sigma^{-2} \langle \lambda_1 \rangle + \varepsilon^2 \sigma^{-3} \langle \sqrt{\lambda_1} \rangle}{\langle \lambda_1 [1 - A\delta_1] \rangle} \right] , \\ \frac{\text{cap}(\mathcal{A}, \mathcal{B})}{C_0} &\geq 1 - M_- \left[\sigma^2 + \frac{\varepsilon \ell(\sigma)^2 \sigma^{-2} \langle \lambda_1 \rangle + \varepsilon^2 \sigma^{-7/2} \langle \sqrt{\lambda_1} \rangle}{\langle \lambda_1 [1 - A\delta_1] \rangle} \right] . \end{aligned}$$

The proof of this result is naturally divided into two parts. We will prove the upper bound in Section 6.1, and the lower bound in Section 6.2.

6.1 Upper bound on the capacity

The upper bound on the capacity will follow from the defective-flow Dirichlet principle (3.14). The expressions for the minimisers given in Proposition 3.6 suggest taking as test functions

$$\begin{aligned} f &= \frac{1}{2} (\tilde{h}_0 + \tilde{h}_0^*) , \\ \varphi &= \Phi_f - \Psi_{\tilde{h}_0} = \Psi_{(\tilde{h}_0^* - \tilde{h}_0)/2} - \pi f c . \end{aligned}$$

The defective-flow Dirichlet principle then reads

$$\text{cap}(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}(\Psi_{\tilde{h}_0}) - 2 \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{A}\mathcal{B}} \, dx \, dy .$$

Proposition 6.2. *There exists a constant M_+ such that*

$$\mathcal{D}(\Psi_{\tilde{h}_0}) \leq \frac{1}{4\varepsilon} \left[\langle \lambda_1 [1 - A\delta_1] \rangle + \frac{\varepsilon \ell(\sigma) M_+}{\sigma^2} \langle \lambda_1 \rangle + \frac{\varepsilon^2 \sqrt{\ell(\sigma)} M_+}{\sigma^4} \langle \sqrt{\lambda_1} \rangle \right] [1 + \mathcal{O}(\sigma^2)] .$$

PROOF: We introduce a probability measure μ on $\tilde{D} = (a(y), b(y))$ with density (see (5.6))

$$\mu(x|y) = \frac{1}{\tilde{N}(y)} e^{2V_0(x,y)/\sigma^2} = -\partial_x \tilde{h}_0(x) . \quad (6.1)$$

Note that

$$\langle \mu, h_0 \rangle = -\frac{N(y)}{\tilde{N}(y)} \int_{a(y)}^{b(y)} h_0 \partial_x h_0 \, dx = \frac{1}{2} [1 + \mathcal{O}(e^{-\kappa/\sigma^2})] , \quad (6.2)$$

where κ can be made large by taking ρ in (2.19) large. Applying the definition (3.11) of \mathcal{D} to $\Psi_{\tilde{h}_0}$, we obtain

$$\mathcal{D}(\Psi_{\tilde{h}_0}) = \frac{\sigma^2}{2\varepsilon} \int_0^1 [\langle \pi, (\partial_x \tilde{h}_0)^2 \rangle + \varepsilon \varrho^2 \langle \pi, (\partial_y \tilde{h}_0)^2 \rangle] \, dy .$$

In order to estimate the first inner product, we use the decomposition $\pi = \pi_0(\Phi_0 + \Phi_\perp^* + \Phi_\perp^1)$, expand, and consider the resulting terms separately. For the first term, using (4.10a) to compute $\langle \mu, \phi_1 \rangle$, we find

$$\begin{aligned} \langle \pi_0, \Phi_0(\partial_x \tilde{h}_0)^2 \rangle &= \frac{1}{\tilde{N}(y)Z_0(y)} [1 + \alpha_1(y)\langle \mu, \phi_1 \rangle] \\ &= \frac{1}{\tilde{N}(y)Z_0(y)} \frac{1 - A(y)\delta_1(y)}{B(y)^2} [1 + \mathcal{O}(e^{-\kappa/\sigma^2})] \\ &= \frac{1}{2\sigma^2} \lambda_1(y) [1 - A(y)\delta_1(y)] [1 + \mathcal{O}(\sigma^2)] . \end{aligned}$$

The second term can be directly bounded via Proposition 4.14 by

$$|\langle \pi_0, \Phi_\perp^* (\partial_x \tilde{h}_0)^2 \rangle| \lesssim \frac{\varepsilon}{\sigma^4} \lambda_1(y) \ell .$$

As for the third term, it satisfies

$$\begin{aligned} |\langle \pi_0, \Phi_\perp^1 (\partial_x \tilde{h}_0)^2 \rangle| &\leq \langle \pi_0, (\Phi_\perp^1)^2 \rangle^{1/2} \langle \pi_0, (\partial_x \tilde{h}_0)^4 \rangle^{1/2} \\ &\lesssim \frac{\varepsilon^2}{\sigma^4 B(y)} \frac{1}{Z_0(y)^{1/2} \tilde{N}(y)^{3/2}} \lesssim \frac{\varepsilon^2}{\sigma^6} \sqrt{\lambda_1(y)} . \end{aligned}$$

It remains to estimate the contribution of $\langle \pi, (\partial_y \tilde{h}_0)^2 \rangle$. We split it into two parts, which satisfy

$$|\langle \pi_0, \Phi_0 (\partial_y \tilde{h}_0)^2 \rangle| \lesssim \frac{1}{B(y)^2} \langle \pi_0, (\partial_y \tilde{h}_0)^2 \rangle \lesssim \frac{\eta(y)}{\sigma^4 B(y)^2} \lesssim \frac{\lambda_1(y) \ell}{\sigma^2} ,$$

(where we used the sharper estimate (C.3) to bound $\partial_y \tilde{h}_0$), and

$$|\langle \pi_0, \Phi_\perp (\partial_y \tilde{h}_0)^2 \rangle| \leq \langle \pi_0, \Phi_\perp^2 \rangle^{1/2} \langle \pi_0, (\partial_y \tilde{h}_0)^4 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^5} \sqrt{\lambda_1(y) \ell} .$$

Collecting all terms gives the claimed result. \square

To complete the proof of the upper bound on the capacity, it remains to control the error due to the fact that φ is not exactly divergence-free. Note that in view of (3.5), we have

$$\nabla \cdot (\pi f c) = \pi (\nabla f \cdot c) ,$$

This yields

$$-2\nabla \cdot \varphi = \left[\frac{\sigma^2}{2\varepsilon} \nabla \cdot (\pi D \nabla \tilde{h}_0) + \pi \nabla \tilde{h}_0 \cdot c \right] - \left[\frac{\sigma^2}{2\varepsilon} \nabla \cdot (\pi D \nabla \tilde{h}_0^*) - \pi \nabla \tilde{h}_0^* \cdot c \right] . \quad (6.3)$$

The contributions of the two brackets to the error term can be estimated separately. They are small because \tilde{h}_0 and \tilde{h}_0^* are both approximately harmonic with respect to \mathcal{L} and \mathcal{L}^* .

Proposition 6.3. *We have the bound*

$$\left| -2 \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{AB}} \, dx \, dy \right| \lesssim \frac{\ell(\sigma)}{\sigma^2} \langle \lambda_1 \rangle + \frac{\varepsilon}{\sigma^3} \langle \sqrt{\lambda_1} \rangle .$$

PROOF: We will consider the contribution of the first bracket in (6.3). The expression (5.4) for c_x shows that the derivatives with respect to x cancel exactly, while the expression (3.4) for c_y shows that the remaining part is equal to

$$\frac{1}{2} \varrho^2 \sigma^2 \partial_y (\pi \partial_y \tilde{h}_0) + \pi \partial_y \tilde{h}_0 c_y = \pi \left[\partial_y \tilde{h}_0 + \frac{\varrho^2 \sigma^2}{2} \partial_{yy} \tilde{h}_0 \right] .$$

The first error term is thus given by

$$\int_0^1 \langle \pi_0, \Phi \left[\partial_y \tilde{h}_0 + \frac{\varrho^2 \sigma^2}{2} \partial_{yy} \tilde{h}_0 \right] h_{\mathcal{AB}} \rangle \, dy .$$

Bounding $h_{\mathcal{AB}}$ by 1, and using

$$\begin{aligned} \langle \pi_0, |\Phi_0| |\partial_y \tilde{h}_0| \rangle &\lesssim \frac{\eta(y)}{\sigma^2 B(y)^2} \lesssim \frac{1}{\sigma^2} \lambda_1(y) \ell , \\ \langle \pi_0, |\Phi_\perp| |\partial_y \tilde{h}_0| \rangle &\leq \langle \pi_0, (\Phi_\perp)^2 \rangle^{1/2} \langle \pi_0, (\partial_y \tilde{h}_0)^2 \rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^3} \sqrt{\lambda_1(y)} , \end{aligned}$$

we find that the contribution of $\partial_y \tilde{h}_0$ satisfies the claimed bound. The contribution of $\partial_{yy} \tilde{h}_0$ is bounded similarly. This proves the result for the first bracket in (6.3), and the proof for the second bracket is similar. \square

6.2 Lower bound on the capacity

To obtain a lower bound on the capacity, we will apply the defective Thomson principle (3.15). Since (5.4) implies that the drift terms b and b^* are close to each other for x near $x_0^*(y)$, Proposition 3.7 suggests taking $f = 0$ and a test flow φ approximately proportional to $-\Psi_{\tilde{h}_0}$, where \tilde{h}_0 is the static committor (5.6). In fact, by (4.6) the \mathbf{e}_x -component of $\Psi_{\tilde{h}_0}$ is close to $-\lambda_1(y) B(y)^2 \Phi(x, y)$. We thus choose as test flow

$$\varphi(x, y) = \frac{1}{4\varepsilon C} \lambda_1(y) B(y)^2 \Phi(x, y) \mathbf{e}_x , \quad (6.4)$$

where the constant C is chosen in such a way that the unit flux condition

$$\int_{\partial \mathcal{A}} (\varphi \cdot \mathbf{n}) \, d\lambda = -1$$

is met. This amounts to requiring

$$\begin{aligned} 4\varepsilon C &= \int_0^1 \lambda_1(y) B(y)^2 \Phi(a(y), y) \, dy \\ &= \int_0^1 \lambda_1(y) B(y)^2 [\Phi_0(a(y), y) + \Phi_\perp(a(y), y)] \, dy \\ &= \int_0^1 \lambda_1(y) \left[1 - A(y) \delta_1(y) + \mathcal{O}\left(\frac{\varepsilon \ell}{\sigma^2}\right) + \mathcal{O}\left(\frac{\varepsilon^3 \ell^{3/2}}{\sigma^6 \sqrt{\lambda_1(y)}}\right) \right] \, dy \\ &\quad + \mathcal{O}\left(\frac{\varepsilon}{\sigma^{3/2}} \int_0^1 \sqrt{\lambda_1(y)} e^{-h_-(y)/\sigma^2} \, dy\right) \\ &= \langle \lambda_1 [1 - A \delta_1] \rangle + \mathcal{O}\left(\frac{\varepsilon \ell}{\sigma^2} \langle \lambda_1 \rangle\right) + \mathcal{O}\left(\frac{\varepsilon^3 \ell^{3/2}}{\sigma^6} \langle \sqrt{\lambda_1} \rangle\right) , \end{aligned}$$

where we have used the expression (5.3) for $B^2\Phi_0$, the expression (4.30) for $\langle \lambda_1[A - \delta_1] \rangle$, Corollary 4.13 to estimate the contribution of Φ_\perp , as well as (2.4).

Proposition 6.4. *There exists a constant M_- such that*

$$\mathcal{D}(-\varphi) \leq \frac{1}{4\varepsilon C^2} \left[\langle \lambda_1[1 - A\delta_1] \rangle + \frac{\varepsilon \ell(\sigma)^2 M_-}{\sigma^2} \langle \lambda_1 \rangle + \frac{\varepsilon^2 M_-}{\sigma^{7/2}} \langle \sqrt{\lambda_1} \rangle \right] [1 + \mathcal{O}(\sigma^2)] .$$

PROOF: Substituting the expression (6.4) of the test flow in the definition (3.11) of \mathcal{D} and using (4.6), we obtain

$$\begin{aligned} \mathcal{D}(-\varphi) &= \frac{1}{8\varepsilon\sigma^2 C^2} \int_0^1 \lambda_1(y)^2 B(y)^4 \int_{a(y)}^{b(y)} \frac{\Phi(x, y)^2}{\pi(x, y)} dx dy \\ &= \frac{1}{4\varepsilon C} \int_0^1 \lambda_1(y) B(y)^2 \langle \mu, \Phi \rangle dy [1 + \mathcal{O}(\sigma^2)] , \end{aligned}$$

where μ is the probability density μ introduced in (6.1). The leading contribution comes from the term (cf. (5.3))

$$\langle \mu, \Phi_0 \rangle = \frac{1}{B(y)^2} \left[1 - A(y)\delta_1(y) + (A(y) - \delta_1(y)) \langle \mu, 2h_0 - 1 \rangle \right] [1 + \mathcal{O}(\lambda_1 \ell)] .$$

Therefore, when integrating against $\lambda_1(y)B(y)^2$, (6.2) and (4.30) imply that the contribution of the term in $(A(y) - \delta_1(y))$ satisfies the claimed bound. Furthermore, it follows from Proposition 4.14 that

$$|\langle \mu, \Phi_\perp^* \rangle| = \frac{Z_0(y)}{\tilde{N}(y)} |\langle \pi_0, e^{4V_0/\sigma^2} \Phi_\perp^* \rangle| \lesssim \frac{\varepsilon}{\sigma^2} \frac{Z_0(y)\lambda_1(y)\ell}{\tilde{N}(y)} \lesssim \frac{\varepsilon \ell}{\sigma^2 B(y)^2} . \quad (6.5)$$

Combining this with the bound

$$|\langle \mu, \Phi_\perp^1 \rangle| \lesssim \frac{\varepsilon^2}{\sigma^{7/2} B(y)^2 \sqrt{\lambda_1(y)}} ,$$

which follows from (4.32), yields the result. \square

Since the test flow φ is not exactly divergence-free, to complete the proof of the lower bound it remains to control the error term in (3.15).

Proposition 6.5. *The error term satisfies*

$$\left| \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{AB}} dx dy \right| \lesssim \frac{1}{4\varepsilon C} \left[\frac{\varepsilon \ell(\sigma)^2}{\sigma^2} \langle \lambda_1 \rangle + \frac{\varepsilon^2}{\sigma^5} \langle \sqrt{\lambda_1} \rangle \right] . \quad (6.6)$$

PROOF: The definition (6.4) of the test flow implies

$$\int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{AB}} dx dy = \frac{1}{4\varepsilon C} \int_0^1 \lambda_1(y) B(y)^2 \langle \partial_x \Phi, h_{\mathcal{AB}} \rangle dy .$$

We decompose the inner product as

$$\langle \partial_x \Phi, h_{\mathcal{AB}} \rangle = \langle \partial_x \Phi_0, \tilde{h}_0 \rangle + \langle \partial_x \Phi_0, g \rangle + \langle \partial_x \Phi_\perp^*, \tilde{h}_0 \rangle + \langle \partial_x \Phi_\perp^1, \tilde{h}_0 \rangle + \langle \partial_x \Phi_\perp, g \rangle \quad (6.7)$$

and estimate the resulting terms separately. For the first term, we note that

$$\langle \partial_x \Phi_0, \tilde{h}_0 \rangle = 2 \frac{A(y) - \delta_1(y)}{B(y)^2} \int_{a(y)}^{b(y)} \tilde{h}_0 \partial_x h_0 dx [1 + \mathcal{O}(\lambda_1 \ell)] .$$

As in (6.2) above, the integral is exponentially close to $\frac{1}{2}$, which results in a negligible term when integrating against $\lambda_1(y)B(y)^2$, owing to (4.30).

Regarding the second term, we observe that

$$|\langle \partial_x \Phi_0, g \rangle| \lesssim \frac{|A(y) - \delta_1(y)|}{B(y)^2} |\langle \partial_x h_0, g \rangle| ,$$

where

$$\langle \partial_x h_0, g \rangle^2 \leq \langle \pi_0^{-1}, (\partial_x h_0)^2 \rangle \langle \pi_0, g^2 \rangle \leq \frac{Z_0(y)}{N(y)} \frac{\varepsilon^2}{\sigma^3 Z_0(y)} \lesssim \frac{\varepsilon^2}{\sigma^4}$$

by (5.7). The contribution of this term is thus of the order of $(\varepsilon/\sigma^2)\langle \lambda_1 \rangle$.

The third term in (6.7) satisfies (cf. (6.5))

$$\langle \partial_x \Phi_\perp^*, \tilde{h}_0 \rangle^2 = \langle \Phi_\perp^*, \partial_x \tilde{h}_0 \rangle^2 = \langle \mu, \Phi_\perp^* \rangle^2 \lesssim \frac{\varepsilon^2 \ell^2}{\sigma^4 B(y)^4} ,$$

which results in a contribution of order $(\varepsilon \ell / \sigma^2)\langle \lambda_1 \rangle$. The fourth term in (6.7) can be bounded using (4.31a) by

$$\langle \partial_x \Phi_\perp^1, \tilde{h}_0 \rangle^2 = \langle \Phi_\perp^1, \partial_x \tilde{h}_0 \rangle^2 \leq \langle \pi_0, (\Phi_\perp^1)^2 \rangle \langle \pi_0^{-1}, (\partial_x \tilde{h}_0)^2 \rangle \lesssim \frac{\varepsilon^4}{\sigma^8 B(y)^2} \frac{Z_0(y)}{N(y)}$$

Using (4.6) and integrating against $\lambda_1 B^2$, we obtain indeed a quantity bounded by the second term on the right-hand side of (6.6), though with a slightly better power of σ . For the last term in (6.7), we use the quick-and-dirty bound

$$\langle \partial_x \Phi_\perp, g \rangle^2 \leq Z_0(y)^2 \langle \pi_0, (\partial_x \Phi_\perp)^2 \rangle \langle \pi_0, g^2 \rangle \lesssim Z_0(y)^2 \frac{\varepsilon^2}{\sigma^8 B(y)^2} \frac{\varepsilon^2}{\sigma^3 Z_0(y)} = \frac{\varepsilon^4 Z_0(y)}{\sigma^{11} B(y)^2}$$

implied by (4.31b), which accounts for the second summand in (6.6). \square

7 Proof of the main result

Theorem 2.3 follows immediately from Proposition 3.4, Corollary 5.2 and the estimate of the capacity given in Theorem 6.1.

We thus proceed with the proof of Theorem 2.4. To this end, we start by evaluating more precisely the expression (2.20) of the expected transition time integrated with respect to the equilibrium measure.

Using (2.17) and (2.18), we obtain

$$\delta_1(y) = \frac{1}{\langle \lambda_1 \rangle} \left[\langle \lambda_1 A \rangle \left(1 + \mathcal{O}\left(\frac{\langle \lambda_1 \rangle}{\varepsilon}\right) \right) + R_\delta \right] ,$$

where

$$R_\delta = \mathcal{O}\left(\frac{\varepsilon \ell^3}{\sigma^2} \langle \lambda_1 \rangle\right) + \mathcal{O}\left(\frac{\varepsilon^3 \ell^{3/2}}{\sigma^6} \langle \sqrt{\lambda_1} \rangle\right) .$$

Using the expressions (2.22) for $\langle \lambda_1 \rangle$ and $\langle \lambda_1 A \rangle$, we obtain

$$1 - \langle \delta_1 \rangle = \frac{2\langle r_+ \rangle}{\langle \lambda_1 \rangle} \left[1 + \mathcal{O}\left(\frac{\langle \lambda_1 \rangle \langle \lambda_1 A \rangle}{\varepsilon \langle r_+ \rangle}\right) + \mathcal{O}\left(\frac{R_\delta}{\langle r_+ \rangle}\right) \right] [1 + \mathcal{O}(\sigma^2)] . \quad (7.1)$$

In a similar way, we get

$$\langle \lambda_1 [1 - A\delta_1] \rangle = \frac{4\langle r_- \rangle \langle r_+ \rangle}{\langle \lambda_1 \rangle} \left[1 + \mathcal{O}\left(\frac{\langle \lambda_1 \rangle \langle \lambda_1 A \rangle^2}{\varepsilon \langle r_- \rangle \langle r_+ \rangle}\right) + \mathcal{O}(\langle \lambda_1 A \rangle R_\delta) \right] [1 + \mathcal{O}(\sigma^2)] . \quad (7.2)$$

Substituting in (2.20) yields

$$\int_{\partial\mathcal{A}} \mathbb{E}_x[\tau_{\mathcal{B}}] \, d\nu_{\mathcal{AB}} = \frac{\varepsilon}{\langle r_- \rangle} [1 + R_1(\varepsilon, \sigma)] ,$$

where

$$|R_1(\varepsilon, \sigma)| \lesssim \frac{\langle \lambda_1 \rangle \langle \lambda_1 A \rangle}{\varepsilon \langle r_+ \rangle} + \frac{\langle \lambda_1 \rangle \langle \lambda_1 A \rangle^2}{\varepsilon \langle r_- \rangle \langle r_+ \rangle} + \frac{R_\delta}{\langle r_+ \rangle} + |R_0| ,$$

and R_0 is defined in (2.21). Discarding error terms already accounted for in the previous estimate, and using (7.1) and (7.2), we obtain

$$|R_0| \lesssim \sigma^2 + \frac{\varepsilon \ell}{\sigma^2} \left[\frac{\langle \lambda_1 \rangle \langle \sqrt{\lambda_1} \rangle}{\langle r_+ \rangle} + \frac{\ell \langle \lambda_1 \rangle^2}{\langle r_- \rangle \langle r_+ \rangle} \right] + \frac{\varepsilon^2 \ell}{\sigma^{7/2}} \frac{\langle \lambda_1 \rangle \langle \sqrt{\lambda_1} \rangle}{\langle r_- \rangle \langle r_+ \rangle} .$$

In order to simplify the expression of R_1 , we start by noting that Jensen's inequality implies

$$\langle \sqrt{\lambda_1} \rangle^2 \leq \langle \lambda_1 \rangle .$$

Then we define

$$\mathcal{R}_- = \frac{\langle r_- \rangle}{\langle r_+ \rangle} , \quad \mathcal{R} = \frac{\langle r_- \rangle}{\langle r_+ \rangle} + \frac{\langle r_+ \rangle}{\langle r_- \rangle} .$$

A short computation shows that

$$|R_1(\varepsilon, \sigma)| \lesssim \sigma^2 + \left(\frac{\varepsilon \ell^3}{\sigma^2} + \frac{\varepsilon^2 \ell}{\sigma^{7/2} \langle \lambda_1 \rangle^{1/2}} + \frac{\langle \lambda_1 \rangle^2}{\varepsilon} \right) \mathcal{R} + \frac{\langle \lambda_1 \rangle}{\varepsilon} (1 + \mathcal{R}_-) .$$

This expression is indeed equivalent to (2.26), since we have

$$\mathcal{R} \lesssim e^{2H/\sigma^2} , \quad \mathcal{R}_- \lesssim e^{2H_-/\sigma^2}$$

for the constants H and H_- introduced in (2.24).

For R_1 to be small, ε has to satisfy the condition

$$\langle \lambda_1 \rangle^2 \mathcal{R} \vee \langle \lambda_1 \rangle \mathcal{R}_- \ll \varepsilon \ll \frac{1}{\mathcal{R}} \wedge \frac{\langle \lambda_1 \rangle^{1/4}}{\mathcal{R}^{1/2}} .$$

Since $\langle \lambda_1 \rangle$ has order $e^{-2(h_-^{\min} \vee h_+^{\min})/\sigma^2}$, where h_{\pm}^{\min} have been defined in (2.23), by treating separately the cases $h_+^{\min} > h_-^{\min}$ and $h_+^{\min} < h_-^{\min}$, one readily obtains that this condition can be satisfied for a non-empty interval of values of ε if and only if Condition (2.28) is met.

It remains to show that we can replace the expectation when starting in the equilibrium measure $\nu_{\mathcal{AB}}$ by the expectation when starting in a single point on $\partial\mathcal{A}$. This will follow if we can show that $\mathbb{E}_z[\tau_{\mathcal{B}}]$ depends little on the starting point $z \in \partial\mathcal{A}$. We will do this by adapting an argument used in the proof of [10, Proposition 3.6].

We first fix a point $z \in \partial\mathcal{A}$, and show that $\mathbb{E}_{\bar{z}}[\tau_{\mathcal{B}}]$ is close to $\mathbb{E}_z[\tau_{\mathcal{B}}]$ for all \bar{z} in a ball of small radius of order 1 centred in z . Let Ω be an event of probability close to 1, on which

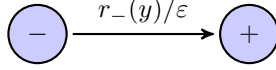


FIGURE 6. Absorbing variant of the two-state markovian jump process.

$\tau_{\mathcal{B}^+}^z \leq \tau_{\mathcal{B}}^{\bar{z}} \leq \tau_{\mathcal{B}^-}^z$, where the sets $\mathcal{B}^- \subset \mathcal{B} \subset \mathcal{B}^+$ have boundaries close to each other. Using the Cauchy–Schwarz inequality, we obtain

$$\mathbb{E}_z[\tau_{\mathcal{B}^+}] - \sqrt{\mathbb{E}_z[\tau_{\mathcal{B}^+}^2]} \sqrt{\mathbb{P}(\Omega^c)} \leq \mathbb{E}_{\bar{z}}[\tau_{\mathcal{B}}] \leq \mathbb{E}_z[\tau_{\mathcal{B}^-}] + \sqrt{\mathbb{E}_z[\tau_{\mathcal{B}^-}^2]} \sqrt{\mathbb{P}(\Omega^c)}.$$

Using a standard large-deviation estimate on $\mathbb{P}_{\bar{z}}\{\tau_{\mathcal{B}} > T\}$ for a fixed $T > 0$, one easily obtains a bound of the form

$$\mathbb{E}_{\bar{z}}[\tau_{\mathcal{B}}^2] \leq T_1(\eta) e^{\eta/\sigma^2} \mathbb{E}_z[\tau_{\mathcal{B}}]^2 \quad (7.3)$$

with $T_1(\eta) < \infty$ for all $\eta > 0$ (see for instance [10, Sections 5.2 to 5.4], which applies to a much harder infinite-dimensional setting). It thus suffices to show that $\mathbb{P}(\Omega^c) \leq e^{-\kappa/\sigma^2}$ for some $\kappa > 0$ and to apply (7.3) with $\eta < \kappa/2$ to show that $\mathbb{E}_{\bar{z}}[\tau_{\mathcal{B}}]$ and $\mathbb{E}_z[\tau_{\mathcal{B}}]$ are exponentially close to each other. We do this by choosing

$$\Omega = \left\{ \frac{\|\bar{z}_t - z_t\|}{\|\bar{z} - z\|} \leq c e^{-mt} \quad \forall t \geq 0 \right\}.$$

Indeed, in [36] it is shown that this event has a probability exponentially close to 1 for appropriate values of $c, m > 0$ (see also [39] for a more streamlined version of the proof of [36] in a more general setting).

It remains to show that $\mathbb{E}_z[\tau_{\mathcal{B}}]$ changes little when z moves along the boundary $\partial\mathcal{A}$. To do this, we fix, say, $\bar{z} = (y, a(y))$ with $0 < y < 1$ and $z = (1, a(1))$ on $\partial\mathcal{A}$. Let

$$\tau_1(\bar{z}) = \inf\{t > 1 : y_t = 1\}$$

be the first time at which the sample path starting in \bar{z} hits the line $\{y = 1\}$. Using for instance [11, Proposition 6.3], one easily obtains that with probability exponentially close to 1, $\tau_1(\bar{z})$ is bounded by a constant of order 1, and $\|\bar{z}_{\tau_1} - z\|$ is smaller than an arbitrary constant of order 1. From that we deduce as above that

$$\mathbb{E}_{\bar{z}}[\tau_{\mathcal{B}}] = \mathbb{E}_z[\tau_{\mathcal{B}}] \left[1 + \mathcal{O}\left(\frac{\langle r_- \rangle}{\epsilon}\right) \right],$$

which concludes the proof of Theorem 2.4.

A The two-state jump process

PROOF OF PROPOSITION 2.1. Consider a modified two-state process in which the $+$ state has been made absorbing (Figure 6). Its first-hitting time τ_+ , starting from the $-$ state at a fixed time y_0 , agrees with the corresponding first-hitting time of the original process. The occupation probability $p_-(y)$ of the $-$ state satisfies

$$\epsilon p'_-(y) = -r_-(y)p_-(y)$$

with initial value $p_-(y_0) = 1$, and is thus given by

$$p_-(y) = \mathbb{P}_{y_0}\{\tau_+ > y\} = e^{-R_-(y, y_0)/\epsilon}.$$

The expectation of τ_+ is thus given by

$$\begin{aligned}\mathbb{E}_{y_0}[\tau_+] &= \int_0^\infty \mathbb{P}_{y_0}\{\tau_+ > y_0 + y\} dy \\ &= \int_{y_0}^\infty e^{-R_-(y, y_0)/\varepsilon} dy .\end{aligned}$$

Noting that by periodicity,

$$R_-(y_0 + n + \bar{y}, y_0) = nR_-(1, 0) + R_-(y_0 + \bar{y}, y_0) ,$$

we obtain

$$\mathbb{E}_{y_0}[\tau_+] = \sum_{n=0}^\infty e^{-nR_-(1, 0)/\varepsilon} \int_0^1 e^{-R_-(y_0 + \bar{y}, y_0)/\varepsilon} d\bar{y} .$$

Summing the geometric series yields the claimed result. \square

B Proofs of the potential-theoretic results

In this section, we provide quick proofs of the potential-theoretic results stated in Section 3. Except for a small addition in the case of test flows which are not divergence-free, all these proofs are contained in [30]. We provide them here for convenience, as we use slightly different notations and scalings.

B.1 Invariant density

PROOF OF LEMMA 3.1. The adjoint in $L^2(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}))$ of \mathcal{L} is given by

$$\mathcal{L}^\dagger \mu = \frac{\sigma^2}{2\varepsilon} (\partial_{xx} \mu + \varepsilon \varrho^2 \partial_{yy} \mu) - \frac{1}{\varepsilon} \partial_x [b\mu] - \partial_y \mu .$$

The condition $\mathcal{L}^\dagger e^{-2V/\sigma^2} = 0$ is equivalent to (3.2). \square

PROOF OF LEMMA 3.2. Relation (3.3) follows from a short computation. Using the explicit form (3.4) of c , one obtains

$$\begin{aligned}\frac{\sigma^2}{2} e^{2V/\sigma^2} \nabla \cdot (e^{-2V/\sigma^2} c) &= -\nabla V \cdot c + \frac{\sigma^2}{2} \nabla \cdot c \\ &= -\frac{1}{\varepsilon} (\partial_x V)^2 - \varrho^2 (\partial_y V)^2 - \frac{1}{\varepsilon} \partial_x V b - \partial_y V \\ &\quad + \frac{\sigma^2}{2\varepsilon} (\partial_{xx} V + \varepsilon \varrho^2 \partial_{yy} V + \partial_x b) ,\end{aligned}$$

which vanishes by (3.2). \square

B.2 Capacity

PROOF OF LEMMA 3.3. The fact that $\text{cap}(\mathcal{A}, \mathcal{B}) = \text{cap}(\mathcal{B}, \mathcal{A})$ follows from the relation

$$h_{\mathcal{A}\mathcal{B}}(x, y) = 1 - h_{\mathcal{B}\mathcal{A}}(x, y) .$$

To prove the first equality in (3.8), we use integration by parts, that is,

$$\begin{aligned} \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla \cdot (h_{\mathcal{AB}} e^{-2V/\sigma^2} D\nabla h_{\mathcal{AB}}) \frac{dx dy}{Z} &= \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla h_{\mathcal{AB}} \cdot (D\nabla h_{\mathcal{AB}}) d\pi \\ &+ \int_{(\mathcal{A} \cup \mathcal{B})^c} h_{\mathcal{AB}} \nabla \cdot (e^{-2V/\sigma^2} D\nabla h_{\mathcal{AB}}) \frac{dx dy}{Z}. \end{aligned} \quad (\text{B.1})$$

By the divergence theorem and the boundary conditions for $h_{\mathcal{AB}}$,

$$\int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla \cdot (h_{\mathcal{AB}} e^{-2V/\sigma^2} D\nabla h_{\mathcal{AB}}) \frac{dx dy}{Z} = \int_{\partial \mathcal{A}} (D\nabla h_{\mathcal{AB}} \cdot \mathbf{n}) \pi d\lambda.$$

Since $\mathcal{L}h_{\mathcal{AB}}$ vanishes on $(\mathcal{A} \cup \mathcal{B})^c$, the second term on the right-hand side of (B.1) is equal to $2\varepsilon/\sigma^2$ times

$$\int_{(\mathcal{A} \cup \mathcal{B})^c} h_{\mathcal{AB}} (\mathcal{L}_s h_{\mathcal{AB}}) d\pi = - \int_{(\mathcal{A} \cup \mathcal{B})^c} h_{\mathcal{AB}} (\mathcal{L}_a h_{\mathcal{AB}}) d\pi = - \int_{(\mathcal{A} \cup \mathcal{B})^c} h_{\mathcal{AB}} (c \cdot \nabla h_{\mathcal{AB}}) d\pi$$

The same skew-symmetry argument as in (3.6) implies that the last integral vanishes. Since the first term on the right-hand side of (B.1) is proportional to the capacity, the first equality in (3.8) follows. To prove the second equality, we use the fact that owing to the vanishing divergence condition (3.5), we have

$$0 = \int_{\mathcal{A}^c} e^{2V/\sigma^2} \nabla \cdot (e^{-2V/\sigma^2} c) d\pi = \int_{\mathcal{A}^c} \nabla \cdot (e^{-2V/\sigma^2} c) \frac{dx dy}{Z} = \int_{\partial \mathcal{A}} (c \cdot \mathbf{n}) \pi d\lambda.$$

Since $h_{\mathcal{AB}} = 1$ on $\partial \mathcal{A}$, the integral of $h_{\mathcal{AB}} (c \cdot \mathbf{n}) \pi d\lambda$ indeed vanishes. To prove the first equality in (3.9), we use a similar computation as in (B.1) to obtain

$$\int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla h_{\mathcal{AB}}^* \cdot (D\nabla h_{\mathcal{AB}}) d\pi = \int_{\partial \mathcal{A}} (D\nabla h_{\mathcal{AB}}) \cdot \mathbf{n} \pi d\lambda + \varepsilon \int_{(\mathcal{A} \cup \mathcal{B})^c} h_{\mathcal{AB}}^* (c \cdot \nabla h_{\mathcal{AB}}) d\pi.$$

The first term on the right-hand side is proportional to the capacity, yielding the claimed result. The second equality in (3.9) then follows from (3.6), while the last equality is obtained by exchanging the roles of $h_{\mathcal{AB}}$ and $h_{\mathcal{AB}}^*$ in the above computation. \square

B.3 Equilibrium measure and mean hitting time

PROOF OF PROPOSITION 3.4. The function $w_{\mathcal{B}}(x) = \mathbb{E}_x[\tau_{\mathcal{B}}]$ satisfies the Poisson problem

$$\begin{cases} (\mathcal{L}w_{\mathcal{B}})(x, y) = -1 & (x, y) \in \mathcal{B}^c, \\ w_{\mathcal{B}}(x, y) = 0 & (x, y) \in \mathcal{B}. \end{cases} \quad (\text{B.2})$$

By the divergence theorem, we have

$$\begin{aligned} \frac{\sigma^2}{2\varepsilon} \int_{\partial \mathcal{A}} w_{\mathcal{B}} (D\nabla h_{\mathcal{AB}}^* \cdot \mathbf{n}) \pi d\lambda &= \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} \nabla \cdot (w_{\mathcal{B}} e^{-2V/\sigma^2} D\nabla h_{\mathcal{AB}}^*) \frac{dx dy}{Z} \\ &= \int_{(\mathcal{A} \cup \mathcal{B})^c} \left[\frac{\sigma^2}{2\varepsilon} \nabla w_{\mathcal{B}} \cdot D\nabla h_{\mathcal{AB}}^* + w_{\mathcal{B}} \mathcal{L}_s h_{\mathcal{AB}}^* \right] d\pi \\ &= \int_{\mathcal{B}^c} \left[\frac{\sigma^2}{2\varepsilon} \nabla w_{\mathcal{B}} \cdot D\nabla h_{\mathcal{AB}}^* + w_{\mathcal{B}} (c \cdot \nabla h_{\mathcal{AB}}^*) \right] d\pi, \end{aligned} \quad (\text{B.3})$$

where we have used the facts that $(\mathcal{L}_s - \mathcal{L}_a)h_{\mathcal{A}\mathcal{B}}^*$ vanishes on $(\mathcal{A} \cup \mathcal{B})^c$, while $\nabla h_{\mathcal{A}\mathcal{B}}^* = 0$ on \mathcal{A} . Furthermore, since $h_{\mathcal{A}\mathcal{B}}^*$ vanishes on \mathcal{B} , we have

$$\begin{aligned} 0 &= \frac{\sigma^2}{2\varepsilon} \int_{\mathcal{B}^c} \nabla \cdot (h_{\mathcal{A}\mathcal{B}}^* e^{-2V/\sigma^2} D\nabla w_{\mathcal{B}}) \frac{dx dy}{Z} \\ &= \frac{\sigma^2}{2\varepsilon} \int_{\mathcal{B}^c} [\nabla h_{\mathcal{A}\mathcal{B}}^* \cdot D\nabla w_{\mathcal{B}}] d\pi + \int_{\mathcal{B}^c} h_{\mathcal{A}\mathcal{B}}^* \mathcal{L}_s w_{\mathcal{B}} d\pi . \end{aligned}$$

Since $w_{\mathcal{B}}$ solves the Poisson problem (B.2), we have $\mathcal{L}_s w_{\mathcal{B}} = -1 - c \cdot \nabla w_{\mathcal{B}}$, so that substitution in (B.3) yields

$$\frac{\sigma^2}{2\varepsilon} \int_{\partial\mathcal{A}} w_{\mathcal{B}} (D\nabla h_{\mathcal{A}\mathcal{B}}^* \cdot \mathbf{n}) d\pi = \int_{\mathcal{B}^c} [h_{\mathcal{A}\mathcal{B}}^* + h_{\mathcal{A}\mathcal{B}}^* (c \cdot \nabla w_{\mathcal{B}}) + w_{\mathcal{B}} (c \cdot \nabla h_{\mathcal{A}\mathcal{B}}^*)] d\pi .$$

By the skew-symmetry property (3.6) and the boundary conditions, the contribution of the last two summands in the integral on the right-hand side vanishes. \square

B.4 Variational principles

PROOF OF LEMMA 3.5. We start by noting that

$$\mathcal{D}(\Phi_f, \Psi_{h_{\mathcal{A}\mathcal{B}}}) = \int_{(\mathcal{A} \cup \mathcal{B})^c} \left(\frac{\sigma^2}{2\varepsilon} D\nabla f - fc \right) \cdot \nabla h_{\mathcal{A}\mathcal{B}} d\pi . \quad (\text{B.4})$$

Integrating by parts with respect to ∇f , we obtain

$$\begin{aligned} \frac{\sigma^2}{2\varepsilon} \int_{(\mathcal{A} \cup \mathcal{B})^c} D\nabla f \cdot \nabla h_{\mathcal{A}\mathcal{B}} d\pi &= \frac{\sigma^2}{2\varepsilon} \int_{\partial\mathcal{A}} \alpha (D\nabla h_{\mathcal{A}\mathcal{B}} \cdot \mathbf{n}) d\pi - \int_{(\mathcal{A} \cup \mathcal{B})^c} f (\mathcal{L}_s h_{\mathcal{A}\mathcal{B}}) d\pi \\ &= \alpha \text{cap}(\mathcal{A}, \mathcal{B}) + \int_{(\mathcal{A} \cup \mathcal{B})^c} f (c \cdot \nabla h_{\mathcal{A}\mathcal{B}}) d\pi . \end{aligned}$$

The second term on the right-hand side cancels the c -dependent term in (B.4). Furthermore, we have

$$\mathcal{D}(\varphi, \Psi_{h_{\mathcal{A}\mathcal{B}}}) = \int_{(\mathcal{A} \cup \mathcal{B})^c} \varphi \cdot \nabla h_{\mathcal{A}\mathcal{B}} dx dy = \int_{\partial\mathcal{A}} (\varphi \cdot \mathbf{n}) d\lambda - \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{A}\mathcal{B}} dx dy .$$

The first term on the right-hand side is equal to $-\gamma$ by (3.12), while the second one vanishes since φ is divergence-free. \square

Remark B.1. If φ is only approximately divergence-free, the above proof yields

$$\mathcal{D}(\Phi_f - \varphi, \Psi_{h_{\mathcal{A}\mathcal{B}}}) = \alpha \text{cap}(\mathcal{A}, \mathcal{B}) + \gamma + \int_{(\mathcal{A} \cup \mathcal{B})^c} (\nabla \cdot \varphi) h_{\mathcal{A}\mathcal{B}} dx dy . \quad (\text{B.5})$$

This can be used to obtain bounds from flows that are not exactly divergence-free. \diamond

PROOF OF PROPOSITION 3.6. Pick $f \in \mathcal{H}_{\mathcal{A}\mathcal{B}}^{1,0}$ and $\varphi \in \mathcal{F}_{\mathcal{A}\mathcal{B}}^0$. By (3.13) with $\alpha = 1$ and $\gamma = 0$ and the Cauchy–Schwarz inequality, we have

$$\text{cap}(\mathcal{A}, \mathcal{B})^2 = \mathcal{D}(\Phi_f - \varphi, \Psi_{h_{\mathcal{A}\mathcal{B}}})^2 \leq \mathcal{D}(\Phi_f - \varphi) \mathcal{D}(\Psi_{h_{\mathcal{A}\mathcal{B}}}) = \mathcal{D}(\Phi_f - \varphi) \text{cap}(\mathcal{A}, \mathcal{B}) ,$$

showing that $\text{cap}(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}(\Phi_f - \varphi)$. Furthermore, $\mathcal{D}(\Phi_{\bar{f}} - \bar{\varphi}) = \mathcal{D}(\Psi_{h_{AB}}) = \text{cap}(\mathcal{A}, \mathcal{B})$. Since clearly $\bar{f} \in \mathcal{H}_{AB}^{1,0}$, it remains to show that $\bar{\varphi} \in \mathcal{F}_{AB}^0$. Noting that

$$\bar{\varphi} = \frac{\sigma^2}{4\varepsilon} \pi D[\nabla h_{AB}^* - \nabla h_{AB}] - \frac{1}{2} \pi c[h_{AB} + h_{AB}^*],$$

we obtain

$$\begin{aligned} \nabla \cdot \bar{\varphi} &= \frac{\sigma^2}{4\varepsilon Z} \nabla \cdot [e^{-2V/\sigma^2} D(\nabla h_{AB}^* - \nabla h_{AB})] - \frac{1}{2Z} \nabla \cdot [e^{-2V/\sigma^2} c(h_{AB} + h_{AB}^*)] \\ &= \frac{1}{2} \pi [\mathcal{L}_s h_{AB}^* - \mathcal{L}_s h_{AB}] - \frac{1}{2} \pi c \cdot (\nabla h_{AB} + \nabla h_{AB}^*) \\ &= \frac{1}{2} \pi [\mathcal{L}^* h_{AB}^* - \mathcal{L} h_{AB}] = 0, \end{aligned}$$

where we have used the fact that $\nabla \cdot (e^{-2V/\sigma^2} c) = 0$ in the second line. Furthermore,

$$\begin{aligned} \int_{\partial\mathcal{A}} \bar{\varphi} \cdot \mathbf{n} \, d\lambda &= \frac{1}{2} \int_{\partial\mathcal{A}} \left[\frac{\sigma^2}{2\varepsilon} D\nabla h_{AB}^* - c h_{AB}^* \right] \cdot \mathbf{n} \, d\lambda - \frac{1}{2} \int_{\partial\mathcal{A}} \left[\frac{\sigma^2}{2\varepsilon} D\nabla h_{AB} + c h_{AB} \right] \cdot \mathbf{n} \, d\lambda \\ &= \frac{1}{2} \text{cap}^*(\mathcal{A}, \mathcal{B}) - \frac{1}{2} \text{cap}(\mathcal{A}, \mathcal{B}), \end{aligned}$$

which vanishes by Lemma 3.3. The bound (3.14) is obtained by using (B.5) instead of (3.13). \square

PROOF OF PROPOSITION 3.7. Pick $f \in \mathcal{H}_{AB}^{0,0}$ and $\varphi \in \mathcal{F}_{AB}^1$. By (3.13) with $\alpha = 0$ and $\gamma = 1$ and the Cauchy–Schwarz inequality, we have

$$1 = \mathcal{D}(\Phi_f - \varphi, \Psi_{h_{AB}})^2 \leq \mathcal{D}(\Phi_f - \varphi) \mathcal{D}(\Psi_{h_{AB}}) = \mathcal{D}(\Phi_f - \varphi) \text{cap}(\mathcal{A}, \mathcal{B}),$$

showing that $\text{cap}(\mathcal{A}, \mathcal{B}) \geq 1/\mathcal{D}(\Phi_f - \varphi)$. By bilinearity of \mathcal{D} , we have $\mathcal{D}(\Phi_{\bar{f}} - \bar{\varphi}) = 1/\text{cap}(\mathcal{A}, \mathcal{B})$. Since $\bar{f} \in \mathcal{H}_{AB}^{0,0}$, it remains to show that $\bar{\varphi} \in \mathcal{F}_{AB}^1$. This time, we have

$$\bar{\varphi} = -\frac{\pi}{2 \text{cap}(\mathcal{A}, \mathcal{B})} \left(\frac{\sigma^2}{2\varepsilon} D[\nabla h_{AB} + \nabla h_{AB}^*] - c[h_{AB} - h_{AB}^*] \right).$$

Using the fact that $\nabla \cdot (e^{-2V/\sigma^2} c) = 0$, we obtain

$$\nabla \cdot \bar{\varphi} = -\frac{\pi}{2 \text{cap}(\mathcal{A}, \mathcal{B})} \left(\mathcal{L}_s[h_{AB} + h_{AB}^*] + c \nabla \cdot [h_{AB} - h_{AB}^*] \right) = 0,$$

and

$$\begin{aligned} \text{cap}(\mathcal{A}, \mathcal{B}) \int_{\partial\mathcal{A}} \bar{\varphi} \cdot \mathbf{n} \, d\lambda &= -\frac{1}{2} \int_{\partial\mathcal{A}} \left[\frac{\sigma^2}{2\varepsilon} D\nabla h_{AB} + c h_{AB} \right] \cdot \mathbf{n} \, d\lambda \\ &\quad - \frac{1}{2} \int_{\partial\mathcal{A}} \left[\frac{\sigma^2}{2\varepsilon} D\nabla h_{AB}^* - c h_{AB}^* \right] \cdot \mathbf{n} \, d\lambda \\ &= -\frac{1}{2} \text{cap}(\mathcal{A}, \mathcal{B}) - \frac{1}{2} \text{cap}^*(\mathcal{A}, \mathcal{B}) = -\text{cap}(\mathcal{A}, \mathcal{B}), \end{aligned}$$

showing that $\bar{\varphi}$ has flux -1 as required. The bound (3.15) is again obtained by using (B.5) instead of (3.13). \square

C Estimates on static eigenfunctions

C.1 Bounds on h_0 , h_1 and ϕ_1

In this section, to lighten notations, we will often drop the dependence of the functions on y .

PROOF OF PROPOSITION 4.1. First note that $\partial_y h_0(x)$ vanishes for $x \notin (x_-^*, x_+^*)$. We thus assume henceforth that $x \in (x_-^*, x_+^*)$. Taking the derivative with respect to y of the expression (2.5) for the committor, we obtain

$$\partial_y h_0(x) = \frac{2}{\sigma^2} \frac{1}{N} [(1 - h_0(x))I(x) - h_0(x)(J - I(x))] , \quad (\text{C.1})$$

where

$$\begin{aligned} I(x) &= \int_x^{x_+^*} \partial_y V_0(\bar{x}) e^{2V_0(\bar{x})/\sigma^2} d\bar{x} + \frac{\sigma^2}{2} \frac{dx_+^*}{dy} e^{-2h_+/\sigma^2} \\ J &= \frac{\sigma^2}{2} N'(y) = I(x_-^*) - \frac{\sigma^2}{2} \frac{dx_-^*}{dy} e^{-2h_-/\sigma^2} . \end{aligned}$$

By standard Laplace asymptotics (see Appendix D), we obtain

$$h_0(x) \asymp \begin{cases} 1 & \text{if } x \leq x_0^* , \\ \frac{\sigma e^{2V_0(x)/\sigma^2}}{\sigma + |x - x_0^*|} & \text{if } x > x_0^* , \end{cases} \quad 1 - h_0(x) \asymp \begin{cases} \frac{\sigma e^{2V_0(x)/\sigma^2}}{\sigma + |x - x_0^*|} & \text{if } x \leq x_0^* , \\ 1 & \text{if } x > x_0^* . \end{cases} \quad (\text{C.2})$$

Similarly, using the fact that $x \mapsto V_0(x)$ is increasing on (x_-^*, x_0^*) and decreasing on (x_0^*, x_+^*) , we get

$$|I(x)| \lesssim \begin{cases} \sigma^3 & \text{if } x \leq x_0^* , \\ \sigma^2 e^{2V_0(x)/\sigma^2} & \text{if } x > x_0^* , \end{cases} \quad |J - I(x)| \lesssim \begin{cases} \sigma^2 e^{2V_0(x)/\sigma^2} & \text{if } x \leq x_0^* , \\ \sigma^3 & \text{if } x > x_0^* . \end{cases}$$

Substituting in (C.1) yields

$$|\partial_y h_0(x)| \lesssim \frac{1}{\sigma} e^{2V_0(x)/\sigma^2} ,$$

which implies (4.7a). The bound (4.7b) follows in an analogous way from the fact that

$$\partial_{yy} h_0(x) = \frac{2}{\sigma^2} \frac{1}{N} [-2J\partial_y h_0(x) + (1 - h_0(x))\partial_y I(x) - h_0(x)\partial_y (J - I(x))] .$$

The bound (4.8) on $\eta(y)$ is a consequence of the fact that (C.2) yields

$$\eta(y) \lesssim \frac{1}{Z_0} \int_{x_-^*}^{x_+^*} e^{-2V_0(x)/\sigma^2} \frac{\sigma e^{2V_0(x)/\sigma^2}}{\sigma + |x - x_0^*|} dx \lesssim \frac{\sigma \log(\sigma^{-1})}{Z_0}$$

combined with (4.3) and (4.6). □

Remark C.1. Actually, Lemma D.3 provides the sharper bound

$$|I(x)| \lesssim \begin{cases} \sigma^3 & \text{for } x \leq x_0^* , \\ \sigma^2 (\sigma + |x - x_0^*|) e^{2V_0(x)/\sigma^2} & \text{for } x > x_0^* , \end{cases} \quad (\text{C.3})$$

and similarly for $J - I(x)$. ◇

In order to prove Proposition 4.2, we will use the fact that h_1 and its derivatives satisfy certain Poisson boundary value problems. For this purpose, we will repeatedly use the following lemma.

Lemma C.2. *Let $\mathcal{D} = (x_-^*, x_+^*)$, and let φ satisfy the Poisson problem*

$$\begin{cases} ((\mathcal{L}_x + \lambda_1)\varphi)(x) = \psi(x) & x \in \mathcal{D} , \\ \varphi(x) = 0 & x \in \partial\mathcal{D} . \end{cases} \quad (\text{C.4})$$

Assume that there exists a constant c such that $|\psi(x)| \leq ch_0(x)$ for all $x \in \mathcal{D}$. Then

$$\begin{aligned} |\varphi(x)| &\lesssim c\ell(\sigma)h_0(x) \\ |\partial_x \varphi(x)| &\lesssim \frac{1}{\sigma^2} c\ell(\sigma)h_0(x) \end{aligned}$$

holds for all $x \in \mathcal{D}$. Analogous bounds hold with $h_0(x)$ replaced by $1 - h_0(x)$ throughout.

PROOF: Consider first the simpler Poisson problem $\mathcal{L}_x \varphi = \psi$, with zero boundary conditions as in (C.4). Using the second expression for \mathcal{L}_x in (2.3), it is easy to check that its solution is given by

$$\begin{aligned} (\mathcal{L}_x^{-1}\psi)(x) = \frac{2}{\sigma^2} \int_{x_-^*}^x e^{2V_0(x_1)/\sigma^2} \left[\int_{x_-^*}^{x_1} e^{-2V_0(x_2)/\sigma^2} (1 - h_0(x_2)) \psi(x_2) dx_2 \right. \\ \left. - \int_{x_1}^{x_+^*} e^{-2V_0(x_2)/\sigma^2} h_0(x_2) \psi(x_2) dx_2 \right] dx_1 . \end{aligned} \quad (\text{C.5})$$

Using the assumption on ψ and the bounds (C.2) on h_0 , one obtains that for $x \leq x_0^*$,

$$|(\mathcal{L}_x^{-1}\psi)(x)| \lesssim c\ell(\sigma) \lesssim c\ell(\sigma)h_0(x) .$$

A similar conclusion is obtained for $x \geq x_0^*$ using the equivalent expression

$$\begin{aligned} (\mathcal{L}_x^{-1}\psi)(x) = \frac{2}{\sigma^2} \int_x^{x_+^*} e^{2V_0(x_1)/\sigma^2} \left[\int_{x_1}^{x_+^*} e^{-2V_0(x_2)/\sigma^2} h_0(x_2) \psi(x_2) dx_2 \right. \\ \left. - \int_{x_-^*}^{x_1} e^{-2V_0(x_2)/\sigma^2} (1 - h_0(x_2)) \psi(x_2) dx_2 \right] dx_1 . \end{aligned}$$

Corresponding bounds on $\partial_x(\mathcal{L}_x^{-1}\psi)$ are obtained in a similar way, using the derivative with respect to x of (C.5). It thus remains to extend the bounds to $(\mathcal{L}_x + \lambda_1)^{-1}\psi$. This follows readily from the Neumann-type series

$$(\mathcal{L}_x + \lambda_1)^{-1}\psi = \sum_{k \geq 0} (-\lambda_1)^k ((\mathcal{L}_x)^{-1})^k \psi ,$$

bounding each term by repeatedly applying the bounds on \mathcal{L}_x^{-1} and summing the resulting geometric series. \square

PROOF OF PROPOSITION 4.2. Taking the difference of the equations $(\mathcal{L}_x + \lambda_1)(h_0 + h_1) = 0$ and $\mathcal{L}_x h_0 = 0$, we find that h_1 satisfies the Poisson problem (C.4) with

$$\psi(x) = -\lambda_1 h_0(x) .$$

Lemma C.2 thus immediately yields

$$|h_1(x)| \lesssim \lambda_1 \ell(\sigma) h_0(x) , \quad |\partial_x h_1(x)| \lesssim \frac{1}{\sigma^2} \lambda_1 \ell(\sigma) h_0(x) . \quad (\text{C.6})$$

Taking the derivative with respect to y of the equation for h_1 , we obtain that $\partial_y h_1$ satisfies (C.4) with

$$\psi(x) = -\lambda_1'(h_0(x) + h_1(x)) - \lambda_1 \partial_y h_0(x) + \partial_{xy} V_0(x) \partial_x h_1(x) .$$

The bounds on h_0 , $\partial_y h_0$ and (C.6) imply that $\psi(x)$ has order $\ell \lambda_1 h_0(x) / \sigma^2$, so that Lemma C.2 yields

$$|\partial_y h_1(x)| \lesssim \frac{1}{\sigma^2} \lambda_1 \ell(\sigma) h_0(x) , \quad |\partial_{xy} h_1(x)| \lesssim \frac{1}{\sigma^4} \lambda_1 \ell(\sigma) h_0(x) .$$

The bound on $\partial_{yy} h_1$ is obtained in an analogous way, by taking one more derivative with respect to y . \square

PROOF OF PROPOSITION 4.3. We introduce the variables

$$u(y) = -\frac{\phi_+(y)}{\phi_-(y)} , \quad v(y) = -\phi_+(y) \phi_-(y) . \quad (\text{C.7})$$

The orthogonality condition

$$0 = \langle \pi_0, \phi_1 \rangle = \phi_-(y) \langle \pi_0, h_0 + h_1 \rangle + \phi_+(y) \langle \pi_0, 1 - h_0 + \bar{h}_1 \rangle$$

yields

$$u(y) = \frac{\langle \pi_0, h_0 + h_1 \rangle}{\langle \pi_0, 1 - h_0 + \bar{h}_1 \rangle} = e^{-2\bar{\Delta}(y)/\sigma^2} [1 + \mathcal{O}(\lambda_1 \ell)] , \quad (\text{C.8})$$

where we have used (4.5) and Proposition 4.2 to obtain the last equality. The function $v(y)$ is then determined via the normalisation condition

$$\begin{aligned} 1 &= \langle \pi_1, \phi_1 \rangle = \langle \pi_0, \phi_1^2 \rangle \\ &= \frac{v(y)}{u(y)} X(y) + u(y) v(y) Y(y) - 2v(y) Z(y) , \end{aligned} \quad (\text{C.9})$$

where

$$\begin{aligned} X(y) &:= \langle \pi_0, [h_0 + h_1]^2 \rangle = \langle \pi_0, h_0 \rangle [1 + \mathcal{O}(\lambda_1 \ell)] + \mathcal{O}(\eta) , \\ Y(y) &:= \langle \pi_0, [1 - h_0 + \bar{h}_1]^2 \rangle = \langle \pi_0, 1 - h_0 \rangle [1 + \mathcal{O}(\lambda_1 \ell)] + \mathcal{O}(\eta) , \\ Z(y) &:= \langle \pi_0, [h_0 + h_1][1 - h_0 + \bar{h}_1] \rangle = \mathcal{O}(\eta) , \end{aligned}$$

owing to Propositions 4.1 and 4.2. Substituting in (C.9), using the expressions (4.5) of $\langle \pi_0, h_0 \rangle$ and $\langle \pi_0, 1 - h_0 \rangle$ and solving for $v(y)$ yields

$$v(y) = 1 + \mathcal{O}(\lambda_1(y) \ell)$$

thanks in particular to the bound (4.8) on $\eta(y)$. Expressing $\phi_{\pm}(y)$ in terms of $u(y)$ and $v(y)$ yields (4.9a).

The other relations then follow essentially by taking derivatives with respect to y of the above expressions. Differentiating (C.7), we obtain

$$\phi'_+(y) = -\frac{1}{2} \left(\frac{v'(y)}{\phi_-(y)} + \phi_-(y) u'(y) \right) , \quad \phi'_-(y) = -\frac{v'(y) - \phi_-(y)^2 u'(y)}{2\phi_+(y)} . \quad (\text{C.10})$$

Differentiating (C.8) yields

$$u'(y) = \partial_y \frac{\langle \pi_0, h_0 \rangle + \langle \pi_0, h_1 \rangle}{\langle \pi_0, 1 - h_0 \rangle + \langle \pi_0, \bar{h}_1 \rangle} = -2 \frac{\bar{\Delta}'(y)}{\sigma^2} e^{-2\bar{\Delta}(y)/\sigma^2} [1 + \mathcal{O}(\lambda_1(y)\ell)] ,$$

while the derivative of (C.9) gives

$$v'(y) = \frac{u^{-2}u'X - u'Y - u^{-1}X' - uY' + 2Z'}{u^{-1}X + uY' - 2Z} v(y) = \mathcal{O}\left(\frac{\lambda_1(y)\ell}{\sigma^2}\right) .$$

Substituting in (C.10) yields (4.9b). In the same spirit, one obtains

$$u''(y) = \frac{4}{\sigma^4} \left[\bar{\Delta}'(y)^2 - \frac{1}{2} \sigma^2 \Delta''(y) + \mathcal{O}(\lambda_1(y)\ell^3) \right] u(y) , \quad v''(y) = \mathcal{O}\left(\frac{\lambda_1(y)\ell^3}{\sigma^2}\right) ,$$

and plugging this into the derivative of (C.10) yields (4.9c). \square

PROOF OF PROPOSITION 4.4. The expression for $f_{10} = -\sigma^2 \langle \pi_0, \partial_y \phi_1 \rangle$ follows from the expression (4.10b) for $\partial_y \phi_1$, the definition (4.5) of $\bar{\Delta}(y)$, and the fact that $\sigma^2 \langle \pi_0, |\partial_y h_0| \rangle$ has order $\eta(y)$ by Proposition 4.1. A similar argument applies to $f_{11} = -\sigma^2 \langle \pi_0, \phi_1 \partial_y \phi_1 \rangle$.

The expression for g_{10} is obtained by evaluating separately the two summands on the right-hand side of (4.1). Proceeding as for f_{10} , we obtain

$$\sigma^4 \langle \pi_0, \partial_{yy} \phi_1 \rangle = B [\sigma^2 \bar{\Delta}'' + \mathcal{O}(\lambda_1 \ell^3)] .$$

In order to determine $\langle \partial_y \pi_0, \partial_y \phi_1 \rangle$, we note that on one hand,

$$\partial_y \langle \pi_0, h_0 \rangle = \langle \partial_y \pi_0, h_0 \rangle + \langle \pi_0, \partial_y h_0 \rangle = \langle \partial_y \pi_0, h_0 \rangle + \mathcal{O}\left(\frac{\lambda_1 B^2}{\sigma^2}\right) ,$$

while on the other hand, (4.5) implies

$$\partial_y \langle \pi_0, h_0 \rangle = -\frac{2\bar{\Delta}'}{\sigma^2(e^{-\bar{\Delta}/\sigma^2} + e^{\bar{\Delta}/\sigma^2})^2} .$$

This yields

$$\sigma^4 \langle \partial_y \pi_0, \partial_y \phi_1 \rangle = B [-2A(\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2)] ,$$

and implies the stated expression for g_{10} . The computation of g_{11} is similar. \square

For further reference, we list here a few more expressions of particular inner products, which can be derived in the same way as in the above proof:

$$\sigma^4 \langle \pi_0, (\partial_y \phi_1)^2 \rangle = (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2) , \tag{C.11a}$$

$$\sigma^4 \langle \partial_y \pi_1, \partial_y \phi_1 \rangle = -A^2 (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2) , \tag{C.11b}$$

$$\sigma^8 \langle \pi_0, (\partial_{yy} \phi_1)^2 \rangle = (\bar{\Delta}')^4 + 2\sigma^2 A (\bar{\Delta}')^2 \bar{\Delta}'' + \sigma^4 (\bar{\Delta}'')^2 + \mathcal{O}(\lambda_1 \ell^3) . \tag{C.11c}$$

C.2 Bounds on other eigenfunctions

To obtain estimates involving other eigenfunctions than ϕ_1 , it will sometimes be useful to take advantage of the fact that \mathcal{L}_x is conjugated to the Schrödinger operator

$$\tilde{\mathcal{L}}_x = e^{-V_0/\sigma^2} \mathcal{L}_x e^{V_0/\sigma^2} = \frac{\sigma^2}{2} \partial_{xx} - \frac{1}{2\sigma^2} U_0, \quad (\text{C.12})$$

where U_0 is the three-well potential

$$U_0(x, y) = (\partial_x V_0(x, y))^2 - \sigma^2 \partial_{xx} V_0(x, y).$$

In particular, $\tilde{\mathcal{L}}_x$ has the same eigenvalues $-\lambda_n$ as \mathcal{L}_x , and its eigenfunctions ψ_n satisfy

$$\psi_n(x) = \frac{1}{\sqrt{Z_0}} e^{-V_0/\sigma^2} \phi_n(x) = \sqrt{Z_0} e^{V_0/\sigma^2} \pi_n(x). \quad (\text{C.13})$$

In particular, we have the relations

$$\partial_y \psi_n = \frac{1}{\sqrt{\pi_0}} \left[\partial_y \pi_n - \frac{1}{\sigma^2} W \pi_n \right] = \sqrt{\pi_0} \left[\partial_y \phi_n + \frac{1}{\sigma^2} W \phi_n \right] \quad (\text{C.14})$$

between derivatives of eigenfunctions, where we have used

$$\partial_y \pi_0 = \frac{2}{\sigma^2} W \pi_0, \quad W = \langle \pi_0, \partial_y V_0 \rangle - \partial_y V_0. \quad (\text{C.15})$$

Note that by the Feynman–Hellmann theorem, we have

$$\lambda'_n(y) = -\langle \psi_n, \partial_y \tilde{\mathcal{L}}_x \psi_n \rangle = \frac{1}{2\sigma^2} \langle \psi_n, \partial_y U_0 \psi_n \rangle, \quad (\text{C.16})$$

while first-order perturbation theory shows that if $\lambda_n \neq \lambda_m$, then

$$2\sigma^2 \langle \psi_n, \partial_y \psi_m \rangle = \frac{1}{\lambda_m - \lambda_n} \langle \psi_n, \partial_y U_0 \psi_m \rangle. \quad (\text{C.17})$$

This entails in particular the following useful estimate.

Lemma C.3. *For any function $f \in L^2(\pi_0[1 + W^2 + \sigma^2 \partial_y W]^2 dx)$,*

$$\begin{aligned} \sigma^4 \sum_{n \geq 1} \langle \partial_y \pi_n, f \rangle^2 &\lesssim \langle \pi_0, [1 + W^2] f^2 \rangle, \\ \sigma^8 \sum_{n \geq 1} \langle \partial_{yy} \pi_n, f \rangle^2 &\lesssim \langle \pi_0, [1 + W^2 + \sigma^2 \partial_y W]^2 f^2 \rangle. \end{aligned}$$

PROOF: By (C.14), we have

$$\sigma^4 \langle \partial_y \pi_n, f \rangle^2 \leq 2 \langle \sqrt{\pi_0}, W f \psi_n \rangle^2 + 2\sigma^4 \langle \sqrt{\pi_0}, \partial_y \psi_n f \rangle^2.$$

Summing over n yields two terms, the first one being equal to

$$2 \langle \sqrt{\pi_0}, W^2 f^2 \sqrt{\pi_0} \rangle = 2 \langle \pi_0, W^2 f^2 \rangle.$$

As for the second sum, the Cauchy–Schwarz inequality and (C.17) show that it is bounded by

$$2\sigma^4 \sum_{n \geq 1} \langle \partial_y \psi_n, \partial_y \psi_n \rangle \langle \pi_0, f^2 \rangle \leq \frac{1}{2} \langle \pi_0, f^2 \rangle \sum_{n \neq m} \frac{\langle \psi_n, \partial_y U_0 \psi_m \rangle^2}{(\lambda_m - \lambda_n)^2}.$$

The last sum is bounded, because λ_n grows like n^2 , while $\langle \psi_n, \partial_y U_0 \psi_m \rangle$ is bounded uniformly in n and m . This proves the first inequality, and the second one is proved in a similar way, taking one more derivative with respect to y . \square

Remark C.4. It may happen that two eigenvalue $\lambda_n(y)$ and $\lambda_m(y)$ cross for particular value of y . In that case, the bound (C.17) becomes useless, but an equivalent result can be obtained by locally modifying the basis of eigenfunctions. For simplicity, we will not give details of this procedure here, but refer the reader to [27, 28]. \diamond

In the same spirit, the following lemma allows to estimate derivatives of functions expanded in the eigenbasis.

Lemma C.5. Recall that $\mathcal{D} = (x_-^*, x_+^*)$ and let

$$\Phi(x) = \sum_{n \geq 2} \alpha_n \phi_n(x) .$$

Then

$$\langle \pi_0, (\partial_x \Phi)^2 \mathbf{1}_{\mathcal{D}} \rangle \lesssim \frac{1}{\sigma^4} \sum_{n \geq 2} \alpha_n^2 + \frac{1}{\sigma^2} \sum_{n \geq 2} \lambda_n \alpha_n^2 .$$

PROOF: By (C.13), we have

$$\partial_x \Phi(x) = \frac{1}{\sqrt{\pi_0(x)}} \sum_{n \geq 2} \alpha_n \left[\frac{1}{\sigma^2} \partial_x V_0(x) \psi_n(x) + \partial_x \psi_n(x) \right] =: \frac{1}{\sqrt{\pi_0(x)}} [\Psi_1(x) + \Psi_2(x)] .$$

This implies

$$\langle \pi_0, (\partial_x \Phi)^2 \mathbf{1}_{\mathcal{D}} \rangle \leq 2 \langle \Psi_1 \mathbf{1}_{\mathcal{D}}, \Psi_1 \mathbf{1}_{\mathcal{D}} \rangle + 2 \langle \Psi_2 \mathbf{1}_{\mathcal{D}}, \Psi_2 \mathbf{1}_{\mathcal{D}} \rangle .$$

The first term on the right-hand side satisfies the claimed bound since $\partial_x V_0$ is bounded on \mathcal{D} . As for the second term, the fact that ψ_m is an eigenfunction of \mathcal{L}_x implies

$$\langle \partial_x \psi_n, \partial_x \psi_m \rangle = - \langle \psi_n, \partial_{xx} \psi_m \rangle = \frac{2}{\sigma^2} \lambda_n \delta_{nm} - \frac{1}{\sigma^4} \langle \psi_n, U_0 \psi_m \rangle .$$

This yields

$$\langle \Psi_2 \mathbf{1}_{\mathcal{D}}, \Psi_2 \mathbf{1}_{\mathcal{D}} \rangle = \frac{2}{\sigma^2} \sum_{n \geq 2} \lambda_n \alpha_n^2 - \frac{1}{\sigma^4} \langle \Psi, \mathbf{1}_{\mathcal{D}} U_0 \Psi \rangle , \quad \Psi(x) = \sum_{n \geq 2} \alpha_n \psi_n(x) ,$$

which also satisfies the claimed bound. \square

PROOF OF PROPOSITION 4.6. To prove the first two bounds, we note that owing to the completeness of the set of eigenfunctions, one has

$$\sum_{n \geq 0} f_{n0}^2 = \sigma^4 \sum_{n \geq 0} \langle \partial_y \pi_0, \phi_n \rangle \langle \pi_n, W \rangle = 4 \langle \pi_0, W^2 \rangle ,$$

which has order 1. In a similar way, we obtain

$$\sum_{n \geq 0} f_{n1}^2 = 4 \langle \pi_0, \phi_1 W^2 \rangle + 2 \sigma^2 \langle \pi_0, \partial_y \phi_1 W \rangle .$$

Using the Cauchy–Schwarz inequality, (4.11) and Remark 4.5, one obtains that both terms have again order 1. The proof of the bounds involving g_{ni} are similar.

The last two bounds then follow directly from Lemma C.3 with n and m interchanged, taking $f = \phi_n$, since W is bounded uniformly on compact sets, while for large $|x|$, the decay of $\pi_0(x)$ dominates any polynomially growing term. \square

PROOF OF PROPOSITION 4.7. Using again the completeness of the set of eigenfunctions, we have

$$\sum_{n \geq 0} \langle \pi_n, h_0 \rangle^2 = \sum_{n \geq 0} \langle \pi_0, h_0 \phi_n \rangle \langle \pi_n, h_0 \rangle = \langle \pi_0, h_0^2 \rangle = \langle \pi_0, h_0 \rangle - \eta(y) .$$

At the same time, we also have

$$\begin{aligned} \sum_{n=0}^1 \langle \pi_n, h_0 \rangle^2 &= \langle \pi_0, h_0 \rangle^2 + \langle \pi_0, \phi_1 h_0 \rangle^2 \\ &= \langle \pi_0, h_0 \rangle^2 + \left[\phi_-(y) \langle \pi_0, h_0^2 \rangle + \phi_+(y) \langle \pi_0, h_0(1-h_0) \rangle \right]^2 [1 + \mathcal{O}(\lambda_1 \ell)] \\ &= \langle \pi_0, h_0 \rangle^2 [1 + e^{2\bar{\Delta}(y)/\sigma^2}] + \mathcal{O}(\eta(y)) . \end{aligned}$$

The result follows by subtracting the two sums, and using the definition (4.5) of $\bar{\Delta}(y)$. \square

PROOF OF PROPOSITION 4.8. We will spell out the proofs in the case $i = 1$, since the case $i = 0$ is similar, though slightly easier. The first sum can be estimated by noting that

$$\sum_{m \geq 0} f_{1m}^2 = \sigma^4 \sum_{m \geq 0} \langle \pi_0, \partial_y \phi_1 \phi_m \rangle \langle \pi_m, \partial_y \phi_1 \rangle = \sigma^4 \langle \pi_0, (\partial_y \phi_1)^2 \rangle = (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2) ,$$

where we have used (C.11a) in the last step. Since Proposition 4.4 also yields

$$f_{10}^2 + f_{11}^2 = (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2) ,$$

we conclude that the first sum indeed has order $\lambda_1 \ell^2$. In the same spirit,

$$\sum_{m \geq 0} f_{1m} f_{m1} = -\sigma^4 \sum_{m \geq 0} \langle \partial_y \pi_1, \phi_m \rangle \langle \pi_m, \partial_y \phi_1 \rangle = -\sigma^4 \langle \partial_y \pi_1, \partial_y \phi_1 \rangle = A^2 (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2)$$

by (C.11b), while $f_{10} f_{01} + f_{11}^2 = A^2 (\bar{\Delta}')^2 + \mathcal{O}(\lambda_1 \ell^2)$, showing the result for the second sum.

Regarding the third sum, we use the decomposition $g_{1m} = -\ell_{1m} - 2k_{1m}$ given in (4.1) and estimate separately the sums of squares of ℓ_{1m} and k_{1m} . Noting that

$$\sum_{m \geq 0} \ell_{1m}^2 = \sigma^8 \sum_{m \geq 0} \langle \pi_0, \partial_{yy} \phi_1 \phi_m \rangle \langle \pi_m, \partial_{yy} \phi_1 \rangle = \sigma^8 \langle \pi_0, (\partial_{yy} \phi_1)^2 \rangle$$

and using (C.11c), we find that this sum is indeed equal to $\ell_{10}^2 + \ell_{11}^2 + \mathcal{O}(\lambda_1 \ell^3)$. As for the sum of k_{1m}^2 , we note that (4.11) implies

$$\begin{aligned} k_{1m} &= \sigma^2 \bar{\Delta}' \langle \partial_y \pi_m, A \phi_1 + B \rangle + \sigma^4 \langle \partial_y \pi_m, R_1 \rangle \\ &= \bar{\Delta}' A f_{1m} + \sigma^2 \langle \partial_y \pi_m, R_1 \rangle . \end{aligned} \tag{C.18}$$

We have already bounded $\sum_{m \geq 2} f_{1m}^2$, and the sum involving the error term R_1 can be bounded using Lemma C.3 and (4.12). The proof is similar for the other sums. \square

In order to prove Proposition 4.9, we introduce two linear operators Π_\perp and \mathcal{L}_\perp^{-1} defined by

$$\begin{aligned} (\Pi_\perp f)(x) &= \sum_{m \geq 2} \langle \pi_m, f \rangle \phi_m(x) , \\ (\mathcal{L}_\perp^{-1} f)(x) &= - \sum_{m \geq 2} \frac{1}{\lambda_m} \langle \pi_m, f \rangle \phi_m(x) . \end{aligned}$$

The operator Π_\perp is the projection on the complement of the span of ϕ_0 and ϕ_1 , while \mathcal{L}_\perp^{-1} is the Green function of \mathcal{L}_x restricted to this complement. Note that $\mathcal{L}_\perp^{-1} = \mathcal{L}_\perp^{-1} \Pi_\perp = \Pi_\perp \mathcal{L}_\perp^{-1}$.

Lemma C.6. Let \mathcal{G}_0 be the Green function with Dirichlet boundary conditions, given by (C.5) for $x \in \mathcal{D} = (x_-^*, x_+^*)$, and by

$$(\mathcal{G}_0 f)(x) = \begin{cases} -\frac{2}{\sigma^2} \int_x^{x_-^*} e^{2V_0(x_1)/\sigma^2} \int_{-\infty}^{x_1} e^{-2V_0(x_2)/\sigma^2} f(x_2) dx_2 dx_1 & \text{if } x < x_-^*, \\ -\frac{2}{\sigma^2} \int_{x_+^*}^x e^{2V_0(x_1)/\sigma^2} \int_{x_1}^{\infty} e^{-2V_0(x_2)/\sigma^2} f(x_2) dx_2 dx_1 & \text{if } x > x_+^*. \end{cases}$$

Then we have the representation

$$(\mathcal{L}_\perp^{-1} f)(x) = f_- h_0(x) + f_+(1 - h_0(x)) + (\mathcal{G}_0 \Pi_\perp f)(x), \quad (\text{C.19})$$

where the boundary values f_\pm are given by

$$\begin{aligned} f_- &= -\langle \pi_0, \mathcal{G}_0 \Pi_\perp f \rangle - e^{\bar{\Delta}/\sigma^2} \langle \pi_1, \mathcal{G}_0 \Pi_\perp f \rangle [1 + \mathcal{O}(\lambda_1 \ell)], \\ f_+ &= -\langle \pi_0, \mathcal{G}_0 \Pi_\perp f \rangle + e^{-\bar{\Delta}/\sigma^2} \langle \pi_1, \mathcal{G}_0 \Pi_\perp f \rangle [1 + \mathcal{O}(\lambda_1 \ell)]. \end{aligned}$$

PROOF: We view $\mathcal{L}_\perp^{-1} f$ as the solution, on each of the intervals $(-\infty, x_-^*)$, \mathcal{D} and (x_+^*, ∞) , of a Dirichlet–Poisson problem similar to (C.4), but with boundary values f_\pm . The expression (C.19) is checked in the same way as in Lemma C.2, recalling that h_0 is constant outside \mathcal{D} . The boundary values f_\pm follow from the conditions $\langle \pi_0, \mathcal{L}_\perp^{-1} f \rangle = \langle \pi_1, \mathcal{L}_\perp^{-1} f \rangle = 0$, which are equivalent to the linear system

$$\begin{pmatrix} \langle \pi_0, h_0 \rangle & \langle \pi_0, 1 - h_0 \rangle \\ \langle \pi_1, h_0 \rangle & \langle \pi_1, 1 - h_0 \rangle \end{pmatrix} \begin{pmatrix} f_- \\ f_+ \end{pmatrix} = - \begin{pmatrix} \langle \pi_0, \mathcal{G}_0 \Pi_\perp f \rangle \\ \langle \pi_1, \mathcal{G}_0 \Pi_\perp f \rangle \end{pmatrix}.$$

Solving this system, using (4.5) and the fact that

$$\langle \pi_1, h_0 \rangle = -\langle \pi_1, 1 - h_0 \rangle = \frac{1 + \mathcal{O}(\lambda_1 \ell)}{e^{\bar{\Delta}/\sigma^2} + e^{-\bar{\Delta}/\sigma^2}}$$

as a consequence of Propositions 4.1, 4.2 and (4.10a) yields the result. \square

PROOF OF PROPOSITION 4.9. The first sum can be written

$$S_1 := \sum_{m \geq 2} \frac{1}{\lambda_m} f_{1m} f_{m1} = \sigma^4 \langle \partial_y \pi_1, \mathcal{L}_\perp^{-1} \partial_y \phi_1 \rangle.$$

Applying Lemma C.6 and using the representation (4.11) of $\partial_y \phi_1$, we obtain

$$\begin{aligned} S_1 &= \sigma^4 (f_- - f_+) \langle \partial_y \pi_1, h_0 \rangle + \sigma^4 \langle \partial_y \pi_1, \mathcal{G}_0 \Pi_\perp R_1 \rangle \\ &= -\sigma^4 (e^{\bar{\Delta}/\sigma^2} + e^{-\bar{\Delta}/\sigma^2}) \langle \partial_y \pi_1, h_0 \rangle \langle \pi_1, \mathcal{G}_0 \Pi_\perp R_1 \rangle + \sigma^4 \langle \partial_y \pi_1, \mathcal{G}_0 \Pi_\perp R_1 \rangle. \end{aligned}$$

By the expressions (4.10a) of ϕ_1 , we have

$$f_{11} = \sigma^2 \langle \partial_y \pi_1, \phi_1 \rangle = \sigma^2 (e^{\bar{\Delta}/\sigma^2} + e^{-\bar{\Delta}/\sigma^2}) \langle \partial_y \pi_1, h_0 \rangle [1 + \mathcal{O}(\lambda_1 \ell)] + \mathcal{O}(\lambda_1 \ell),$$

which yields

$$\sigma^2 (e^{\bar{\Delta}/\sigma^2} + e^{-\bar{\Delta}/\sigma^2}) \langle \partial_y \pi_1, h_0 \rangle = f_{11} + \mathcal{O}(\lambda_1 \ell) = -A\Delta' + \mathcal{O}(\lambda_1 \ell).$$

Using the fact that $\partial_y \pi_1 = \partial_y \pi_0 \phi_1 + \pi_0 \partial_y \phi_1$ and the expression (C.15) for $\partial_y \pi_0$, we arrive at

$$\begin{aligned} S_1 &= \sigma^2 \langle 2\pi_1 (W + \bar{\Delta}' A + \mathcal{O}(\lambda_1 \ell^2)) + \pi_0 (\bar{\Delta}' B + R_1), \mathcal{G}_0 \Pi_\perp R_1 \rangle \\ &\lesssim \langle \pi_0, |\mathcal{G}_0 \Pi_\perp R_1| \rangle + \langle |\pi_1|, |\mathcal{G}_0 \Pi_\perp R_1| \rangle. \end{aligned}$$

It remains to estimate $\mathcal{G}_0 \Pi_\perp R_1$. The remainder R_1 is a sum of several terms, but the leading contribution comes from $(\phi_- - \phi_+) \sigma^2 \partial_y h_0$. We have

$$\Pi_\perp(\sigma^2 \partial_y h_0) = \sigma^2 \partial_y h_0 + c_0 + c_1 \phi_1 ,$$

where $c_0 = -\langle \pi_0, \sigma^2 \partial_y h_0 \rangle = \mathcal{O}(\lambda_1 \ell B^2)$ and $c_1 = -\langle \pi_0, \sigma^2 \partial_y h_0 \rangle = \mathcal{O}(\lambda_1 \ell B)$. By Lemma C.2, we obtain

$$|\mathcal{G}_0 \Pi_\perp(\sigma^2 \partial_y h_0)| \lesssim h_0(1 - h_0) + c_0 \ell + c_1 \ell |\phi_1| .$$

Thanks to Remark 4.5, we conclude that

$$\begin{aligned} \langle \pi_0, |\mathcal{G}_0 \Pi_\perp(\sigma^2 \partial_y h_0)| \rangle &\lesssim \lambda_1 \ell^2 B^2 , \\ \langle |\pi_1|, |\mathcal{G}_0 \Pi_\perp(\sigma^2 \partial_y h_0)| \rangle &\lesssim \lambda_1 \ell^2 B . \end{aligned}$$

After estimating the other terms of R_1 , we arrive at the bound $S_1 \lesssim \lambda_1 \ell^2$.

The second sum can be written

$$\sum_{m \geq 2} \frac{1}{\lambda_m} f_{1m} g_{m1} = \sigma^6 \langle \partial_{yy} \pi_1, \mathcal{L}_\perp^{-1} \partial_y \phi_1 \rangle ,$$

and can be estimated in a similar way, expressing $\partial_{yy} \pi_1$ in terms of $\partial_y \phi_1$ and $\partial_{yy} \phi_1$, where the latter can be written in terms of ϕ_1 and a remainder using (4.10c).

The third sum can be written, using (C.18), as

$$- \sum_{m \geq 2} \frac{1}{\lambda_m} [\ell_{1m} + 2k_{1m}] f_{m1} = \sigma^6 \langle \partial_y \pi_1, \mathcal{L}_\perp^{-1} \partial_{yy} \phi_1 \rangle + 2\sigma^2 S_1 - 2\sigma^6 \sum_{m \geq 2} \frac{f_{m1}}{\lambda_m} \langle \partial_y \pi_m, R_1 \rangle ,$$

where the last sum can be estimated via the Cauchy–Schwarz inequality. The case of the last sum is similar. \square

PROOF OF PROPOSITION 4.14. The expression (4.24) of α_\perp^* can be rewritten as

$$\alpha_n^*(y) = -\varepsilon \frac{1}{\lambda_n} \langle \partial_y \pi_0 + \alpha_1 \partial_y \pi_1, \phi_n \rangle .$$

Therefore, we have

$$\langle \pi_0, \Phi_\perp^* f \rangle = \sum_{n \geq 2} \alpha_n^*(y) \langle \pi_n, f \rangle = -\varepsilon \langle \partial_y \pi_0 + \alpha_1 \partial_y \pi_1, \mathcal{L}_\perp^{-1} f \rangle .$$

This quantity can be estimated in a similar way as S_1 in the previous proof, by noting that the only thing that really matters is the fact that f can be bounded by a constant times $h_0(1 - h_0)$. \square

D Laplace asymptotics

In this appendix, we gather a few standard results on Laplace asymptotics, which can be obtained from those in [38].

Lemma D.1. *Let $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ satisfy the following conditions:*

- $|f(x)|$ has at most polynomial growth for large x ;
- f is bounded away from 0 in neighbourhoods I_\pm of $x_-^*(y)$ and $x_+^*(y)$, whose size does not depend in σ ;

- f'/f and f''/f are bounded uniformly in σ in I_{\pm} .

Then

$$\langle \pi_0, f \rangle = \frac{f(x_{-}^{*}(y)) e^{-\bar{\Delta}(y)/\sigma^2} + f(x_{+}^{*}(y)) e^{\bar{\Delta}(y)/\sigma^2}}{e^{-\bar{\Delta}(y)/\sigma^2} + e^{\bar{\Delta}(y)/\sigma^2}} [1 + \mathcal{O}(\sigma^2)] .$$

PROOF: Using the change of variables $x = x_{\pm}^{*}(y) + \sigma z / (\sqrt{2}\omega_{-}(y))$, one obtains

$$\begin{aligned} & \int_{I_{\pm}} e^{-2V_0(x,y)/\sigma^2} f(x) dx \\ &= \frac{\sigma}{\sqrt{2}\omega_{-}(y)} \int_{\tilde{I}_{\pm}} \left[f(x_{\pm}^{*}) + \frac{\sigma}{\sqrt{2}\omega_{-}(y)} z f'(x_{\pm}^{*}(y)) + \frac{\sigma^2}{4\omega_{-}(y)^2} z^2 f''(x_{\pm}^{*}(y) + \theta) \right] e^{-z^2/2} dz \\ &= \frac{\sigma\sqrt{\pi}}{\omega_{-}(y)} f(x_{\pm}^{*}(y)) [1 + \mathcal{O}(\sigma^2)] , \end{aligned} \quad (\text{D.1})$$

where $\tilde{I}_{\pm} = \sqrt{2}\omega_{\pm}(y)(x - x_{\pm}^{*}(y))/\sigma$. Furthermore, the integral over $\mathbb{R} \setminus (I_{-} \cup I_{+})$ is negligible with respect to the sum of these two integrals. The result then follows from applying (D.1) first to $f = 1$ to estimate $Z_0(y)$, and then to general f satisfying the stated assumptions. \square

We will also need estimates involving the integral of e^{2V_0/σ^2} against a function vanishing polynomially at $x = x_0^{*}$. To ease notation, we will assume that $x_0^{*} = 0$, and write

$$V_0(x) = -\frac{1}{2}\omega_0^2 x^2 + W(x)$$

where $W(x) = \mathcal{O}(x^3)$. Consider the integrals

$$\begin{aligned} I_n &= \int_{-\delta}^{\delta} x^n e^{2V_0(x)/\sigma^2} dx , \\ J_n(x) &= e^{-2V_0(x)/\sigma^2} \int_x^{\delta} x_1^n e^{2V_0(x_1)/\sigma^2} dx_1 , \end{aligned}$$

where $n \in \mathbb{N}_0$ and δ has order 1.

Lemma D.2. *We have the asymptotics*

$$I_n = \begin{cases} \Gamma\left(\frac{n+1}{2}\right) \frac{\sigma^{n+1}}{\omega_0^{n+1}} [1 + \mathcal{O}(\sigma^2)] & \text{if } n \text{ is even} , \\ \mathcal{O}(\sigma^{n+2}) & \text{if } n \text{ is odd} . \end{cases} \quad (\text{D.2})$$

PROOF: The case of even n follows from a direct application of [38, Theorem 8.1], where the fact that the error has order σ^2 is due to the leading term of W being odd. When n is odd, we use integration by parts to obtain

$$I_n = -\frac{\sigma^2}{2\omega_0^2} e^{-\omega_0^2 x^2/\sigma^2} x^{n-1} e^{2W(x)/\sigma^2} \Big|_{-\delta}^{\delta} + \frac{\sigma^2}{2\omega_0^2} \int_{-\delta}^{\delta} e^{-\omega_0^2 x^2/\sigma^2} \frac{d}{dx} \left[x^{n-1} e^{2W(x)/\sigma^2} \right] dx . \quad (\text{D.3})$$

If $n = 1$, the integral has order σ^3 by (D.2) with $n = 2$, while the boundary terms are negligible. For $n \geq 3$, we obtain $I_n = \mathcal{O}(\sigma^2 I_{n-1}) + \mathcal{O}(\sigma^{n+2})$, so that the result follows by induction. \square

In particular, we have

$$I_0 = \frac{\sqrt{\pi}}{\omega_0} \sigma [1 + \mathcal{O}(\sigma^2)] , \quad I_2 = \frac{\sqrt{\pi}}{2\omega_0^3} \sigma^3 [1 + \mathcal{O}(\sigma^2)] .$$

Lemma D.3. *There is a constant $M > 0$, independent of σ and δ , such that*

$$|J_n(x)| \leq M\sigma^2(\sigma + |x|)^{n-1} \quad (\text{D.4})$$

holds for any $n \in \mathbb{N}$ and any $x \in [-\delta, \delta]$. In particular,

$$J_1(x) = \frac{\sigma^2}{2\omega_0^2} + \mathcal{O}(\sigma^2(|x| + \sigma)) , \quad (\text{D.5})$$

$$J_3(x) = \frac{\sigma^4}{2\omega_0^4} + \mathcal{O}(\sigma^2(x^2 + \sigma^3)) . \quad (\text{D.6})$$

PROOF: For $x = 0$ and for $|x|$ of order 1, (D.4) follows from [38, Theorem 8.1]. For intermediate x , we can use the fact that

$$\begin{aligned} \frac{\sigma^2}{2} J'_n(x) &= -\partial_x V_0 J_n(x) - \frac{\sigma^2}{2} x^n \\ &= [\omega_0^2 x - \partial_x W(x)] J_n(x) - \frac{\sigma^2}{2} x^n , \end{aligned}$$

whose right-hand side vanishes for $J_n(x) = J_n^*(x) = \mathcal{O}(\sigma^2 x^{n-1})$. Since $J_n(x) - J_n^*(x)$ is a decreasing function of x for $x < 0$, and (D.4) is satisfied for negative x of order 1, it holds for all $x \in [0, \delta]$. A similar argument applies for $x > 0$ by changing x into $-x$. To prove (D.5), we use a similar integration-by-parts argument as in (D.3) to obtain

$$e^{2V_0(x)/\sigma^2} J_1(x) = \frac{\sigma^2}{2\omega_0^2} [e^{2V_0(x)/\sigma^2} - e^{2V_0(\delta)/\sigma^2}] + \frac{1}{\omega_0^2} \int_x^\delta \partial_x W e^{2V_0(x_1)/\sigma^2} dx_1 .$$

The integral on the right-hand side can be bounded using (D.4) with $n = 2$, while the term $e^{2V_0(\delta)/\sigma^2}$ is negligible. Finally, (D.6) follows from the integration-by-parts relation

$$e^{2V_0(x)/\sigma^2} J_1(x) = -\frac{x^2}{2} [e^{2V_0(x)/\sigma^2} - e^{2V_0(\delta)/\sigma^2}] - \frac{1}{\sigma^2} \int_x^\delta x_1^2 \partial_x V_0 e^{2V_0(x_1)/\sigma^2} dx_1 ,$$

expressing the integral on the right-hand side in terms of $J_3(x)$. □

References

- [1] Yuri Bakhtin. Gumbel distribution in exit problems. [arXiv:1307.7060](#), 2013.
- [2] Yuri Bakhtin. On Gumbel limit for the length of reactive paths. *Stochastics and Dynamics*, 15(1):1550001, 2015.
- [3] Florent Barret. Sharp asymptotics of metastable transition times for one dimensional SPDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(1):129–166, 2015.
- [4] Nils Berglund. Kramers’ law: validity, derivations and generalisations. *Markov Process. Related Fields*, 19(3):459–490, 2013.
- [5] Nils Berglund. Noise-induced phase slips, log-periodic oscillations, and the Gumbel distribution. *Markov Process. Related Fields*, 22(3):467–505, 2016.
- [6] Nils Berglund. An introduction to singular stochastic PDEs: Allen–Cahn equations, metastability and regularity structures. Lecture notes, Sarajevo Stochastic Analysis Winter School. [arXiv/1901.07420](#), January 2019.

- [7] Nils Berglund, Giacomo Di Gesù, and Hendrik Weber. An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two. *Electron. J. Probab.*, 22:1–27, 2017.
- [8] Nils Berglund and Barbara Gentz. Universality of first-passage and residence-time distributions in non-adiabatic stochastic resonance. *Europhys. Letters*, 70:1–7, 2005.
- [9] Nils Berglund and Barbara Gentz. *Noise-induced phenomena in slow-fast dynamical systems. A sample-paths approach*. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2006.
- [10] Nils Berglund and Barbara Gentz. Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers’ law and beyond. *Electron. J. Probab.*, 18:no. 24, 58, 2013.
- [11] Nils Berglund and Barbara Gentz. On the noise-induced passage through an unstable periodic orbit II: General case. *SIAM J. Math. Anal.*, 46(1):310–352, 2014.
- [12] Freddy Bouchet and Julien Reygner. Generalisation of the Eyring–Kramers transition rate formula to irreversible diffusion processes. *Annales Henri Poincaré*, pages 1–34, 2016.
- [13] Anton Bovier and Frank den Hollander. *Metastability. A potential-theoretic approach*, volume 351 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2015.
- [14] Anton Bovier, Michael Eckhoff, Véronique Gayrard, and Markus Klein. Metastability in reversible diffusion processes. I. Sharp asymptotics for capacities and exit times. *J. Eur. Math. Soc. (JEMS)*, 6(4):399–424, 2004.
- [15] Anton Bovier, Véronique Gayrard, and Markus Klein. Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues. *J. Eur. Math. Soc. (JEMS)*, 7(1):69–99, 2005.
- [16] Frédéric Cérou, Arnaud Guyader, Tony Lelièvre, and Florent Malrieu. On the length of one-dimensional reactive paths. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 10(1):359–389, 2013.
- [17] Martin V. Day. On the exponential exit law in the small parameter exit problem. *Stochastics*, 8:297–323, 1983.
- [18] Martin V. Day. Some phenomena of the characteristic boundary exit problem. In *Diffusion processes and related problems in analysis, Vol. I (Evanston, IL, 1989)*, volume 22 of *Progr. Probab.*, pages 55–71. Birkhäuser Boston, Boston, MA, 1990.
- [19] Martin V. Day. Conditional exits for small noise diffusions with characteristic boundary. *Ann. Probab.*, 20(3):1385–1419, 1992.
- [20] Martin V. Day. Cycling and skewing of exit measures for planar systems. *Stoch. Stoch. Rep.*, 48:227–247, 1994.
- [21] Martin V. Day. Exit cycling for the van der Pol oscillator and quasipotential calculations. *J. Dynam. Differential Equations*, 8(4):573–601, 1996.
- [22] Weinan E and Eric Vanden-Eijnden. Towards a theory of transition paths. *J. Stat. Phys.*, 123(3):503–523, 2006.
- [23] H. Eyring. The activated complex in chemical reactions. *Journal of Chemical Physics*, 3:107–115, 1935.
- [24] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.
- [25] Mark I. Freidlin. On stable oscillations and equilibriums induced by small noise. *J. Stat. Phys.*, 103:283–300, 2001.
- [26] Bernard Helffer, Markus Klein, and Francis Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. *Mat. Contemp.*, 26:41–85, 2004.

- [27] Alain Joye. An adiabatic theorem for singularly perturbed hamiltonians. *Annales de l'I.H.P. Physique théorique*, 63(2):231–250, 1995.
- [28] Alain Joye, François Monti, Stéphane Guérin, and Hans R. Jauslin. Adiabatic evolution for systems with infinitely many eigenvalue crossings. *J. Math. Phys.*, 40(11):5456–5472, 1999.
- [29] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7:284–304, 1940.
- [30] Claudio Landim, Mauro Mariani, and Insuk Seo. Dirichlet’s and Thomson’s principles for non-selfadjoint elliptic operators with application to non-reversible metastable diffusion processes. *Arch. Ration. Mech. Anal.*, 231(2):887–938, 2019.
- [31] Dorian Le Peutrec. Local WKB construction for Witten Laplacians on manifolds with boundary. *Anal. PDE*, 3(3):227–260, 2010.
- [32] Dorian Le Peutrec. Small eigenvalues of the Witten Laplacian acting on p -forms on a surface. *Asymptot. Anal.*, 73(4):187–201, 2011.
- [33] Dorian Le Peutrec and Laurent Michel. Sharp spectral asymptotics for non-reversible metastable diffusion processes. *arXiv:1907.09166*, 2019.
- [34] Dorian Le Peutrec, Francis Nier, and Claude Viterbo. Precise Arrhenius law for p -forms: the Witten Laplacian and Morse-Barannikov complex. *Ann. Henri Poincaré*, 14(3):567–610, 2013.
- [35] Jianfeng Lu and James Nolen. Reactive trajectories and the transition path process. *Probab. Theory Related Fields*, 161(1-2):195–244, 2015.
- [36] Fabio Martinelli, Enzo Olivieri, and Elisabetta Scoppola. Small random perturbations of finite- and infinite-dimensional dynamical systems: unpredictability of exit times. *J. Statist. Phys.*, 55(3-4):477–504, 1989.
- [37] Boris Nectoux and Dorian Le Peutrec. Small eigenvalues of the Witten Laplacian with Dirichlet boundary conditions: the case with critical points on the boundary. *arXiv:1907.07517*, 2019.
- [38] F.W.J. Olver. *Asymptotics and Special Functions*. Academic Press, 1974.
- [39] Pavlos Tsatsoulis and Hendrik Weber. Exponential loss of memory for the 2-dimensional Allen–Cahn equation with small noise. *Probability Theory and Related Fields*, 177:257–322, 2020. *arXiv:1808.04171*.

Contents

1	Introduction	1
2	Results	4
2.1	Set-up	4
2.2	Static system	5
2.3	Two-state jump process	6
2.4	Invariant measure	8
2.5	Main results: expected transition time	9
2.6	Discussion	11
2.7	Outline of the proof	12
3	Non-reversible potential theory	13
3.1	Invariant density	13
3.2	Adjoint process	14
3.3	Committor and capacity	15
3.4	Equilibrium measure and mean hitting time	15
3.5	Variational principles	15
4	The invariant measure	17
4.1	Eigenfunctions of the static system	17
4.2	Proof of Theorem 2.2	20
5	Adjoint process and committors	27
6	Estimating the capacity	30
6.1	Upper bound on the capacity	30
6.2	Lower bound on the capacity	32
7	Proof of the main result	34
A	The two-state jump process	36
B	Proofs of the potential-theoretic results	37
B.1	Invariant density	37
B.2	Capacity	37
B.3	Equilibrium measure and mean hitting time	38
B.4	Variational principles	39
C	Estimates on static eigenfunctions	41
C.1	Bounds on h_0 , h_1 and ϕ_1	41
C.2	Bounds on other eigenfunctions	45
D	Laplace asymptotics	49

Institut Denis Poisson (IDP)
 Université d'Orléans, Université de Tours, CNRS – UMR 7013
 Bâtiment de Mathématiques, B.P. 6759
 45067 Orléans Cedex 2, France
E-mail address: nils.berglund@univ-orleans.fr