Topics in Gaussian Wiener chaos expansion

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Lecture notes

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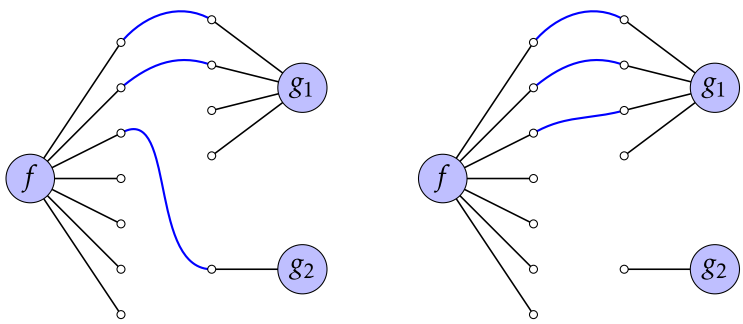

— Version of May 20, 2026 —

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Preface

These notes have been written for a series of lectures to be given at the 44th Finnish Summer School on Probability and Statistics in Lammi, Finland, from 25th to 29th May, 2026. They contain an introduction to Wiener chaos decomposition in finite dimension, a construction of Gaussian fields on the torus, including white noise and the Gaussian free field, and applications to the Φ^4 model. They do *not* cover other important aspects of the topic, such as stochastic integration, stochastic PDEs and Malliavin calculus. Sections with a $*$ are more technical, and can safely be skipped in a first reading.

The material included in these notes is mostly based on the monograph [Nua06], the lecture notes [Hai26], and the monograph [Ber22]. Other useful resources on the topic include [PT11, Jan08, SS05]. This is a preliminary version of the notes, that may contain mistakes and typos. Feel free to let me know if you find any.

Thanks are due to Dario Gasbarra for organising the Summer School and inviting me to give these lectures, thereby providing the motivation to compile these notes, as well as supporting institutions.

Chapter 1

The one-dimensional case

We start this exposition with the very simple situation of a one-dimensional Gaussian random variables, since this allows us to introduce many objects that will become important in higher dimension in a relatively simple setting.

1.1 Gaussian random variables

Our fundamental objects are Gaussian random variables, whose definition we recall here.

Definition 1.1.1: Gaussian random variable

Let $\mathbb{R}$ be the real line, equipped with the σ-algebra $\mathscr{B}$ of Borel sets and Lebesgue measure $\mathrm{d}x$. A random variable $X : \mathbb{R} \to \mathbb{R}$ is a (one-dimensional) Gaussian random variable with mean m and variance σ^2 if its law is

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi\sigma^2}} \,\mathrm{e}^{-(x-m)^2/(2\sigma^2)} \,\mathrm{d}x \,.$$

In that case, we write $X \sim \mathcal{N}(m, \sigma^2)$.

We summarise some fundamental properties of Gaussian random variables as follows.

Proposition 1.1.2: Basic properties of normal random variables

1. One has $X \sim \mathcal{N}(m, \sigma^2)$ if, and only if, $X = m + \sigma Y$ with $Y \sim \mathcal{N}(0, 1)$.
2. Assume $X \sim \mathcal{N}(m_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(m_2, \sigma_2^2)$ are defined on a common probability space, and let $Z = X + Y$. Then Z is Gaussian, with parameters

$$Z \sim \mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{Cov}(X, Y)) \,.$$

3. Two Gaussian variables X and Y are independent if, and only if, they are uncorrelated, that is, $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

The first property states that all one-dimensional Gaussian random variables are equivalent by an affine transformation. The second one states that Gaussian random variables are stable, and is at the core of the central limit theorem. The third property is only true for vary special random variables: while independence always implies non-correlation, the converse is false in general.

Because of the first property, we will almost always focus on the case $\mu = 0$, that is, when X is *centered*. Whenever possible, we will assume $\sigma^2 = 1$, although it will sometimes be useful to allow for different variances.

Our main interest will be expectations of functions of Gaussian random variables. Assume $X \sim \mathscr{N}(0,1)$, and let $f : \mathbb{R} \to \mathbb{R}$. Then

$$\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\mu(\mathrm{d}x)\,,$$

provided the integral is absolutely convergent.

Example 1.1.3: Laplace transform

Let $f(x) = \mathrm{e}^{tx}$ for $t \in \mathbb{R}$. Then, using completion of squares to write

$$tx - \frac{x^2}{2} = -\frac{1}{2}(x-t)^2 + \frac{t^2}{2}\,,$$

we find

$$\mathbb{E}[\mathrm{e}^{tX}] = \int_{-\infty}^{\infty} \mathrm{e}^{tx-x^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \mathrm{e}^{t^2/2} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-t)^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \mathrm{e}^{t^2/2}\,. \tag{1.1.1}$$

For general functions f, explicit expressions of $\mathbb{E}[f(X)]$ are not available. One possible strategy to compute expectations efficiently is to compute expectations of a set of appropriate basis functions. One choice of basis functions is given by monomials.

Proposition 1.1.4: Moments of Gaussian random variables

Let $X \sim \mathscr{N}(0,1)$. For any $n \in \mathbb{N}$, one has

$$\mathbb{E}[X^n] = \begin{cases} (n-1)!! & \text{if } n \text{ is even}\,, \\ 0 & \text{if } n \text{ is odd}\,, \end{cases} \tag{1.1.2}$$

where

$$(n-1)!! = \prod_{k=0}^{n/2-1} (2k+1) = 1\cdot 3\cdot 5 \dots (n-3)(n-1)$$

is the double factorial.

Exercise 1.1.5

Prove Proposition 1.1.4 in two different ways:

1. by using (1.1.1);
2. by showing, using integration by parts, that

$$\mathbb{E}[X^{n+1}] = n\mathbb{E}[X^{n-1}] \qquad \text{for all } n \geqslant 1\,.$$

If f admits an entire series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with positive radius of convergnce R, then Proposition 1.1.4 allows to compute

$$\mathbb{E}[f(X)] = \sum_{n\geqslant 0} a_n \mathbb{E}[X^n]\,. \tag{1.1.3}$$

1.2 Hermite polynomials

Our main workhorse will be Hermite polynomials. In this section, we review several of their definitions, and how they are related to geometry/liner algebra, probability theory, analysis, algebra, and combinatorics.

1.2.1 Gram–Schmidt orthogonalisation

One drawback of using moments to compute expectations is that the basis $(X^n)_{n\geqslant 0}$ is not orthogonal. Here orthogonality is defined with respect to the inner product

$$\langle f, g\rangle = \mathbb{E}[fg] ,$$

meaning that random variables are orthogonal if, and only if, they are independent. Indeed, Propsition 1.1.4 shows that X^n and X^m are orthogonal, according to this definition, if and only if $n+m$ is odd.

However, a basis can always be turned into an orthogonal basis by the *Gram–Schmidt procedure*. Let $(v_n)_{n\geqslant 0}$ be an arbitrary basis of a vector space. Then the Gram–Schmidt procedure provides an orthogonal basis $(u_n)_{n\geqslant 0}$ defined inductively by $u_0 = v_0$ and

$$u_n = v_n - \sum_{k=0}^{n-1} \frac{\langle v_n, u_k\rangle}{\langle u_k, u_k\rangle} u_k , \qquad n \geqslant 1 .$$

This means that u_n is obtained by subtracting from v_n its projection on the plane spanned by the $n-1$ first u_k. It is easy to show by induction that u_n is orthogonal to all vectors $u_0, \dots, u_{n-1}$. Indeed, the base case $n = 0$ is trivially true, while for $n \geqslant 1$, one has, for any $\ell \in \{0, \dots, n-1\}$,

$$\langle u_n, u_\ell\rangle = \langle v_n, v_\ell\rangle - \frac{\langle v_n, u_\ell\rangle}{\langle u_\ell, u_\ell\rangle}\langle u_\ell, u_\ell\rangle = 0 .$$

Let us apply this procedure to the basis $(X^n)_{n\geqslant 0}$, denoting the resulting orthogonal basis by $(H_n(X))_{n\geqslant 0}$. The first steps are

$$\begin{aligned}
H_0(X) &= X^0 = 1 , \\
H_1(X) &= X - \frac{\langle X, 1\rangle}{\langle 1, 1\rangle} 1 = X , \\
H_2(X) &= X^2 - \frac{\langle X^2, X\rangle}{\langle X, X\rangle} X - \frac{\langle X^2, 1\rangle}{\langle 1, 1\rangle} 1 = X^2 - 0 - \frac{\mathbb{E}[X^2]}{\mathbb{E}[1]} 1 = X^2 - 1 , \\
H_3(X) &= X^3 - \frac{\langle X^3, H_2(X)\rangle}{\langle H_2(X), H_2(X)\rangle} H_2(X) - \frac{\langle X^3, X\rangle}{\langle X, X\rangle} X - \frac{\langle X^3, 1\rangle}{\langle 1, 1\rangle} 1 = X^3 - 0 - \frac{\mathbb{E}[X^4]}{\mathbb{E}[1]} X = X^3 - 3X .
\end{aligned}$$

Exercise 1.2.1

Show by this method that

$$H_4(X) = X^4 - 6X^2 + 3 .$$

Check that the random variables $(H_n(X))_{0\leqslant n\leqslant 4}$ are mutually independent.

Table 1.1 shows the first 11 Hermite polynomials. Clearly, while the Gram–Schmidt procedure does produce an orthogonal basis, the computations are not efficient. We are thus going to look for more efficient methods. One of them, which we discuss in the next section, uses the notion of *cumulant expansion*.

n	$H_n(x)$
0	1
1	x
2	x^2-1
3	x^3-3x
4	x^4-6x^2+3
5	x^5-10x^3+15x
6	$x^6-15x^4+45x^2-15$
7	$x^7-21x^5+105x^3-105x$
8	$x^8-28x^6+210x^4-420x^2+105$
9	$x^9-36x^7+378x^5-1260x^3+945$
10	$x^{10}-45x^8+630x^6-3150x^4+472x^2-945$

Table 1.1 – List of the first Hermite polynomials.

Remark 1.2.2: Scaling conventions

One finds several conventions for Hermite polynomials in the literature. What we use here are the "probabilists' Hermite polynomials". Another convention, called the "physicists' Hermite polynomials", uses the scaled version

$$\widetilde{H}_n(x) = 2^{n/2} H_n(\sqrt{2}x)\,.$$

Yet another convention, used in [Nua06], is to multiply $H_n(x)$ by $1/n!$.

1.2.2 Hermite polynomials and cumulants

Definition 1.2.3: Cumulant expansion

Let X be a random variable such that $\mathbb{E}[\mathrm{e}^{tX}]$ exists for all t in an open interval $(-\delta,\delta)$, and write

$$\mathbb{E}[\mathrm{e}^{tX}] = \sum_{n\geqslant 0} \mu_n \frac{t^n}{n!}\,, \qquad \mu_n = \mathbb{E}[X^n]\,. \tag{1.2.1}$$

Then the *cumulant expansion* of X is given by

$$K_X(t) = \log \mathbb{E}[\mathrm{e}^{tX}] = \sum_{n\geqslant 0} \kappa_n \frac{t^n}{n!}\,. \tag{1.2.2}$$

The coefficients μ_n are called *moments* of X, while the κ_n are called *cumulants*.

In the case of a Gaussian $X \sim \mathscr{N}(0,1)$, the situation is particularly simple. Indeed, we have by (1.1.1)

$$K_X(t) = \log(\mathrm{e}^{t^2/2}) = \frac{t^2}{2}\,,$$

so that the cumulants are given by

$$\kappa_n = \begin{cases} 1 & \text{if } n = 2\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.2.3}$$

Consider now the function

$$G(t,x) = \frac{e^{tx}}{\mathbb{E}[e^{tX}]} = e^{tx-t^2/2} . \tag{1.2.4}$$

where $t \geqslant 0$ and $x \in \mathbb{R}$.

Proposition 1.2.4: Generating function

G is the generating function of Hermite polynomials, that is

$$G(t,x) = \sum_{n\geqslant 0} \frac{t^n}{n!} H_n(x) . \tag{1.2.5}$$

We will proceed in several steps to prove this result. First, we make an easy observation: if $X \sim \mathcal{N}(0,1)$ we have

$$1 = \mathbb{E}[G(t,X)] = \sum_{n\geqslant 0} \frac{t^n}{n!} \mathbb{E}[H_n(X)] .$$

Since this is valid for all t in $(-\delta, \delta)$, uniqueness of coefficients of power series shows that

$$\mathbb{E}[H_n(X)] = \delta_{n0} = \begin{cases} 1 & \text{if } n = 0 , \\ 0 & \text{otherwise} . \end{cases}$$

In other terms, all $H_n(X)$ with $n \geqslant 1$ are centered, and therefore orthogonal to $H_0(X) = 1$. The following result is a generalisation of this observation.

Proposition 1.2.5: Orthogonality of the $H_n(X)$

For any $n, m \in \mathbb{N}_0$, one has

$$\mathbb{E}[H_n(X)H_m(X)] = n!\delta_{nm} = \begin{cases} n! & \text{if } n = m , \\ 0 & \text{otherwise} . \end{cases} \tag{1.2.6}$$

Proof: We compute the expectation of $G(t,X)G(s,X)$ in two different ways. The first way is

$$\begin{aligned} \mathbb{E}[G(t,X)G(s,X)] &= \mathbb{E}[e^{tX-t^2/2} e^{sX-s^2/2}] \\ &= e^{-(t^2+s^2)/2} \mathbb{E}[e^{(t+s)X}] \\ &= e^{-(t^2+s^2)/2} e^{(t+s)^2/2} \\ &= e^{ts} \\ &= \sum_{n\geqslant 0} \frac{t^n s^n}{n!} . \end{aligned}$$

The second way to perform the computation is, using (1.2.5),

$$\mathbb{E}[G(t,X)G(s,X)] = \sum_{n\geqslant 0} \sum_{m\geqslant 0} \frac{t^n}{n!} \frac{s^m}{m!} \mathbb{E}[H_n(X)H_m(X)] .$$

Comparing the two obtained power series yields the result, by uniqueness of the coefficients of a series. □

This result shows that the $H_n(X)$ defined via (1.2.5) form *an* orthogonal basis. It remains to show that this basis is identical with the one obtained by the Gram–Schmidt procedure. We do this with the following result.

Lemma 1.2.6: Recursive relation between Hermite polynomials

For any $n \geqslant 0$, one has

$$H_{n+1}(x) = xH_n(x) - H_n'(x)\,. \tag{1.2.7}$$

Proof: This follows from the relation

$$\frac{\partial}{\partial t}G(t,x) = (x-t)G(t,x) = xG(t,x) - \frac{\partial}{\partial x}G(t,x)\,.$$

Indeed, we have

$$\frac{\partial}{\partial t}G(t,x) = \sum_{n\geqslant 1}\frac{t^{n-1}}{(n-1)!}H_n(x) = \sum_{n\geqslant 0}\frac{t^n}{n!}H_{n+1}(x)\,,$$

while

$$xG(t,x) - \frac{\partial}{\partial x}G(t,x) = \sum_{n\geqslant 0}\frac{t^n}{n!}(xH_n(x) - H_n'(x))\,.$$

Comparing the coefficients of the last two power series yields the result. □

The recursive relation (1.2.7) provides a quicker way to compute Hermite polynomials than the Gram–Schmidt procedure, starting with $H_0(x) = 1$. It also allows us to complete the proof of Proposition 1.2.4.

Proof of Proposition 1.2.4. We have already shown that the $(H_n)_{n\geqslant 0}$ form an orthogonal family. It remains to show that they coïncide with the polynomials constructed by the Gram–Schmidt procedure. Evaluating (1.2.5) at $t = 0$ yields $1 = G(0,x) = H_0(x)$. From (1.2.7), it follows by induction on n that $H_n(x)$ has degree n, with x^n having coefficient 1, which is also the case for the H_n obtained via the Gram–Schmidt procedure. □

Another useful consequence of the expression (1.2.5) of the generating function is the following generalisation of Proposition 1.2.5, which allows to transform products of Hermite functions into sums of such functions. One can think of it as an analogue of product-sum formulas in trigonometry, which are useful to compute Fourier series.

Proposition 1.2.7: Product–sum formula

For any $n, m \geqslant 0$, one has

$$H_n(x)H_m(x) = \sum_{p=0}^{n\wedge m} p!\binom{n}{p}\binom{m}{p}H_{n+m-2p}(x)\,, \tag{1.2.8}$$

where $n \wedge m$ denotes the minimum of n and m.

Proof: We start by observing that

$$\begin{aligned} G(t,x)G(s,x) &= \mathrm{e}^{(t+s)x-(t^2+s^2)/2} \\ &= \mathrm{e}^{ts}\,\mathrm{e}^{(t+s)x-(t+s)^2/2} \\ &= \mathrm{e}^{ts}\,G(t+s,x)\,. \end{aligned}$$

Expanding the exponential and $G(t+s,x)$, and using Newton's binomial formula yields

$$
\begin{aligned}
G(t,x)G(s,x) &= \sum_{p=0}^{\infty} \frac{(ts)^p}{p!} \sum_{q=0}^{\infty} \frac{(t+s)^q}{q!} H_q(x) \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{t^{p+r} s^{p+q-r}}{p!r!(q-r)!} H_q(x) .
\end{aligned} \tag{1.2.9}
$$

On the other hand, we have

$$
G(t,x)G(s,x) = \sum_{n\geqslant 0} \sum_{m\geqslant 0} \frac{t^n}{n!} \frac{s^m}{m!} H_n(x) H_m(x) .
$$

The result follows by determining the coefficient of $t^n s^m$ in (1.2.9), which is obtained by summing over all triples (p,q,r) such that $p+r=n$ and $p+q-r=m$. □

Note that when taking the expectation on both sides of (1.2.8) when $x=X$, we recover the orthogonalilty relation (1.2.6).

Exercise 1.2.8

Use (1.2.8) to write $H_4(X)^2$ as a sum of Hermite polynomials, and compute $\mathbb{E}[H_4(X)^3]$.

1.2.3 Hermite polynomials and differential operators

Lemma 1.2.6 has revealed a link between Hermite polynomials and differential operators. To make this connection more precise, we indroduce the linear operators

$$
a = \frac{\mathrm{d}}{\mathrm{d}x} , \qquad a^\dagger = x - \frac{\mathrm{d}}{\mathrm{d}x} , \qquad \mathscr{L} = -a^\dagger a = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} \tag{1.2.10}
$$

acting on $\mathscr{C}^\infty$ functions in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}, \mu(\mathrm{d}x))$, where $\mu(\mathrm{d}x)$ is the Gaussian measure

$$
\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \, \mathrm{d}x .
$$

The notation $a^\dagger$ is motivated by the following result.

Lemma 1.2.9

The operators a and $a^\dagger$ are mutually adjoint, while $\mathscr{L}$ is self-adjoint in $\mathscr{H}$. Furthermore,

$$
aa^\dagger - a^\dagger a = \mathrm{id} . \tag{1.2.11}
$$

Proof: An elegant proof of the first claim consists in rewriting a and $a^\dagger$ as

$$
(af)(x) = \mathrm{e}^{x^2/2}\Big(x + \frac{\mathrm{d}}{\mathrm{d}x}\Big)(\mathrm{e}^{-x^2/2} f(x)) , \qquad (a^\dagger f)(x) = -\mathrm{e}^{x^2/2} \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x^2/2} f(x)) . \tag{1.2.12}
$$

One indeed checks that this is equivalent to (1.2.10) by applying Leibniz' rule. Now for any $f,g \in \mathscr{H}$, integration by parts gives

$$
\begin{aligned}
\langle a^\dagger f, g\rangle &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x^2/2} f(x)) g(x) \frac{\mathrm{d}x}{\sqrt{2\pi}} \\
&= \mathrm{e}^{-x^2/2} f(x) g(x) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2} f(x) g'(x) \frac{\mathrm{d}x}{\sqrt{2\pi}} \\
&= \langle f, ag\rangle ,
\end{aligned}
$$

since the boundary term vanishes because $\langle f,g\rangle < \infty$ by the Cauchy–Schwarz inequality. As a consequence, we also have

$$-\langle f,\mathscr{L}g\rangle = \langle f,a^\dagger ag\rangle = \langle af,ag\rangle = \langle a^\dagger af,g\rangle = -\langle \mathscr{L}f,g\rangle\,,$$

which implies that $\mathscr{L}$ is self-adjoint. Finally,

$$(aa^\dagger f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(xf(x)-f'(x)) = f(x)+xf'(x)-f''(x) = (a^\dagger af)(x)+f(x)\,,$$

which proves (1.2.11). □

The link with Hermite polynomials is as follows.

Corollary 1.2.10: Hermite polynomials and differential operators

The Hermite polynomials are eigenfunctions of $\mathscr{L}$. More precisely,

$$(\mathscr{L}H_n)(x) = -nH_n(x) \qquad \forall n\geqslant 0\,. \tag{1.2.13}$$

Furthermore, one has

$$a^\dagger H_{n-1} = H_n\,, \qquad aH_n = nH_{n-1} \qquad \forall n\geqslant 1\,. \tag{1.2.14}$$

Proof: First note that the first relation in (1.2.14) is just a rewriting of (1.2.7). This allows us to prove (1.2.13) by induction on n. For $n=0$, we clearly have $\mathscr{L}H_0(x)=0$, proving the base case. Assuming (1.2.13) holds for some $n\geqslant 0$, (1.2.11) yields

$$-\mathscr{L}H_{n+1} = a^\dagger aa^\dagger H_n = a^\dagger(a^\dagger a+\mathrm{id})H_n = (n+1)a^\dagger H_n = (n+1)H_{n+1}\,.$$

Finally, we also have

$$aH_{n+1} = aa^\dagger H_n = (a^\dagger a+\mathrm{id})H_n = (n+1)H_n\,,$$

which proves the second relation in (1.2.14). □

Note that this result provides another proof of orthogonality of Hermite polynomials, since eigenfunctions of a self-adjoint operator corresponding to different eigenvalues are known to be orthogonal. The second relation in (1.2.14) can also be written

$$H_n'(x) = nH_{n-1}(x)\,.$$

Together with the recursive relation (1.2.7), this yields

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x) \qquad \forall n\geq 1\,. \tag{1.2.15}$$

Yet another relation, following from the representation (1.2.12) of $a^\dagger$ is

$$H_n(x) = ((a^\dagger)^n H_0)(x) = (-1)^n\,\mathrm{e}^{x^2/2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2/2})\,.$$

The operator $\mathscr{L}$ defined in (1.2.10) occurs in several applications. In particular, define the *Ornstein–Uhlenbeck process* as the solution of the stochastic differential equation

$$\mathrm{d}x_t = -x_t\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t\,,$$

which can be written in terms of an Itô integral as

$$x_t = x_0 \,\mathrm{e}^{-t} + \sqrt{2} \int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}W_s \,,$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion. Then $\mathscr{L}$ is the infinitesimal generator of the process, meaning that for any sufficiently regular test function f, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^{x_0}[f(x_t)]\Big|_{t=0} = (\mathscr{L}f)(x_0) \,.$$

This is a consequence of Itô's formula.

There also is a connection with quantum physics. Indeed, one finds that the conjugated operator

$$H = \mathrm{e}^{-x^2/4} \mathscr{L} \,\mathrm{e}^{x^2/4}$$

has the expression

$$(Hf)(x) = \Big(\frac{1}{2} - \frac{x^2}{4}\Big) f(x) + f''(x) \,,$$

which is equivalent, up to a scaling, to the Hamiltonian of the quantum harmonic oscillator. The operator H is self-adjoint in $L^2(\mathbb{R}, \mathrm{d}x)$, and its eigenfunctions are conjugated to the Hermite polynomials. As the operators $a^\dagger$ and a allow to move between eigenfunctions, and these eigenfunctions are interpreted as n-particle states in quantum field theory, they as known as *creation operator* and *annihilation operator*.

1.2.4 Convolution algebra*

Some of the above computations required to perform operations on power series, such as multiplication, division, and taking the logarithm. There exists an algebraic framework that makes these computations particularly easy.

Let $\mathbb{R}[x]$ denote the vector space of polynomials in one variable x, with the canonical basis $\{x^n\}_{n\geqslant 0}$. This is also an algebra for the usual product

$$x^n \cdot x^m = x^{n+m} \,.$$

Let $\mathbb{R}[[t]]$ be the space of formal power series, that is, expressions of the form

$$\sum_{n\geqslant 0} \varphi_n \frac{t^n}{n!} \,, \qquad \varphi_n \in \mathbb{R} \,,$$

endowed with pointwise multiplication. By formal we mean that at this point, we are not concerned about convergence of the series.

We can view the coefficients φ_n as the images of the x^n by a linear map $\varphi : \mathbb{R}[x] \to \mathbb{R}$. Denoting by $\mathscr{L}(\mathbb{R}[x], \mathbb{R})$ the space of all these maps, we can define a linear map

$$\begin{aligned} \Lambda \colon \mathscr{L}(\mathbb{R}[x], \mathbb{R}) &\longrightarrow \mathbb{R}[[t]] \\ \varphi &\longmapsto \sum_{n\geqslant 0} \varphi(x^n) \frac{t^n}{n!} \,. \end{aligned} \tag{1.2.16}$$

This map associates a power series with the map φ defining its coefficients. The interest of this construction is that due to the Cauchy product formula, multiplication of power series is equivalent to a convolution operation of maps. More precisely, for two maps $\varphi, \psi \in \mathscr{L}(\mathbb{R}[x], \mathbb{R})$, define a map $\varphi * \psi \in \mathscr{L}(\mathbb{R}[x], \mathbb{R})$ by

$$(\varphi * \psi)(x^n) = \sum_{k=0}^{n} \binom{n}{k} \varphi(x^k)\psi(x^{n-k}) . \tag{1.2.17}$$

We then have the following result.

Theorem 1.2.11: Isomorphism between convolution algebra and algebra of power series

The map Λ is an isomorpism between $\mathscr{L}(\mathbb{R}[x], \mathbb{R})$ and $\mathbb{R}[[t]]$.

Proof: By the Cauchy product formula, for any $\varphi, \psi \in \mathscr{L}(\mathbb{R}[x], \mathbb{R})$,

$$\begin{aligned}
\Lambda(\varphi)(t)\Lambda(\psi)(t) &= \Big(\sum_{n\geqslant 0} \varphi(x^n)\frac{t^n}{n!}\Big)\Big(\sum_{n\geqslant 0} \psi(x^n)\frac{t^n}{n!}\Big) \\
&= \sum_{p\geqslant 0}\Big(\sum_{k=0}^{p} \frac{\varphi(x^k)}{k!}\frac{\psi(x^{p-k})}{(p-k)!}\Big)t^p \\
&= \sum_{p\geqslant 0} (\varphi * \psi)(x^p)\frac{t^p}{p!} \\
&= \Lambda(\varphi * \psi)(t) .
\end{aligned}$$

This shows that Λ is indeed an algebra morphism. Bijectivity follows from uniqueness of the coefficients of power series. □

This result allows us to work with convolution of linear maps instead of multiplication of power series. More generally, we define the p-fold convolution by

$$\varphi^{*p}(x^n) = \sum_{\substack{n_1,\dots,n_p\geqslant 0 \\ n_1+\dots+n_p=n}} \frac{n!}{n_1!\dots n_p!}\varphi(x^{n_1})\dots\varphi(x^{n_p}) .$$

It till turn out to be useful to work with linear instead of multilinear maps. This is achieved by introducing a linear map $\Delta : \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x]$, given by

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} . \tag{1.2.18}$$

Here the tensor product $\mathbb{R}[x] \otimes \mathbb{R}[x]$ is the vector space spanned by all $x^n \otimes x^m$ with $n, m \geqslant 0$. Indeed (1.2.18) allows us to rewrite the convolution product (1.2.17) as

$$\varphi * \psi = \mathscr{M}(\varphi \otimes \psi)\Delta ,$$

where $\mathscr{M}$ denotes the multiplication map, $\mathscr{M}(a \otimes b) = ab$. More generally, for $p \geqslant 3$ we set

$$\Delta^{(p-1)}(x^n) = \sum_{\substack{n_1,\dots,n_p\geqslant 0 \\ n_1+\dots+n_p=n}} \frac{n!}{n_1!\dots n_p!} x^{n_1} \otimes \dots \otimes x^{n_p} .$$

Writing $\mathscr{M}_p(a_1 \otimes \dots \otimes a_p) = a_1 \dots a_p$, the n-fold convolution product becomes

$$\varphi^{*p} = \mathscr{M}_p(\varphi^{\otimes p})\Delta^{(p-1)} .$$

Remark 1.2.12: Hopf algebra

The space $\mathbb{R}[x]$ endowed with the maps $\cdot$, Δ, and a *counit* $\mathbf{1}^\star : \mathbb{R}[x] \to \mathbb{R}$ defined by $\mathbf{1}^\star(x^n) = \delta_{n0}$ is a so-called *bi-algebra*. When adding a linear map $\mathscr{A}$ defined by $\mathscr{A}(x^n) = (-1)^n x^n$ and called *antipode*, it becomes a *Hopf algebra*. The map Δ is called a *co-product*, because it enjoys a property called *co-associativity*, saying that applying Δ to the left or to the right of the tensor product in $\Delta(x^n)$ yields the same result, namely

$$\Delta^{(2)}(x^n) = (\Delta \otimes \mathrm{id})\Delta(x^n) = (\mathrm{id} \otimes \Delta)\Delta(x^n) = \sum_{\substack{n_1,n_2,n_3 \geqslant 0 \\ n_1+n_2+n_3=n}} \frac{n!}{n_1!n_2!n_3!} x^{n_1} \otimes x^{n_2} \otimes x^{n_3}$$

and similarly for higher powers.

Let us now introduce two special subsets of $\mathscr{L}(\mathbb{R}[x],\mathbb{R})$, given by

$$\begin{aligned} \mathscr{L}_1 &= \{\varphi \in \mathscr{L}(\mathbb{R}[x],\mathbb{R}) \colon \varphi(1) = 1\}, \\ \mathscr{L}_0 &= \{\varphi \in \mathscr{L}(\mathbb{R}[x],\mathbb{R}) \colon \varphi(1) = 0\}. \end{aligned}$$

Elements of $\mathscr{L}_1$ can be inverted, via the Neumann series

$$\varphi^{-1} = \sum_{k=0}^{\infty} (\mathbf{1}^\star - \varphi)^{*k}.$$

One has explicitly

$$\varphi^{-1}(x^n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \varphi(x^{n_1})\dots\varphi(x^{n_k}). \tag{1.2.19}$$

Exercise 1.2.13

Check that one has indeed $(\varphi * \varphi^{-1})(x^n) = \delta_{n0}$ for all $n \geqslant 0$.
Hints: Recall that $\varphi(1) = 1$, and therefore $\varphi^{-1}(1) = 1$. Check the cases $n \in \{0,1,2\}$ first. For the general case, the term $k = 0$ in (1.2.17) should be treated separately.

We can now define an exponential map $\exp_* : \mathscr{L}_0 \to \mathscr{L}_1$ and its inverse $\log_* : \mathscr{L}_1 \to \mathscr{L}_0$ by

$$\exp_*(\varphi) = \sum_{k \geqslant 0} \frac{1}{k!} \varphi^{*k}, \qquad \log_*(\varphi) = \sum_{k \geqslant 1} \frac{(-1)^k}{k} (\varphi - \mathbf{1}^*)^{*k}.$$

There is no issue of convergence, since the sums are always finite when evaluated on a basis element. In fact,

$$\exp_*(\varphi)(x^n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \varphi(x^{n_1})\dots\varphi(x^{n_k}), \tag{1.2.20}$$

$$\log_*(\varphi)(x^n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \varphi(x^{n_1})\dots\varphi(x^{n_k}). \tag{1.2.21}$$

The following corollary of Theorem 1.2.11 clarifies the link between these objects and the usual inverse, exponential and logarithm.

Proposition 1.2.14

For any $\varphi \in \mathscr{L}_0$ and $\psi \in \mathscr{L}_1$, one has the relations

$$\begin{aligned}\Lambda(\psi^{-1})(t) &= [\Lambda(\psi)(t)]^{-1}\,,\\ \Lambda(\exp_* \varphi)(t) &= \exp(\Lambda(\varphi)(t))\,,\\ \Lambda(\log_* \psi)(t) &= \log(\Lambda(\psi)(t))\,.\end{aligned}$$

Proof: We prove the second relation. Setting $\psi = \exp_* \varphi$, we have

$$\Lambda(\psi)(t) = \sum_{k\geqslant 0} \frac{1}{k!}\Lambda(\varphi^{*k})(t) = \sum_{k\geqslant 0} \frac{1}{k!}\Lambda(\varphi)(t)^k = \exp(\Lambda(\varphi)(t))\,.$$

The other relations are proved in a similar way. □

Returning to the topic of Hermite polynomials, we consider the special case where X is a real-valued random variable, admitting exponential moments, and associate with it the linear map $\mu_X : \mathbb{R}[x] \to \mathbb{R}$ given by

$$\mu_X(x^n) = \mathbb{E}[X^n]\,.$$

Note that $\mu_X \in \mathscr{L}_1$, since $\mathbb{E}[1] = 1$. The associated power series

$$\Lambda(\mu_X)(t) = \sum_{n\geqslant 0} \frac{t^n}{n!}\mathbb{E}[X^n] = \mathbb{E}[\mathrm{e}^{tX}] \tag{1.2.22}$$

is the *moment generating function* of X introduced in (1.2.1). Proposition 1.2.14 implies that the cumulant generating function (cf. (1.2.2)) can be written as

$$K_X(t) = \log \mathbb{E}[\mathrm{e}^{tX}] = \Lambda(\log_* \mu_X)(t) = \Lambda(\kappa_X)(t)\,,$$

where

$$\kappa_X = \log_* \mu_X\,, \qquad \mu_X = \exp_* \kappa_X\,.$$

Applying (1.2.20) and (1.2.21) implies

$$\mu_X(x^n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geqslant 1\\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!}\kappa_X(x^{n_1})\dots\kappa_X(x^{n_k})\,, \tag{1.2.23}$$

$$\kappa_X(x^n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1,\dots,n_k\geqslant 1\\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!}\mu_X(x^{n_1})\dots\mu_X(x^{n_k})\,.$$

These are the so-called *Leonov–Shiraev moment-cumulant relations*. We now define the *Wick exponential* associated to X as the linear map $\mathrm{W} : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$\mathrm{W} = (\mu_X^{-1} \otimes \mathrm{id})\Delta = (\exp_*(-\kappa_X) \otimes \mathrm{id})\Delta\,. \tag{1.2.24}$$

One easily checks that $\mathrm{W}(1) = 1$, while for $n \geqslant 1$, (1.2.19) and (1.2.20) imply

$$\mathrm{W}(x^n) = \sum_{k=0}^{n}\sum_{j=1}^{k} \frac{(-1)^j}{j!} \sum_{\substack{n_1,\dots,n_j\geqslant 1\\ n_1+\dots+n_j=k}} \frac{n!}{(n-k)!n_1!\dots n_j!}\kappa_X(x^{n_1})\dots\kappa_X(x^{n_j})x^{n-k}\,. \tag{1.2.25}$$

Moreover, we have an explicit expression for the inverse of W.

Lemma 1.2.15

The inverse of W is the map
$$\mathrm{W}^{-1} = (\mu_X \otimes \mathrm{id})\Delta\,. \tag{1.2.26}$$

Proof: For a linear map $g : \mathbb{R}[x] \to \mathbb{R}$, write
$$M^g(x^n) = (g \otimes \mathrm{id})\Delta(x^n)\,.$$
Note that for any linear form $f : \mathbb{R}[x] \to \mathbb{R}$, we have
$$\langle f, M^g(x^n)\rangle = (g \otimes f)\Delta(x^n) =: \langle g \circ f, x^n\rangle\,,$$
where we use an "inner product" notation for the action of f for clarity. It follows that if $h : \mathbb{R}[x] \to \mathbb{R}$ is yet another linear form, then
$$\langle f, M^h M^g(x^n)\rangle = \langle h \circ f, M^g(x^n)\rangle = \langle g \circ h \circ f, x^n\rangle = \langle f, M^{g\circ h}(x^n)\rangle\,.$$
In short, $M^h M^g = M^{g\circ h}$. Now we observe that $\mathrm{W} = M^{\mu_X^{-1}}$ and $\mathrm{W}^{-1} = M^{\mu_X}$. Therefore
$$\mathrm{W}\,\mathrm{W}^{-1}(x^n) = M^{\mu_X \circ \mu_X^{-1}}(x^n) = ((\mu_X \circ \mu_X^{-1}) \otimes \mathrm{id})\Delta(x^n)\,,$$
where
$$\begin{aligned}(\mu_X \circ \mu_X^{-1})(x^k) &= (\mu_X \otimes \mu_X^{-1})\Delta(x^k)\\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} \mu_X(x^\ell)\mu_X^{-1}(x^{k-\ell})\\ &= (\mu_X * \mu_X^{-1})(x^k)\\ &= \delta_{k0}\,.\end{aligned}$$
This implies $(\mu_X \circ \mu_X^{-1} \otimes \mathrm{id})\Delta(x^n) = x^n$ for all n, proving that W^{-1} is indeed the inverse of W. □

The following result shows that $\mathrm{W}(t,x) := \Lambda(\mathrm{W})(t)$ is nothing but the generating function that we encountered in (1.2.4).

Proposition 1.2.16

One has the relation
$$\mathrm{W}(t,x) = \frac{\mathrm{e}^{tx}}{\mathbb{E}[\mathrm{e}^{tX}]} = \mathrm{e}^{tx - K_X(t)}\,.$$

Proof: Observe that
$$\mathrm{W}(x^n) = \sum_{k=0}^{n} \binom{n}{k} \mu_X^{-1}(x^k) x^{n-k} = (\mu_X^{-1} * \mathrm{id})(x^n)\,.$$
Therefore, Theorem (1.2.11) implies
$$\Lambda(\mathrm{W})(t) = \Lambda(\mu_X^{-1} * \mathrm{id})(t) = \Lambda(\mu_X^{-1})(t)\Lambda(\mathrm{id})(t) = \frac{\Lambda(\mathrm{id})(t)}{\Lambda(\mu_X)(t)} = \frac{\mathrm{e}^{tx}}{\mathbb{E}[\mathrm{e}^{tX}]}\,,$$
where we have used Proposition 1.2.14 and (1.2.22). □

So far, the construction works for any random variable X with exponential moments. Let us now particularise to the case $X \sim \mathcal{N}(0,1)$. Since $\kappa_X(x^n) = \delta_{n2}$, cf. (1.2.3), all n_i in the first Leonov–Shiraev relation (1.2.23) must have value 2. This is only possible if n is even and $k = n/2$. Therefore, we obtain

$$\mathbb{E}[X^{2k}] = \mu_X(x^{2k}) = \frac{(2k)!}{k!2^k} . \tag{1.2.27}$$

This is equal to $(k-1)!!$, as one checks by separating even and odd factors in $(2k)!$. We thus recover Proposition 1.1.4. In addition, we obtain the following explicit expressions for Hermite polynomials, as well as the inverse relations between monomials and Hermite polynomials.

Proposition 1.2.17: Explicit expression of Hermite polynomials

For any $n \in \mathbb{N}_0$, one has

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^k k!(n-2k)!} x^{n-2k} .$$

The inverse relation is given by

$$x^n = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^k k!(n-2k)!} H_{n-2k}(x) . \tag{1.2.28}$$

Proof: Since

$$\mathrm{W}(t,x) = \Lambda(\mathrm{W})(t) = \sum_{n\geqslant 0} \mathrm{W}(x^n)\frac{t^n}{n!}$$

by the definition (1.2.16) of Λ, we have $H_n(x) = \mathrm{W}(x^n)$. We can thus apply (1.2.25) when all n_i are equal to 2. This is only possible if $2j = k$, and the result follows by the index shift $k \mapsto k/2$. The inverse relation (1.2.28) follows in the same way from the expression (1.2.26) for W^{-1}. □

1.2.5 Hermite polynomials and combinatorics

The coefficients of Hermite polynomials also have a nice combinatorial interpretation. Given a finite set E_n of cardinal n, say $E_n = [\![1,n]\!] := \{1,2,\dots,n\}$, we will call *pairwise matching*, or *pairing*, a partition of E_n into sets of cardinality 1 or 2, the former being called *singletons* and the latter being called *pairs*.

Proposition 1.2.18: Combinatorial interpretation of Hermite polynomials

Let $0 \leqslant 2k \leqslant n$. The coefficient of x^{n-2k} of $H_n(x)$ is equal to the number of pairwise matchings of E_n with k pairs.

Proof: Given a pairwise matching of E_n with k pairs, we associate to each singleton the label x, and to each pair the label -1. The value of this matching is defined as the product of all labels, namely

$$(-1)^k x^{n-2k} ,$$

see Figure 1.1 for an example. We claim that $H_n(x)$ is equal to the sum of the values of all pairwise matchings of E_n.

We proceed by induction, taking $n = 1$ as base case. Then the only pairwise matching is $\{\{1\}\}$, which has value $x = H_1(x)$.

Assume now that $n \geqslant 1$. The pairwise matchings of E_{n+1} are of two types. The first type are those in which $\{n+1\}$ is a singleton. These have value $xH_n(x)$ by induction hypothesis.

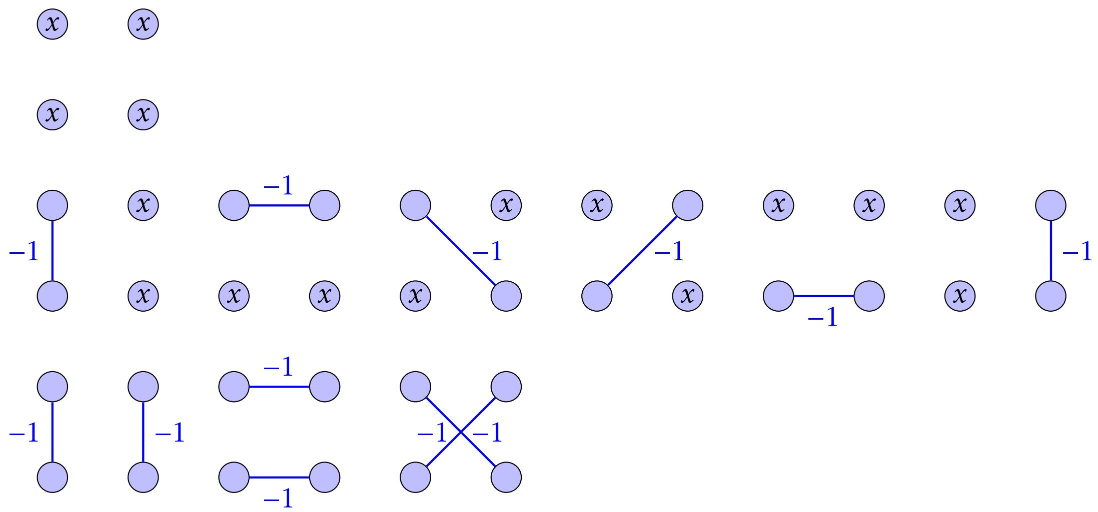


Figure 1.1 – Pairwise matchings of $E_4 = \{1,2,3,4\}$. There is one matching with no pairs, of value x^4. There are 6 matchings with one pair, of total value $-6x^2$, and 3 matchings with two pairs, of total value 3. The sum of all values is $x^4 - 6x^2 + 3 = H_4(x)$.

The second type of pairings are those in which $n+1$ belongs to a pair. There are n choices for the partner of $n+1$, and for each of these choices, the value of the matching of the remaining elements is $H_{n-1}(x)$. This shows that

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)\,,$$

which is exactly the recurrence relation (1.2.15). □

The number of pairwise matchings can also be computed exactly, which yields the following alternative proof of the first relation in Proposition 1.2.17.

Proposition 1.2.19: Explicit expression of Hermite polynomials

For any $n \in \mathbb{N}_0$, one has

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^k k!(n-2k)!} x^{n-2k}\,.$$

PROOF: It is sufficient to count the number of matchings with k pairs. There are $\binom{n}{n-2k}$ choices for the $n-2k$ singletons. The number of pairings of the remaining $2k$ elements is

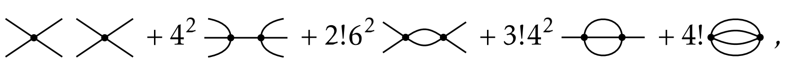
$$(2k-1)!! = \frac{(2k)!}{k!2^k}\,. \tag{1.2.29}$$

This is because the first element has $2k-1$ possible partners. Having chosen the first part, it remains to pair $2k-2$ elements, so that the claim follows by induction. Multiplying the two numbers gives the claimed coefficient of x^{n-2k}. □

Pairings appear at other places in computations with Hermite polynomials. For instance, Proposition 1.2.7 yields

$$H_4(x)^2 = H_8(x) + 4^2 H_6(x) + 2!6^2 H_4(x) + 3!4^2 H_2(x) + 4!H_0(x)\,.$$

The right-hand side can be represented graphically by



where the numerical coefficients count the number of ways to pair $2p$ legs from the two different diagrams having 4 legs each.

1.3 Wiener chaos expansion

Recall that we have introduced the Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}, \mu(\mathrm{d}x)) ,$$

which consists in random variables of the form $f(X)$ which have a finite variance, with $X \sim \mathscr{N}(0,1)$. We begin with a preparatory lemma, which is a special case of [Nua06, Lemma 1.1.1].

Lemma 1.3.1

The random variables $\{\mathrm{e}^{tX} : t \in \mathbb{R}\}$ form a total subset of $\mathscr{H}$.

Proof: Let $Z \in \mathscr{H}$ be such that $\mathbb{E}[Z\,\mathrm{e}^{tX}] = 0$ for all $t \in \mathbb{R}$. We have to show that $Z = 0$. Define a signed measure ν by

$$\nu(B) = \mathbb{E}[Z 1_B(X)]$$

for any Borel set B of $\mathbb{R}$. The fact that $\mathbb{E}[Z\,\mathrm{e}^{tX}] = 0$ means that the Laplace transform of ν is identically zero on $\mathbb{R}$. Therefore, this measure is zero, so that $\mathbb{E}[Z 1_B] = 0$ for any Borel set B. This implies that Z is indeed equal to 0. □

Definition 1.3.2: Wiener chaos

For any $n \geqslant 1$, we denote by $\mathscr{H}_n$ the one-dimensional subspace of $\mathscr{H}$ spanned by the random variable $H_n(X)$. For $n = 0$, $\mathscr{H}_0$ is the set of constants, which is isomorphic to $\mathbb{R}$. Then $\mathscr{H}_n$ is called the *homogeneous Wiener chaos of order n*. The *inhomogeneous Wiener chaos of order n* is defined as

$$\mathscr{H}_{\leqslant n} = \bigoplus_{k=0}^{n} \mathscr{H}_k .$$

Note that Proposition 1.2.5 implies that the subspaces $\mathscr{H}_n$ are mutually orthogonal.

The main theorem is then as follows (this is a particular case of [Nua06, Theorem 1.1.1]:

Theorem 1.3.3: Wiener chaos decomposition

The Hilbert space $\mathscr{H}$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathscr{H}_n$:

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n .$$

Proof: Let $Z \in \mathscr{H}$ be orthogonal to $\mathscr{H}_n$ for all $n \geqslant 0$. This means that

$$\mathbb{E}[Z H_n(X)] = 0 \qquad \text{for all } n \geqslant 0 .$$

By (1.2.28), X^n can be expressed as a linear combination of $H_k(X)$, with $0 \leqslant k \leqslant n$. Therefore, we also have

$$\mathbb{E}[Z X^n] = 0 \qquad \text{for all } n \geqslant 0 .$$

This implies that $\mathbb{E}[Z\,\mathrm{e}^{tX}] = 0$ for all $t \in \mathbb{R}$. By Lemma 1.3.1, this means that $Z = 0$, which completes the proof. □

In a sense, this theorem is quite remarkable, because f being in $L^2(\mathbb{R},\mu(\mathrm{d}x))$ is a rather weak requirement. In particular, f needs not be continuous, nor does it need to be bounded. We started this chapter by remarking that $\mathbb{E}[f(X)]$ can be computed if f admits a power series expansion, with strictly positive radius of convergence, see (1.1.3). Theorem 1.3.3 shows that this assumption of f is not at all necessary.

Remark 1.3.4

If $f \in \mathscr{H}$, we have

$$f(X) = \sum_{n=0}^{\infty} \mathbb{E}[f(X)H_n(X)]H_n(X)\,. \tag{1.3.1}$$

Taking expectations shows that

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X)H_0(X)]\,,$$

which does not yield any new information. However, in applications $f(X)$ often admits a more or less explicit decomposition of the form (1.3.1), which makes it possible to compute expectations.

Exercise 1.3.5

Consider the function

$$f(X) = \mathrm{e}^{-\lambda H_4(X)}$$

where $\lambda \in [0,\infty)$. Our aim is to obtain an asymptotic expansion

$$\mathbb{E}[f(X)] \asymp \sum_{n\geqslant 0} a_n\lambda^n\,,$$

which means that

$$\mathbb{E}[f(X)] = \sum_{n=0}^{N} a_n\lambda^n + \mathcal{O}(\lambda^{N+1})$$

for any $N \geqslant 1$.

- Compute explicitly a_0, a_1, a_2 and a_3.
- Give a combinatorial interpretation of a_n for any n (in terms of certain types of graphs).
- Find an upper bound b_n for $|a_n|$. What is the radius of convergence of the series $\sum_n b_n\lambda^n$?

Exercise 1.3.6

Consider the function

$$f(X) = \mathrm{e}^{-\lambda H_2(X)}$$

where $\lambda \in [0,\infty)$.

- Compute $\mathbb{E}[f(X)]$ explicitly.
- What is the radius of convergence of the expansion of $\mathbb{E}[f(X)]$ into powers of λ?
- Give a combinatorial interpretation of the nth coefficient of this series. Check this for the first values of n.

Chapter 2

The multi-dimensional case

In this chapter, we extend results from the previous chapter to n-dimensional Gaussian random variables.

2.1 Wick calculus

2.1.1 Multivariate Gaussian random variables

We start by a quick recapitulation of basic properties of N-dimensional Gaussian random variables.

Definition 2.1.1: Multivariate Gaussian

For $N \geqslant 1$, let $\mathbb{R}^N$ be equipped with the σ-algebra $\mathscr{B}$ of Borel sets and Lebesgue measure $\mathrm{d}x$. Let $m \in \mathbb{R}^N$ and let $C \in \mathbb{R}^{N\times N}$ be a symmetric, positive definite matrix. A random variable $X : \mathbb{R}^n \to \mathbb{R}$ is a (multivariate) Gaussian random variable with mean m and covariance matrix C if its law is

$$\mu(\mathrm{d}x) = \frac{1}{(2\pi)^{N/2}\det(C)^{1/2}}\,\mathrm{e}^{-\langle (x-m), C^{-1}(x-m)\rangle/2}\,\mathrm{d}x\,. \tag{2.1.1}$$

In that case, we write $X \sim \mathscr{N}(m, C)$.

The following result generalises (1.1.1) to the multivariate case.

Proposition 2.1.2: Laplace transform

Let C be symmetric, positive definite. Then $X \sim \mathscr{N}(0, C)$ if, and only if, for any $t \in \mathbb{R}^n$, one has

$$\mathbb{E}[\mathrm{e}^{\langle t, X\rangle}] = \mathrm{e}^{\langle t, Ct\rangle/2}\,. \tag{2.1.2}$$

Proof: We proceed by diagonalisation. Since C is symmetric, positive definite, there exists an orthogonal matrix U such that $U^\top C U = \Lambda$ is diagonal, with real, strictly positive diagonal elements $\lambda_1, \dots, \lambda_N$. Since $\det(\Lambda) = \det(C)\det(UU^\top) = \det(C)$, the fact that $X \sim \mathscr{N}(0, C)$ is equivalent, by the change of variables formula (or transfer theorem), to Y having law

$$\hat{\mu}(\mathrm{d}y) = \frac{1}{(2\pi)^{N/2}\det(\Lambda)^{1/2}}\,\mathrm{e}^{-\langle U^\top y, C^{-1}U^\top Y\rangle/2}\,\mathrm{d}y = \frac{1}{(2\pi)^{N/2}\det(\Lambda)^{1/2}}\,\mathrm{e}^{-\langle y, \Lambda^{-1}y\rangle/2}\,\mathrm{d}y = \prod_{i=1}^N \frac{\mathrm{e}^{-\lambda_i y_i^2/2}}{\sqrt{2\pi\lambda_i}}\,.$$

This means that the components Y_i of Y are independent, with $Y_i \sim \mathcal{N}(0,\lambda_i)$. By (1.1.1), this implies that for any $s \in \mathbb{R}^n$,

$$\mathbb{E}[\mathrm{e}^{\langle s,Y\rangle}] = \prod_{i=1}^N \mathbb{E}[\mathrm{e}^{\langle s_i,Y_i\rangle}] = \prod_{i=1}^N \mathrm{e}^{\lambda_i s_i^2/2} = \mathbb{E}[\mathrm{e}^{\langle s,\Lambda s\rangle}]\,.$$

The converse is actually true, because $\mathbb{E}[\mathrm{e}^{\langle s_i,Y_i\rangle}] = 1$ if $s_i = 0$. Setting $s = tU^\top$, this is equivalent to

$$\mathbb{E}[\mathrm{e}^{\langle t,X\rangle}] = \mathbb{E}[\mathrm{e}^{\langle t,UY\rangle}] = \mathbb{E}[\mathrm{e}^{\langle U^\top t,Y\rangle}] = \mathrm{e}^{\langle U^\top t,\Lambda U^\top t\rangle/2} = \mathrm{e}^{\langle t,U\Lambda U^\top t\rangle/2} = \mathrm{e}^{\langle t,Ct\rangle/2}\,,$$

which completes the proof. □

One consequence of this result is that one can easily justify the name "covariance matrix" (though there are other ways to verify this).

Corollary 2.1.3

If $X \sim \mathcal{N}(0,C)$, then $\mathbb{E}[X_iX_j] = C_{ij}$ for all $i,j \in [\![1,N]\!]$.

Proof: This follows by taking the derivative of (2.1.2) with respect to t_i and t_j, and evaluating at $t = 0$. □

2.1.2 Isserlis' theorem

Isserlis' theorem (also known as Wick's theorem in physics) is a generalisation of the expression (1.1.2) for the moments of a Gaussian random variable to the multivariate case. We first show a preparatory lemma, which is a simple instance of the Schwinger–Dyson equations in quantum field theory.

Lemma 2.1.4: Integation by parts

Assume $X \sim \mathcal{N}(0,C)$. For any $i \in [\![1,N]\!]$, the equality

$$\mathbb{E}[X_i f(X)] = \sum_{j=1}^N C_{ij}\mathbb{E}[\partial_j f(X)] \tag{2.1.3}$$

holds for all differentiable $f : \mathbb{R}^N \to \mathbb{R}$ such that both sides of the equality are well-defined.

Proof: Leibniz' rule yields

$$\frac{\partial}{\partial x_j}\Big(f(x)\,\mathrm{e}^{-\langle x,C^{-1}x\rangle/2}\Big) = \biggl[\frac{\partial f}{\partial x_j}(x) - \frac12 f(x)\frac{\partial}{\partial x_j}\langle x,C^{-1}x\rangle\biggr]\mathrm{e}^{-\langle x,C^{-1}x\rangle/2}\,.$$

Since we have

$$\frac12\sum_{j=1}^N C_{ij}\frac{\partial}{\partial x_j}\langle x,C^{-1}x\rangle = \sum_{j,k=1}^N C_{ij}C^{-1}_{jk}x_k = \sum_{k=1}^N \delta_{ik}x_k = x_i\,,$$

it follows that

$$x_i f(x)\,\mathrm{e}^{-\langle x,C^{-1}x\rangle/2} = \sum_{j=1}^N C_{ij}\biggl[-\frac{\partial}{\partial x_j}\Big(f(x)\,\mathrm{e}^{-\langle x,C^{-1}x\rangle/2}\Big) + \frac{\partial f}{\partial x_j}\,\mathrm{e}^{-\langle x,C^{-1}x\rangle/2}\biggr]\,.$$

Integrating over the whole space, we see that the boundary terms vanish, proving (2.1.3). □

An immediate consequence of this result is the following theorem, due to Leon Isserlis.

Theorem 2.1.5: Isserlis

For any $1 \leqslant k \leqslant \frac{N}{2}$, we have

$$\mathbb{E}[X_1 \dots X_{2k}] = \sum_{\mathscr{P}} \prod_{\{i,j\} \in \mathscr{P}} \mathbb{E}[X_i X_j] ,$$
$$\mathbb{E}[X_1 \dots X_{2k-1}] = 0 ,$$

where the sum runs over all *perfect matchings* $\mathscr{P}$ of $[\![1, 2k]\!]$, meaning that each $\mathscr{P}$ is a partition $\mathscr{P} = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\}$ of this set into disjoint subsets of two elements.

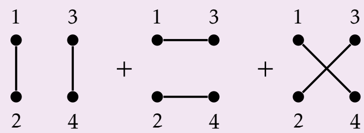
Proof: The proof proceeds by induction on the number of factors, applying (2.1.3) with $i = 1$ and $f(X)$ of the form $X_2 \dots X_m$. □

Example 2.1.6

In the case $2k = 4$, we obtain

$$\mathbb{E}[X_1X_2X_3X_4] = \mathbb{E}[X_1X_2]\mathbb{E}[X_3X_4] + \mathbb{E}[X_1X_3]\mathbb{E}[X_2X_4] + \mathbb{E}[X_1X_4]\mathbb{E}[X_2X_3] .$$

A convenient graphical way of representing this relation is the following:

$\mathbb{E}[X_1X_2X_3X_4] =$ 

Remark 2.1.7

Theorem 2.1.5 remains true if the covariance matrix C is only semi-definite positive. We may thus allow for the case $X = (X_1, X_1, \dots, X_1)$, in which case it yields

$$\mathbb{E}[X_1^{2k}] = (2k-1)!!$$

since this is the number of perfect matchings of $[\![1, 2k]\!]$. We thus recover Proposition 1.1.4.

Exercise 2.1.8

Provide an alternative proof of Isserlis' theorem, using the expression (2.1.2) of the Laplace transform of $X = (X_1, \dots, X_n)$.

2.2 Hermite polynomials for multivariate Gaussians

In this section, we derive some additional properties of Hermite polynomials, when they are evaluated in linear combinations of Gaussian random variables.

2.2.1 Scaling

In Chapter 1, we defined Hermite polynomials for centred Gaussian random variables of variance 1. While centering is never a problem, it will often be useful to allow for general variances. This can be achieved by a simple scaling.

n	$H_n(x;\sigma^2)$
0	1
1	x
2	$x^2-\sigma^2$
3	$x^3-3\sigma^2 x$
4	$x^4-6\sigma^2x^2+3\sigma^4$
5	$x^5-10\sigma^2x^3+15\sigma^4x$

Table 2.1 – List of the first Hermite polynomials of variance σ^2.

Definition 2.2.1: Scaled Hermite polynomials

The *Hermite polynomial of degree n with variance* σ^2 is defined as

$$H_n(x;\sigma^2) = \sigma^n H_n(x/\sigma)\,. \tag{2.2.1}$$

Table 2.1 shows the expressions of the first six scaled Hermite polynomials.
One easily checks the following properties:

- scaled Hermite polynomials admit the generating function
$$G(t,x) = \mathrm{e}^{tx-\sigma^2t^2/2}\,;$$
- scaled Hermite polynomials satisfy the same orthogonality (1.2.6) as unscaled ones;
- scaled Hermite polynomials satisfy the recursive relations
$$\begin{aligned}H_{n+1}(x;\sigma^2) &= xH_n(x;\sigma^2) - \sigma^2\partial_x H_n(x;\sigma^2)\,,\\ \partial_x H_n(x;\sigma^2) &= nH_{n-1}(x;\sigma^2)\,;\end{aligned}$$
- scaled Hermite polynomials admit the explicit expression
$$H_n(x;\sigma) = n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{2^k k!(n-2k)!}\sigma^{2k}x^{n-2k}\,. \tag{2.2.2}$$

2.2.2 Binomial formula

The following binomial formula allows to relate Hermite polynomials with different variances.

Lemma 2.2.2: Binomial formula for Hermite polynomials

For any $x,y\in\mathbb{R}$, any $\sigma_1,\sigma_2\in\mathbb{R}$ and any $n\in\mathbb{N}_0$, one has

$$H_n(x+y;\sigma_1^2+\sigma_2^2) = \sum_{m=0}^{n}\binom{n}{m}H_m(x;\sigma_1^2)H_{n-m}(y;\sigma_2^2)\,.$$

Proof: Expanding the identity

$$\mathrm{e}^{t(x+y)-(\sigma_1^2+\sigma_2^2)t^2/2} = \mathrm{e}^{tx-\sigma_1t^2/2}\,\mathrm{e}^{ty-\sigma_2^2t^2/2}$$

yields

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x+y;\sigma_1^2+\sigma_2^2) = \sum_{m=0}^{\infty}\frac{t^m}{m!}H_m(x;\sigma_1^2)\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(y;\sigma_2^2)\,.$$

Comparing coefficients of t^n gives the result. □

A very useful generalisation of this property, spelled out here for Hermite polynomials of unit variance, is the following result.

Lemma 2.2.3: Multinomial formula for Hermite polynomials

Let $a \in \ell^2$ be a sequence of real numbers such that $\sum_{i\geqslant 0} a_i^2 = 1$. Then for any sequence $(x_i)_{i\geqslant 0}$ such that $\sum_{i\geqslant 0} a_i x_i$ converges, one has

$$H_n\Big(\sum_{i\geqslant 0} a_i x_i\Big) = \sum_{|k|=n} \frac{n!}{k!} a^k \prod_{i\geqslant 0} H_{k_i}(x_i)\,, \tag{2.2.3}$$

where the sum runs over all $k \in \mathbb{N}_0^{\mathbb{N}_0}$ such that $|k| = \sum_{i\geqslant 0} k_i = n$, and

$$k! := \prod_{i\geqslant 0} k_i!\,, \quad a^k := \prod_{i\geqslant 0} a_i^{k_i}\,. \tag{2.2.4}$$

Proof: First note that the condition $|k| = n$ implies that k has only finitely many nonzero indices, so that all products are well-defined. Consider now $x, y, a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Then Lemma 2.2.2 and the scaling property (2.2.1) imply

$$\begin{aligned} H_n(ax+by) &= H_n(ax+by; a^2+b^2) \\ &= \sum_{m=0}^{n} \binom{n}{m} H_m(ax;a^2) H_{n-m}(by;b^2) \\ &= \sum_{m=0}^{n} \binom{n}{m} a^n b^{n-m} H_m(x) H_{n-m}(y)\,. \end{aligned}$$

The result then follows by applying this identity repeatedly. □

Exercise 2.2.4

Use these results to compute $H_2\big(\frac{x+y}{\sqrt{2}}\big)$ and $H_4\big(\frac{x+y}{\sqrt{2}}\big)$ in terms of $H_n(x)$ and $H_n(y)$.

2.3 Wiener chaos expansion

2.3.1 Main result

Consider now the case where $X_1, \dots, X_N$ are independent Gaussian random variables, of zero expectation and unit variance, defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We are interested in the Hilbert space

$$\mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$$

of random variables $F = f(X_1, \dots, X_N)$ admitting a finite variance.

In order to deal with linear combinations of the X_i, it turns out to be useful to introduce the Hilbert space $\mathbf{H} = \mathbb{R}^N$ endowed with the canonical inner product $\langle\cdot,\cdot\rangle_{\mathbf{H}}$. Define a linear map $W : \mathbf{H} \to \mathscr{H}$ by

$$W(h) = \sum_{i=1}^{N} h_i X_i\,. \tag{2.3.1}$$

Then we have, for any $h, g \in \mathbf{H}$,

$$\mathbb{E}[W(h)W(g)] = \sum_{i=1}^{N}\sum_{j=1}^{N} h_i g_j \mathbb{E}[X_i X_j] = \sum_{i=1}^{N} h_i h_j = \langle h, g\rangle_H . \tag{2.3.2}$$

This means that the map W is an isometry from $\mathbf{H}$ onto $\mathscr{H}$.

Definition 2.3.1: Wiener chaos, N-dimensional version

For any $n \geqslant 1$, we denote by $\mathscr{H}_n$ the subspace of $\mathscr{H}$ spanned by the random variables

$$\{H_n(W(h)) \colon h \in \mathbf{H}, \|h\|_{\mathbf{H}} = 1\} .$$

For $n = 0$, $\mathscr{H}_0$ is the set of constants, which is isomorphic to $\mathbb{R}$. Then $\mathscr{H}_n$ is called the *homogeneous Wiener chaos of order n*. The *inhomogeneous Wiener chaos of order n* is defined as

$$\mathscr{H}_{\leqslant n} = \bigoplus_{k=0}^{n} \mathscr{H}_k .$$

We can now extend Lemma 1.3.1 and Theorem 1.3.3 to the situation at hand. These are again particular cases of [Nua06, Lemma 1.1.2 and Theorem 1.1.1].

Lemma 2.3.2

The random variables $\{e^{W(h)} \colon h \in \mathbf{H}\}$ form a total subset of $\mathscr{H}$.

Proof: Let $Z \in \mathscr{H}$ be such that $\mathbb{E}[Z\, e^{W(h)}] = 0$ for all $h \in \mathbf{H}$. The linearity of the map $h \mapsto W(h)$ implies that

$$\mathbb{E}\biggl[Z \exp\biggl\{\sum_{i=1}^{m} t_i W(h_i)\biggr\}\biggr] = 0 \tag{2.3.3}$$

for any choice of $t_1, \dots, t_m \in \mathbb{R}$, any $h_1, \dots, h_m \in \mathbf{H}$ and $m \geqslant 1$. We now fix m and $h_1, \dots, h_m$, and define a signed measure ν by

$$\nu(B) = \mathbb{E}[Z \mathbb{1}_B(W(h_1), \dots, W(h_m))]$$

for any Borel set B of $\mathbb{R}^N$. Then (2.3.3) means that the Laplace transform of ν is identically zero on $\mathbb{R}$. Therefore, this measure is zero, so that $\mathbb{E}[Z\mathbb{1}_B] = 0$ for any Borel set B. This implies that Z is equal to 0, and therefore the completeness if the system. □

Theorem 2.3.3: Wiener chaos decomposition

The Hilbert space $\mathscr{H}$ can be decomposed into the infinite orthogonal sum

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n .$$

Proof: Let $Z \in \mathscr{H}$ be orthogonal to $\mathscr{H}_n$ for all $n \geqslant 0$. This means that

$$\mathbb{E}[Z H_n(W(h))] = 0$$

for all $n \geqslant 0$ and all $h \in \mathbf{H}$ with $\|h\|_{\mathbf{H}} = 1$. By (1.2.28), we also have $\mathbb{E}[Z W(h)^n] = 0$ for all $n \geqslant 0$, and therefore $\mathbb{E}[Z\, e^{tW(h)}] = 0$ for all $t \in \mathbb{R}$ and all $h \in \mathbf{H}$ such that $\|h\|_{\mathbf{H}} = 1$. By Lemma 2.3.2, this means that $Z = 0$, which completes the proof. □

2.3.2 Wiener isometry and Fock space

We now construct an orthogonal basis of each Wiener chaos $\mathscr{H}_n$. For $k \in \mathbb{N}_0^N$, we define

$$\Phi_k = \prod_{i=1}^{N} H_{k_i}(X_i) . \tag{2.3.4}$$

Then Proposition 1.2.5 (orthogonality of the H_n) implies that

$$\mathbb{E}[\Phi_k \Phi_\ell] = \prod_{i=1}^{N} \mathbb{E}[H_{k_i}(X_i) H_{\ell_i}(X_i)] = \prod_{i=1}^{N} k_i! \delta_{k_i \ell_i} = k! \delta_{k\ell}$$

with $k!$ as in (2.2.4).

We denote by $\mathbf{H}^{\otimes n}$ the n-fold tensor product of $\mathbf{H}$, and by $\mathbf{H}^{\otimes_s n}$ the subspace of symmetric tensors. Any element of $\mathbf{H}^{\otimes n}$ can be canonically projected on $\mathbf{H}^{\otimes_s n}$ via

$$\Pi(h_1 \otimes \cdots \otimes h_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)} , \tag{2.3.5}$$

where $\mathfrak{S}_n$ denotes the set of all permutations of $\{1, \dots, n\}$. Let $(e_1, \dots, e_N)$ denote any orthonormal basis of $\mathbf{H}$ (for instance the canonical basis). Then for any $k \in \mathbb{R}^N$,

$$e_k := \Pi \bigotimes_{i=1}^{N} e_i^{\otimes k_i} \tag{2.3.6}$$

is an element of $\mathbf{H}^{\otimes_s |k|}$, where $|k| = \sum_{i=1}^{N} |k_i|$, with the convention that only strictly positive k_i count in the product (2.3.6). Moreover, the set $\{e_k : |k| = n\}$ forms an orthogonal basis of $\mathbf{H}^{\otimes_s n}$, with

$$\langle e_k, e_\ell \rangle = \frac{k!}{n!} \delta_{k\ell} .$$

This is because among the $n!$ permutations defining e_k, there are $k!$ permutations that yield the same term.

Example 2.3.4

Assume $N = 3$, and let $k = (2, 1, 0)$. Then $k! = 2$, $n = |k| = 3$, and

$$\begin{aligned} e_k &= \Pi(e_1 \otimes e_1 \otimes e_2) \\ &= \frac{1}{3}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) , \end{aligned}$$

because the $3! = 6$ permutations of $\{1, 2, 3\}$ have pairwise the same image. It follows that

$$\langle e_k, e_k \rangle = \frac{1}{3} = \frac{k!}{n!} .$$

Definition 2.3.5: Wiener isometry

For $k \in \mathbb{N}_0^N$, let $n = |k|$. The map

$$I_n : e_k \longmapsto \frac{1}{\sqrt{n!}} \Phi_k \tag{2.3.7}$$

is called the nth *Wiener isometry*.

Exercise 2.3.6

1. Show that I_n is indeed an isometry between $\mathbf{H}^{\otimes_s n}$ and $\mathscr{H}_n$.
2. If $N = 3$, what are the dimensions of $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$? Provide orthogonal bases of these spaces.
3. What is the dimension of $\mathscr{H}_n$ for general N?
   Hint: Use the method of stars and bars.

The space

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathbf{H}^{\otimes_s n}$$

is called *Fock space*. The Wiener isometry thus provides an isometry between Fock space and $\mathscr{H}$. Fock space is known from quantum physics, where it describes interacting bosons. The space $\mathbf{H}^{\otimes_s n}$ describes the set of states with given number n of particles. It is identified with the space of symmetric functions of n variables.

Two important particular cases of Wiener isometries are

$$I_0 = 1 , \qquad I_1(h) = W(h) ,$$

where the second relation follows from $I_1(e_i) = X_i$, see (2.3.1). The following lemma generalises these relations.

Lemma 2.3.7

For any $n \geqslant 1$ and $h \in \mathbf{H}$ with $\|h\|_{\mathbf{H}} = 1$, one has

$$I_n(h^{\otimes n}) = \frac{1}{\sqrt{n!}} H_n(W(h)) , \tag{2.3.8}$$

independently of the basis $\{e_i\}_{1 \leqslant i \leqslant N}$ of $\mathbf{H}$.

Proof: If $h = \sum_{i=1}^N h_i e_i$, then

$$h^{\otimes n} = \Pi(h^{\otimes n}) = \sum_{1 \leqslant i_1, \dots, i_n \leqslant N} h_{i_1} \dots h_{i_n} \Pi(e_{i_1} \otimes \dots \otimes e_{i_n}) = \sum_{|k|=n} \frac{n!}{k!} h^k e_k ,$$

where the combinatorial factor counts the number of ways a tuple $(i_1, \dots, i_n)$ can be mapped to a given k, defined by the fact that k_j is the number of indices equal to j. The result then follows from the definitions (2.3.7) of I_n and (2.3.4) of Φ_k and the multinomial formula (2.2.3) (see Lemma 2.2.3). □

Remark 2.3.8

If h is not normalised, one has

$$I_n(h^{\otimes n}) = \frac{\|h\|_{\mathbf{H}}^n}{\sqrt{n!}} H_n\biggl(W\biggl(\frac{h}{\|h\|_{\mathbf{H}}}\biggr)\biggr) = \frac{1}{\sqrt{n!}} H_n(W(h); \|h\|_{\mathbf{H}}^2)$$

by the scaling property (2.2.1) of Hermite polynomials.

In general, we can compute I_n on an element of $\mathbf{H}^n$ by projecting it on $\mathbf{H}^{\otimes_s n}$, decomposing it in the basis of e_k, and applying the definition (2.3.7) of I_n.

Exercise 2.3.9

Assume $h = \sum_{i=1}^N h_i e_i$ and $g = \sum_{i=1}^N g_i e_i$ belong to $\mathbf{H}$. Show that

$$I_2(h \otimes g) = \frac{1}{\sqrt{2!}} \Bigl[\sum_{i \neq j = 1}^N h_i g_j H_1(X_i) H_1(X_j) + \sum_{i=1}^N h_i g_i H_2(X_i) \Bigr] . \tag{2.3.9}$$

Check that one recovers (2.3.8) when $h = g$ and $\|h\|_{\mathbf{H}} = \|g\|_{\mathbf{H}} = 1$.

2.3.3 Multiplication and Wick product

Now that we have defined $I_n(f)$ for general $f \in \mathbf{H}^{\otimes_s n}$ (or $\mathbf{H}^{\otimes n}$, using symmetrisation), it is of interest to compute products of such quantities, in the same spirit as for the product-sum formulas seen in Proposition 1.2.7. Here it will be more convenient to use the normalisation

$$\hat{I}_n(f) = \sqrt{n!} I_n(f) .$$

In order not to confuse components of tensor products in $\mathbf{H}^n$ and components of elements of $\mathbf{H}$, we will write the latter as

$$f = \sum_{i=1}^N f(i) e_i \in \mathbf{H} .$$

In this way, elements of $\mathbf{H}^{\otimes n}$ can be written as

$$f_1 \otimes \dots \otimes f_n = \sum_{i_1,\dots,i_n=1}^N f_1(i_1) \dots f_n(i_n) e_{i_1} \otimes \dots \otimes e_{i_n} =: \sum_{i_1,\dots,i_n=1}^N f(i_1,\dots,i_n) e_{i_1} \otimes \dots \otimes e_{i_n} .$$

We can thus view f as a map from $[\![1,N]\!]$ to $\mathbb{R}$.

Let us start by computing $\hat{I}_1(f)\hat{I}_1(g)$. This is given by

$$\hat{I}_1(f)\hat{I}_1(g) = W(f)W(g) = \sum_{i,j=1}^N f(i)g(j) X_i X_j .$$

Comparing with (2.3.9), we see that

$$\hat{I}_1(f)\hat{I}_1(g) - \hat{I}_2(f \otimes g) = \sum_{i=1}^N f(i)g(i)[X_i^2 - H_2(X_i)] = \langle f, g \rangle_{\mathbf{H}} \in \mathscr{H}_0 .$$

We have thus obtained

$$\hat{I}_1(f)\hat{I}_1(g) = \hat{I}_2(f \otimes g) + \langle f, g \rangle_{\mathbf{H}} . \tag{2.3.10}$$

Let $\mathfrak{S}(p,n) \subset \mathfrak{S}(n)$ denote the set of permutations of $[\![1,n]\!]$ preserving the order of the first p and the last $n-p$ elements, also called *shuffles*. A first generalisation of (2.3.10) is as follows.

Lemma 2.3.10: Multiplication between the nth and first chaos

Assume $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}$. Then

$$\hat{I}_n(f)\hat{I}_1(g) = \hat{I}_{n+1}(f \otimes g) + \hat{I}_{n-1}(f \star_1 g) ,$$

where $\star_1$ denotes the *contraction operation*

$$(f \star_1 g)(i_1,\dots,i_{n-1}) = \sum_{\Sigma \in \mathfrak{S}(1,n)} \sum_{j=1}^N f(\Sigma(j,i_1,\dots,i_{n-1})) g(j) . \tag{2.3.11}$$

Proof: By linearity, it suffices to check the relation for basis vectors, that is, when one has $f = e_{i_1} \otimes \cdots \otimes e_{i_n}$ and $g = e_j$. We distinguish between two cases.

1. If $j \notin \{i_1, \dots, i_n\}$, then $f \star_1 g = 0$, because all components of f containing one index j are zero. One easily sees that
$$\hat{I}_{n+1}(f \otimes e_j) = \hat{I}_n(f) X_j \,.$$
2. If $j \in \{i_1, \dots, i_n\}$, say $j = i_1$, let m be the number of indices equal to i_1. We have
$$\hat{I}_n(f) = H_m(X_{i_1}) P_{n-m} \,, \qquad \hat{I}_1(g) = H_1(X_{i_1}) \,,$$
where P_{n-m} is a polynomial of degree $n-m$ that does not contain X_{i_1}. By the product-sum formula (1.2.9), we obtain
$$H_m(X_{i_1}) H_1(X_{i_1}) = \sum_{p=0}^{1} \binom{m}{p} \binom{1}{p} H_{m+1-p}(X_{i_1}) = H_{m+1}(X_{i_1}) + m H_{m-1}(X_{i_1}) \,,$$
so that
$$\hat{I}_n(f) \hat{I}_1(g) = [H_{m+1}(X_{i_1}) + m H_{m-1}(X_{i_1})] P_{n-m} \,. \tag{2.3.12}$$
On the other hand, we have
$$\hat{I}_{n+1}(f \otimes g) = H_{m+1}(X_i) P_{n-m} \,, \tag{2.3.13}$$
while, using $g(\ell) = \delta_{\ell j}$,
$$(f \star_1 g)(i_1, \dots, i_{n-1}) = \sum_{\Sigma \in \mathfrak{S}(1,m)} f(\Sigma(i_1, i_1, \dots, i_{n-1})) \,.$$
There are m ways to insert the first i_1 among the other indices, which all yield the same value upon applying $\hat{I}_{n-1}$. Therefore,
$$\hat{I}_{n-1}(f \star_1 g) = m H_{m-1}(X_{i_1}) P_{n-m} \,. \tag{2.3.14}$$
The sum of (2.3.13) and (2.3.14) is indeed equal to (2.3.12). □

Example 2.3.11

Consider the case $n = 3$. Then
$$\hat{I}_3(f) \hat{I}_1(g) = \hat{I}_4(f \otimes g) + \hat{I}_2(f \star_1 g) \,,$$
where
$$(f \otimes g)(i_1, i_2, i_3, j) = f(i_1, i_2, i_3) g(j) \,.$$
Denoting a permutation $\Sigma \in \mathfrak{S}(1,3)$ by the image $(\Sigma(1), \Sigma(2), \Sigma(3))$ of $(1,2,3)$, we have
$$\mathfrak{S}(1,3) = \{(1,2,3), (2,1,3), (2,3,1)\}$$
and therefore
$$(f \star_1 g)(i_1, i_2) = \sum_{j=1}^{N} [f(j, i_1, i_2) + f(i_1, j, i_2) + f(i_1, i_2, j)] g(j) \,.$$

The generalisation of this to arbitrary f and g is as follows.

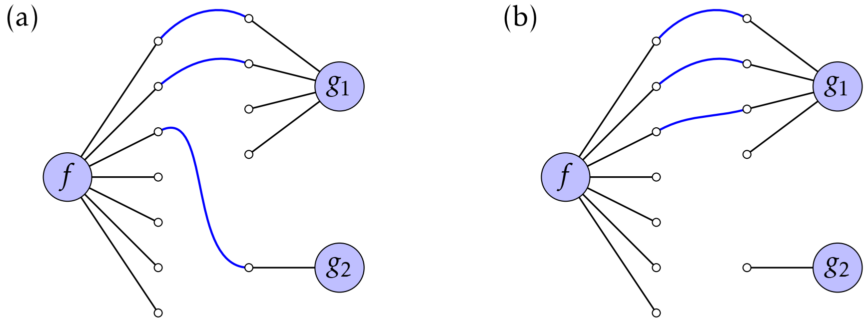


Figure 2.1 – Examples of pairings corresponding to $(f \star_{p-1} g_1) \star_1 g_2$ (a) and to $(f \star_p g_1) \otimes g_2$ (b), when $n = 7$, $m = 5$ and $p = 3$.

Proposition 2.3.12: Multiplication between nth and mth chaos

Assume $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$. Then

$$\hat{I}_n(f)\hat{I}_m(g) = \sum_{p=0}^{n \wedge m} \hat{I}_{n+m-2p}(f \star_p g) \tag{2.3.15}$$

where $\star_0 = \otimes$ and the contraction $\star_p$ is defined for $\mathbf{i} = (i_1, \dots, i_{n-p})$ and $\mathbf{j} = (j_1, \dots, j_{m-p})$ by

$$(f \star_p g)(\mathbf{i}, \mathbf{j}) = \sum_{\substack{\Sigma \in \mathfrak{S}(p,n) \\ \bar{\Sigma} \in \mathfrak{S}(p,m)}} \sum_{\sigma \in \mathfrak{S}(p)} \sum_{\mathbf{k} \in [\![1,N]\!]^p} f(\Sigma(\mathbf{k}, \mathbf{i})) g(\bar{\Sigma}(\mathbf{k}, \sigma(\mathbf{j}))) \,. \tag{2.3.16}$$

Before giving a proof of this result, we provide some intuition for the meaning of contractions, see Figure 2.1. We think of f as a vertex with n legs, representing its components, while g is a vertex with m legs. The contraction $\star_p$ represents all ways of pairing p legs of f with p legs of g, where Σ represents the choice of legs of f, $\bar{\Sigma}$ represents the choice of legs of g, and σ counts all ways of pairing the chosen legs. As for the sum over ℓ, it can be viewed as an inner product in $\mathbf{H}$. With this picture in mind, we can show the following lemma.

Lemma 2.3.13

Assume $f \in \mathbf{H}^{\otimes n}$, and $g = g_1 \otimes g_2 \in \mathbf{H}^{\otimes m}$ with $g_1 \in \mathbf{H}^{\otimes(m-1)}$ and $g_2 \in \mathbf{H}$. Then one has

$$f \star_p g = \mathbb{1}_{p \neq 0} (f \star_{p-1} g_1) \star_1 g_2 + \mathbb{1}_{p \neq m} (f \star_p g_1) \otimes g_2 \qquad \text{for } 0 \leqslant p \leqslant m \,.$$

Proof: We give a graphical proof. The first term on the right-hand side represents pairing $p-1$ legs of f with $p-1$ legs of g_1, and one leg of f with one leg of g_2, see Figure 2.1 (a). This is only possible if $p \geqslant 1$. The second term on the right-hand side represents pairing p legs of f with p legs of g_1, and leaving g_2 alone, see Figure 2.1 (b). This is only possible if $p \leqslant m-1$. □

We can now give the proof of Proposition 2.3.12.

Proof of Proposition 2.3.12. We may assume $n \geqslant m$. The proof is by induction on m. The base case $m = 1$ is Lemma 2.3.10. For the induction step, we can restrict by linearity to the case where $g = g_1 \otimes g_2$ with $g_1 \in \mathbf{H}^{\otimes(m-1)}$ and $g_2 \in \mathbf{H}$. Assume that g_2 is orthogonal to all components of g_1. Then we claim that

$$g_1 \star_1 g_2 = 0 \,.$$

Indeed, taking a g_2 as a basis vector of $\mathbf{H}$, g_1 will be a linear combination of basis vectors different from g_2, so that the inner product in (2.3.11) vanishes. It follows that

$$\hat{I}_{m-1}(g_1)\hat{I}_1(g_2) = \hat{I}_m(g_1 \otimes g_2)\,.$$

Therefore,

$$\begin{aligned}\hat{I}_n(f)\hat{I}_m(g) &= \hat{I}_n(f)\hat{I}_{m-1}(g_1)\hat{I}_1(g_2)\\ &= \sum_{p=0}^{m-1} \hat{I}_{n+m-1-2p}(f \star_p g_1)\hat{I}_1(g_2)\\ &= \sum_{p=0}^{m-1} \hat{I}_{n+m-2p}((f \star_p g_1)\otimes g_2) + \sum_{p=0}^{m-1} \hat{I}_{n+m-1-2p}((f \star_p g_1)\star_1 g_2)\,,\end{aligned}$$

where we have used the induction hypothesis in the second line, and Lemma 2.3.10 in the third one. Making the index shift $p \mapsto p+1$ in the second sum and using Lemma 2.3.13 yields the result.

In case it is not possible to find an orthogonal decomposition $g = g_1 \otimes g_2$, by linearity it suffices to consider the case where $g = g_2^{\otimes m}$. This case can be reduced to an explicit computation, see Exercise 2.3.15 below. □

Remark 2.3.14

If f and g are symmetric under permutations of their arguments, since $\mathfrak{S}(p,n)$ has cardinality $\binom{n}{p}$, we have

$$(f \star_p g)(\mathbf{i},\mathbf{j}) = p!\binom{n}{p}\binom{m}{p}\sum_{\mathbf{k}\in[[1,N]]^p} f(\mathbf{k},\mathbf{i})g(\mathbf{k},\mathbf{j})\,.$$

Exercise 2.3.15

Compute $\hat{I}_n(f)\hat{I}_m(g)$ when $f = h_1^{\otimes n}$ and $g = h_2^{\otimes m}$ with $h_1, h_2 \in \mathbf{H}$. Argue that for the case left out in the proof of Proposition 2.3.12, is suffices to consider the case $h_1 = h_2$. Why is the result true in that case?

The leading term in the decomposition (2.3.15) plays a special role, and is therefore given a name.

Definition 2.3.16: Wick product

The *Wick product* of $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$ is defined by

$$\hat{I}_n(f) \diamond \hat{I}_m(g) = \hat{I}_{n+m}(f \otimes g)\,.$$

Relation (2.3.15) can thus be written

$$\hat{I}_n(f)\hat{I}_m(g) = \hat{I}_n(f) \diamond \hat{I}_m(g) + \sum_{p=1}^{n\wedge m} \hat{I}_{n+m-2p}(f \star_p g)\,.$$

2.4 Equivalence of moments

The purpose of this section is to give a proof of the following, very useful result.

Theorem 2.4.1: Equivalence of moments

Assume F belongs to the nth Wiener chaos $\mathscr{H}_n$. Then for any $p > 1$, one has

$$\mathbb{E}[F^{2p}]^{1/2p} \leqslant (2p-1)^{n/2}\mathbb{E}[F^2]^{1/2} . \tag{2.4.1}$$

This result states that the variance of a random variable $F = f(X_1,\dots,X_N)$ controls all L^p norms of F for $p > 1$.

We will follow a proof given in [Nua06, Section 1.4], based on hypercontractivity of the Ornstein–Uhlenbeck semigroup. Other proofs can be found in [DPT07, Section 4] and in the lecture notes [Hai26, Section 7].

2.4.1 Ornstein–Uhlenbeck semigroup

Definition 2.4.2: Ornstein–Uhlenbeck semigroup

The Ornstein–Uhlenbeck semigroup is the one-parameter semigroup $\{T_t \colon t \geqslant 0\}$ of contraction operators on $\mathscr{H}$ defined by

$$T_t(F) = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} P_n F \tag{2.4.2}$$

for any $F \in \mathscr{H}$, where $P_n : \mathscr{H} \to \mathscr{H}_n$ denotes the orthogonal projection on the nth Wiener chaos.

We have already encountered the Ornstein–Uhlenbeck process in Section 1.2.3. Consider first the one-dimensional case ($N = 1$). Then this process is defined as the solution of the stochastic differential equation in $\mathbb{R}$

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t , \qquad X_0 = x , \tag{2.4.3}$$

where $(W_t)_{t\geqslant 0}$ denotes standard Brownian motion. By Ito's formula, for any $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^2$, the process $Y_t = f(x_t)$ satisfies

$$\mathrm{d}Y_t = [-X_t f'(X_t) + f''(X_t)]\,\mathrm{d}t + \sqrt{2} f'(X_t)\,\mathrm{d}W_t .$$

In particular, if $f(x) = H_n(x)$, then the term in brackets is $-nH_n(X_t)$ (see (1.2.13) in Corollary 1.2.10), so that

$$\mathrm{d}Y_t = -nY_t\,\mathrm{d}t + \sqrt{2} f'(X_t)\,\mathrm{d}W_t .$$

Integrating from 0 to t, we find

$$H_n(X_t) = H_n(x) - n\int_0^t H_n(X_s)\,\mathrm{d}s + \sqrt{2}\int_0^t f'(X_s)\,\mathrm{d}W_s .$$

Since the expectation of the stochastic integral is zero, it follows that

$$\mathbb{E}[H_n(X_t)] = H_n(x) - n\int_0^t \mathbb{E}[H_n(X_s)]\,\mathrm{d}s ,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[H_n(X_t)] = -n\mathbb{E}[H_n(X_t)] .$$

The solution of this ordinary differential equation is simply

$$\mathbb{E}[H_n(X_t)] = H_n(x)\,\mathrm{e}^{-nt} .$$

Consider now a general f of the form

$$f(x) = \sum_{n\geqslant 0} c_n H_n(x) .$$

Then by linearity, we obtain

$$\mathbb{E}[f(X_t)] = \sum_{n\geqslant 0} c_n H_n(x)\,\mathrm{e}^{-nt} = \sum_{n\geqslant 0} (P_n f))(x)\,\mathrm{e}^{-nt} = T_t(f)(x) ,$$

explaining why T_t in (2.4.2) is called Ornstein–Uhlenbeck semigroup.

Consider now the N-dimensional case. The N-dimensional Ornstein–Uhlenbeck is defined as the solution of the stochastic differential equation (2.4.3) when $x_t \in \mathbb{R}^N$ and W_t is N-dimensional Brownian motion. If $Y_t = f(X_t)$ for a twice differentiable $f : \mathbb{R}^2 \to \mathbb{R}$, Ito's formula now reads

$$\mathrm{d}Y_t = [-\langle \nabla f(X_t), X_t\rangle + \Delta f(X_t)]\,\mathrm{d}t + \sqrt{2}\langle \nabla f(X_t), \mathrm{d}W_t\rangle .$$

The fact that the associated semigroup is again of the form (2.4.2) is a consequence of the following observation.

Lemma 2.4.3: Eigenfunctions of the Ornstein–Uhlenbeck generator in $\mathbb{R}^N$

For $k \in \mathbf{H} = \mathbb{R}^N$ let

$$f(x) = \Phi_k = \prod_{i=1}^{N} H_{k_i}(x_i) .$$

Then

$$-\langle \nabla f(x), x\rangle + \Delta f(x) = -|k| f(x) . \tag{2.4.4}$$

Proof: This follows directly from the fact that

$$\frac{\partial f}{\partial x_i} = H'_{k_i}(x_i)\prod_{j\neq i} H_{k_j}(x_j) , \qquad \frac{\partial f^2}{\partial x_i^2} = H''_{k_i}(x_i)\prod_{j\neq i} H_{k_j}(x_j)$$

for any $i \in [[1, N]]$. □

Exercise 2.4.4

Check that (2.4.4) implies that the semigroup of the N-dimensional Ornstein–Uhlenbeck process is given by (2.4.2).

Using variation of constants, the solution of (2.4.3) can be represented as

$$X_t = \mathrm{e}^{-t} x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)}\,\mathrm{d}W_s ,$$

as can be verified by applying Ito's formula. The second term on the right-hand side is Gaussian, centered, and by Ito's isometry, its variance is

$$2\int_0^t \mathrm{e}^{-2(t-s)}\,\mathrm{d}s = 1-\mathrm{e}^{-2t}\ .$$

The Ornstein–Uhlenbeck process can thus be written as

$$X_t = \mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}X'_t\,,$$

where $X'_t \sim \mathcal{N}(0,1)$ for any $t \geqslant 0$. This observation is the intuition behind the following result.

Proposition 2.4.5: Mehler's formula

Let $W' = \{W'(h)\colon h \in \mathbf{H}\}$ be an independent copy of $W = \{W(h)\colon h \in \mathbf{H}\}$, where W and W' are defined on a product probability space $(\Omega\times\Omega', \mathscr{F}\otimes\mathscr{F}', \mathbb{P}\times\mathbb{P}')$. For $t > 0$, consider the process $Z = \{Z(h)\colon h \in \mathbf{H}\}$ defined by

$$Z(h) = \mathrm{e}^{-t}W(h) + \sqrt{1-\mathrm{e}^{-2t}}W'(h)\,.$$

Then for any $F \in \mathscr{H}$ of the form $F = f(W)$, one has

$$T_t(F) = \mathbb{E}'[f(Z)] \tag{2.4.5}$$

where $\mathbb{E}'$ denotes the expectation with respect to the law $\mathbb{P}'$ of W'.

Proof: The process $Z(h)$ is Gaussian, centered, with covariance

$$\begin{aligned}\mathbb{E}[Z(h_1)Z(h_2)] &= \mathrm{e}^{-2t}\,\mathbb{E}[W(h_1)W(h_2)] + (1-\mathrm{e}^{-2t})\mathbb{E}[W'(h_1)W'(h_2)]\\ &= \langle h_1,h_2\rangle_{\mathbf{H}}\\ &= \mathbb{E}[W(h_1)W(h_2)]\end{aligned} \tag{2.4.6}$$

by (2.3.2). The right-hand side of (2.4.5) defines a linear contraction on any $L^p(\Omega)$ with $p > 1$, because by Jensen's inequality

$$\begin{aligned}\mathbb{E}[|T_t(F)|^p] &= \mathbb{E}\Big[\Big|\mathbb{E}'[f(\mathrm{e}^{-t}W(h) + \sqrt{1-\mathrm{e}^{-2t}}W'(h))]\Big|^p\Big]\\ &\leqslant \mathbb{E}\Big[\mathbb{E}'[|f(\mathrm{e}^{-t}W(h) + \sqrt{1-\mathrm{e}^{-2t}}W'(h))|^p]\Big] = \mathbb{E}[|F|^p]\,.\end{aligned}$$

Therefore, it is sufficient to check that (2.4.5) holds for the generating function of Hermite polynomials $F = f(W) = \exp\{W(h) - \frac{1}{2}\|h\|^2_{\mathbf{H}}\}$ with any $h \in \mathbf{H}$. On one hand, we have

$$\begin{aligned}F = G(1,W(h)) &= \sum_{n\geqslant 0}\frac{1}{n!}H_n(W(h);\|h\|_H)\\ &= \sum_{n\geqslant 0}\frac{1}{n!}\hat{I}_n(h^{\otimes n})\end{aligned}$$

by Remark 2.3.8. This implies

$$T_t(F) = \sum_{n\geqslant 0}\frac{\mathrm{e}^{-nt}}{n!}\hat{I}_n(h^{\otimes n})\,. \tag{2.4.7}$$

On the other hand, we have

$$\mathbb{E}'[f(Z)] = \mathbb{E}'[\mathrm{e}^{a+bW'(h)}]\,, \qquad a = \mathrm{e}^{-t}W(h) - \frac{1}{2}\|h\|^2_{\mathbf{H}}\,, \qquad b = \sqrt{1-\mathrm{e}^{-2t}}\,.$$

By Proposition 2.1.2 (Laplace transform) and the expression (2.4.6) for the covariance of Z, this yields

$$\begin{aligned}
\mathbb{E}'[f(Z)] &= e^{a}\,\mathbb{E}'[e^{\langle b,W(h)\rangle}] \\
&= \exp\left\{e^{-t}\,W(h) - \frac{1}{2}\|h\|_{\mathbf{H}}^2 + \frac{1}{2}(1-e^{-2t})\|h\|_{\mathbf{H}}^2\right\} \\
&= \exp\left\{e^{-t}\,W(h) - \frac{1}{2}\,e^{-2t}\|h\|_{\mathbf{H}}^2\right\} \\
&= G(e^{-t}, W(h)) \\
&= \sum_{n\geqslant 0} \frac{e^{-nt}}{n!} H_n(W(h);\|h\|_{\mathbf{H}}) \\
&= \sum_{n\geqslant 0} \frac{1}{n!}\hat{I}_n(h^{\otimes n})\,.
\end{aligned}$$

This is equal to (2.4.7), which concludes the proof. □

2.4.2 Hypercontractivity

We will denote by $L^p(\Omega,\mathscr{F},\mathbb{P})$ the Banach space of random variables $F:\Omega\to\mathbb{R}$ satisfying

$$\|F\|_p = \mathbb{E}[F^p]^{1/p} < \infty\,.$$

The following result, originally due to Nelson, says that the Ornstein–Uhlenbeck semigroup is *hypercontractive*, meaning that if $t>0$, then T_t maps $L^p(\Omega,\mathscr{F},\mathbb{P})$ into $L^q(\Omega,\mathscr{F},\mathbb{P})$ for some $q=q(t)$ strictly larger than p. We follow essentially the proof of [Nua06, Theorem 1.4.1].

Theorem 2.4.6: Hypercontractivity of the Ornstein–Uhlenbeck semigroup

For $p>1$ and $t>0$, let

$$q(t) = e^{2t}(p-1)+1 > p\,. \tag{2.4.8}$$

Then for any $F\in L^p(\Omega,\mathscr{F},\mathbb{P})$, one has

$$\|T_tF\|_{q(t)} \leqslant \|F\|_p\,. \tag{2.4.9}$$

Proof: Let q' be the Hölder congujate of $q=q(t)$, that is,

$$\frac{1}{q}+\frac{1}{q'} = 1\,.$$

By duality, we have

$$\|T_tF\|_q = \sup_{G\in L^{q'}} \frac{\langle T_tF, G\rangle}{\|G\|_{q'}} = \sup_{G\in L^{q'}} \frac{\mathbb{E}[(T_tF)G]}{\|G\|_{q'}}\,.$$

We will thus have proved (2.4.9) if we manage to prove that

$$\mathbb{E}[(T_tF)G] \leqslant \|F\|_p\|G\|_{q'}$$

for all $G\in L^{q'}(\Omega,\mathscr{F},\mathbb{P})$. Since T_t is non-negative, and thus $|T_tF|\leqslant T_t(|F|)$, we may assume that F and G are both non-negative. In fact, by an approximation argument, we may assume that $a\leqslant F,G\leqslant b<\infty$ for some $b\geqslant a>0$.

The main idea is to use a kind of interpolation argument. Recall that both F and G are functions of the random variables $W(e_i) = X_i$ with $i \in [\![1,N]\!]$. By Mehler's identity (2.4.5),

$$T_t F = \mathbb{E}'[f(Z_1,\dots,Z_N)]\,,$$

where

$$Z_i = \mathrm{e}^{-t} X_i + \sqrt{1-\mathrm{e}^{-2t}} X_i'\,,$$

the X_i' being independent copies of the X_i. We can represent these variables as

$$X_i = \int_0^1 \mathrm{d}W_i(s)\,, \qquad X_i' = \int_0^1 \mathrm{d}W_i'(s)\,,$$

where $(W_i(s))_{0\leqslant s\leqslant 1}$ and $(W_i'(s))_{0\leqslant s\leqslant 1}$ are independent Brownian motions. Then we have

$$Z_i = \int_0^1 B_i(s)\,\mathrm{d}s\,, \qquad B_i(s) = \mathrm{e}^{-t} W_i(s) + \sqrt{1-\mathrm{e}^{-2t}} W_i'(s)\,.$$

Therefore, we can write

$$\mathbb{E}[(T_t F)G] = \mathbb{E}[PQ]\,,$$

where

$$P = f(B_1,\dots,B_N)\,, \qquad Q = g(W_1,\dots,W_N)\,.$$

By Ito's formula, P^p and Q^p have integral representations of the form

$$P^p = \mathbb{E}[P^p] + \int_0^1 \varphi(s)\,\mathrm{d}B(s)\,, \qquad Q^{q'} = \mathbb{E}[Q^{q'}] + \int_0^1 \psi(s)\,\mathrm{d}W(s)$$

for some bounded, positive φ and ψ. We introduce two bounded, positive martingales

$$M(s) = \mathbb{E}[P^p] + \int_0^s \varphi(u)\,\mathrm{d}B(u)\,, \qquad N(s) = \mathbb{E}[Q^{q'}] + \int_0^s \psi(u)\,\mathrm{d}W(u)\,,$$

which satisfy

$$\begin{aligned} M(0) &= \mathbb{E}[P^p]\,, & N(0) &= \mathbb{E}[Q^{q'}]\,, \\ M(1) &= P^p\,, & N(1) &= Q^{q'}\,. \end{aligned}$$

Define $f(x,y) = x^\alpha y^\beta$, with $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q'}$. Then $U(s) = f(M(s),N(s))$ satisfies

$$U(0) = \|P\|_p \|Q\|_{q'}\,, \qquad U(1) = PQ\,.$$

By Ito's formula, we have

$$\mathrm{d}U(s) = \frac{\partial f}{\partial x}\,\mathrm{d}M(s) + \frac{\partial f}{\partial y}\,\mathrm{d}N(s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,\mathrm{d}M(s)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\,\mathrm{d}N(s)^2 + \frac{\partial^2 f}{\partial x^2}\,\mathrm{d}M(s)\,\mathrm{d}N(s)\,,$$

where

$$\begin{aligned} \mathrm{d}M(s)^2 = \varphi(s)^2\,\mathrm{d}s\,, \qquad \mathrm{d}N(s)^2 = \psi(s)^2\,\mathrm{d}s\,, \\ \mathrm{d}M(s)\,\mathrm{d}N(s) = \varphi(s)\psi(s)\,\mathrm{d}B(s)\,\mathrm{d}W'(s) = \varphi(s)\psi(s)\,\mathrm{e}^{-s}\,\mathrm{d}s\,. \end{aligned}$$

Computing the partial derivatives of f, this leads to

$$\mathrm{d}U(s) = \alpha M(s)^{\alpha-1} N(s)^\beta\,\mathrm{d}M(s) + \beta M(s)^\alpha N(s)^{\beta-1}\,\mathrm{d}N(s) + \frac{1}{2} M(s)^\alpha N(s)^\beta A(s)\,\mathrm{d}s\,,$$

where

$$A(s) = \alpha(\alpha-1)M(s)^{-2}\varphi(s)^2 + \beta(\beta-1)N(s)^{-2}\psi(s)^2 + 2\alpha\beta M(s)^{-1}N(s)^{-1}\varphi(s)\psi(s)\,\mathrm{e}^{-t}$$
$$= \begin{pmatrix}\varphi(s)M(s)^{-1} & \psi(s)N(s)^{-1}\end{pmatrix}\begin{pmatrix}\alpha(\alpha-1) & \alpha\beta\,\mathrm{e}^{-t} \\ \alpha\beta\,\mathrm{e}^{-t} & \beta(\beta-1)\end{pmatrix}\begin{pmatrix}\varphi(s)M(s)^{-1} \\ \psi(s)N(s)^{-1}\end{pmatrix}. \tag{2.4.10}$$

We have

$$PQ = U(1) = U(0) + \int_0^1 \mathrm{d}U(s) = \|P\|_p\|Q\|_{q'} + \int_0^1 \mathrm{d}U(s)\,.$$

Taking expectations, since $M(s)$ and $N(s)$ are martingales, we obtain

$$\mathbb{E}[(T_tF)G] = \mathbb{E}[PQ] = \|P\|_p\|Q\|_{q'} + \frac{1}{2}\int_0^1 \mathbb{E}[M(s)^\alpha N(s)^\beta A(s)]\,\mathrm{d}s\,.$$

The result will thus be proved if we manage to show that $A(s) \leqslant 0$ for all $s \in [0,1]$. Note that if $p > 1$, then one finds $\alpha(1-\alpha) < 0$ and $\beta(1-\beta) < 0$. It thus suffices to show that the determinant of the matrix in (2.4.10) is non-negative. This is equivalent to (2.4.8). □

It is now easy to prove equivalence of moments.

Proof of Theorem 2.4.1: We make the change of index $q(t) \mapsto 2p$ and $p \mapsto 2$. Then (2.4.9) becomes

$$\|T_tF\|_{2p} \leqslant \|F\|_2\,,$$

where

$$\mathrm{e}^{2t} = 2p-1\,.$$

In particular, for $F \in \mathscr{H}_n$, we have

$$\|T_tF\|_{2p} = \mathrm{e}^{-nt}\|F\|_{2p}\,.$$

This yields

$$\|F\|_{2p} = \mathrm{e}^{nt}\|T_tF\|_{2p} \leqslant \mathrm{e}^{nt}\|F\|_2 = (2p-1)^{n/2}\|F\|_2\,,$$

which is equivalent to (2.4.1). □

Chapter 3

Gaussian fields

In this chapter, we extend the results of the previous chapter to the infinite-dimensional case, and discuss two particularly important cases of Gaussian fields, namely white noise and the Gaussian free field.

3.1 Isonormal Gaussian processes

In this chapter, we will mostly be concerned with the following set-up. Let $\Lambda = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ be the d-dimensional torus, for some $d \geqslant 1$. Let

$$\mathbf{H} = L^2(\Lambda, \mathrm{d}x)$$

be the Hilbert space of square-integrable functions $h : \mathbb{T}^d \to \mathbb{R}$. Let $(e_i)_{i\geqslant 0}$ be an orthonormal basis of $\mathbf{H}$ – we will typically choose a Fourier basis. We are then interested in random fields of the form

$$\phi(x) = \sum_{i\geqslant 0} X_i e_i(x)\,,$$

where the X_i are independent, centred jointly Gaussian random variables of unit variance, defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We will see that this is related to the following general construction.

Definition 3.1.1: Isonormal Gaussian process

Let $\mathbf{H}$ be a separable Hilbert space. A stochastic process $W = \{W(h) \colon h \in \mathbf{H}\}$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is an *isonormal Gaussian process* if W is a centred Gaussian family of random variables such that

$$\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2\rangle_{\mathbf{H}}$$

for all $h_1, h_2 \in \mathbf{H}$.

Note that the map $h \mapsto W(h)$ is necessarily linear. This is because for any $h_1, h_2 \in \mathbf{H}$, and any $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}[(W(\lambda h_1 + \mu h_2) &- \lambda W(h_1) - \mu W(h_2))^2] \\
&= \|\lambda h_1 + \mu h_2\|_{\mathbf{H}}^2 + \lambda^2 \|h_1\|_{\mathbf{H}}^2 + \mu^2 \|h_2\|_{\mathbf{H}}^2 \\
&\quad - 2\lambda\langle \lambda h_1 + \mu h_2, h_1\rangle_{\mathbf{H}} - 2\mu\langle \lambda h_1 + \mu h_2, h_2\rangle_{\mathbf{H}} + 2\lambda\mu\langle h_1, h_2\rangle_{\mathbf{H}}^2 = 0\,.
\end{aligned}$$

As a consequence, if each random variable is Gaussian and centred, the set $\{W(h)\colon h \in \mathbf{H}\}$ is automatically a family of jointly Gaussian random variables. The existence of the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ follows from Kolmogorov's continuity theorem.

3.1.1 Wiener chaos expansion and Wiener isometry

The definition of Wiener chaoses is the same in infinite dimension as in finite dimension.

Definition 3.1.2: Wiener chaos, infinite-dimensional version

For any $n \geqslant 1$, we denote by $\mathscr{H}_n$ the subspace of $\mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$ spanned by the random variables

$$\{H_n(W(h))\colon h \in \mathbf{H}, \|h\|_{\mathbf{H}} = 1\} .$$

For $n = 0$, $\mathscr{H}_0$ is the set of constants, which is isomorphic to $\mathbb{R}$. Then $\mathscr{H}_n$ is called the *homogeneous Wiener chaos of order n*. The *inhomogeneous Wiener chaos of order n* is defined as

$$\mathscr{H}_{\leqslant n} = \bigoplus_{k=0}^{n} \mathscr{H}_k .$$

We also have an analogue of Theorem 2.3.3, with the same proof (see also [Nua06, Theorem 1.1.1]).

Theorem 3.1.3: Wiener chaos decomposition

The Hilbert space $\mathscr{H}$ can be decomposed into the infinite orthogonal sum

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n .$$

The construction of the Wiener isometry proceeds in the same way as we have seen in Section 2.3.2, except that one now assumes that the multiindex $k \in \mathbb{N}_0^{\mathbb{N}}$ has only finitely many non-zero entries. In this way,

$$\Phi_k = \prod_{i \geqslant 0\colon k_i > 0} H_{k_i}(X_i)$$

is well-defined, as are

$$|k| = \sum_{i \geqslant 0\colon k_i > 0} |k_i| \qquad \text{and} \qquad k! = \prod_{i \geqslant 0\colon k_i > 0} k_i! .$$

The Wiener isometry, with its two normalisation conventions, is again defined as

$$I_n(e_k) = \frac{1}{\sqrt{n!}} \Phi_k , \qquad \hat{I}_n(e_k) = \Phi_k$$

for all $k \in \mathbb{N}_0^{\mathbb{N}}$ with $|k| = n$, where

$$e_k = \Pi \bigotimes_{i \geqslant 0} e_i^{\otimes k_i} ,$$

Π being the symmetrisation operator (2.3.5). Note that we have slightly overloaded the notation for e, as we use the same letter for elements of $\mathbf{H}$ and elements of Fock space. It should always be clear from the context which basis vector is meant.

Let us give some simple examples for clarity.

Example 3.1.4

- Assume that k has only one nonzero entry $k_i = 1$. Then $e_k = e_i$ and $\hat{I}_1(e_k) = H_1(X_i) = X_i$.
- If k has two nonzero entries $k_i = k_j = 1$ with $i \neq j$, then $e_k = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$, and $\hat{I}_2(e_k) = X_i X_j$.
- If k has one nonzero entry $k_i = 2$, then $e_k = e_i \otimes e_i$ and $\hat{I}_2(e_k) = H_2(X_i)$.

The bound showing equivalence of moments, cf. Theorem 2.4.1, can be proved in essentially the same way as we did in finite dimension, so we just repeat it here.

Theorem 3.1.5: Equivalence of moments

Assume F belongs to the nth Wiener chaos $\mathscr{H}_n$. Then for any $p > 1$, one has

$$\mathbb{E}[F^{2p}]^{1/2p} \leqslant (2p-1)^{n/2} \mathbb{E}[F^2]^{1/2} . \tag{3.1.1}$$

Exercise 3.1.6

- Let $f, g \in \mathbf{H}$. Compute $\mathbb{E}[\hat{I}_1(f)\hat{I}_1(g)]$.
- Do the same for $\mathbb{E}[\hat{I}_n(f)\hat{I}_m(g)]$, when $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$.

3.1.2 The case of $L^2(\mathbb{T}^d)$

As mentioned above, we will mainly be concerned with the case $\mathbf{H} = L^2(\Lambda, \mathrm{d}x)$, with $\Lambda = \mathbb{T}^d$, endowed with a Fourier basis $(e_i)_{i \geqslant 0}$. Elements of $\mathbf{H}$ are thus functions $h : \Lambda \to \mathbb{R}$ that can be written as a Fourier series

$$h(x) = \sum_{i \geqslant 0} \hat{h}(i) e_i(x) . \tag{3.1.2}$$

Recall that (3.1.2) defines an isometry between $\mathbf{H}$ and the Hilbert space

$$\widehat{\mathbf{H}} = \ell^2 = \left\{ \hat{h} \in \mathbb{R}^{\mathbb{N}_0} : \sum_{i \geqslant 0} \hat{h}(i)^2 < \infty \right\} .$$

Indeed, by Parseval's relation, we have

$$\|h\|^2_{\mathbf{H}} = \int_\Lambda h(x)^2 \, \mathrm{d}x = \sum_{i \geqslant 0} \hat{h}(i)^2 = \|\hat{h}\|^2_{\widehat{\mathbf{H}}} .$$

It is thus equivalent to work in $\widehat{\mathbf{H}}$, which is essentially the same as what we did in the finite-dimensional case, or in $\mathbf{H}$.

Elements of $\mathbf{H}^{\otimes n}$ can be written either as

$$h = h_1 \otimes \cdots \otimes h_n = \sum_{i_1 \geqslant 0, \dots, i_n \geqslant 0} \hat{h}_1(i_1) \dots \hat{h}_n(i_n) e_{i_1} \otimes \cdots \otimes e_{i_n} ,$$

or, equivalently, as

$$h(x_1, \dots, x_n) = \sum_{i_1 \geqslant 0, \dots, i_n \geqslant 0} \hat{h}(i_1, \dots, i_n) e_{i_1}(x_{i_1}) \dots e_{i_n}(x_{i_n}) , \tag{3.1.3}$$

where $\hat{h}(i_1, \dots, i_n) = \hat{h}_1(i_1) \dots \hat{h}_n(i_n)$. In this way, elements of $\mathbf{H}^{\otimes_s n}$ are represented by functions h that are symmetric in all their arguments, that is,

$$h(\sigma(x_1, \dots, x_n)) = h(x_1, \dots, x_n) \qquad \forall \sigma \in \mathfrak{S}_n .$$

The definition of contractions can then be rewritten in the following way.

Lemma 3.1.7

Let $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$. For any $p \leqslant n \wedge m$, all $x \in \Lambda^{n-p}$ and all $y \in \Lambda^{m-p}$, one has

$$(f \star_p g)(x,y) = \sum_{\substack{\Sigma \in \mathfrak{S}(p,n) \\ \bar\Sigma \in \mathfrak{S}(p,m)}} \sum_{\sigma \in \mathfrak{S}(p)} \int_{\Lambda^p} f(\Sigma(z,x)) g(\bar\Sigma(z,\sigma(x)))\, \mathrm{d}z\,.$$

Proof: Consider the case $m = 1$. Then (2.3.16) yields

$$(\hat f \star_1 \hat g)(i_1,\dots,i_{n-1}) = \sum_{\Sigma \in \mathfrak{S}(1,n)} \sum_{j \geqslant 0} \hat f(\Sigma(j,i_1,\dots,i_{n-1}))\hat g(j)\,. \tag{3.1.4}$$

By (3.1.3), we have

$$(f \star_1 g)(x_1,\dots,x_{n-1}) = \sum_{i_1,\dots,i_{n-1} \geqslant 0} (\hat f \star_1 \hat g)(i_1,\dots,i_{n-1}) e_{i_1}(x_{i_1}) \dots e_{i_{n-1}}(x_{i_{n-1}})\,.$$

Consider the contribution of the identity permutation to (3.1.4), which is

$$\sum_{j,i_1,\dots,i_{n-1} \geqslant 0} \hat f_1(j) \hat f_2(i_1) \dots \hat f_n(i_{n-1}) \hat g(j) e_{i_1}(x_{i_1}) \dots e_{i_{n-1}}(x_{i_{n-1}})\,. \tag{3.1.5}$$

On the other hand, we have

$$\begin{aligned}
&\int_\Lambda f(z,x_1,\dots,x_{n-1}) g(z)\,\mathrm{d}z \qquad (3.1.6)\\
&= \int_\Lambda \sum_{j,i_1,\dots,i_{n-1},k \geqslant 0} \hat f_1(j) \hat f_2(i_1) \dots \hat f_n(i_{n-1}) \hat g(k) e_j(z) e_k(z) e_{i_1}(x_{i_1}) \dots e_{i_{n-1}}(x_{i_{n-1}})\,\mathrm{d}z\,.
\end{aligned}$$

Orthogonality of eigenfunctions implies

$$\int_\Lambda e_j(z) e_k(z)\,\mathrm{d}z = \delta_{jk}\,.$$

Therefore, (3.1.6) is equal to (3.1.5). The argument is similar for other permutations, and for $m > 1$. □

3.1.3 A construction of Gaussian fields

Consider now the following construction. For $h = \sum_{i \geqslant 0} \hat h(i) e_i \in \mathbf{H}$, we define

$$\Psi(h) = \sum_{i \geqslant 0} \hat h(i) W(e_i) e_i = \sum_{i \geqslant 0} \hat h(i) X_i e_i\,. \tag{3.1.7}$$

This is now an $\mathbf{H}$-valued random variable, whose value at point $x \in \Lambda$ is

$$\Psi(h)(x) = \sum_{i \geqslant 0} \hat h(i) X_i e_i(x)\,.$$

Note furthermore that

$$\begin{aligned}
\|\Psi(h)\|_{\mathbf{H}}^2 &= \int_\Lambda \Psi(h)(x)^2\,\mathrm{d}x \\
&= \sum_{i,j \geqslant 0} \hat h(i) \hat h(j) X_i X_j \int_\Lambda e_i(x) e_j(x)\,\mathrm{d}x \\
&= \sum_{i \geqslant 0} \hat h(i)^2 X_i^2
\end{aligned}$$

by orthonormality of the e_i. Therefore,

$$\mathbb{E}[\|\Psi(h)\|_{\mathbf{H}}^2] = \sum_{i\geqslant 0} \hat{h}(i)^2 = \|h\|_{\mathbf{H}}^2 .$$

This shows that the map Ψ is an isometry from the Hilbert space $\mathbf{H}$ to a subset of the Hilbert space $\widetilde{\mathscr{H}}$ of $\mathbf{H}$-valued random variables that have finite variance.

In terms of Wiener chaos, since $\Psi(h)$ is defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as the X_i, if makes sense to decompose $\widetilde{\mathscr{H}}$ into Wiener chaoses, by viewing them as random variables inded by $x \in \Lambda$. In particular, $\Psi(h)$ belongs to the first Wiener chaos. This allows us to work within the framework of separable Hilbert space. For a general construction of Gaussian measures on separable Banach spaces, see Chapter 3 of Martin Hairer's lecture notes [Hai09].

3.2 Gaussian white noise

3.2.1 Definition and basic properties

Consider the case where

$$\hat{h} = (1, 1, 1, \dots)$$

is the vector all of whose components are equal to 1. Then (3.1.7) becomes

$$\xi(x) := \Psi(h)(x) = \sum_{i\geqslant 0} X_i e_i(x) . \tag{3.2.1}$$

This random variable is called *white noise* on Λ, because it means that every Fourier mode is random with the same variance. The trouble is that h does not belong to $\mathbf{H}$, since $\hat{h}$ is not square-summable. As a result, ξ does not have finite variance.

One way to try to make sense of this definition is to set, for any finite $N \in \mathbb{N}$,

$$\hat{h}_N = (\underbrace{1, 1, 1, \dots, 1}_{N \text{ components}}, 0, 0, \dots) .$$

In this way, we obtain

$$\xi_N(x) := \Psi(h_N)(x) = \sum_{i=0}^{N} X_i e_i(x) .$$

Since h_N is in $\mathbf{H}$ for any finite N, ξ_N is a random function with finite variance for these N. It is called a *mollification with cut-off* N of white noise. Of course, the variance of ξ_N diverges as N goes to infinity.

Another way to make sense of ξ is to view it as a random distribution. Let $\varphi : \Lambda \to \mathbb{R}$ be a sufficiently regular so-called *test function*. Then we have, at least formally,

$$\langle \xi, \varphi \rangle = \int_\Lambda \xi(x)\varphi(x)\,\mathrm{d}x = \sum_{i\geqslant 0} X_i \int_\Lambda e_i(x)\varphi(x)\,\mathrm{d}x = \sum_{i\geqslant 0} X_i \hat{\varphi}(i) .$$

This is indeed well-defined for $\varphi \in \mathbf{H}$. Furthermore, for any $\varphi_1, \varphi_2 \in \mathbf{H}$, we have

$$\begin{aligned}\mathbb{E}[\langle \xi, \varphi_1\rangle\langle \xi, \varphi_2\rangle] &= \sum_{i\geqslant 0}\sum_{j\geqslant 0} \mathbb{E}[X_i X_j]\hat{\varphi}_1(i)\hat{\varphi}_2(j) \\ &= \sum_{i\geqslant 0} \hat{\varphi}_1(i)\hat{\varphi}_2(i) = \langle \varphi_1, \varphi_2\rangle_{\mathbf{H}} .\end{aligned}$$

This motivates the following definition.

Definition 3.2.1: Gaussian white noise on the torus

Gaussian white noise on $\mathbb{T}^d$ is the random distribution ξ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that for any smooth test function $\varphi \in \mathbf{H}$, $\langle \xi, \varphi \rangle$ is a centred Gaussian random variable of variance $\|\varphi\|_{\mathbf{H}}^2$, while the covariance is given by

$$\mathbb{E}[\langle \xi, \varphi_1 \rangle \langle \xi, \varphi_2 \rangle] = \langle \varphi_1, \varphi_2 \rangle_{\mathbf{H}} \tag{3.2.2}$$

for any two smooth test functions $\varphi_1, \varphi_2 \in \mathbf{H}$.

One immediate consequence of this definition is that if φ_1 and φ_2 have disjoint support (meaning that they cannot be different from zero at the same point), then $\langle \varphi_1, \varphi_2 \rangle_{\mathbf{H}} = 0$, and therefore $\langle \xi, \varphi_1 \rangle$ and $\langle \xi, \varphi_2 \rangle$ are independent by Proposition 1.1.2. The relation (3.2.2) is sometimes formally written

$$\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y) ,$$

where $\delta(x-y)$ is the Dirac distribution, that can be formally obtained as a limit of scaled test functions localised at $x-y$.

Another important property of white noise is related to scaling. To this end, define for $\lambda \in (0,1]$ a scaling operator $\mathscr{S}^\lambda$ acting on $\varphi \in \mathbf{H}$ by

$$(\mathscr{S}^\lambda \varphi)(x) = \frac{1}{\lambda^d} \varphi\Big(\frac{x}{\lambda}\Big) . \tag{3.2.3}$$

Let ξ_λ be the distribution defined by

$$\langle \xi_\lambda, \varphi \rangle = \langle \xi, \mathscr{S}^\lambda \varphi \rangle$$

for any test function φ.

Exercise 3.2.2

Show that if $h : \Lambda \to \mathbb{R}$ is an integrable function, then $h_\lambda(x) = h(\lambda x)$.

Proposition 3.2.3: Scaling of white noise

For any $\lambda \in (0,1]$, one has equality in law

$$\xi_\lambda \stackrel{\text{law}}{=} \frac{1}{\lambda^{d/2}} \xi . \tag{3.2.4}$$

Proof: Both processes are Gaussian and centred. Therefore, it suffices to show that they have the same covariance. Given two compactly supported test functions φ_1, φ_2, we have by (3.2.2)

$$\begin{aligned}
\mathbb{E}[\langle \xi_\lambda, \varphi_1 \rangle \langle \xi_\lambda, \varphi_2 \rangle] &= \mathbb{E}[\langle \xi, \mathscr{S}^\lambda \varphi_1 \rangle \langle \xi, \mathscr{S}^\lambda \varphi_2 \rangle] \\
&= \int_\Lambda (\mathscr{S}^\lambda \varphi_1)(x) (\mathscr{S}^\lambda \varphi_2)(x) \, \mathrm{d}x \\
&= \frac{1}{\lambda^d} \int_\Lambda \varphi_1\Big(\frac{x}{\lambda}\Big) \varphi_2\Big(\frac{x}{\lambda}\Big) \mathrm{d}x \\
&= \frac{1}{\lambda^d} \langle \varphi_1, \varphi_2 \rangle_{\mathscr{H}} ,
\end{aligned}$$

where we have not changed the domain of integration, because the φ_i are compactly supported. This is indeed the covariance of $\lambda^{-d/2} \xi$. □

The scaling property (3.2.4) can be interpreted as a form of self-similarity, in a statistical sense.

3.2.2 Regularity of white noise*

Even though white noise is very irregular, one can find function spaces to which it belongs. Two important such function spaces are fractional Sobolev spaces and Besov–Hölder spaces of negative regularity index.

In order to define fractional Sobolev spaces, it is more convenient to index basis elements of $\mathbf{H} = L^2(\Lambda)$ by their wave number. This means that they are of the form

$$e_k(x) = \mathrm{e}^{2\pi \mathrm{i}\langle k,x\rangle}\,, \qquad k \in \mathbb{Z}^d\,,$$

where k should not be confused with the index k used in the Wiener isometry. We then write

$$f(x) = \sum_{k\in\mathbb{Z}^d} \hat f(k) e_k(x)$$

for the Fourier series of a function $f \in \mathbf{H}$.

Remark 3.2.4: Complex versus real Fourier series

When one uses complex Fourier series, because of the reality condition $\overline{\hat f(k)} = \hat f(-k)$, one should not take independent X_k in the underlying probability space. Instead, they should satisfy

$$\mathbb{E}[X_k X_\ell] = \delta_{k,-\ell}\,.$$

It is also possible to use real instead of complex Fourier series and independent X_k, but this makes some computation slightly more tedious.

Note that we have

$$(\mathrm{id}-\Delta)e_k(x) = \lambda_k e_k(x)\,, \qquad \lambda_k = 1 + (2\pi)^d \|k\|^2\,. \tag{3.2.5}$$

The eigenvalues λ_k of the positive operator $\mathrm{id}-\Delta$ play a natural role as weights in Sobolev spaces.

Definition 3.2.5: Fractional Sobolev spaces

For $s \geqslant 0$, the *fractional Sobolev space* $H^s(\Lambda)$ is the space of functions $f \in \mathbf{H} = L^2(\Lambda)$ such that

$$\|f\|^2_{H^s} := \sum_{k\in\mathbb{Z}^d} \lambda_k^s |\hat f(k)|^2 < \infty\,. \tag{3.2.6}$$

In particular, $H^0(\Lambda) = L^2(\Lambda)$. For $s < 0$, $H^s(\Lambda)$ is the closure of $L^2(\Lambda)$ under the norm (3.2.6).

Note that in the definition (3.2.6) of the fractional Sobolev norm, one may replace the weight λ_k^s by $(1+\|k\|^2)^s$. The resulting norm is equivalent.

Proposition 3.2.6: Sobolev regularity of white noise on the torus

White noise ξ belongs to H^s for any $s < -\frac{d}{2}$, in the sense that

$$\mathbb{E}[\|\xi\|^2_{H^s}] < \infty \qquad \text{for all } s < -\frac{d}{2}\,.$$

PROOF: It follows from the Fourier representation (3.2.1) of white noise that

$$\|\xi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^d} \lambda_k^s X_k^2 \,,$$

where the X_k are independent standard Gaussians. Taking the expectation, we obtain

$$\mathbb{E}[\|\xi\|_{H^s}^2] = \sum_{k\in\mathbb{Z}^d} \lambda_k^s \,.$$

The sum is comparable to the integral

$$\int_{\mathbb{R}^d} (1+\|y\|^2)^s \,\mathrm{d}y \asymp \int_1^\infty r^{2s} r^{d-1} \,\mathrm{d}r \,,$$

which is convergent if and only if $s < -\frac{d}{2}$. Here we write $x \asymp y$ to indicate that $c^{-1}x \leqslant y \leqslant cx$ for some $c \geqslant 1$. □

A second scale of function spaces that are useful when working with white noise are so-called Hölder–Besov spaces. To define them, we first introduce a generalisation of the scaling operator (3.2.3) given by

$$(\mathcal{S}_x^\lambda \varphi)(y) = \frac{1}{\lambda^d}\varphi\Big(\frac{y-x}{\lambda}\Big).$$

For $r \in \mathbb{N}$, we denote by B_r the set of smooth test functions $\varphi : \Lambda \to \mathbb{R}$, supported on a ball or radius 1, whose partial derivatives up to order r are bounded by 1.

Definition 3.2.7: Hölder–Besov spaces

For $\alpha < 0$, the space $\mathcal{C}^\alpha(\Lambda)$ consists in all Schwartz distributions $\zeta \in \mathcal{S}'(\Lambda)$ such that

$$\|\zeta\|_{\mathcal{C}^\alpha} = \sup_{x\in\Lambda} \sup_{\varphi\in B_r} \sup_{\lambda\in(0,1]} \left| \frac{\langle \zeta, \mathcal{S}_x^\lambda \varphi\rangle}{\lambda^\alpha} \right| < \infty \,, \tag{3.2.7}$$

where $r = \lceil -\alpha \rceil$.

This definition says that if $\zeta \in \mathcal{C}^\alpha$, then $\langle \zeta, \mathcal{S}_x^\lambda\varphi\rangle$ diverges at most like $\lambda^{-\alpha}$ for any $x \in \Lambda$. It is thus a measure of how far the distribution is from admitting a finite value at x.

Proposition 3.2.8: Hölder–Besov regularity of white noise on the torus

White noise ξ belongs to $\mathcal{C}^\alpha$ for any $\alpha < -\frac{d}{2}$.

SKETCH OF PROOF. We follow the argument outlined in [CW17, Theorem 2.7]. Given $k \in \mathbb{N}_0$, we introduce a dyadic lattice discretisation of Λ on scale 2^{-k}, given by

$$\Lambda_k = (2^{-k}\mathbb{Z})^d \cap \Lambda \,.$$

For any $\alpha \in \mathbb{R}$, one can show the existence of a test function φ and a constant C such that

$$\|\xi\|_{\mathcal{C}^\alpha} \leqslant C \sup_{k\geqslant 0} \sup_{x\in\Lambda_k} 2^{k\alpha} \big|\langle \xi, \mathcal{S}_x^{2^{-k}} \varphi\rangle\big| \,.$$

See for instance [CZ21, Section12]. Bounding the suprema by sums, and taking the pth power, we arrive at

$$(\|\xi\|_{\mathcal{C}^\alpha})^p \leqslant C^p \sum_{k\geqslant 0} \sum_{x\in\Lambda_k} 2^{k\alpha p} \big|\langle \xi, \mathcal{S}_x^{2^{-k}} \varphi\rangle\big|^p \,. \tag{3.2.8}$$

Since $\langle \xi, \mathscr{S}_x^{2^{-k}}\varphi\rangle$ belongs to the first Wiener chaos, its pth power belongs to the pth (inhomogeneous) chaos. Using the scaling property (3.2.4) and (3.2.3), we get

$$\mathbb{E}[\langle \xi, \mathscr{S}_x^{2^{-k}}\varphi\rangle^2] = \frac{1}{\lambda^d}\mathbb{E}[\langle \xi, \phi\rangle^2] = \frac{1}{\lambda^d}\|\varphi\|_{\mathbf{H}}^2 \,.$$

The equivalence of moments (3.1.1) implies

$$\mathbb{E}[\langle \xi, \mathscr{S}_x^{2^{-k}}\varphi\rangle^p] \leqslant \frac{C_p}{\lambda^{dp/2}}\|\varphi\|_{\mathbf{H}}^p$$

for some constant C_p depending only on p. Plugging this into the expectation of (3.2.8) with $\lambda = 2^{-k}$ and using the fact that Λ_k has 2^{kd} points, we obtain

$$\mathbb{E}[(\|\xi\|_{\mathscr{C}^\alpha})^p] \leqslant C_p' \sum_{k\geqslant 0} 2^{kd} 2^{k\alpha p} 2^{kdp/2} = C_p' \sum_{k\geq 0} 2^{k(d+\alpha p+dp/2)}$$

where C_p' depends only on p. If $\alpha p < -d(1+\frac{p}{2})$, the geometric series can be summed. It then follows by a version of Kolmogorov's continuity theorem (see Theorem 3.3.6 below) that there exists a version of ξ with bounded $\mathscr{C}^\alpha$ norm. Since p can be taken arbitrarily large, the condition reduces to $\alpha < -\frac{d}{2}$. □

Remark 3.2.9: Besov spaces

Fractional Sobolev and Hölder–Besov spaces are particular instances of a general class of functional spaces called *Besov spaces*. The Besov space $\mathscr{B}^\alpha_{p,q}$ is a Banach space for all $\alpha \in \mathbb{R}$, and all $p, q \in [1,\infty]$, where α measures regularity, and p and q measure integrablity. Sobolev and Hölder–Besov spaces correspond to the particular cases

$$H^s = \mathscr{B}^s_{2,2} \qquad \text{and} \qquad \mathscr{C}^\alpha = \mathscr{B}^\alpha_{\infty,\infty} \,.$$

3.3 The Gaussian free field

3.3.1 Definition and basic properties

Consider now the case where h is given by

$$\hat{h}(k) = \frac{1}{\sqrt{\lambda_k}}\,, \qquad k \in \mathbb{Z}^d\,,$$

where λ_k is defined in (3.2.5). The associated Gaussian field is

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} e_k(x)\,. \tag{3.3.1}$$

This time, we have

$$\|h\|_{\mathbf{H}}^2 = \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k} \asymp \int_{\mathbb{R}^d} \frac{1}{1+\|y\|^2}\,\mathrm{d}y \asymp \int_1^\infty \frac{r^{d-1}\,\mathrm{d}r}{r^2} = \int_1^\infty \frac{\mathrm{d}r}{r^{3-d}}\,, \tag{3.3.2}$$

which converges if $d < 2$. We conclude that the Gaussian field (3.3.1) has a finite variance in dimension $d = 1$, but not in higher dimension. Still, this is somewhat better than for white noise. Its covariance is given by

$$\begin{aligned}\mathbb{E}[\phi_{\text{GFF}}(x)\phi_{\text{GFF}}(y)] &= \sum_{k,\ell\in\mathbb{Z}^d} \frac{\mathbb{E}[X_k X_\ell]}{\sqrt{\lambda_k\lambda_\ell}} e_k(x)e_\ell(y) \\ &= \sum_{k\in\mathbb{Z}^d} \frac{e_k(x)e_{-k}(y)}{\lambda_k} \\ &= \sum_{k\in\mathbb{Z}^d} \frac{e_k(x-y)}{\lambda_k} =: G(x-y) \end{aligned} \tag{3.3.3}$$

by Remark 3.2.4 and the fact that $e_k(x)e_{-k}(y) = e_k(x-y)$. Note that while $G(0)$ is defined only for $d=1$, one can show that $G(x)$ is defined for all d if $x \neq 0$. The function G has the following property, which states that it can be considered as the inverse of the linear operator $(\mathrm{id} - \Delta)$.

Lemma 3.3.1

For any $g \in \mathbf{H}$, the function f defined by

$$f(x) = \int_\Lambda G(x-y)g(y)\,\mathrm{d}y\,,$$

if it exists, satisfies

$$(\mathrm{id} - \Delta)f(x) = g(x)\,.$$

Proof: Assuming the integral is well-defined, we have

$$\begin{aligned}(\mathrm{id}-\Delta)f(x) &= \int_\Lambda \sum_{k\in\mathbb{Z}^d} \frac{(\mathrm{id}-\Delta)e_k(x)e_{-k}(y)}{\lambda_k} g(y)\,\mathrm{d}y \\ &= \sum_{k\in\mathbb{Z}^d} \int_\Lambda e_k(x)e_{-k}(y)g(y)\,\mathrm{d}y \\ &= \sum_{k\in\mathbb{Z}^d} \hat{g}(k)e_k(x) = g(x)\end{aligned}$$

by Dirichlet's theorem on Fourier series. □

This property motivates the following definition.

Definition 3.3.2: Gaussian free field and Green function

- The function G defined by (3.3.3) is called the *Green function* associated with the operator $(\mathrm{id}-\Delta)$. It is also written $G = (\mathrm{id}-\Delta)^{-1}$.
- The Gaussian field defined by (3.3.1) is called the *Gaussian free field (GFF) with covariance* $(\mathrm{id}-\Delta)^{-1}$.

Exercise 3.3.3

Show that for any test functions $\varphi_1, \varphi_2 \in \mathbf{H}$, one has

$$\mathbb{E}[\langle\phi_{\text{GFF}}, \varphi_1\rangle\langle\phi_{\text{GFF}}, \varphi_2\rangle] = \langle\varphi_1, (\mathrm{id}-\Delta)^{-1}\varphi_2\rangle_{\mathbf{H}}\,.$$

We can now easily adapt the proof of Proposition 3.2.6 to the GFF (3.3.1).

Proposition 3.3.4: Sobolev regularity of the Gaussian free field

The Gaussian free field ϕ_{GFF} belongs to H^s for any $s < 1 - \frac{d}{2}$, in the sense that

$$\mathbb{E}[\|\phi_{\mathrm{GFF}}\|_{H^s}^2] < \infty \qquad \text{for all } s < 1 - \frac{d}{2}\,.$$

Proof: A similar computation as before shows that

$$\mathbb{E}[\|\phi_{\mathrm{GFF}}\|_{H^s}^2] = \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k^{1-s}} \asymp \int_{\mathbb{R}^d} \frac{\mathrm{d}y}{(1+\|y\|^2)^{1-s}} \asymp \int_1^\infty \frac{\mathrm{d}r}{r^{3-d-2s}}\,.$$

This is finite if and only if $s < 1 - \frac{d}{2}$. □

For $d = 1$, we obtain a bit more regularity than $L^2 = H^0$, that we got in the estimate (3.3.2). For dimensions $d \geqslant 2$, on the other hand, we find again that the GFF is less regular than a function. We will examine these two cases more closely in the next two sections.

3.3.2 The Gaussian free field on the circle $\mathbb{T}^1$

Proposition 3.3.4 shows that the one-dimensional GFF is a function that has better regularity properties than being merely square-integable. In fact, one can show that it enjoys some Hölder regularity. Recall the definition of classical Hölder spaces.

Definition 3.3.5: Hölder spaces of regularity $\alpha \in (0,1)$

For $0 < \alpha < 1$, the space $\mathscr{C}^\alpha(\Lambda)$ consists in all functions $f : \Lambda \to \mathbb{R}$ such that

$$\|f\|_{\mathscr{C}^\alpha} = \sup_{x\in\Lambda}|f(x)| + \sup_{\substack{x,y\in\Lambda\\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|^\alpha} < \infty\,. \tag{3.3.4}$$

A classical result allowing the obtain Hölder regularity of a stochastic process is due to Kolmogorov.

Theorem 3.3.6: Kolmogorov continuity criterion

Let $(\phi(x))_{x\in[0,L]}$ be a stochastic process such that

$$\mathbb{E}[\|\phi(y)-\phi(x)\|^\mu] \leqslant C|y-x|^{1+\nu}$$

for all $x, y \in [0,L]$, for some constants $\mu, \nu > 0$. Then there exists a modification of the process $(\phi(x))_{x\in[0,L]}$ whose paths belong to $\mathscr{C}^\alpha$ for all $\alpha < \frac{\nu}{\mu}$.

We then have the following result, which shows that the one-dimensional GFF has the same regularity as Brownian motion. Its proof can be seen as a much easier relative of the proof of Proposition 3.2.8.

Proposition 3.3.7: Hölder regularity of the GFF on $\mathbb{T}^1$

The GFF on the circle belongs to $\mathscr{C}^\alpha(\mathbb{T}^1)$ for all $\alpha < \frac{1}{2}$.

Proof: For $x,y \in \mathbb{T}^1$, we have

$$\begin{aligned}\mathbb{E}[(\phi_{\mathrm{GFF}}(y)-\phi_{\mathrm{GFF}}(x))^2] &= \sum_{k,\ell\in\mathbb{Z}} \frac{\mathbb{E}[X_k X_\ell]}{\sqrt{\lambda_k\lambda_\ell}}(e_k(y)-e_k(x))\overline{(e_\ell(y)-e_\ell(x))} \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k}|e_k(y)-e_k(x)|^2\,.\end{aligned}$$

The trigonometric identity $|e^{i\theta}-1|^2 = 4\sin^2\bigl(\frac{\theta}{2}\bigr)$ yields

$$|e_k(y)-e_k(x)|^2 = |e^{ik(y-x)}-1|^2 = 4\sin^2(k(y-x)) \leqslant 4[k^2(y-x)^2 \vee 1]\,,$$

where $a\vee b = \max\{a,b\}$. Therefore, if $|y-x|\leqslant 1$, one has

$$\mathbb{E}[(\phi_{\mathrm{GFF}}(y)-\phi_{\mathrm{GFF}}(x))^2] \lesssim \sum_{k\in\mathbb{Z}} \frac{(k^2(y-x)^2)\vee 1}{1+k^2} \asymp \int_1^{1/|y-x|}(y-x)^2\,\mathrm{d}r + \int_{1/|y-x|}^{\infty}\frac{\mathrm{d}r}{r^2} \asymp |y-x|\,.$$

On the other hand, if $|y-x|>1$, we can simply bound $|e_k(y)-e_k(x)|^2$ by 4. We conclude that there exists a constant $C_0>0$ such that

$$\mathbb{E}[(\phi_{\mathrm{GFF}}(y)-\phi_{\mathrm{GFF}}(x))^2] \leqslant C_0|y-x|$$

for all $x,y\in\mathbb{T}^1$. By the equivalence of moments bound (3.1.1), it follows that for any $p>0$, there exists a constant $C(p)$ such that

$$\mathbb{E}[(\phi_{\mathrm{GFF}}(y)-\phi_{\mathrm{GFF}}(x))^{2p}] \leqslant C(p)|x-y|^p\,.$$

Kolmogorov's criterion thus applies with $\mu=2p$ and $\nu=p-1$, showing that $\phi_{\mathrm{GFF}}(x)$ is (up to a modification) Hölder continuous with exponent $\frac{1}{2}(1-\frac{1}{p})$. Since p can be taken arbitrarily large, the result follows. □

Remark 3.3.8: Link between $\alpha>0$ and $\alpha<0$

The definition (3.3.4) of the Hölder norm for positive α looks very different from the definition (3.2.7). However, one can show that it is equivalent to

$$\|f\|_{\mathscr{C}^\alpha} = \sup_{x\in\Lambda}\sup_{\varphi\in B_0}\sup_{\lambda\in(0,1]}\left|\frac{\langle f-f(x),\mathscr{S}_x^\lambda\varphi\rangle}{\lambda^\alpha}\right|\,.$$

We can also easily estimate moments of the GFF. By translation invariance, these do not depend on the point x. The second moment is given by

$$\mathbb{E}[\phi_{\mathrm{GFF}}(x)^2] = \sum_{k\in\mathbb{Z}}\frac{1}{\lambda_k} = G(0)\,, \tag{3.3.5}$$

which is finite for $d=1$, cf. (3.3.2). Odd moments of the GFF are equal to zero, while even moments behave as follows.

Proposition 3.3.9: Moments of the GFF on $\mathbb{T}^1$

For any $p>1$, there exists a constant $C(p)$ such that

$$\mathbb{E}[\phi_{\mathrm{GFF}}(x)^{2p}] \leqslant C(p)\mathbb{E}[\phi_{\mathrm{GFF}}(x)^2]^p\,. \tag{3.3.6}$$

Proof: A direct application of the equivalence of moments bound (3.1.1) shows that (3.3.6) holds with $C(p) = (2p-1)^p$. We can actually get a slightly better bound by a direct computation. Indeed, we have

$$\mathbb{E}[\phi_{\text{GFF}}(x)^{2p}] = \sum_{k_1,\dots,k_{2p}\in\mathbb{Z}} \frac{\mathbb{E}[X_{k_1}\dots X_{k_{2p}}]}{\sqrt{\lambda_{k_1}\dots\lambda_{k_{2p}}}} . \tag{3.3.7}$$

By Isserlis' theorem (Theorem 2.1.5), the expectation vanishes unless the k_i are pairwise equal, in which case it has value 1. Since there are $(2p-1)!!$ pairwise matchings, we get

$$\mathbb{E}[\phi_{\text{GFF}}(x)^{2p}] = (2p-1)!! \sum_{k_1,\dots,k_p\in\mathbb{Z}} \frac{1}{\lambda_{k_1}\dots\lambda_{k_p}} = (2p-1)!!\mathbb{E}[\phi_{\text{GFF}}(x)^2]^p .$$

We thus obtain (3.3.6) with $C(p) = (2p-1)!!$. Using (1.2.29) and Stirling's formula, we find that $(2p-1)!!$ behaves asymptotically like $\sqrt{2}(2p)^p\,\mathrm{e}^{-p}$, which is slightly better that $(2p-1)^p$. □

3.3.3 The Gaussian free field on the torus $\mathbb{T}^2$

We have seen that the GFF on the two-dimensional torus has infinite variance, and belongs only to Sobolev spaces H^s with $s < 0$. By an argument similar to the one used in Proposition 3.2.8, one can also show that it belongs to the Besov–Hölder spaces $\mathscr{C}^\alpha$ with $\alpha < 0$. For this reason, it is not possible to define powers of ϕ_{GFF}.

To circumvent this difficulty, we can use the idea introduced at the beginning of Section 3.2.1, which is to work with a cut-off N.

Definition 3.3.10: Truncated two-dimensional Gaussian free field

For $N \geqslant 1$, let $\mathscr{K}_N = \{k \in \mathbb{Z}^2 : |k| \leqslant N\}$, where $|k| = |k_1| + |k_2|$. The *truncated two-dimensional Gaussian free field (GFF)* with covariance $(\text{id} - \Delta_N)^{-1}$ on Λ is defined as

$$\phi_{\text{GFF},N}(x) := \sum_{k\in\mathscr{K}_N} \frac{X_k}{\sqrt{\lambda_k}} e_k(x) .$$

Here Δ_N is the restriction of Δ to the subspace E_N of $\mathbf{H}$ spanned by Fourier basis functions e_k with $|k| \leqslant N$.

The name is justified by the fact that the same computation as in (3.3.3) gives

$$\mathbb{E}[\phi_{\text{GFF},N}(x)\phi_{\text{GFF},N}(y)] = \sum_{k\in\mathscr{K}_N} \frac{1}{\lambda_k} e_k(x-y) =: G_N(x-y) , \tag{3.3.8}$$

where G_N is the Green function of $(\text{id} - \Delta_N)$. In particular, the variance at any x is given by

$$C_N := \mathbb{E}[\phi_{\text{GFF},N}(x)^2] = G_N(0) = \sum_{k\in\mathscr{K}_N} \frac{1}{\lambda_k} = \text{Tr}[(\text{id} - \Delta_N)^{-1}] . \tag{3.3.9}$$

It is not hard to see that C_N diverges like $\log(N)$ as $N \to \infty$.

Exercise 3.3.11

Show that $C_N = \dfrac{\log N}{2\pi} + \mathcal{O}(1)$ as $N \to \infty$, by viewing (3.3.9) as a Riemann sum.

Since the variance C_N does not depend on x, we can define for any $n \geqslant 1$ the quantity

$$:\phi^n_{\mathrm{GFF},N}(x): = :\phi^n_{\mathrm{GFF},N}(x):_{C_N} := H_n(\phi_{\mathrm{GFF},N}(x); C_N)\,,$$

called *nth Wick power* of the truncated GFF, where $H_n(\phi; C_N)$ denotes the scaled Hermite polynomial introduced in (2.2.1). For every $x \in \Lambda$, $:\phi^n_{\mathrm{GFF},N}(x):$ is a random variable belonging to the nth homogeneous Wiener chaos $\mathscr{H}_n$. The function $:\phi^n_{\mathrm{GFF}}:$ is also an $\mathbf{H}$-valued random variable. These random variables are independent for different n, while their variance is bounded uniformly in the cut-off N, as shows the following very useful result.

Proposition 3.3.12: Uniform bound on the variance of Wick powers

For every $n \geqslant 1$,

$$\sup_{N\geqslant 1} \mathbb{E}\biggl[\biggl(\int_\Lambda :\phi^n_{\mathrm{GFF},N}(x):\mathrm{d}x\biggr)^2\biggr] < \infty\,.$$

To prove this result, we will need the following inequality.

Lemma 3.3.13: Young-type inequality

Fix integers $d > n, m > 0$ such that $n+m > d$. Then there exists a constant $C > 0$, independent of k, such that

$$\sum_{\substack{k_1,k_2\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2=k}} \frac{1}{\|k_1\|^n\|k_2\|^m} \leqslant \frac{C}{\|k\|^{n+m-d}}$$

for all $k \in \mathbb{Z}^d$.

Proof: We may restrict the sum to (k_1, k_2) such that $\|k_1\| \geqslant \|k_2\|$, and multiply the end result by 2. Since $k_1 + k_2 = k$, we cannot have both $\|k_1\| < \frac12\|k\|$ and $\|k_2\| < \frac12\|k\|$. The half sum can thus be decomposed as

$$S_1 + S_2 = \sum_{\substack{k_1,k_2\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2=k\\ \|k_2\|\leqslant\|k_1\|\wedge\|k\|/2}} \frac{1}{\|k_1\|^n\|k_2\|^m} + \sum_{\substack{k_1,k_2\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2=k\\ \|k_1\|\geqslant\|k_2\|>\|k\|/2}} \frac{1}{\|k_1\|^n\|k_2\|^m}\,.$$

Since $\|k_2\| \leqslant \frac12\|k\|$ implies $\|k_1\| \geqslant \frac12\|k\|$, we have

$$S_1 \leqslant \frac{2^n}{\|k\|^n} \sum_{\substack{k_2\in\mathbb{Z}^d\setminus\{0\}\\ \|k_2\|\leqslant\|k\|/2}} \frac{1}{\|k_2\|^m} \lesssim \frac{2^n}{\|k\|^n} \int_1^{\|k\|/2} \frac{r^{d-1}\,\mathrm{d}r}{r^m} \lesssim \frac{1}{\|k\|^{n+m-d}}\,.$$

As for S_2, it satisfies

$$S_2 \leqslant \sum_{\substack{k_2\in\mathbb{Z}^d\setminus\{0\}\\ \|k_2\|>\|k\|/2}} \frac{1}{\|k_2\|^{n+m}} \lesssim \int_{\|k\|/2}^\infty \frac{r^{d-1}\,\mathrm{d}r}{r^{n+m}} \lesssim \frac{1}{\|k\|^{n+m-d}}\,,$$

which yields the result. □

PROOF OF PROPOSITION 3.3.12. We have

$$\begin{aligned}
\mathbb{E}\bigg[\Big(\int_\Lambda :\phi^n_{\mathrm{GFF},N}(x):\mathrm{d}x\Big)^2\bigg] &= \int_\Lambda\int_\Lambda \mathbb{E}[:\phi^n_{\mathrm{GFF},N}(x)::\phi^n_{\mathrm{GFF},N}(y):]\,\mathrm{d}x\,\mathrm{d}y \\
&= n!\int_\Lambda\int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF}}(x)\phi_{\mathrm{GFF}}(y)]^n\,\mathrm{d}x\,\mathrm{d}y \\
&= n!\int_\Lambda\int_\Lambda\Big(\sum_{k\in\mathcal{K}_N}\frac{1}{\lambda_k}e_k(x-y)\Big)^n\,\mathrm{d}x\,\mathrm{d}y \\
&= n!\sum_{k_1,\dots,k_n\in\mathcal{K}_N}\frac{1}{\lambda_{k_1}\dots\lambda_{k_n}}\bigg|\int_\Lambda e_{k_1}(x)\dots e_{k_n}(x)\,\mathrm{d}x\bigg|^2 \\
&= n!\sum_{\substack{k_1,\dots,k_n\in\mathcal{K}_N\\ k_1+\dots+k_n=0}}\frac{1}{\lambda_{k_1}\dots\lambda_{k_n}}\,, \qquad (3.3.10)
\end{aligned}$$

where we have used Proposition 1.2.5 to get the second line, and (3.3.8) to get the third line. For $k\in\mathbb{Z}^d$ and $n\geqslant 2$, let

$$S(n,k)=\sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}^d\\ k_1+\dots+k_n=k}}\frac{1}{\lambda_{k_1}\dots\lambda_{k_n}}\,.$$

Then Lemma 3.3.13, together with an index shift, shows that $S(2,k)\leqslant C(2)\lambda_k^{-1}$ for some constant $C(2)$, and by induction one gets

$$S(n,k)=\sum_{k_1\in\mathbb{Z}^d}\frac{1}{\lambda_{k_1}}S(n-1,k-k_1)\leqslant\frac{C(n)}{\lambda_k}$$

for some finite $C(n)$. In particular, $S(n,0)$ is bounded. Since this provides an upper bound uniform in N for (3.3.10), the result is proved. □

By the equivalence of moments bound (3.1.1), we also have

$$\mathbb{E}\bigg[\Big(\int_\Lambda :\phi^n_{\mathrm{GFF},N}(x):\mathrm{d}x\Big)^{2p}\bigg]\leqslant(2p-1)^{np}\,\mathbb{E}\bigg[\Big(\int_\Lambda :\phi^n_{\mathrm{GFF},N}(x):\mathrm{d}x\Big)^2\bigg]^p\,,$$

which is bounded uniformly in the cut-off N for all $p>1$. We thus conclude that all moments of Wick powers of the two-dimensional GFF are well-defined, when viewed as limits as $N\to\infty$ of the truncated GFF. Note that the same holds in dimensions $d>2$, the difference being that the variance C_N computed in (3.3.9) diverges like N^{d-2} instead of $\log(N)$. This faster divergence causes new problems for non-linear fields, as we will see in Section 4.4.

Chapter 4

The Φ^4 model

The Φ^4 model on the d-dimensional torus is arguably the simplest example of non-linear field theory. It originated as a toy model in Euclidean Quantum Field Theory, and its behaviour is very different depending on the dimension d.

4.1 Definition of the model

Let $\Lambda = \mathbb{T}^d$ be the d-dimensional torus for some $d \geqslant 1$. Given constants $\alpha \geqslant 0$, $m > 0$ and a function $\phi : \Lambda \to \mathbb{R}$, we define its energy

$$\mathscr{H}_{d,\alpha}(\phi) = \int_\Lambda \Big[\|\nabla\phi(x)\|^2 + \frac{m^2}{2}\phi(x)^2 + \alpha\phi(x)^4 \Big] \mathrm{d}x \,. \tag{4.1.1}$$

The name Φ^4 *model* is due to the term $\phi(x)^4$ in the integral. It is called the Φ^4_d model if we want to emphasize the value of the dimension. The parameter m has the physical interpretation of a mass. We will take it equal to 1 in what follows.

The Φ^4_d measure is the probability measure $\mu_{d,\alpha}$ on $\mathbf{H} = L^2(\Lambda, \mathrm{d}x)$ (or on a suitable a space of functions $\phi : \Lambda \to \mathbb{R}$), formally defined by

$$\mu_{d,\alpha} \sim \frac{1}{\mathscr{Z}_{d,\alpha}} \mathrm{e}^{-\mathscr{H}_{d,\alpha}(\phi)} \, \mathrm{d}\phi \,,$$

where $\mathscr{Z}_{d,\alpha}$ is the normalisation. This is called a *Gibbs measure*, and $\mathscr{Z}_{d,\alpha}$ is known in statistical physics as the *partition function*.

As such, this definition does not make sense, since there is no such thing as Lebesgue measure on $\mathbf{H}$. Note however that for $\alpha = 0$, we can integrate by parts, to obtain

$$\begin{aligned} \mathscr{H}_{d,0}(\phi) &= \frac{1}{2}\int_\Lambda [-\Delta\phi(x)\phi(x) + \phi(x)^2] \, \mathrm{d}x \\ &= \frac{1}{2}\langle \phi, [\mathrm{id} - \Delta]\phi \rangle_{\mathbf{H}} \,. \end{aligned}$$

In view of the expression (2.1.1) of the density of a finite-dimensional Gaussian measure, we can interpret $\mu_{d,0}(\mathrm{d}\phi)$ as the law of a centred Gaussian field with covariance $(\mathrm{id} - \Delta)^{-1}$ that we have studied in Section 3.3. This means that for a random variable $F : \mathbf{H} \to \mathbb{R}$, we can try to define its expectation under $\mu_{d,0}$ as

$$\mathbb{E}^{\mu_{d,0}}[F] = \mathbb{E}\Big[F\Big(\sum_{k \in \mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} e_k \Big) \Big] \,,$$

where the X_i are independent, identically distributed standard Gaussians, and the $\lambda_k = 1 + (2\pi)^d\|k\|^2$ are eigenvalues of the operator $\mathrm{id} - \Delta$.

Example 4.1.1

- Let $F(\phi) = \|\phi\|_{L^2}^2$. Then

$$\mathbb{E}^{\mu_{d,0}}[\|\phi\|_{L^2}^2] = \mathbb{E}\biggl[\sum_{k\in\mathbb{Z}^d} \frac{X_k^2}{\lambda_k}\biggr] = \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k}\,, \tag{4.1.2}$$

which we have seen is the variance of the Gaussian free field. This is finite for $d = 1$ (cf. (3.3.5)), but not for $d \geqslant 2$. By translation invariance, (4.1.2) is also the expectation of $\phi(x)^2$ for any $x \in \Lambda$.
- Let $F(\phi) = \phi(x)\phi(y)$ for $x, y \in \Lambda$. Then we have seen in (3.3.3) that

$$\mathbb{E}^{\mu_{d,0}}[\phi(x)\phi(y)] = G(x-y) \tag{4.1.3}$$

is the Green function, which is defined for all x, y if $d = 1$, and for all d if $x \neq y$.

This suggests using $\mu_{d,0}$ as a reference measure, instead of the non-existent Lebesgue measure on $\mathbf{H}$, and to define expectations under $\mu_{d,\alpha}$ with $\alpha > 0$ by

$$\mathbb{E}^{\mu_{d,\alpha}}[F] = \frac{\mathcal{Z}_{d,0}}{\mathcal{Z}_{d,\alpha}}\mathbb{E}^{\mu_{d,0}}\biggl[F(\phi)\exp\biggl\{-\alpha\int_\Lambda \phi(x)^4\,\mathrm{d}x\biggr\}\biggr]\,. \tag{4.1.4}$$

In particular, taking $F = 1$, we obtain

$$\frac{\mathcal{Z}_{d,\alpha}}{\mathcal{Z}_{d,0}} = \mathbb{E}^{\mu_{d,0}}\biggl[\exp\biggl\{-\alpha\int_\Lambda \phi(x)^4\,\mathrm{d}x\biggr\}\biggr]\,. \tag{4.1.5}$$

If we manage to compute this ratio, at least perturbatively for small α, it gives us access to more general expectations of the form (4.1.4).

One example of random variables F that are physically relevant is $F = \phi(x_1)\dots\phi(x_n)$, for given $x_1, \dots, x_n \in \Lambda$, leading to so-called n-point functions

$$G_{n,d,\alpha}(x_1, \dots, x_n) = \mathbb{E}^{\mu_{d,\alpha}}[\phi(x_1)\dots\phi(x_n)]\,.$$

But one can also consider F depending on an integral involving ϕ, such as the energy itself.

Exercise 4.1.2

Compute the n-point function $G_{n,d,0}(x_1, \dots, x_n)$ in the case $\alpha = 0$.

In what follows, we will focus on the ratio (4.1.5) of partition functions, since this is the first step in computing expectations of more general random variables.

4.2 The Φ_1^4 model

In this section, we consider the Φ^4 model on the one-dimensional torus $\Lambda = \mathbb{T}^1 = \mathbb{T}$, that is, the circle. As we have seen, all moments of the GFF are well-defined in this case. The ratio of partition functions (4.1.5) becomes

$$\frac{\mathcal{Z}_{1,\alpha}}{\mathcal{Z}_{1,0}} = \mathbb{E}^{\mu_{1,0}}\biggl[\exp\biggl\{-\alpha\int_\Lambda \phi(x)^4\,\mathrm{d}x\biggr\}\biggr]\,. \tag{4.2.1}$$

Figure 4.1 – Two pairwise matchings contributing to the sum (4.2.3). An oriented arrow from vertex i to vertex j means that $k_j = -k_i$.

One way to proceed is to expand the exponential, leading to

$$\frac{\mathcal{Z}_{1,\alpha}}{\mathcal{Z}_{1,0}} \asymp \sum_{n\geqslant 0} \frac{(-\alpha)^n}{n!} \mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda \phi(x)^4 \,\mathrm{d}x\biggr)^n\biggr], \tag{4.2.2}$$

where we use the symbol $\asymp$ because we do not know if this series is convergent (in fact, one can show that it is not!).

The term $n = 1$ in the sum (4.2.2) can be computed using (3.3.7). By Isserlis' theorem, we have

$$\mathbb{E}^{\mu_{1,0}}[\phi(x)^4] = \sum_{k_1,k_2,k_3,k_4\in\mathbb{Z}} \frac{\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}]}{\sqrt{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\lambda_{k_4}}} = 3\sum_{k_1,k_2\in\mathbb{Z}} \frac{1}{\lambda_{k_1}\lambda_{k_2}} = 3G(0)^2\,,$$

where the factor 3 counts the number of pairwise matchings of the four indices. Therefore, we also have

$$\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda \phi(x)^4 \,\mathrm{d}x\biggr)\biggr] = 3G(0)^2\,.$$

For the term $n = 2$, we find

$$\begin{aligned}
\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda \phi(x)^4 \,\mathrm{d}x\biggr)^2\biggr] &= \mathbb{E}^{\mu_{1,0}}\biggl[\int_\Lambda \phi(x)^4 \,\mathrm{d}x \int_\Lambda \phi(y)^4 \,\mathrm{d}y\biggr] \\
&= \sum_{k_1,\dots,k_8\in\mathbb{Z}} \frac{\mathbb{E}[X_{k_1}\dots X_{k_8}]}{\sqrt{\lambda_{k_1}\dots\lambda_{k_8}}} \int_\Lambda\int_\Lambda e_{k_1}(x)\dots e_{k_4}(x)e_{k_5}(y)\dots e_{k_8}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{\substack{k_1,\dots,k_8\in\mathbb{Z}\\ k_1+k_2+k_3+k_4=0\\ k_5+k_6+k_7+k_8=0}} \frac{\mathbb{E}[X_{k_1}\dots X_{k_8}]}{\sqrt{\lambda_{k_1}\dots\lambda_{k_8}}}\,.
\end{aligned} \tag{4.2.3}$$

The combinatorics is now more complicated, since we have to sum over all pairwise matchings of the k_i that satisfy the two sum constraints. Figure 4.1 gives two examples of such pairings.

4.2.1 Wick calculus and Feynman diagrams

So far, we have not used the power of the Wiener chaos decomposition. One way to do this is to modify the energy (4.1.1) to

$$\mathscr{H}^{\mathrm{Wick}}_{1,\alpha}(\phi) = \int_\Lambda \biggl[\|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 + \alpha{:}\phi(x)^4{:}\biggr]\mathrm{d}x\,.$$

This is now a different model, since we have replaced the fourth power $\phi(x)^4$ by the fourth Wick power ${:}\phi(x)^4{:} = H_4(\phi(x)^4;C)$, where C should be taken equal to the covariance $G(0)$. One could transform this into the original model by adding a suitable multiple of the second Wick power $:\phi(x)^2:$ and a suitable constant, but we will not explore this further here, and work with the new energy. The ratio of partition functions (4.2.1) now becomes

$$\frac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} = \mathbb{E}^{\mu_{1,0}}\biggl[\exp\biggl\{-\alpha\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr\}\biggr] \asymp \sum_{n\geqslant 0}\frac{(-\alpha)^n}{n!}\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr)^n\biggr]. \tag{4.2.4}$$

The coefficient $n = 1$ is simply

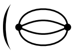
$$\mathbb{E}^{\mu_{1,0}}\biggl[\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr] = 0\,, \tag{4.2.5}$$

since Wick powers are centred. The coefficient $n = 2$ is given by

$$\begin{aligned}\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr)^2\biggr] &= \int_\Lambda\int_\Lambda \mathbb{E}^{\mu_{1,0}}[:\phi(x)^4::\phi(y)^4:]\,\mathrm{d}x\,\mathrm{d}y \\ &= 4!\int_\Lambda\int_\Lambda \mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)]^4\,\mathrm{d}x\,\mathrm{d}y \\ &= 4!\int_\Lambda\int_\Lambda G(x-y)^4\,\mathrm{d}x\,\mathrm{d}y\,, \end{aligned} \tag{4.2.6}$$

where we have used Proposition 1.2.5 and (4.1.3). Note that since G is translation invariant, one can replace the double integral in (4.2.6) by a simple integral of $G(x)^4$, but this will not be important here. We can represent (4.2.6) graphically as

$$\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr)^2\biggr] = 4!\Pi\Bigl(\quad\Bigr)\,,$$



where the two vertices of the graph indicate the two integration variables x and y in (4.2.6), while the four edges indicate the four factors $G(x-y)$. The map Π stands for evaluation of the integral. This is a first example of *Feynman diagram*.

In order to generalise this computation to higher powers, we reformulate it in terms of the Wiener isometry. For given $x\in\Lambda$, considered as a parameter, define

$$h_x = \sum_{k\in\mathbb{Z}}\hat h_x(k)e_k\,, \qquad \text{where } \hat h_x(k) := \frac{e_k(x)}{\sqrt{\lambda_k}}\,.$$

For every $x\in\Lambda$, h_x is an element of $\mathbf{H} = L^2(\Lambda,\mathrm{d}x)$. Furthermore,

$$\hat I_1(h_x) = \sum_{k\in\mathbb{Z}}\hat h_x(k)X_k = \sum_{k\in\mathbb{Z}}\frac{X_k}{\sqrt{\lambda_k}}e_k(x) = \phi(x)$$

is the Gaussian free field. It follows that

$$:\phi(x)^4: = H_4(\phi(x);C) = \hat I_4(h_x^{\otimes 4})\,,$$

so that Proposition 2.3.12 yields

$$\begin{aligned}:\phi(x)^4::\phi(y)^4: &= \hat I_4(h_x^{\otimes 4})\hat I_4(h_y^{\otimes 4}) \\ &= \sum_{p=0}^4 \hat I_{8-2p}(h_x^{\otimes 4}\star_p h_y^{\otimes 4})\,.\end{aligned}$$

Taking the expectation, we obtain

$$\mathbb{E}^{\mu_{1,0}}[:\phi(x)^4::\phi(y)^4:] = \hat I_0(h_x^{\otimes 4}\star_4 h_y^{\otimes 4})\,,$$

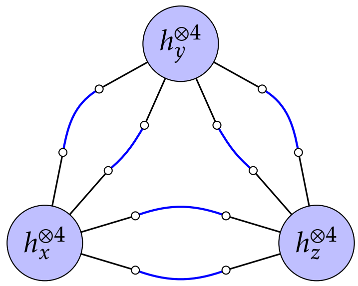


Figure 4.2 – One of the pairings occurring in the expectation (4.2.7).

and the definition (2.3.16) of the contraction $f \star_p g$ yields

$$\begin{aligned}
\hat{I}_0(h_x^{\otimes 4} \star_4 h_y^{\otimes 4}) &= 4! \sum_{k_1,\dots,k_4 \in \mathbb{Z}} \overline{\hat{h}_x^{\otimes 4}(k_1,\dots,k_4)} \hat{h}_y^{\otimes 4}(k_1,\dots,k_4) \\
&= 4! \Big(\sum_{k \in \mathbb{Z}} \overline{\hat{h}_x(k)} \hat{h}_y(k) \Big)^4 \\
&= 4! \Big(\sum_{k \in \mathbb{Z}} \frac{e_k(x-y)}{\lambda_k} \Big)^4 \\
&= 4! G(x-y)^4 \,.
\end{aligned}$$

Here we have used complex conjugates in the inner products because we work with complex Fourier series. We thus recover (4.2.6) in a way that may seem more convoluted, but allows for generalisation to higher powers. Indeed, in the case $n = 3$ we obtain

$$\begin{aligned}
:\phi(x)^4::\phi(y)^4::\phi(z)^4: &= \hat{I}_4(h_x^{\otimes 4}) \hat{I}_4(h_y^{\otimes 4}) \hat{I}_4(h_z^{\otimes 4}) \\
&= \sum_{p=0}^{4} \hat{I}_{8-2p}(h_x^{\otimes 4} \star_p h_y^{\otimes 4}) \hat{I}_4(h_z^{\otimes 4}) \\
&= \sum_{p=0}^{4} \sum_{q=0}^{(8-2p)\wedge 4} \hat{I}_{12-2p-2q}((h_x^{\otimes 4} \star_p h_y^{\otimes 4}) \star_q h_z^{\otimes 4}) \,.
\end{aligned}$$

When taking the expectation, only terms with $2p + 2q = 12$ remain. There is actually only one option, which is to take $p = 2$ and $q = 4$, yielding

$$\mathbb{E}^{\mu_{1,0}}[:\phi(x)^4::\phi(y)^4::\phi(z)^4:] = \hat{I}_0((h_x^{\otimes 4} \star_2 h_y^{\otimes 4}) \star_4 h_z^{\otimes 4}) \,. \tag{4.2.7}$$

The contraction operations can be represented graphically, as we did in Section 2.3.3. The functions $h_x^{\otimes 4}$, $h_y^{\otimes 4}$ and $h_z^{\otimes 4}$ are represented each by a vertex with four legs. The operation $\star_2$ corresponds to pairing two legs of $h_x^{\otimes 4}$ with two legs of $h_y^{\otimes 4}$, while the operation $\star_4$ corresponds to pairing the remaining four legs of $h_x^{\otimes 4}$ and $h_y^{\otimes 4}$ with the four legs of $h_z^{\otimes 4}$, see Figure 4.2. The result is

$$\begin{aligned}
\mathbb{E}^{\mu_{1,0}}[:\phi(x)^4::\phi(y)^4::\phi(z)^4:] &= 2! \binom{4}{2}^2 4! \sum_{k_1,\dots,k_6 \in \mathbb{Z}} h_x^{\otimes 4}(k_1,k_2,k_3,k_4) h_y^{\otimes 4}(k_1,k_2,k_5,k_6) h_z^{\otimes 4}(k_3,k_4,k_5,k_6) \\
&= 1728 \Big(\sum_{k_1 \in \mathbb{Z}} \frac{e_{k_1}(x-y)}{\lambda_{k_1}} \Big)^2 \Big(\sum_{k_2 \in \mathbb{Z}} \frac{e_{k_2}(y-z)}{\lambda_{k_2}} \Big)^2 \Big(\sum_{k_3 \in \mathbb{Z}} \frac{e_{k_3}(x-z)}{\lambda_{k_3}} \Big)^2 \\
&= 1728\, G(x-y)^2 G(y-z)^2 G(x-z)^2 \,.
\end{aligned}$$

It follows that

$$\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr)^3\biggr] = 1728\int_{\Lambda^3} G(x-y)^2 G(y-z)^2 G(x-y)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= 1728\,\Pi\bigl(\quad\bigr)\,.$$

With these examples, the pattern should have become clear. The terms in the expansion (4.2.4) can be written as some combinatorial coefficients, times an integral of a product of Green functions. To formalise this, we make the following definition.

Definition 4.2.1: Vacuum Feynman diagram

A *vacuum diagram* is a multigraph $\Gamma = (\mathscr{V},\mathscr{E})$, meaning there can be multiple edges between vertices. Its *valuation* is defined by

$$\Pi(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e\in\mathscr{E}} G(x_{e_+} - x_{e_-})\,\mathrm{d}x\,,$$

where $e_\pm$ are the vertices connected by the edge e.

The general principle behind the above examples is as follows.

Proposition 4.2.2: Expansion of moments into Feynman diagrams

For any $n \geqslant 2$,

$$\mathbb{E}^{\mu_{1,0}}\biggl[\biggl(\int_\Lambda :\phi(x)^4:\mathrm{d}x\biggr)^n\biggr] = \sum_k \Pi(\Gamma_{n,k})\,, \tag{4.2.8}$$

where the sum runs over all vacuum diagrams $\Gamma_{n,k}$ with n vertices and $2n$ edges, obtained as perfect pairwise matchings of n vertices of arity 4 (each vertex belongs to four edges), when matchings of different legs are counted as different terms.

Proof: For any $n \geqslant 2$, we write

$$\prod_{i=1}^n :\phi(x_i)^4: = \prod_{i=1}^n \hat{I}_4(h_{x_i}^{\otimes 4})\,.$$

Taking the expectation, we are left with the component in the zeroth Wiener chaos, which by a repeated application of Proposition 2.3.12 can be represented as the sum over all pairwise matchings of n vertices with 4 legs each. Each pairing gives rise to a Green function, and the result follows by integrating over all x_i. □

Exercise 4.2.3

Show that for g periodic, the unique periodic solution of $f''(x) = f(x) - g(x)$ is given by

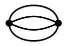
$$\begin{pmatrix} f(x) \\ f'(x)\end{pmatrix} = -[U(-1)-\mathbb{1}]^{-1}\int_x^{x+1} U(x-y)\begin{pmatrix} 0 \\ g(x+y)\end{pmatrix}\mathrm{d}y\,, \qquad U(x) = \begin{pmatrix}\cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x)\end{pmatrix}.$$

Deduce that the Green function $G = (\mathrm{id} - \Delta)^{-1}$ is bounded in dimension $d = 1$.

Exercise 4.2.4

Give an upper bound on the number of vacuum diagrams $\Gamma_{n,k}$ occurring in (4.2.8) for given n, by allowing the graphs to have loops (edges connecting a vertex with itself). Assuming this bound has the right order of magnitude, what does this say about the convergence of the series (4.2.4)?

4.2.2 The linked-cluster theorem

The vacuum diagrams occurring in the nth power of the expansion need not be connected. For instance, the 4th power contains terms of the form  ^{2}, that arise from pairing the legs of two vertices between each other, and of the other two vertices among themselves. However, a rather remarkable result known in quantum field theory as *linked-cluster theorem* [Bro09, Riv09, Sal99] states that the *logarithm* of the ratio of partition functions, that is, its cumulant expansion, contains only the connected graphs. More precisely, it is obtained by keeping only the connected diagrams in the expansion.

Theorem 4.2.5: Linked-cluster theorem

The cumulant expansion of the ratio of partition functions is given by

$$\log \frac{\mathcal{Z}_{1,\alpha}}{\mathcal{Z}_{1,0}} \asymp \sum_{n\geqslant 0} \frac{(-\alpha)^n}{n!} \sum_{k\colon \Gamma_{n,k} \text{connected}} \Pi(\Gamma_{n,k})\,. \tag{4.2.9}$$

Proof: This elegant proof is due to Dimitri Faure. We use the formalism of convolution algebras introduced in Section 1.2.4. For an abstract variable x (unrelated to coordinates on Λ), we write $\psi(x^n)$ for the coefficient of $(-\alpha)^n/n!$ in the cumulant expansion (4.2.9). This means that

$$\log \frac{\mathcal{Z}_{1,\alpha}}{\mathcal{Z}_{1,0}} \asymp \Lambda(\psi)(-\alpha)\,,$$

where Λ is the map introduced in (1.2.16). By Proposition 1.2.14, we have

$$\frac{\mathcal{Z}_{1,\alpha}}{\mathcal{Z}_{1,0}} \asymp \Lambda(\varphi)(-\alpha) = \sum_{n\geqslant 0} \frac{(-\alpha)^n}{n!} \varphi(x^n)\,,$$

where

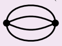
$$\varphi(x^n) = \exp_*(\psi)(x^n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geqslant 1\\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \psi(x^{n_1})\dots\psi(x^{n_k})\,. \tag{4.2.10}$$

To deal with combinatorics, we assume that all vertices and edges of the diagrams are numbered. Assuming (4.2.9) is true, each $\psi(x^n)$ is a linear combination of connected graphs with n vertices. Then (4.2.10) says that the coefficient of $(-\alpha)^n/n!$ in the expansion (4.2.8) is obtained by all possible disjoint unions of connected graphs such that the total number of vertices is n. The multinomial coefficient accounts for the choices of vertices in the subgraphs, while the factor $1/k!$ accounts for the fact that the order of the subgraphs is irrelevant. Since this yields *all* pairings of n vertices, the result follows from uniqueness of coefficients of power series. □

Example 4.2.6

Since $\psi(x) = 0$ by (4.2.5), we obtain

$$\exp_*(\psi)(x^4) = \psi(x^4) + \binom{4}{2}\psi(x^2)^2 ,$$

since the only allowed decompositions of 4 are 4 and 2 + 2. This means that the term of order α^4 in the expansion (4.2.8) differs from the corresponding term in the cumulant expansion by a term $\frac{1}{2}\binom{4}{2}$  ^{2}. The factor $\frac{1}{2}\binom{4}{2} = 3$ is precisely the number of pairwise matchings of four vertices.

4.2.3 Asymptotic expansion

So far, we have not shown that the expansion (4.2.8) is a genuine asymptotic expansion. We do this now, by proving the following result.

Proposition 4.2.7: Asymptotic series

For every $n \geqslant 0$ there exists a constant M_n such that the ratio of partition functions satisfies

$$\left|\frac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} - \sum_{m=0}^{n} \frac{(-\alpha)^m}{m!}\mathbb{E}^{\mu_{1,0}}\left[\left(\int_\Lambda :\phi(x)^4: \mathrm{d}x\right)^m\right]\right| \leqslant M_n \alpha^{n+1} . \qquad (4.2.11)$$

We start by showing an a-priori bound on the Laplace transform. To lighten notations, we will write

$$\mathbf{X} = \int_\Lambda :\phi(x)^4: \mathrm{d}x . \qquad (4.2.12)$$

Lemma 4.2.8

There exists $\alpha_0 > 0$ such that for all $\alpha \in [0, \alpha_0)$, one has

$$0 \leqslant \mathbb{E}^{\mu_{1,0}}[\mathrm{e}^{-\alpha\mathbf{X}}] \leqslant 1 + \mathcal{O}(\alpha) .$$

Proof: In order to exploit signs, we write $\mathbf{X} = \mathbf{X}_0 - 6C\mathbf{Y}_0 + 3C^2$, where $C = G(0)$ and

$$\mathbf{X}_0 = \int_\Lambda \phi(x)^4 \,\mathrm{d}x \geqslant 0 , \qquad \mathbf{Y}_0 = \int_\Lambda \phi(x)^2 \,\mathrm{d}x \geqslant 0 .$$

Therefore we have

$$\mathbb{E}^{\mu_{1,0}}[\mathrm{e}^{-\alpha\mathbf{X}}] = \mathrm{e}^{-3C^2\alpha}\,\mathbb{E}^{\mu_{1,0}}[\mathrm{e}^{-\alpha\mathbf{X}_0 + 6C\alpha\mathbf{Y}_0}] = \mathrm{e}^{-3C^2\alpha}\frac{\widetilde{\mathscr{Z}}(\alpha)}{\widetilde{\mathscr{Z}}(0)}\mathbb{E}^{\tilde{\mu}(\alpha)}[\mathrm{e}^{-\alpha\mathbf{X}_0}] ,$$

where $\tilde{\mu}(\alpha)$ is a GFF with covariance $((1-6C\alpha)\mathrm{id} - \Delta)^{-1}$, and $\widetilde{\mathscr{Z}}(\alpha)$ denotes the normalisation of this measure. Then we have

$$\begin{aligned}
\log\frac{\widetilde{\mathscr{Z}}(0)}{\widetilde{\mathscr{Z}}(\alpha)} &= \frac{1}{2}\log\prod_{k\in\mathbb{Z}}\frac{1-6\alpha+2\pi k^2}{1+2\pi k^2} \\
&= \frac{1}{2}\log\prod_{k\in\mathbb{Z}}\left(1 - \frac{6\alpha}{1+2\pi k^2}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\log\left(1 - \frac{6\alpha}{1+2\pi k^2}\right) \asymp -3\alpha\sum_{k\in\mathbb{Z}}\frac{1}{1+2\pi k^2} .
\end{aligned}$$

Since the sum converges, we obtain $\widetilde{\mathscr{Z}}(\alpha) = \widetilde{\mathscr{Z}}(0)[1+\mathcal{O}(\alpha)]$, where $\widetilde{\mathscr{Z}}(0) = \mathscr{Z}_{1,0}$. As a consequence,

$$\mathbb{E}^{\mu_{1,0}}[\mathrm{e}^{-\alpha\mathbf{X}}] = \mathbb{E}^{\tilde{\mu}(\alpha)}[\mathrm{e}^{-\alpha\mathbf{X}_0}](1+\mathcal{O}(\alpha)) .$$

Since $\mathbf{X}_0$ is positive, we have $0 \leqslant \mathbb{E}^{\tilde{\mu}(\alpha)}[\mathrm{e}^{-\alpha\mathbf{X}_0}] \leqslant 1$, which concludes the proof. □

Proof of Proposition 4.2.7. For $n \geqslant 0$ and $t \in \mathbb{R}$, let

$$D_n(t) = \mathrm{e}^{-t} - \sum_{m=0}^{n} \frac{(-t)^m}{m!} .$$

If n is even, we have

$$\begin{cases} 0 \leqslant D_n(t) \leqslant \dfrac{(-t)^n}{n!} & \text{if } t \geqslant 0 , \\ 0 \leqslant D_n(t) \leqslant \dfrac{(-t)^n}{n!}\,\mathrm{e}^{-t} & \text{if } t < 0 . \end{cases} \tag{4.2.13}$$

It follows that

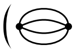
$$\mathbb{E}^{\mu_{1,0}}[|D_n(\alpha\mathbf{X})|\mathbb{1}_{\mathbf{X}\geqslant 0}] \leqslant \frac{\alpha^n}{n!}\mathbb{E}[\mathbf{X}^n] ,$$

while

$$\begin{aligned}\mathbb{E}^{\mu_{1,0}}[|D_n(\alpha\mathbf{X})|\mathbb{1}_{\mathbf{X}<0}] &\leqslant \frac{\alpha^n}{n!}\mathbb{E}[\mathbf{X}^n\,\mathrm{e}^{-\alpha\mathbf{X}}\,\mathbb{1}_{\mathbf{X}<0}] \\ &\leqslant \frac{\alpha^n}{n!}\sqrt{\mathbb{E}[\mathbf{X}^{2n}]\mathbb{E}[\mathrm{e}^{-2\alpha\mathbf{X}}\,\mathbb{1}_{\mathbf{X}<0}]} ,\end{aligned}$$

by the Cauchy–Schwarz inequality. We know from Exercise 4.2.3 that $\mathbb{E}[\mathbf{X}^{2n}]$ is finite, while Lemma 4.2.8 shows that $\mathbb{E}[\mathrm{e}^{-2\alpha\mathbf{X}}\,\mathbb{1}_{\mathbf{X}<0}]$ is bounded as well. A similar argument applies for odd n, with some signs reversed in (4.2.13). □

With the computations made in Section 4.2.1, we have thus obtained that the ratio of partition functions satisfies

$$\frac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} = 1 + 12\alpha^2\Pi\Big(\;$$  $$\Big) + 288\alpha^3\Pi\Big(\;$$  $$\Big) + \mathcal{O}(\alpha^4) ,$$

and more terms can be computed if needed. This expansion does not converge, however, since the number of pairwise matchings grows like $(4n-1)!!$ (cf. Exercise 4.2.4), which by (1.2.27) and Stirling's formula behaves like $(n!)^2$. The constant M_n in (4.2.11) thus grows like $n!$. Such an expansion is called *Gevrey*-1. The non-convergence of the expansion does not mean that it is useless, but it means that there is an optimal value of n, depending on α, at which the expansion should be stopped to obtain the smallest possible error bound.

Exercise 4.2.9

Let $r(n) = n!\alpha^n$. By extending r to real arguments via $n! = \Gamma(n+1)$ and using Stirling's formula, estimate the minimal value of $r(n)$ for small α. For what n is this minimal value reached?

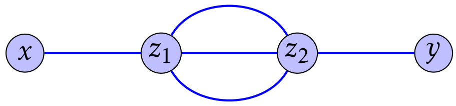


Figure 4.3 – Pairing corresponding to the term $n = 2$ in the expansion of the two-point function.

4.2.4 Two-point function

Let us briefly outline how the two-point function

$$G_{2,1,\alpha}(x,y) = \mathbb{E}^{\mu_{1,\alpha}}[\phi(x)\phi(y)]$$

can be computed by a similar procedure. By (4.1.4) we have

$$G_{2,1,\alpha}(x,y) = \frac{\mathscr{Z}_{1,0}}{\mathscr{Z}_{1,\alpha}}\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\,\mathrm{e}^{-\alpha\mathbf{X}}]\,,$$

where $\mathbf{X}$ denotes the integral of the fourth Wick power, cf. (4.2.12). Expanding the exponential, we get

$$\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\,\mathrm{e}^{-\alpha\mathbf{X}}] \asymp \sum_{n\geqslant 0}\frac{(-\alpha)^n}{n!}\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\mathbf{X}^n]\,.$$

We already know that the term $n = 0$ is equal to $G(x-y)$. For $n = 1$, we find

$$\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\mathbf{X}] = \int_\Lambda \mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y){:}\,\phi(z)^4{:}]\,\mathrm{d}z = \int_\Lambda \mathbb{E}^{\mu_{1,0}}[\hat{I}_1(h_x)\hat{I}_1(h_y)\hat{I}_4(h_z^{\otimes 4})]\,\mathrm{d}z\,.$$

This is equal to zero, because there is no perfect matching, leaving no free legs, of two vertices with one leg each and one vertex with four legs. For $n = 2$, we obtain

$$\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\mathbf{X}^2] = \int_\Lambda\int_\Lambda \mathbb{E}^{\mu_{1,0}}[\hat{I}_1(h_x)\hat{I}_1(h_y)\hat{I}_4(h_{z_1}^{\otimes 4})\hat{I}_4(h_{z_2}^{\otimes 4})]\,\mathrm{d}z_1\,\mathrm{d}z_2\,.$$

This is now different from zero, because there exist perfect matchings, as show in Figure 4.3. We thus conclude that the expectation of $\phi(x)\phi(y)\,\mathrm{e}^{-\alpha\mathbf{X}}$ can be computed in a similar way as for the ratio of partition functions, except that it now involves Feynman diagrams with two free legs, labeld x and y.

Remark 4.2.10

There exists a slightly different representation of the two-point function, that avoids having to divide by the ratio of partition functions, based on the Schwinger–Dyson equations (2.1.3) [BFS83b].

4.3 The Φ_2^4 model

In this section, we consider the Φ^4 model on the two-dimensional torus $\Lambda = \mathbb{T}^2$. We have seen in Section 3.3.1 that the GFF in dimension 2 has infinite variance. In fact, one can show that the Green function behaves like

$$G(x) \asymp \big|\log(\|x\|)\big|\,. \tag{4.3.1}$$

This can be seen by computing the Green function $G_{\mathbb{R}^2}$ of the full plane $\mathbb{R}^2$ using polar coordinates, and then periodising it via

$$G_{\mathbb{T}^2}(x) = \sum_{k\in\mathbb{Z}^2} G_{\mathbb{R}^2}(x-k)\,.$$

We know however that the truncated two-dimensional GFF has finite moments. This suggests considering the modified energy

$$\mathscr{H}^{\text{Wick}}_{2,\alpha,N}(\phi_N) = \int_\Lambda \Big[\|\nabla\phi_N(x)\|^2 + \frac{1}{2}\phi_N(x)^2 + \alpha\!:\!\phi_N(x)^4\!:_{C_N}\Big]\,\mathrm{d}x\,, \tag{4.3.2}$$

defined for the truncated field

$$\phi_N(x) = \sum_{k\in\mathscr{K}_N} \frac{X_k}{\sqrt{\lambda_k}} e_k(x)\,,$$

which has variance

$$C_N = \sum_{k\in\mathscr{K}_N} \frac{1}{\lambda_k} \asymp \log(N)\,. \tag{4.3.3}$$

We recall that $\mathscr{K}_N = \{k\in\mathbb{Z}^2 \colon |k| \leqslant N\}$, where $|k| = |k_1| + |k_2|$.

The situation is now similar to the one we have encountered in dimension 1, except for the important difference that the constant C_N occuring in (4.3.2) depends on the cut-off N. The model thus changes with N. This is an instance of what is called *renormalisation* in quantum field theory, and C_N is known as a *counterterm*.

The computations from Section 4.2.4 can now be repeated in the same way, and result in an expansion of the form

$$\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} \asymp \sum_{n\geqslant 0} \frac{(-\alpha)^n}{n!} \mathbb{E}^{\mu_{2,0,N}}\bigg[\bigg(\int_\Lambda :\!\phi_N(x)^4\!:_{C_N}\mathrm{d}x\bigg)^n\bigg]$$

of the ratio of partition functions, where the expectations of powers of the fourth Wick power are given by a sum of valuations of the same Feynman vacuum diagrams as in (4.2.8). The only difference is that the diagrams involve the Green function satisfying (4.3.1). It is not immediately obvous that the diagrams all have a finite valuation, but we will show in Section 4.4.1 below that this is indeed the case, uniformly in the cut-off N.

The linked-cluster theorem is also true in this case. However, the proof of the a priori bound on the Laplace transform given in Lemma 4.2.8 does not work here, because of the diverging constant C_N in the energy (4.3.2). Fortunately, there is an alternative proof of that bound, due to Nelson.

4.3.1 Nelson's estimate

In order to bound the Laplace transform of the fourth Wick power $\mathbf{X}$, we first derive the following consequence of the equivalence of moments bound (3.1.1).

Lemma 4.3.1

Fix two cut-offs $M > N \geqslant 1$. Then for any $p > 1$ and $n \geqslant 2$, there exists a constant K_n depending only on n such that

$$\mathbb{E}^{\mu_{2,0,N}}\bigg[\bigg(\int_\Lambda :\!\phi_M(x)^n\!:_{C_M}\mathrm{d}x - \int_\Lambda :\!\phi_N(x)^n\!:_{C_N}\mathrm{d}x\bigg)^{2p}\bigg]^{1/(2p)} \leqslant K_n(2p-1)^{n/2}\frac{(\log N)^{n-2}}{N}\,.$$

Proof: By the same computation as in the proof of Proposition 3.3.12, see (3.3.10), we have

$$\mathbb{E}\bigg[\int_\Lambda :\!\phi_M^n(x)\!:_{C_M}\mathrm{d}x \int_\Lambda :\!\phi_N^n(x)\!:_{C_N}\mathrm{d}x\bigg] = n! \sum_{\substack{k_1,\dots,k_n\in\mathscr{K}_N\\ k_1+\dots+k_n=0}} \frac{1}{\lambda_{k_1}\dots\lambda_{k_n}}\,.$$

It follows by expanding the square that

$$\mathbb{E}\biggl[\biggl(\int_\Lambda :\phi_M^n(x):_{C_M}\,\mathrm{d}x - \int_\Lambda :\phi_N^n(x):_{C_N}\,\mathrm{d}x\biggr)^2\biggr] = n! \sum_{\substack{k_1,\dots,k_n\in\mathcal{K}_M\setminus\mathcal{K}_N\\ k_1+\dots+k_n=0}} \frac{1}{\lambda_{k_1}\dots\lambda_{k_n}}\,.$$

By a similar argument as in Lemma 3.3.13, one can show by induction on n that

$$\sum_{\substack{k_1,\dots,k_n\in\mathcal{K}_M\setminus\mathcal{K}_N\\ k_1+\dots+k_n=k}} \frac{1}{\lambda_{k_1}\dots\lambda_{k_n}} \leqslant K_n \min\biggl\{\frac{(\log N)^{n-2}}{N^2}, \frac{(\log\|k\|)^{n-2}}{\|k\|^2}\biggr\}\,,$$

where K_n does not depend on N or M. Therefore, the result follows by taking $k=0$ in the above bound and using equivalence of moments (Theorem (3.1.1)). □

Proposition 4.3.2: Nelson's estimate

For any $\alpha\geqslant 0$, there exists a constant $K>0$, independent of N, such that for all $N\in\mathbb{N}$, one has

$$0\leqslant \mathbb{E}^{\mu_{2,0,N}}\biggl[\exp\biggl\{-\alpha\int_\Lambda :\phi_N(x)^4:\mathrm{d}x\biggr\}\biggr]\leqslant K\,.$$

Proof: By the definition of scaled Hermite polynomials,

$$H_4(\phi_N(x);C_N) = (\phi_N(x)^2-3C_N^2)^2 - 6C_N^2\,,$$

which shows that

$$\mathbf{X}_N := \int_\Lambda :\phi_N(x)^4:_{C_N}\mathrm{d}x \geqslant -6C_N^2 =: -D_N\,.$$

Since $\mathbb{E}^{\mu_{2,0,N}}[\mathrm{e}^{-\alpha\mathbf{X}_N}\,\mathbb{1}_{\mathbf{X}_N\geqslant 0}]\leqslant\mathbb{P}^{\mu_{2,0,N}}\{\mathbf{X}_N\geqslant 0\}\leqslant 1$, it is sufficient to bound

$$\begin{aligned}\mathbb{E}^{\mu_{2,0,N}}[\mathrm{e}^{-\alpha\mathbf{X}_N}\,\mathbb{1}_{\mathbf{X}_N<0}] &= \mathbb{P}^{\mu_{2,0,N}}\{\mathbf{X}<0\} + \alpha\int_0^\infty \mathrm{e}^{\alpha t}\,\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N>t\}\,\mathrm{d}t\\ &\leqslant \mathrm{e}^\alpha + \int_1^\infty \mathrm{e}^{\alpha t}\,\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N>t\}\,\mathrm{d}t\,.\end{aligned} \tag{4.3.4}$$

If $t\geqslant D_N$, then $\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N>t\}=0$. Otherwise we have, for any $M\in\mathbb{N}$ and any choice of even $p(t)$,

$$\begin{aligned}\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N>t\} &\leqslant \mathbb{P}^{\mu_{2,0,N}}\{\mathbf{X}_M-\mathbf{X}_N>t-D_M\}\\ &\leqslant \mathbb{P}^{\mu_{2,0,N}}\{|\mathbf{X}_M-\mathbf{X}_N|^{2p(t)}>|t-D_M|^{2p(t)}\}\,.\end{aligned}$$

We now apply this inequality with $M=M(t)$ satisfying

$$t-D_{M(t)}\geqslant 1\,,$$

which by (4.3.3) is possible with $\log(M(t))$ of order $t^{1/2}$, and implies $M(t)<N$. By Markov's inequality and Lemma 4.3.1 with $n=4$,

$$\begin{aligned}\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N>t\} &\leqslant \mathbb{E}^{\mu_{2,0,N}}[|\mathbf{X}_M(t)-\mathbf{X}_N|^{p(t)}]\\ &\leqslant K_4(p(t)-1)^{2p(t)}\frac{(\log N)^{4p(t)}}{N^{2p(t)}}\\ &\leqslant K(\eta)\frac{(p(t)-1)^{2p(t)}}{M(t)^{2(1-\eta)p(t)}}\end{aligned}$$

with a finite $K(\eta)$ for any $\eta > 0$. Choosing $p(t)$ of order t^β for $\beta > \frac{1}{2}$, we obtain

$$\log\Big(\mathrm{e}^{\alpha t}\,\mathbb{P}^{\mu_{2,0,N}}\{-\mathbf{X}_N > t\}\Big) \leqslant \alpha t + c_1\beta t^\beta \log(t) - c_2(1-\eta)t^{\beta+1/2}$$

for constants $c_1, c_2 > 0$. Since the term $-c_2(1-\eta)t^{\beta+1/2}$ dominates for large t, this leads to a convergent integral in (4.3.4). □

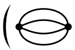
As a consequence, we obtain the following analogue of Proposition 4.2.7, by the same proof.

Corollary 4.3.3: Asymptotic series

For every $n \geqslant 0$ and $N \geqslant 1$, there exists a constant M_n, independent of N, such that the ratio of partition functions satisfies

$$\left|\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} - \sum_{m=0}^{n}\frac{(-\alpha)^m}{m!}\mathbb{E}^{\mu_{2,0,N}}\bigg[\Big(\int_\Lambda :\phi_N(x)^4\colon \mathrm{d}x\Big)^m\bigg]\right| \leqslant M_n\alpha^{n+1}\,.$$

As a consequence, we obtain again an expansion of the form

$\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} = 1 + 12\alpha^2\Pi_N\Big($  $\Big) + 288\alpha^3\Pi_N\Big($  $\Big) + \mathcal{O}(\alpha^4)$ (4.3.5)

for the ration of partition functions, and a similar expansion holds for the two-point function and other random variables. The main difference with the one-dimensional case, besides the fact that the model itself depends on C_N, is that the valuations of Feynman diagrams depend on N, via the cut-off Green function G_N, given by (3.3.8). One can however show that these valuations all converge to a finite limit as $N \to \infty$.

Remark 4.3.4

A non-perturbative proof of the existence of the ratio of partition functions (4.3.5), based on a Girsanov formula, has been obtained by Barashkov and Gubinelli [BG20].

4.4 The Φ^4_3 model*

We consider now the Φ^4 model on the three-dimensional torus $\Lambda = \mathbb{T}^3$. One can show that the Green function now behaves as

$$G(x) \asymp \frac{1}{\|x\|}\,,$$

while the variance of the Gaussian free field satisfies

$$C_N = G_N(0) = \sum_{k\in\mathscr{K}_N}\frac{1}{\lambda_k} \asymp N$$

with $\mathscr{K}_N = \{k \in \mathbb{Z}^3 : |k| \leqslant N\}$. One also checks that the truncated Green function satisfies

$$G_N(x) = \sum_{k\in\mathscr{K}_N}\frac{1}{\lambda_k}e_k(x) \asymp \frac{1}{\|x\| + N^{-1}}\,. \qquad (4.4.1)$$

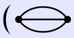
A natural guess would be that the energy (4.3.2), transposed to the three-dimensional setting, would still lead to well-defined asymptotic expansions of expectations. Unfortunately, this is not the case. The actual result is as follows.

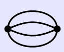
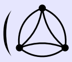
Theorem 4.4.1: Renormalisation of the Φ^4_3 model

Define the energy by

$$\mathscr{H}^{\Phi^4}_{3,\alpha,N}(\phi_N) = \int_\Lambda \Big[\|\nabla\phi_N(x)\|^2 + \frac{1}{2}\phi_N(x)^2 + \alpha{:}\phi_N(x)^4{:}_{C_N} + \beta_N(\alpha){:}\phi_N(x)^2{:}_{C_N} + \gamma_N(\alpha) \Big]\,\mathrm{d}x\,,$$

where the additional counterterms are

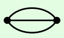
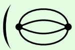
$\beta_N(\alpha) = -48\alpha^2\Pi_N($  $)\,,$

$\gamma_N(\alpha) = 12\alpha^2\Pi_N\big($  $\big) - 288\alpha^3\Pi_N\big($  $\big)\,.$

Then the n-point functions admit asymptotic expansions in α, all of whose terms are uniformly bounded in the cut-off N.

The counterterm $\beta_N(\alpha)$ is called *mass renormalisation*, while the counterterm $\gamma_N(\alpha)$ is called *energy renormalisation*. The latter is not crucial for most computations, since it will cancel out when taking ratios of partition functions. These counterterms were not needed in dimension 2, because their value actually converges to finite limits as $N \to \infty$. However, as we shall see in Section 4.4.1, they diverge in dimension 3, either like N or like $\log N$. This is a symptom of the fact that the Φ^4_3 measure is not absolutely continuous with respect to the three-dimensional Gaussian free field.

Exercise 4.4.2

Determine how the values $\Pi_N($  $)$ and $\Pi_N\big($  $\big)$ scale with N as $N \to \infty$, for $d = 3$, by using (4.4.1) and spherical coordinates. Compare with the case $d = 2$.

Theorem 4.4.1 is an important result in Euclidean Quantum Field theory, which has a long history. The earliest works by Glimm and Jaffe and by Feldman approached the problem via a detailed combinatorial analysis of Feynman diagrams [GJ68, GJ73, Fel74, GJ81]. The works [BCG+78, BCG+80] introduced the idea of using a renormalisation group approach, consisting in a decomposition of the covariance of the GFF into scales, which then allows to integrate successively over one scale after the other. This method was further perfected in [BDH95], using polymers to control error terms, an approach based on ideas from Statistical Physics [GK71].

In another direction, the approach provided in [BFS83a, BFS83b] allows to bound correlation functions without having to compute the partition function explicitly, by using the Schwinger–Dyson equations (see also Remark 4.2.10). This involves the derivation of so-called skeleton inequalities, which were obtained up to third order in [BFS83a], and later extended to all orders in [BF84]. A relatively compact derivation of bounds on the partition function based on a Girsanov formula was recently obtained in [BG20].

We are not going to present a full proof of Theorem 4.4.1 here, as all its versions are quite technical. However, in the next sections, we shall outline some key ideas of a proof.

4.4.1 Hepp sectors and subdivergences*

In this section, we provide a simple way to determine whether the value of a vacuum diagram diverges as $N \to \infty$, or not. Given a diagram $\Gamma = (\mathscr{V}, \mathscr{E})$, define its degree by

$$\deg(\Gamma) := d(|\mathscr{V}| - 1) - (d-2)|\mathscr{E}|\,. \tag{4.4.2}$$

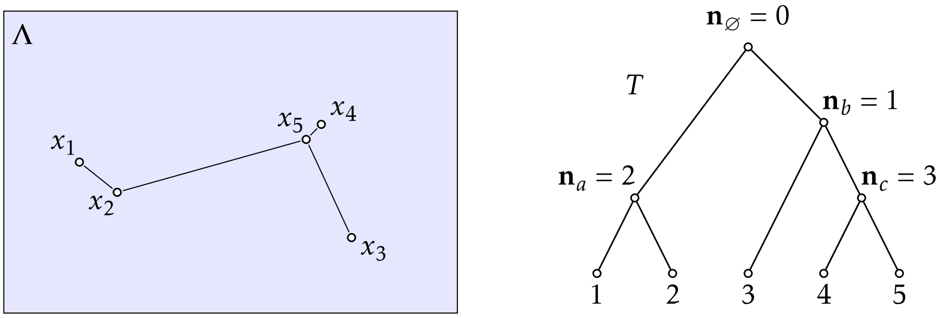


Figure 4.4 – A point configuration $x \in \Lambda^5$, its minimal spanning tree (left), and the associated Hepp tree $\mathbf{T} = (T, \mathbf{n})$ (right). The inner nodes of T are labelled $\varnothing, a, b, c$. For instance, $\mathbf{n}_{1\wedge 2} = \mathbf{n}_a = 2$, so that x_1 and x_2 are at a distance of order 2^{-2}, while $\mathbf{n}_{3\wedge 5} = \mathbf{n}_b = 1$, so that x_3 and x_5 are at a distance of order 2^{-1}.

We will call a diagram *divergent* if $\deg(\Gamma) \leqslant 0$. Here we have defined the degree for a general dimension d, though the main focus of this section is the case $d = 3$.

Exercise 4.4.3

Compute the degrees of the diagrams occurring in Theorem 4.4.1 for $d = 2$ and $d = 3$.

The following theorem was obtained by Weinberg [Wei60].

Theorem 4.4.4: Criterion for non-divergence

Assume $G_N(x) \asymp (\|x\|+N^{-1})^{d-2}$. If Γ satisfies $\deg(\bar\Gamma) > 0$ for any subgraph $\bar\Gamma$ of Γ, then $\Pi_N(\Gamma)$ is bounded uniformly in N.

This result can be proved quite easily, using an idea introduced by Hepp [Hep66].

Definition 4.4.5: Hepp sector

Fix a constant $C > 1$. Let T be a binary tree with $m = |\mathscr{V}|$ leaves, and let $\mathbf{n}$ be a node decoration on the vertices of T, which is non-decreasing on any path from the root of T to a leaf of T. We write $\mathbf{T} = (T, \mathbf{n})$ and define a subset of Λ^m by

$$D_{\mathbf{T}} := \Bigl\{x \in \Lambda^m \colon C^{-1}2^{-\mathbf{n}_{i\wedge j}} \leqslant \|x_i - x_j\| \leqslant C2^{-\mathbf{n}_{i\wedge j}}\ \forall i,j \in \{1,\dots,m\}\Bigr\}\,,$$

where $i \wedge j$ denotes the last common ancestor of i and j in the tree T. Then $D_{\mathbf{T}}$ is called the *Hepp sector* associated with $\mathbf{T}$.

Given a point configuration $x = (x_1, \dots, x_m) \in \Lambda^m$, one can associate with it a Hepp sector $D_{\mathbf{T}}$ in the following way (Figure 4.4). One starts by finding a minimal spanning tree of $x_1, \dots, x_m$. Pairs of points which are closest are children of a common node in T, which is labelled by the power of 2 which is closest to the length of the edge in the spanning tree. The construction is then iterated until all leaves of T are connected to the root.

Proof of Theorem 4.4.4. One can check that for sufficiently large C, Λ^m is covered by the union of all Hepp sectors $D_{\mathbf{T}}$. The bound on $G_N(x)$ implies

$$|\Pi_N(\Gamma)| \lesssim \sum_{T,\mathbf{n}} \int_{D_{T,\mathbf{n}}} \prod_{e\in\mathscr{E}} \frac{1}{(\|x_{e_+} - x_{e_-}\| + N^{-1})^{d-2}}\,\mathrm{d}x\,. \tag{4.4.3}$$

It follows from the definition of Hepp sectors that the term $\|x_{e_+} - x_{e_-}\|$ in (4.4.3) is bounded below by $C^{-1}2^{(d-2)\mathbf{n}_{e^\uparrow}}$, where $e^\uparrow := e_+ \wedge e_-$ denotes the last common ancestor of the two vertices incident to e. We thus obtain that, uniformly in the cut-off N, one has

$$|\Pi_N(\Gamma)| \lesssim \sum_{T,\mathbf{n}} \prod_{e\in\mathscr{E}} 2^{(d-2)\mathbf{n}_{e^\uparrow}} \int_{D_{T,\mathbf{n}}} \mathrm{d}x\,.$$

The volume of the Hepp sector, given by the integral over $D_{T,\mathbf{n}}$, is easily seen to have order $\prod_{v\in T} 2^{-d\mathbf{n}_v}$, where the product runs over all inner nodes of T. It follows that

$$|\Pi_N(\Gamma)| \lesssim \sum_{T,\mathbf{n}} \prod_{v\in T} 2^{-\eta_v \mathbf{n}_v}\,, \quad \text{where} \quad \eta_v := d - (d-2)\sum_{e\in\mathscr{E}} 1_{e^\uparrow}(v)\,.$$

Let $\geqslant$ be the partial order on inner vertices of T given by descendence: $w \geqslant v$ if and only if the unique path from the root of T to w contains v. We claim that

$$\sum_{w\geqslant v} \eta_w > 0$$

holds uniformly in $v \in T$. Indeed, this expression is the degree of a subgraph of Γ, which is positive by assumption. Using this observation, it is not difficult to show that $|\Pi_N(\Gamma)|$ is uniformly bounded, by induction starting from the leaves of T. □

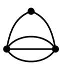
Example 4.4.6

Consider the case of the tree depicted in Figure 4.4, with the inner vertices of T denoted $\varnothing, a, b, c$. We have $\eta_a > 0$, $\eta_c > 0$, $\eta_b + \eta_c > 0$ and $\eta_\varnothing + \eta_a + \eta_b + \eta_c > 0$ (this last sum being the degree of Γ). The sum over all node decorations is given by

$$\sum_{\mathbf{n}_\varnothing\geqslant 0} 2^{-\eta_\varnothing} \sum_{\mathbf{n}_a\geqslant\mathbf{n}_\varnothing} 2^{-\eta_a\mathbf{n}_a} \sum_{\mathbf{n}_b\geqslant\mathbf{n}_\varnothing} 2^{-\eta_b\mathbf{n}_b} \sum_{\mathbf{n}_c\geqslant\mathbf{n}_b} 2^{-\eta_c\mathbf{n}_c}\,.$$

Performing first the sums over $\mathbf{n}_a$ and $\mathbf{n}_c$, then the sum over $\mathbf{n}_b$, and finally the sum over $\mathbf{n}_\varnothing$ yields indeed a finite quantity. Since there are finitely many binary trees with 5 leaves, the result follows.

The examples seen in Exercice 4.4.3 may suggest that the large-N behaviour of $\Pi_N(\Gamma)$ is governed by the degree of Γ, in the sense that $|\Pi_N(\Gamma)|$ is bounded uniformly in N if $\deg(\Gamma) > 0$, diverges like $\log(N)$ if $\deg(\Gamma) = 0$, and diverges like $N^{-\deg(\Gamma)}$ if $\deg(\Gamma) < 0$. Unfortunately, the reality is a bit more complex. For example, we have

deg() $= 2d - 5(d-2) = 10 - 3d$ (4.4.4)

which is strictly positive for $d \leqslant 3$. However, the diagram contains the subgraph , which is divergent in dimension 3. As a result, one can show that the diagram is divergent as well. This is an instance of a *subdivergence*, which is a serious source of complication for the analysis. Note that this does not contradict Theorem 4.4.4.

4.4.2 BPHZ renormalisation*

BPHZ renormalisation, named after Bogoliubov, Parasiuk, Hepp, and Zimmermann [Bog56, Hep66, Zim69] is a procedure allowing to systematically analyse the divergent behavior of diagrams that are either divergent, or contain sub-divergences, and to determine the counterterms needed to obtain well-defined asymptotic expansions. The key result is the following.

Theorem 4.4.7: BPHZ renormalisation

There exists a linear map $\mathscr{A}$, acting on Feynman diagrams, such that

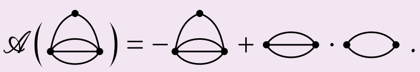
$$\Pi_N(\mathscr{A}(\Gamma)) \asymp \begin{cases} N^{-\deg(\Gamma)} & \text{if } \deg(\Gamma) < 0 \,, \\ \log(N)^\zeta & \text{if } \deg(\Gamma) = 0 \,, \end{cases} \tag{4.4.5}$$

for a finite integer ζ, while $\Pi_N(\mathscr{A}(\Gamma))$ is bounded uniformly in N if $\deg(\Gamma) > 0$.

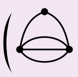
For a modern exposition, see [Hai18]. Slightly sharper bounds have been obtained for a different model in [BB25]. While the proof is quite involved, the basic mechanism can be understood in a simple example.

Example 4.4.8

Consider again the diagram in (4.4.4). In that case,

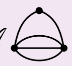


This means that while



$$\Pi_N(\cdot) = \iiint_{\Lambda^3} G_N(x_2 - x_1)^3 G_N(x_3 - x_2) G_N(x_3 - x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \,, \tag{4.4.6}$$

one has



$$\Pi_N(\mathscr{A}\cdot) = -\iiint_{\Lambda^3} G_N(x_2 - x_1)^3 G_N(x_3 - x_2)[G_N(x_3 - x_2) - G_N(x_3 - x_1)]\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \,. \tag{4.4.7}$$

The crucial observation is that by Taylor's formula,

$$|G_N(x_3 - x_2) - G_N(x_3 - x_1)| \lesssim |(x_2 - x_1)\cdot\nabla G_N(x_3 - x_1)| \lesssim \frac{\|x_2 - x_1\|}{(\|x_3 - x_1\| + N^{-1})^2} \,.$$

If $\|x_2 - x_1\| \ll \|x_3 - x_1\|$, this is smaller than the contribution of $G_N(x_3 - x_1)$ to (4.4.6). This gain is enough to make the integral (4.4.7) convergent.

The linear map $\mathscr{A}$ in (4.4.5) has an algebraic meaning: it is actually the so-called *antipode* of the *Connes–Kreimer extraction–contraction Hopf algebra* [CK00, CK01]. To explain its construction, we introduce the following sets:

- **F** is the set of all *connected* multigraphs whose vertices have arity 2, 3 or 4 (that is, 2, 3 or 4 edges meet at each vertex);
- $\mathbf{F}_- \subset \mathbf{F}$ is the subset of all divergent multigraphs in **F**;
- $\mathscr{F}$ is the algebra generated by **F** with respect to the disjoint union product $\cdot$ – note that $\mathscr{F}$ also contains *non-connected* Feynman diagrams;
- $\mathscr{F}_- \subset \mathscr{F}$ is the subalgebra of $\mathscr{F}$ generated by $\mathbf{F}_-$; in particular, for any $\Gamma \in \mathscr{F}_-$, all connected components are divergent.

We also denote by $\langle\mathbf{F}\rangle$, $\langle\mathbf{F}_-\rangle$, $\langle\mathscr{F}\rangle$, $\langle\mathscr{F}_-\rangle$ the linear spans, respectively, of **F**, $\mathbf{F}_-$, $\mathscr{F}$ and $\mathscr{F}_-$. The neutral element for multiplication is the empty graph, which we denote by **1**. Note that the valuation Π_N is multiplicative, meaning that

$$\Pi_N(\Gamma_1 \cdot \Gamma_2) = \Pi_N(\Gamma_1)\Pi_N(\Gamma_2) \qquad \text{for all } \Gamma_1, \Gamma_2 \in \mathscr{F} \,.$$

Definition 4.4.9: Connes–Kreimer extraction-contraction coproduct

The *Connes–Kreimer extraction-contraction coproduct* $\Delta_{\mathrm{CK}} : \langle \mathbf{F} \rangle \to \langle \mathscr{F}_- \rangle \otimes \langle \mathbf{F} \rangle$ is defined by

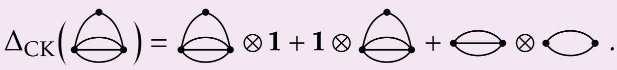
$$\Delta_{\mathrm{CK}}(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\mathbf{1} \neq \overline{\Gamma} \subsetneq \Gamma, \overline{\Gamma} \in \mathbf{F}_-} \overline{\Gamma} \otimes (\Gamma / \overline{\Gamma}) ,$$

where the sum ranges over all *divergent* subgraphs $\overline{\Gamma}$, and $\Gamma/\overline{\Gamma}$ denotes the graph obtained by replacing $\overline{\Gamma}$ by a single vertex. The subgraphs have to be *full*, in the sense that if an edge e belongs to $\overline{\Gamma}$, all edges connecting the same vertices also belong to $\overline{\Gamma}$.

The coproduct can be extended multiplicatively to a map $\Delta_{\mathrm{CK}} : \langle \mathscr{F} \rangle \to \langle \mathscr{F}_- \rangle \otimes \langle \mathscr{F} \rangle$.

Example 4.4.10

We have



We endow $\langle \mathbf{F} \rangle$ with two more linear maps. A *counit* $\mathbf{1}^\star : \langle \mathbf{F} \rangle \to \mathbb{R}$, given by projection on the unit $\mathbf{1}$, and an antipode, defined as follows.

Definition 4.4.11: Antipode

The *antipode* $\mathscr{A} : \langle \mathbf{F} \rangle \to \langle \mathscr{F} \rangle$ is defined inductively by $\mathscr{A}(\mathbf{1}) = \mathbf{1}$ and

$$\begin{aligned} \mathscr{A}(\Gamma) &= -\Gamma - \sum_{\mathbf{1} \neq \overline{\Gamma} \subsetneq \Gamma, \overline{\Gamma} \in \mathbf{F}_-} \mathscr{A}(\overline{\Gamma}) \cdot (\Gamma / \overline{\Gamma}) \\ &= -\Gamma - \mathscr{M}(\mathscr{A} \otimes \mathrm{id}) \mathring{\Delta}_{\mathrm{CK}}(\Gamma) . \end{aligned} \tag{4.4.8}$$

Here $\mathring{\Delta}_{\mathrm{CK}} = \Delta_{\mathrm{CK}} - \Gamma \otimes \mathbf{1} - \mathbf{1} \otimes \Gamma$ denotes the *reduced coproduct*, and the map $\mathscr{M} : \langle \mathscr{F} \rangle \otimes \langle \mathscr{F} \rangle \to \langle \mathscr{F} \rangle$ denotes multiplication, defined by $\mathscr{M}(\Gamma_1 \otimes \Gamma_2) = \Gamma_1 \cdot \Gamma_2$.

Both the counit $\mathbf{1}^\star$ and the antipode $\mathscr{A}$ can be extended multiplicatively to the whole algebra $\mathscr{F}$. The space $(\mathscr{F}, \cdot, \Delta_{\mathrm{CK}}, \mathbf{1}, \mathbf{1}^\star, \mathscr{A})$ constructed in this way is a Hopf algebra, called *Connes–Kreimer extraction-contraction Hopf algebra*. This means in particular that we have

$$\begin{aligned} (\mathrm{id} \otimes \Delta) \Delta \Gamma &= (\Delta \otimes \mathrm{id}) \Delta \Gamma , \\ \mathscr{M}(\mathscr{A} \otimes \mathrm{id}) \Delta \Gamma &= \mathscr{M}(\mathrm{id} \otimes \mathscr{A}) \Delta \Gamma = \mathbf{1}^\star(\Gamma) \mathbf{1} \end{aligned}$$

for all $\Gamma \in \mathscr{F}$. We have already encountered the first property, called *co-associativity*, in the case of polynomials, see Remark 1.2.12.

To define BPHZ renormalisation, we first introduce the *twisted antipode*, defined as

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma) 1_{\deg \Gamma \leqslant 0} .$$

Note that if $\deg(\Gamma) \leqslant 0$, then one has

$$\tilde{\mathscr{A}}(\Gamma) = -\Gamma - \mathscr{M}(\tilde{\mathscr{A}} \otimes \mathrm{id}) \mathring{\Delta}_{\mathrm{CK}}(\Gamma) , \tag{4.4.9}$$

because $\mathring{\Delta}_{\mathrm{CK}}$ produces only divergent terms on the left of the tensor product.

A *character* on $\langle\mathscr{F}\rangle$ is a linear map $g : \langle\mathscr{F}\rangle \to \mathbb{R}$ which is multiplicative in the sense that we have $g(\Gamma_1 \cdot \Gamma_2) = g(\Gamma_1)g(\Gamma_2)$ for all $\Gamma_1, \Gamma_2 \in \langle\mathscr{F}\rangle$. With any character g, one can associate a linear map M^g defined by

$$M^g(\Gamma) = (g \otimes \mathrm{id})\Delta_{\mathrm{CK}}(\Gamma)\,,$$

and the set of these maps is known to form a group. The *BPHZ character* is the linear map $g^{\mathrm{BPHZ}} : \langle\mathscr{F}\rangle \to \mathbb{R}$ given by

$$g^{\mathrm{BPHZ}}(\Gamma) = \Pi_N \tilde{\mathscr{A}}(\Gamma)\,.$$

The fact that g^{BPHZ} is indeed a character follows from multiplicativity of $\mathscr{A}$ and Π_N. The map $M^{g^{\mathrm{BPHZ}}}$ is called *BPHZ renormalisation map*. It defines a *renormalised valuation* given by

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) = (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta_{\mathrm{CK}}(\Gamma) = (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta_{\mathrm{CK}}(\Gamma)\,. \tag{4.4.10}$$

The interest of this construction is the following result.

Lemma 4.4.12

The BPHZ renormalised valuation satisfies

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \begin{cases} 0 & \text{if } \deg\Gamma \leqslant 0\,, \\ -\Pi_N \mathscr{A}(\Gamma) & \text{if } \deg\Gamma > 0\,. \end{cases}$$

Proof: In the case $\deg\Gamma \leqslant 0$, using (4.4.9) we get

$$\begin{aligned}
\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= (\Pi_N \otimes \Pi_N)(\tilde{\mathscr{A}} \otimes \mathrm{id})[\Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \mathring{\Delta}_{\mathrm{CK}}(\Gamma)] \\
&= (\Pi_N \otimes \Pi_N)[\tilde{\mathscr{A}}(\Gamma) \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + (\tilde{\mathscr{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)] \\
&= (\Pi_N \otimes \Pi_N)[-\Gamma \otimes \mathbf{1} - \mathscr{M}(\tilde{\mathscr{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma) \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + (\tilde{\mathscr{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)]\,,
\end{aligned}$$

which vanishes by multiplicativity of Π_N. In the case $\deg\Gamma > 0$, using $\tilde{\mathscr{A}}(\Gamma) = 0$ in the second line of the above computation, we obtain

$$\begin{aligned}
\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= (\Pi_N \otimes \Pi_N)[\mathbf{1} \otimes \Gamma + (\tilde{\mathscr{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)] \\
&= \Pi_N(\Gamma) + (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\mathring{\Delta}_{\mathrm{CK}}(\Gamma) \\
&= \Pi_N(\Gamma) + \Pi_N \mathscr{M}(\tilde{\mathscr{A}} \otimes \mathrm{id})\mathring{\Delta}_{\mathrm{CK}}(\Gamma)
\end{aligned}$$

again by multiplicativity of Π_N. This is equal to $-\Pi_N \mathscr{A}(\Gamma)$ by (4.4.8). □

If follows from Theorem 4.4.7 that the renormalized valuation Π_N^{BPHZ} is bounded uniformly in the cut-off N for any $\Gamma \in \langle\mathscr{F}\rangle$.

4.4.3 Wick map*

We now indicate how the above results allow to prove Theorem 4.4.1. We again focus on the problem of computing the ratio of partition functions. Writing as before

$$\mathbf{X} = \int_\Lambda {:}\phi(x)^4{:}_{C_N}\,\mathrm{d}x\,, \qquad \mathbf{Y} = \int_\Lambda {:}\phi(x)^2{:}_{C_N}\,\mathrm{d}x\,,$$

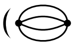
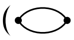
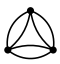
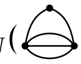
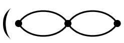
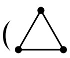
this ratio is given by

$$\frac{\mathscr{Z}_{3,\alpha,N}}{\mathscr{Z}_{3,0,N}} = \mathbb{E}^{\mu_{3,0,N}}[\mathrm{e}^{-\alpha\mathbf{X} - \beta\mathbf{Y} - \gamma}] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_{3,0,N}}[\mathrm{e}^{-\alpha\mathbf{X} - \beta\mathbf{Y}}]\,, \tag{4.4.11}$$

where $\beta = \beta_N(\alpha)$ and $\gamma = \gamma_N(\alpha)$. A naive approach would be to expand first the exponential, and then use Newton's binomial formula to get

$$\begin{aligned}\mathbb{E}^{\mu_{3,0,N}}\big[\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}\big] &\asymp \sum_{n\geqslant 0}\frac{1}{n!}\mathbb{E}^{\mu_{3,0,N}}\big[(-\alpha\mathbf{X}-\beta\mathbf{Y})^n\big] \\ &= \sum_{n\geqslant 0}\sum_{k=0}^{n}\frac{(-\alpha)^k}{k!}\frac{(-\beta)^{n-k}}{(n-k)!}\mathbb{E}^{\mu_{3,0,N}}\big[\mathbf{X}^k\mathbf{Y}^{n-k}\big]\,.\end{aligned}$$

The first terms of this expansion are

$$\mathbb{E}^{\mu_{3,0,N}}\big[\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}\big] \asymp 1 + 4!\alpha^2\,\Pi_N($$  $$) + 2!\beta^2\,\Pi_N($$  $$)$$
$$-\binom{4}{2}^3 2^3\alpha^3\Pi_N($$  $$) - 3(4^2\cdot 2\cdot 3!)\alpha^2\beta\,\Pi_N($$  $$)$$
$$-3\cdot 4!\alpha\beta^2\,\Pi_N($$  $$) - 8\beta^3\,\Pi_N($$  $$) + \dots\,. \qquad (4.4.12)$$

The problem is that some Feynman diagrams are divergent, while others are not, and β, being of order $\alpha^2\log(N)$ is itself divergent. Therefore, the whole asymptotic series can have non-divergent coefficients only if there are many cancellations between divergent terms. The combinatorics of this is very hard to keep track of.

An alternative reformulation is as follows. We introduce a linear map $\mathscr{P} : \mathbb{R}[\mathbf{X},\mathbf{Y}] \to \langle\mathbf{F}\rangle$ associating with a monomial $\mathbf{X}^n\mathbf{Y}^m$ all possible diagrams obtained by perfect pairwise matchings of n vertices with 4 legs each and m vertices with 2 legs each, and projecting on connected diagrams. In this way, by the linked-cluster theorem,

$$\log\mathbb{E}^{\mu_{3,0,N}}\big[\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}\big] = \Pi_N\circ\mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}})\,. \qquad (4.4.13)$$

The following theorem, inspired by ideas in [EFPTZ20, BH25], has been proved in [BKT25].

Theorem 4.4.13: Commutative diagram

There exist linear maps $\mathrm{W} : \mathbb{R}[\mathbf{X}] \to \mathbb{R}[\mathbf{X},\mathbf{Y}]$, and $\Theta_{\mathrm{F}} : \langle\mathbf{F}\rangle \to \langle\mathbf{F}\rangle$ such that the following diagram commutes:

$$\begin{array}{ccc}\mathbb{R}[\mathbf{X}] & \xrightarrow{\ \mathscr{P}\ } & \langle\mathbf{F}\rangle \\ {\scriptstyle \mathrm{W}}\big\downarrow & & \big\downarrow{\scriptstyle (\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta_{\mathrm{CK}}+\Theta_{\mathrm{F}}} \\ \mathbb{R}[\mathbf{X},\mathbf{Y}] & \xrightarrow{\ \mathscr{P}\ } & \langle\mathbf{F}\rangle\end{array} \qquad (4.4.14)$$

The map W, called *Wick map*, satisfies

$$\mathrm{W}(\mathbf{X}^n) = H_n(\mathbf{X};-\beta\mathbf{Y}) \quad \forall n\geqslant 2 \qquad \text{and} \qquad \mathrm{W}(\mathrm{e}^{-\alpha\mathbf{X}}) = \mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}\,,$$

where H_n is the nth scaled Hermite polynomial, while the map Θ_{F} satisfies

$$(\Pi_N\Theta_{\mathrm{F}}\circ\mathscr{P})(\mathrm{e}^{-\alpha\mathbf{X}}) = -(\Pi_N\tilde{\mathscr{A}}\circ\mathscr{P})(\mathrm{e}^{-\alpha\mathbf{X}}) = \gamma\,. \qquad (4.4.15)$$

The main idea of this result is that the complicated map $(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta_{\mathrm{CK}}$, acting on Feynman diagrams, can be replaced by the much simpler map W, acting on polynomials. The Wick map is of the form

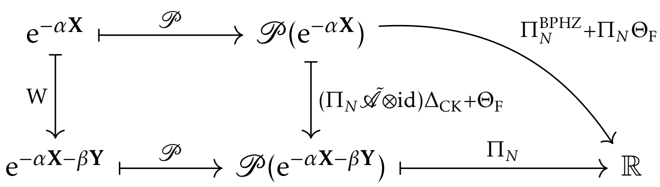
$$\mathrm{W} = (\exp_*(-\kappa)\otimes\mathrm{id})\Delta$$

that we have encountered in (1.2.24), where

$$\kappa(x^n) = \begin{cases} \beta\mathbf{Y} & \text{if } n = 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The intuition behind this is that the only subdivergences are graphs containing one or several "bubbles"  as subgraphs. The effect of the map $(\Pi_N\tilde{\mathscr{A}} \otimes \mathrm{id})\Delta_{\mathrm{CK}}$ is to extract bubbles and replace them by β times a vertex of arity 2, which can be seen as replacing $\mathbf{X}^2$ by $\beta\mathbf{Y}$. This is also compatible with the combinatorial interpretation of Hermite polynomials we have seen in Section 1.2.5. As for the relation $\mathrm{W}(\mathrm{e}^{-\alpha\mathbf{X}}) = \mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}$, it is a consequence of Proposition 1.2.16. One should note that we are working here with a "second level Wick renormalisation", the first level being associated with using Wick powers in the energy, and the second level taking care of the remaining subdivergences.

Together with the definition (4.4.10) of the BPHZ valuation, Theorem 4.4.13 implies that the following diagram commutes:



This has the following consequence. On the one hand, (4.4.11) and (4.4.13) show that

$$\log\frac{\mathscr{Z}_{3,\alpha,N}}{\mathscr{Z}_{3,0,N}} = \Pi_N \circ \mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}) - \gamma\,.$$

On the other hand, by commutativity and (4.4.15), we have

$$\begin{aligned}\Pi_N \circ \mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}-\beta\mathbf{Y}}) &= (\Pi_N^{\mathrm{BPHZ}} + \Pi_N\Theta_{\mathrm{F}}) \circ \mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}}) \\ &= \Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}}) + \gamma\,.\end{aligned}$$

It follows that

$$\log\frac{\mathscr{Z}_{3,\alpha,N}}{\mathscr{Z}_{3,0,N}} = \Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}(\mathrm{e}^{-\alpha\mathbf{X}}) \asymp \sum_{n\geqslant 1}\frac{(-\alpha)^n}{n!}\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}(\mathbf{X}^n)\,. \tag{4.4.16}$$

Now Lemma 4.4.12 shows that $\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}(\mathbf{X}^n)$ is different from 0 only if $\mathscr{P}(\mathbf{X}^n)$ has strictly positive degree, in which case it is equal to $-\Pi_N\mathscr{A} \circ \mathscr{P}(\mathbf{X}^n)$. By Theorem 4.4.7, these terms are bounded uniformly in N. This completes the proof of Theorem 4.4.1 in the case of the ratio of partition functions.

In fact, all Feynman diagrams in $\mathscr{P}(\mathbf{X}^n)$ have n vertices and $2n$ edges, as they result from pairing $4n$ legs, or half-edges. Therefore,

$$\deg(\mathscr{P}(\mathbf{X}^n)) = 3(n-1) - 2n = n-3\,,$$

which is strictly positive for all $n \geqslant 4$. It follows that the sum (4.4.16) starts at $n = 4$. On the other hand, the terms of order α^2 and α^3 in (4.4.12) are compensated by γ.

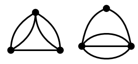
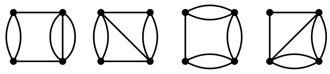
Graphs	Degree	Critical d	Minimal n
	$6-2d$	$3 = d_{\mathrm{m}}^*(2)$	4
	$10-3d$	$\frac{10}{3} = d_{\mathrm{m}}^*(3)$	5
	$14-4d$	$\frac{7}{2} = d_{\mathrm{m}}^*(4)$	6

Table 4.1 – List of the first divergent subdiagrams of the Φ_d^4 model, with their degree, the value of d for which they become divergent, and the minimal value of n such that they occur in $\mathscr{P}(\mathbf{X}^n)$.

4.5 The $\Phi_{4-\varepsilon}^4$ model*

Now that the Φ_d^4 model is understood in dimensions 1, 2 and 3, one may wonder what happens in higher dimensions. In fact, Fröhlich has shown [Frö82] that the model in trivial for any $d > 4$, while Aizenmann and Duminil–Copin have shown [AD21] that it is trivial for $d = 4$ as well, for any reasonable renormalisation procedure. Here *trivial* means that in the limit $N \to \infty$, the n-point functions are the same as for the Gaussian model, as given by Isserlis' theorem.

This leaves the question of what happens in dimensions $d \in (3,4)$. Non-integer dimensions in that interval can be interpreted as working on the three-dimensional torus, but changing the behaviour of the Green function to

$$G(x) \asymp \frac{1}{\|x\|^{d-2}} .$$

The degree of $\mathscr{P}(\mathbf{X}^n)$, computed via (4.4.2), becomes

$$\deg \mathscr{P}(\mathbf{X}^n) = 4n - (n+1)d .$$

This means that more diagrams in $\mathscr{P}(\mathbf{X}^n)$ can become divergent when d increases. In fact, we have

$$\deg \mathscr{P}(X^n) \leqslant 0 \qquad \Leftrightarrow \qquad d \geqslant d_{\mathrm{e}}^*(n) := 4 - \frac{4}{n+1} .$$

Note that these threshold accumulate at $d = 4$ as $n \to \infty$. Furthermore, one can check that new subdivergences appear whenever d crosses one of the thresholds

$$d_{\mathrm{m}}^*(n) = d_{\mathrm{e}}^*(2n-1) = 4 - \frac{2}{n} .$$

Table 4.1 shows the first few of these new subdivergences. We introduce the notations

$$n_{\mathrm{e}}^*(d) = \left\lfloor \frac{d}{4-d} \right\rfloor \qquad \text{and} \qquad n_{\mathrm{m}}^*(d) = \left\lfloor \frac{2}{4-d} \right\rfloor$$

for the inverse thresholds of $d_{\mathrm{m}}^*(n)$ and $d_{\mathrm{e}}^*(n)$.

In order to deal with these subdivergences, we have to allow for more general replacement rules than $\mathbf{X}^2 \mapsto \beta\mathbf{Y}$. Therefore, the relevant Wick map W no longer involves Hermite polynomials, but more general so-called *Bell polynomials*.

4.5.1 Bell polynomials*

Bell polynomials can be defined by starting with cumulants

$$\kappa(x^n) = \begin{cases} 0 & \text{if } n = 1 , \\ y_n & \text{otherwise ,} \end{cases}$$

where the y_n are considered for now as parameters. The associated Wick map is

$$\mathrm{W}(t,x) = \mathrm{e}^{tx-K(t)} = \exp\Big\{tx - \sum_{n\geqslant 2} y_n \frac{t^n}{n!}\Big\},$$

where $K(t) = \Lambda(\kappa)(t)$, see (1.2.16).

Definition 4.5.1: Bell polynomials

The Wick map $\mathrm{W}(t,x)$ is the generating function of Bell polynomials, in the sense that

$$\mathrm{W}(t,x) = \sum_{n\geqslant 0} B_n(x,-y_2,\dots,-y_n)\frac{t^n}{n!}.$$

B_n is called the *nth complete Bell polynomial*. It can be decomposed as

$$B_n(x,-y_2,\dots,-y_n) = \sum_{k=1}^{n} B_{n,k}(x,-y_2,\dots,-y_{n-k+1}),$$

where each $B_{n,k}$ is the homogeneous part of degree k of B_n, and is called *incomplete Bell polynomial* of order (n,k).

Using the fact (see Section 1.2.4) that

$$B_n(x,-y_2,\dots,-y_n) = \mathrm{W}(x^n) = \sum_{k=0}^{n}\binom{n}{k}\mu^{-1}(x^k)x^{n-k}$$

where $\mu^{-1} = \exp_*(-\kappa)$, one finds the explicit expression

$$B_n(x,-y_2,\dots,-y_n) = n!\sum_{k=0}^{n}\sum_{p=1}^{k}\frac{(-1)^p}{p!}\sum_{\substack{n_1,\dots,n_p\geqslant 2\\ n_1+\dots+n_p=k}}\frac{y_{n_1}}{n_1!}\cdots\frac{y_{n_p}}{n_p!}\frac{x^{n-k}}{(n-k)!}.$$

The incomplete Bell polynomial $B_{n,k}$ is simply the kth term in this sum. In particular, comparing with the explicit expression (2.2.2) of scaled Wick polynomials, we recover

$$B_n(x,-y_2,0,\dots,0) = H_n(x;y_2).$$

Bell polynomials also have a simple combinatorial interpretation: the coefficients of $B_{n,k}$ count the number of partitions of a set of cardinality n into k subsets, whose sizes are encoded into the monomial.

Example 4.5.2

One has

$$B_{5,3}(x,y_2,y_3) = 15xy_2^2 + 10x^2y_3.$$

Therefore, there are 15 ways of partitioning a set of 5 elements into 3 subsets of sizes 1, 2 and 2, and 10 ways of partitioning it into 3 subsets of sizes 1, 1 and 3. The polynomial $B_{5,3}(x,y_2,y_3)$ is also obtained by applying the substitutions $x^2 \mapsto y_2$ and $x^3 \mapsto y_3$ to the monomial x^5 in all possible ways, and keeping only terms of degree 3.

4.5.2 Wick map*

Theorem 4.4.13 admits the following generalisation, also shown in [BKT25].

Theorem 4.5.3: Commutative diagram

There exist linear maps $\mathrm{W} : \mathbb{R}[\mathbf{X}] \to \mathbb{R}[\mathbf{X}, \mathbf{Y}]$ and $\Theta_{\mathrm{F}} : \langle \mathbf{F} \rangle \to \langle \mathbf{F} \rangle$, and constants $\sigma_n(N)$, diverging like $N^{2-(4-d)n}$, such that the diagram (4.4.14) commutes. The map Wick map W satisfies

$$\mathrm{W}(\mathrm{e}^{-\alpha \mathbf{X}}) = \mathrm{e}^{-\alpha \mathbf{X} - \beta \mathbf{Y}}$$

where

$$\beta = \beta_{N,d}(\alpha) = \sum_{n=2}^{n^*_{\mathrm{m}}(d)} \frac{(-\alpha)^n}{n!} \sigma_n(N) \,.$$

Furthermore,

$$\mathrm{W}(\mathbf{X}^n) = B_n(\mathbf{X}, -\sigma_2(N)\mathbf{Y}, \dots, -\sigma_n(N)\mathbf{Y})$$

holds for any $n \geqslant 2$, where B_n is the nth complete Bell polynomial, while the map Θ_{F} satisfies

$$(\Pi_N \Theta_{\mathrm{F}} \circ \mathscr{P})(\mathrm{e}^{-\alpha \mathbf{X}}) = -(\Pi_N \tilde{\mathscr{A}} \circ \mathscr{P})(\mathrm{e}^{-\alpha \mathbf{X}}) = \gamma \,.$$

The counterterms $\sigma_n(N)$ can again be written explicitly in terms of valuations of divergent Feynman diagrams, the first of which are shown in Table 4.1. For given n, these diagrams have n vertices, which are either two vertices of arity 3 and $n-2$ vertices of arity 4, or one vertex of arity 2, and $n-1$ vertices of arity 4. Using the same arguments as in Section 4.4.3, we arrive at the following result.

Corollary 4.5.4: Renormalisation of the Φ^4_d model with $d \in (3,4)$

Let

$$\gamma = \gamma_{N,d}(\alpha) = -\sum_{n=2}^{n^*_{\mathrm{e}}(d)} \frac{(-\alpha)^n}{n!} \Pi_N \tilde{\mathscr{A}}(\mathscr{P}(\mathbf{X}^n)) \,.$$

Then the Φ^4_d model with $d \in (3,4)$, mass counterterm β and energy counterterm γ satisfies

$$\log \frac{\mathscr{Z}_{d,\alpha,N}}{\mathscr{Z}_{d,0,N}} \asymp -\sum_{n \geqslant n^*_{\mathrm{e}}(d)} \frac{(-\alpha)^n}{n!} \Pi_N \mathscr{A}(\mathscr{P}(\mathbf{X}^n)) \,,$$

where the coefficients are bounded uniformly in the cut-off N.

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