Metastability
in irreversible diffusion processes
and stochastic resonance

Nils Berglund
Centre de Physique Théorique - CNRS
Marseille Luminy
http://berglund.univ-tln.fr

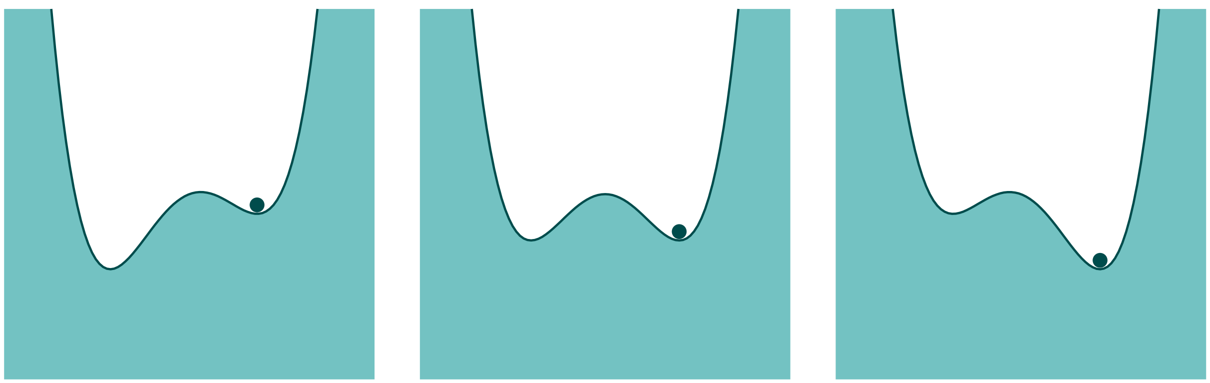
Joint work with Barbara Gentz, WIAS, Berlin

Max-Planck-Institut Leipzig, May 2005

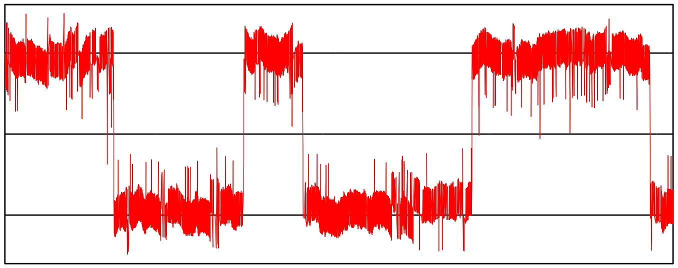
Stochastic resonance: typical example

$$\mathrm{d}x_t = \underbrace{\Big[x_t - x_t^3 + A\cos\varepsilon t\Big]}_{= -\frac{\partial}{\partial x}V(x_t,t)}\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

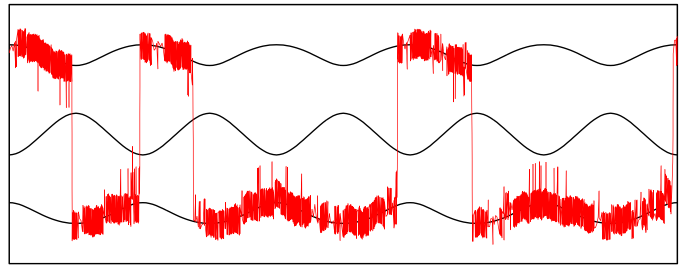
Potential: $$V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos\varepsilon t.$$



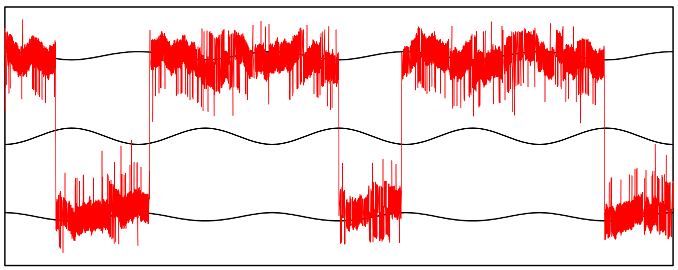
Sample paths $\{x_t\}_t$ $\qquad \varepsilon = 0.001$



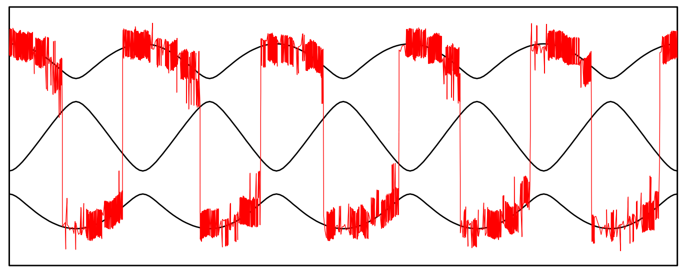
$A = 0$, $\sigma = 0.3$



$A = 0.24$, $\sigma = 0.2$



$A = 0.1$, $\sigma = 0.27$



$A = 0.35$, $\sigma = 0.2$

Qualitative measures:

- Power spectrum, signal-to-noise ratio.
- Residence-time distribution: law of interwell transition times.

Theory of large deviations [Freidlin, Wentzell]

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Rate function: $I_{[0,T]}(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \|\dot\varphi_t - f(\varphi_t)\|^2\,\mathrm{d}t & \text{for } \varphi \in H_1 \\ +\infty & \text{otherwise} \end{cases}$

Probability of staying close to $\varphi \sim \mathrm{e}^{-I(\varphi)/\sigma^2}$.

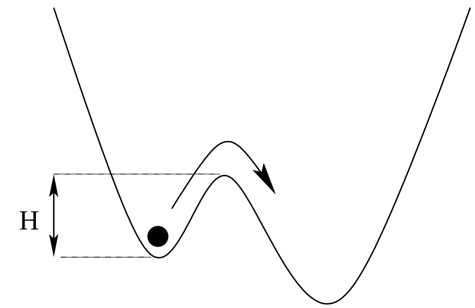
Large-deviation principle: for Γ set of paths,

$$\mathbb{P}\bigl\{(x_t)_{0\leqslant t\leqslant T} \in \Gamma\bigr\} \sim \mathrm{e}^{-\inf_\Gamma I/\sigma^2} \text{ as } \sigma \to 0.$$

Meaning:

$$-\inf_{\Gamma^\circ} I \leqslant \liminf_{\sigma\to0} \sigma^2 \log \mathbb{P}\bigl\{(x_t)_t \in \Gamma\bigr\} \leqslant \limsup_{\sigma\to0} \sigma^2 \log \mathbb{P}\bigl\{(x_t)_t \in \Gamma\bigr\} \leqslant -\inf_{\overline{\Gamma}} I$$

Gradient case (reversible)



$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- $I(\varphi)$ minimized on paths with $\dot\varphi_t = -f(\varphi_t)$.
- Cost of leaving potential well
  $\inf\{I(\varphi)\colon \varphi_0 = \text{bottom}, \exists t : \varphi_t \notin \text{well}\} = 2H$.
- Expected time to leave well: $\mathbb{E}(\tau) \sim \mathrm{e}^{2H/\sigma^2}$
  [Eyring, Kramers, Freidlin, Wentzell, . . .]
- Law of $\tau/\mathbb{E}(\tau) \to \mathrm{Exp}(1)$ as $\sigma \to 0$ [Day]
- Subexponential behaviour known, related to small eigenvalues of generator of diffusion
  [Bovier, Eckhoff, Gayrard, Klein], [Helffer, Klein, Nier]

Stochastic resonance: quasistatic regime

$$\mathrm{d}x_t = \Big[x_t - x_t^3 + A\cos\varepsilon t\Big]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \qquad \text{(for simplicity)}$$

Take $\varepsilon = \dfrac{2\pi}{T(\sigma)}$, $\qquad T(\sigma) \sim \mathrm{e}^{\lambda/\sigma^2}$

x_t crosses barrier whenever $\qquad T(\sigma) \gg \mathrm{e}^{2H(t)/\sigma^2}$
$H(t) =$ instantaneous barrier height

$\phi(t,\lambda)$: deterministic function, switches to deeper well whenever $2H(t) < \lambda$. ($\Rightarrow$ possibility of hysteresis.)

Theorem: [Freidlin, 2000]

For $S, p, \delta > 0$,

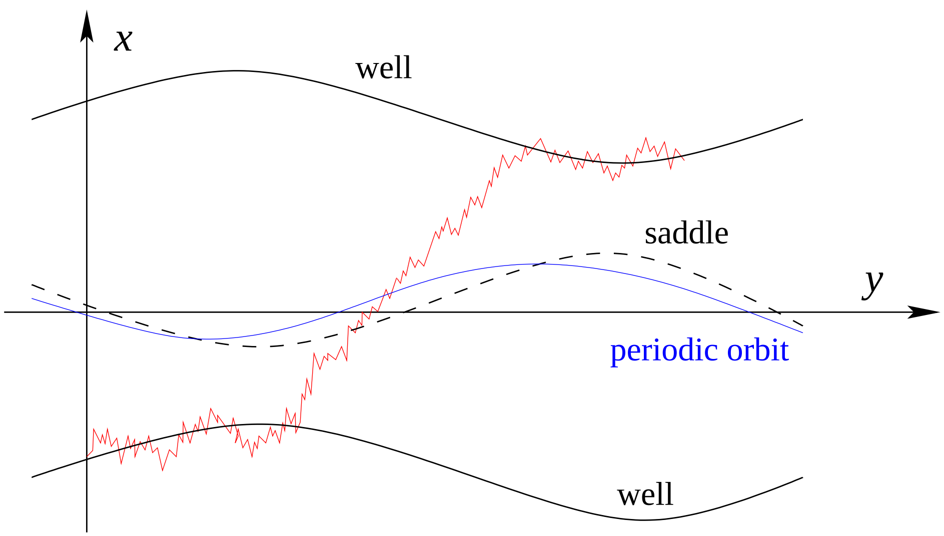
$$\lim_{\sigma\to 0} \mathbb{P}\bigg\{\int_0^S \Big|x_{sT(\sigma)} - \phi(sT(\sigma),\lambda)\Big|^p\,\mathrm{d}s > \delta\bigg\} = 0.$$

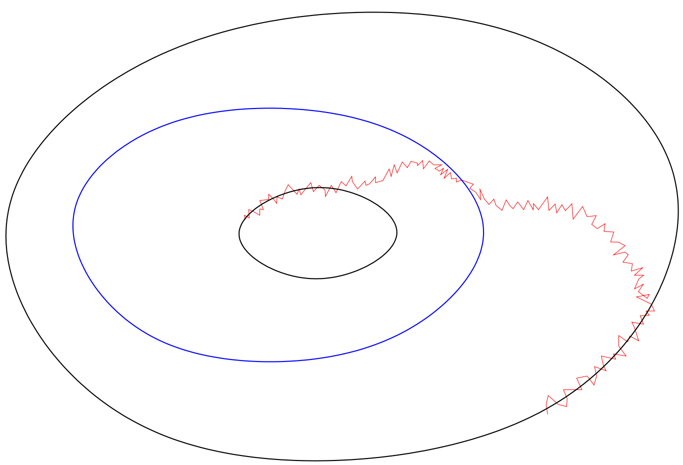
What about non-quasistatic regime?

$$\begin{aligned} \mathrm{d}x_t &= \Bigl[x_t - x_t^3 + A\cos\Omega y_t\Bigr]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ \mathrm{d}y_t &= \mathrm{d}t \end{aligned}$$

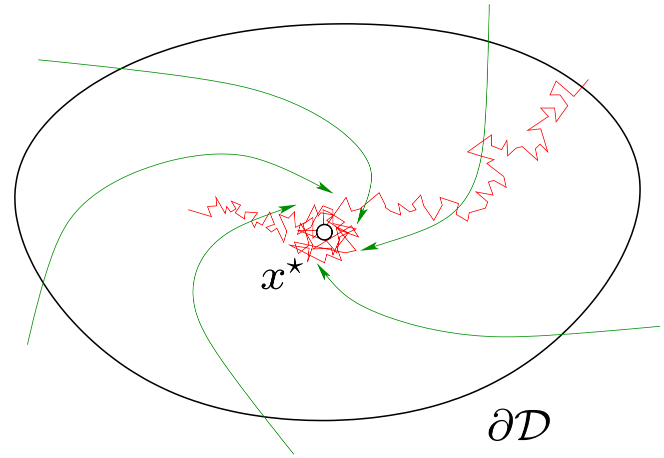
Irreversible (degenerate) diffusion process.

Interwell transition $\to$ crossing unstable periodic orbit tracking the saddle. Distribution of transition locations?





Exit problem



$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

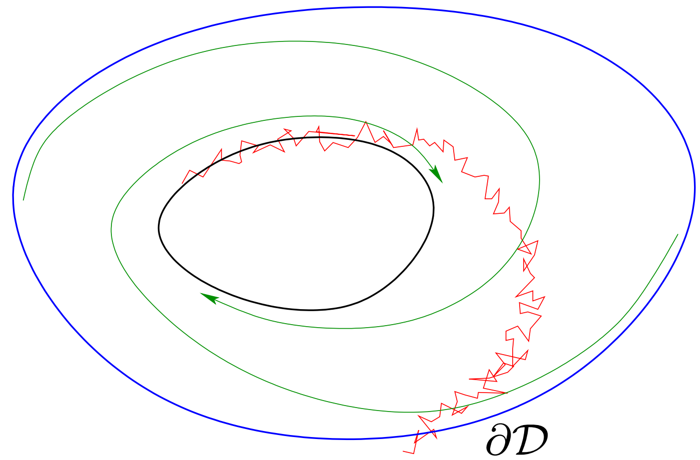
$x_0 \in \mathcal{D}$, open, bdd set, $\partial\mathcal{D}$ smooth.
First-exit time: $\tau = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$.
First-exit location: $x_\tau \in \partial\mathcal{D}$.

Case 1: $\bar{\mathcal{D}} \subset$ basin of attraction of asympt. stable equil. point $x^\star$

Quasipotential
$$V(x^\star, y) = \inf_{t>0}\Bigl\{I(\varphi) \colon \varphi_0 = x^\star,\ \varphi_t = y\Bigr\}.$$
$$\overline{V} = \inf_{y\in\partial\mathcal{D}} V(x^\star, y).$$

- $\mathbb{E}(\tau) \sim \mathrm{e}^{\overline{V}/\sigma^2}$
- $\lim\limits_{\sigma\to0} \mathbb{P}\Bigl\{\mathrm{e}^{(\overline{V}-\delta)/\sigma^2} \leqslant \tau \leqslant \mathrm{e}^{(\overline{V}+\delta)/\sigma^2}\Bigr\} = 1$.
- If $y \mapsto V(x^\star, y)$ has non-degenerate global minimum in $x_1 \in \partial\mathcal{D}$,
  $\lim\limits_{\sigma\to0} \mathbb{P}\Bigl\{\|x_\tau - x_1\| > \delta\Bigr\} = 0$. [Freidlin, Wentzell]

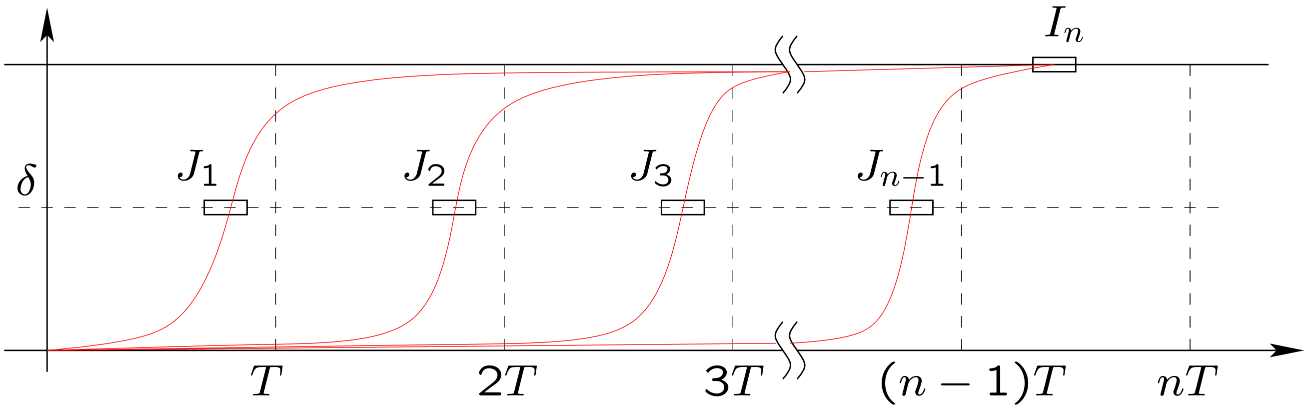
Exit problem



Case 2: $\partial\mathcal{D}$ is unstable periodic orbit (characteristic boundary)

- Quasipotential $V(x^\star, y) = \overline{V}$ is constant on $\partial\mathcal{D}$.
  $\Rightarrow$ no concentration of exit location?
- $\mathbb{E}(\tau) \sim \mathrm{e}^{\overline{V}/\sigma^2}$ still holds. [Day]
- As $\sigma \to 0$, density of x_τ is translated along $\partial\mathcal{D}$ proportionally to $|\log\sigma|$: cycling. [Day]

How to compute law of exit location



If exit occurs in $I_n = [y, y+\Delta] \subset [nT, (n+1)T]$:
Rate function has n minimizers of comparable value.

$$\mathbb{P}^{0,0}\Bigl\{y_\tau \in I_n\Bigr\} \simeq \sum_{\ell=1}^{n} \underbrace{\mathbb{P}^{\delta,J_\ell}\Bigl\{y_\tau \in I_n\Bigr\}}_{Q_{n-\ell}(y)} \underbrace{\mathbb{P}^{0,0}\Bigl\{y_{\tau'} \in J_\ell\Bigr\}}_{P_\ell}$$

$$P_\ell \simeq \textit{const}\, \mathrm{e}^{-\ell\varepsilon} \exp\biggl\{-\frac{\overline{V}_1}{\sigma^2}\Bigl(1-\mathrm{e}^{-2\ell\lambda T}\Bigr)\biggr\}, \qquad \varepsilon = T\,\mathrm{e}^{-\overline{V}/\sigma^2}$$

$$Q_k(y) \simeq C(y)\,\mathrm{e}^{-2k\lambda T} \exp\biggl\{-\frac{\overline{V}_2}{\sigma^2}\Bigl(1-c(y)\,\mathrm{e}^{-2k\lambda T}\Bigr)\biggr\}$$

$\lambda =$ Lyapunov exponent of unstable orbit.

Theorem:

$\partial\mathcal{D}$: unstable periodic orbit, Lyapunov exponent λ.

$\theta(y)$: "natural" parametrization of boundary, $\theta(y+T) = \theta(y)+\lambda T$.

$\forall\Delta > \sigma^{1/2}$,

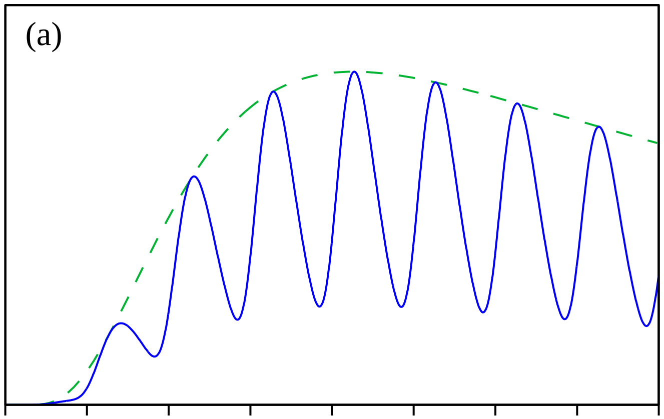
$$\mathbb{P}^{0,\theta^{-1}(\theta_0)}\Big\{\theta(y_{\tau_{\mathcal{D}}}) \in [\theta_1, \theta_1+\Delta]\Big\} = \int_{\theta_1}^{\theta_1+\Delta} p(\theta|\theta_0)\,\mathrm{d}\theta\,[1+\mathcal{O}(\sigma^{1/2})]$$

$$p(\theta|\theta_0) = \frac{1}{N} f_{\text{trans}}(\theta,\theta_0) \frac{\mathrm{e}^{-(\theta-\theta_0)/\lambda T_{\text{K}}}}{\lambda T_{\text{K}}} P_{\lambda T}(\theta - |\log\sigma|)$$

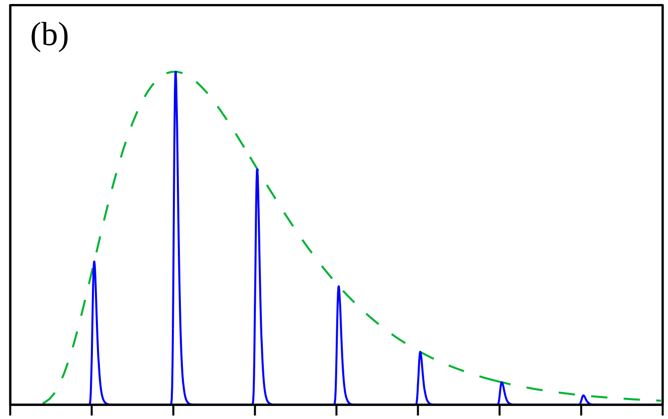
- $f_{\text{trans}}(\theta,\theta_0)$ grows from 0 to 1 for $\theta-\theta_0$ of order $|\log\sigma|$.
- $T_{\text{K}} = T_{\text{K}}(\sigma) = \dfrac{C}{\sigma}\,\mathrm{e}^{\overline{V}/\sigma^2}$, Kramers time.
- $P_{\lambda T}(x) = \displaystyle\sum_{k=-\infty}^{\infty} A(x-k\lambda T)$ $\qquad A(z) = \dfrac{1}{2}\mathrm{e}^{-2z}\exp\Big\{-\dfrac{1}{2}\mathrm{e}^{-2z}\Big\}$.

First-passage-time distributions

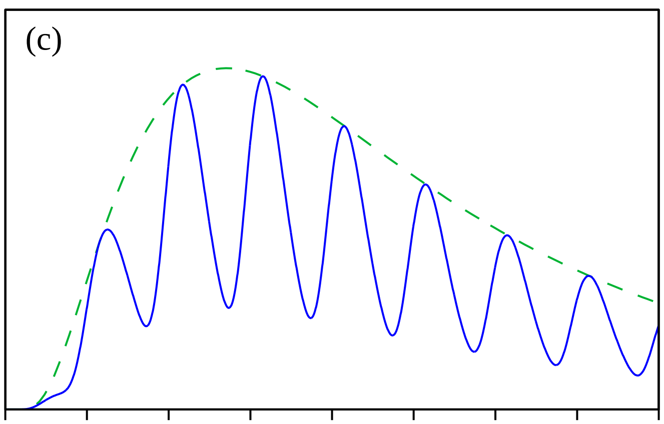
$\overline{V} = 0.5,\ \lambda = 1$



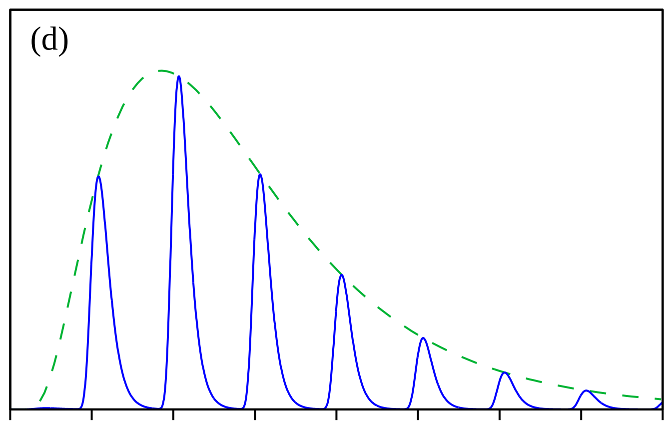
$\sigma = 0.4,\ T = 2$



$\sigma = 0.4,\ T = 20$



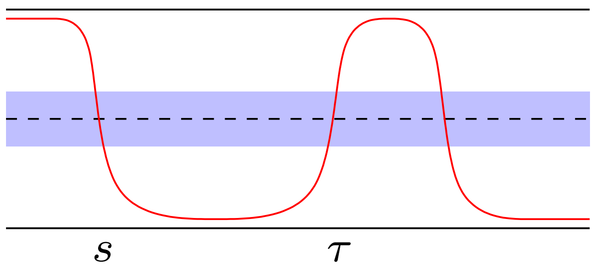
$\sigma = 0.5,\ T = 2$



$\sigma = 0.5,\ T = 5$

Application to residence-time distribution

x_t crosses $x^{\text{per}}(t)$ at time s.
τ: time of first crossing back after s.



- First-passage-time density:

$$p(t|s) = \frac{\partial}{\partial t}\mathbb{P}^{s,x^{\text{per}}(s)}\Big\{\tau < t\Big\}.$$

- Asymptotic transition-phase density:

$$\psi(t) = \int_{-\infty}^{t} p(t|s)\psi(s - T/2)\,\mathrm{d}s = \psi(t+T).$$

- Residence-time distribution:

$$q(t) = \int_0^T p(s+t|s)\psi(s-T/2)\,\mathrm{d}s.$$

Computation of residence-time distribution

Without forcing $(A = 0)$:

$p(t|s) \sim$ exponential, $\psi(t)$ uniform $\Rightarrow q(t) \sim$ exponential.

With forcing $(A \gg \sigma^2)$: time change $t \mapsto \theta(t)/\lambda$

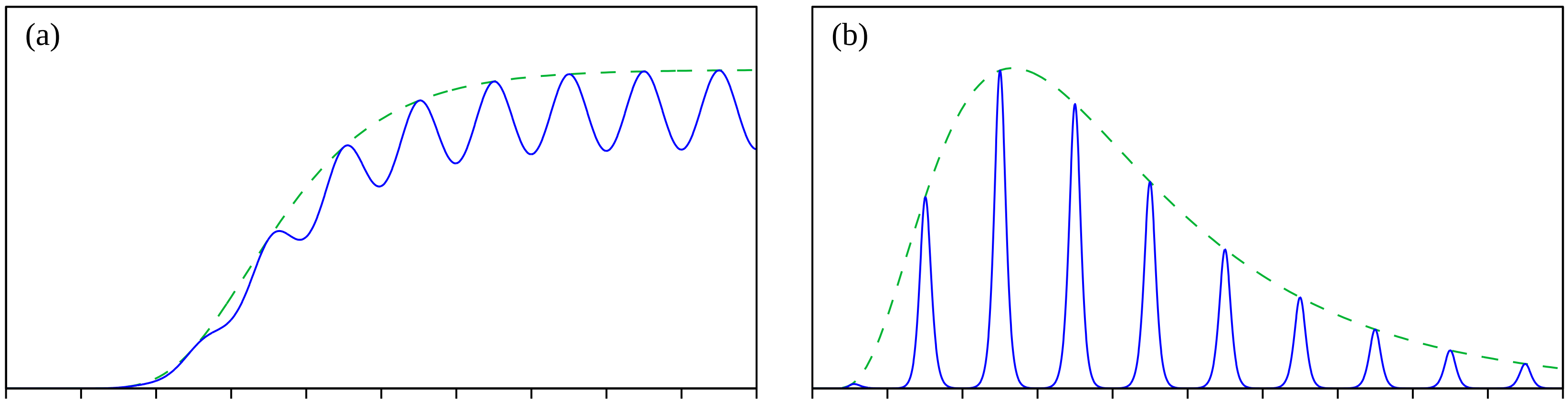
$$p(t|s) \simeq \frac{1}{N} f_{\text{trans}}(t,s) \frac{\mathrm{e}^{-(t-s)/T_{\mathsf{K}}}}{T_{\mathsf{K}}} P_{\lambda T}(\lambda t - |\log \sigma|)$$

$$\psi(s) \simeq \frac{1}{T} P_{\lambda T}(\lambda t - |\log \sigma|) \Big[1 + \mathcal{O}(T/T_{\mathsf{K}})\Big]$$

$$q(t) \simeq \tilde{f}_{\text{trans}}(t) \frac{\mathrm{e}^{-t/T_{\mathsf{K}}}}{T_{\mathsf{K}}} \frac{\lambda T}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(\lambda(t + T/2 - kT))}$$

Residence-time distributions

$\overline{V} = 0.5, \ \lambda = 1$



$\sigma = 0.2, \ T = 2$ $\qquad$ $\sigma = 0.4, \ T = 10$

References

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- In preparation . . .