Geometric singular perturbation theory applied to stochastic climate models

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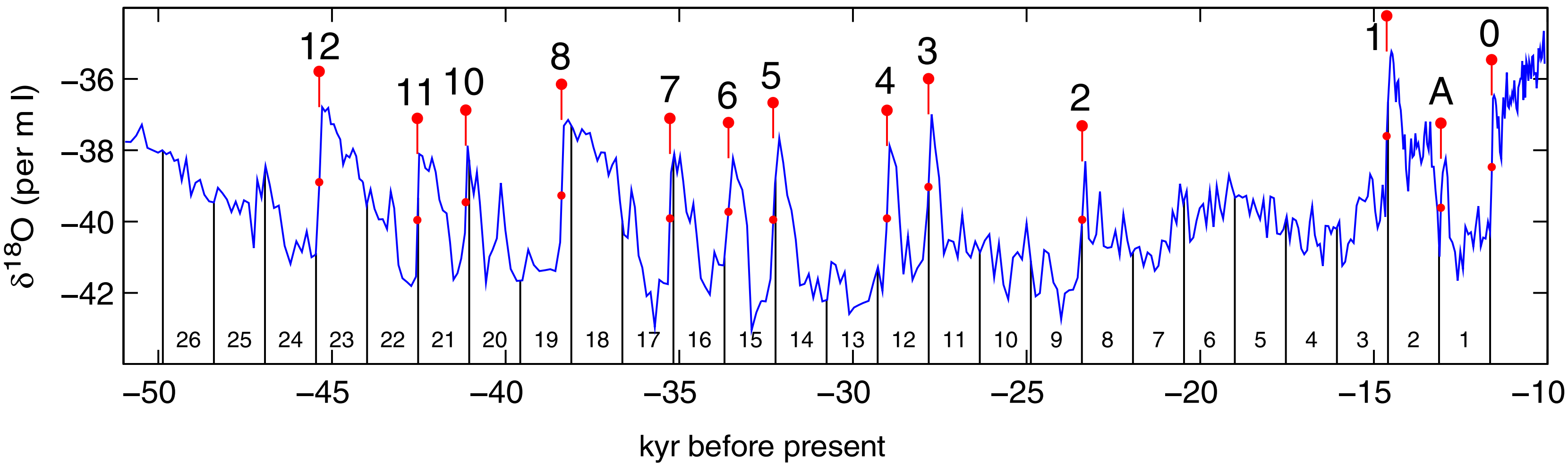
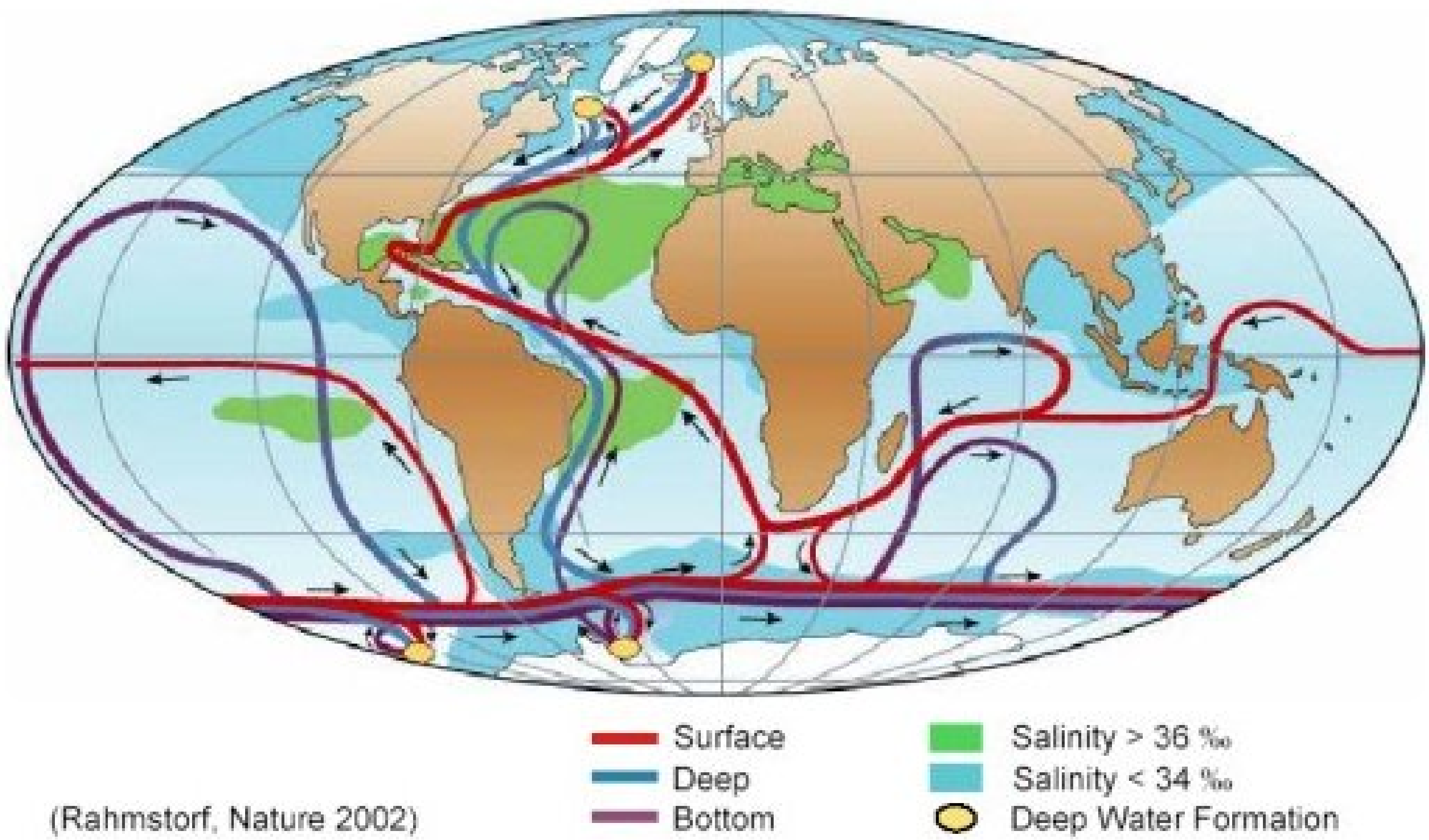
Example: Dansgaard-Oeschger events



- "Little Ice Ages"
- 1470-year cycle
- Some cycles are left out
- More time spent in cold (stadial) than warm (interstadial) state
- Fast transition to interstadial, slower return to stadial

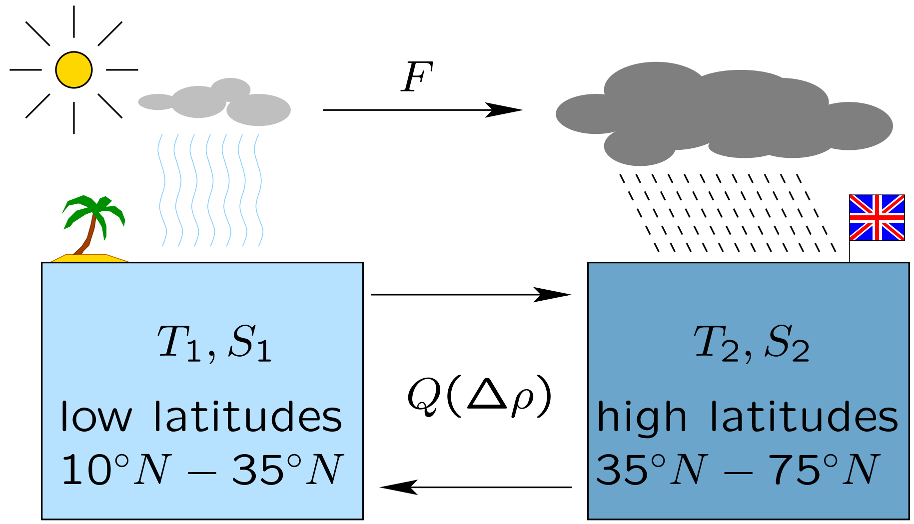
Thermohaline Circulation (THC)



- "Realistic" models (GCMs, EMICs): numerics
- Simple conceptual models (box models): analytical results

North-Atlantic THC: Stommel's Box Model ('61)

- T_i: temperatures
- S_i: salinities
- F: freshwater flux
- $Q(\Delta\rho)$: mass exchange
- $\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$



$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S$$

Model for Q (Cessi): $Q(\Delta\rho) = \frac{1}{\tau_d} + \frac{q}{V}\Delta\rho^2$.

Slow–fast systems

Separation of time scales: $\tau_r \ll \tau_d$
Scaling: $x = \Delta T/\theta$, $y = \Delta S \alpha_S/(\alpha_T \theta)$, $s = \tau_d t$, ...

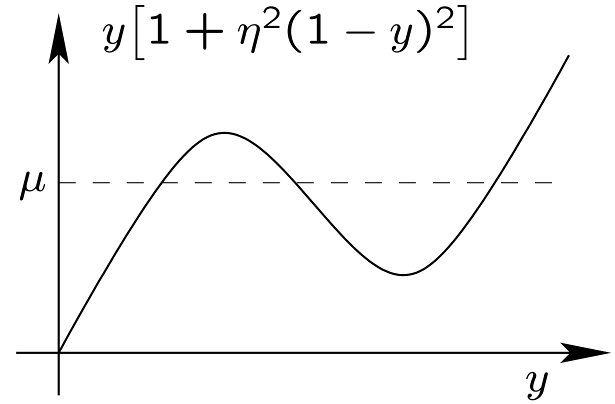
$$\begin{aligned} \varepsilon \dot{x} &= -(x-1) - \varepsilon x \left[1 + \eta^2 (x-y)^2\right] \\ \dot{y} &= \mu - y \left[1 + \eta^2 (x-y)^2\right] \end{aligned}$$

$\varepsilon = \tau_r/\tau_d \ll 1$

Slow manifold: $x = 1 + \mathcal{O}(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0$.

Reduced equation on slow manifold:

$$\dot{y} = \mu - y \left[1 + \eta^2 (1-y)^2 + \mathcal{O}(\varepsilon)\right]$$



One or two stable equilibria,
depending on μ (and η).

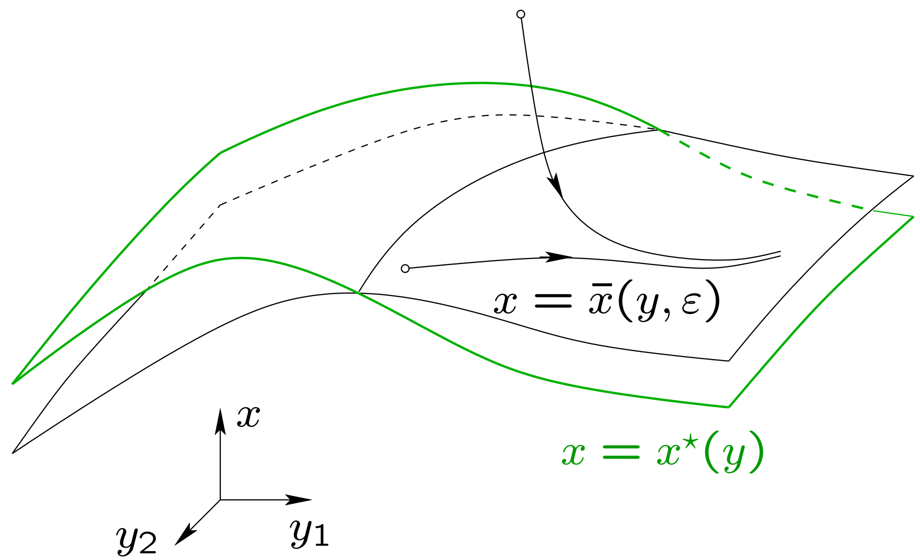
Geometric singular perturbation theory

$$\varepsilon\dot{x} = f(x,y) \qquad x \in \mathbb{R}^n, \text{ fast variable}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m, \text{ slow variable}$$

- Slow manifold: $f = 0$ for $x = x^\star(y)$
- Stability: Eigenvalues of $\partial_x f(x^\star(y), y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]
$\exists$ *adiabatic manifold* $x = \bar{x}(y,\varepsilon)$
s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



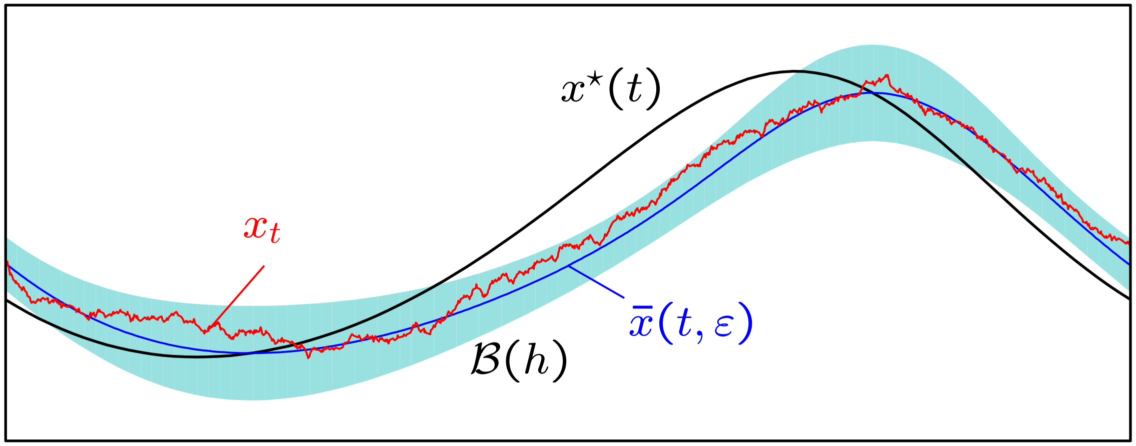
Stochastic perturbation: one-dimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Stable equil. branch: $f(x^\star(t), t) = 0$, $a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0$

Adiabatic solution: $\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$

$\mathcal{B}(h)$: strip of width $\simeq h/|a^\star(t)|$ around $\bar{x}(t, \varepsilon)$.



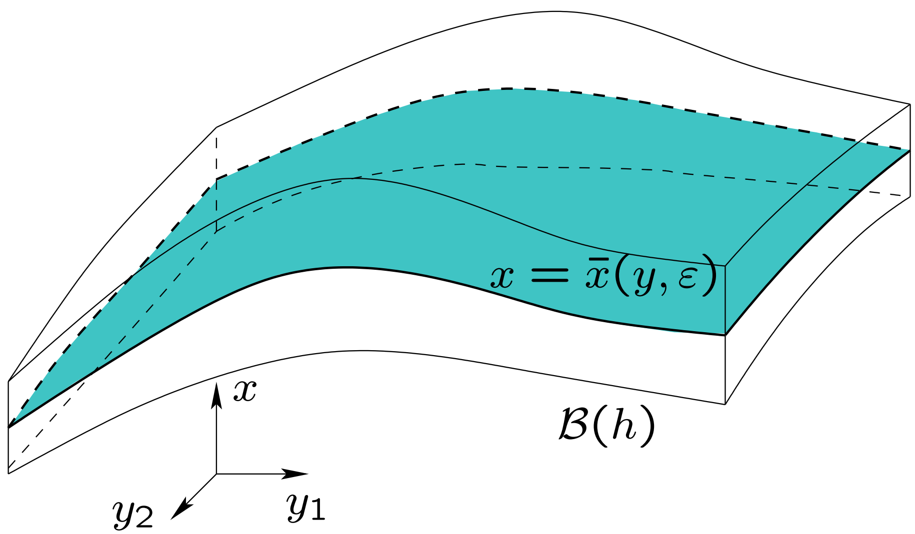
Theorem: [B. & G., PTRF 2002]

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \simeq \sqrt{\frac{2}{\pi}}\frac{1}{\varepsilon}\left|\int_0^t a^\star(s)\,\mathrm{d}s\right|\frac{h}{\sigma}\,\mathrm{e}^{-h^2/2\sigma^2}$$

Stochastic perturbation: n-dimensional case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\, G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

Stable slow manifold: $f(x^\star(y), y) = 0$, $A(y) = \partial_x f(x^\star(y), y)$ stable



$$\mathcal{B}(h) := \Big\{ (x, y) \colon \Big\langle \big[x - \bar{x}(y, \varepsilon)\big], X^\star(y)^{-1} \big[x - \bar{x}(y, \varepsilon)\big] \Big\rangle < h^2 \Big\}$$

$X^\star(y)$ solution of $A(y) X^\star + X^\star A(y)^{\mathsf{T}} + F(x^\star, y) F(x^\star, y)^{\mathsf{T}} = 0$

Stochastic perturbation: n-dimensional case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\, G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

Theorem [B. & G., JDE 2003]

- $\mathbb{P}\Big\{$leaving $\mathcal{B}(h)$ before time $t\Big\} \simeq C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$
  $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$.
- Projection on $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}W_t$$

  y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y} = g(\bar{x}(y,\varepsilon)y)$.

Example: Stommel's model with Ornstein-Uhlenbeck noise

$$\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}\Big[-(x_t-1) - \varepsilon x_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^1_t \\
\mathrm{d}\xi^1_t &= -\frac{\gamma_1}{\varepsilon}\xi^1_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W^1_t \\
\mathrm{d}y_t &= \Big[\mu - y_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^2_t \\
\mathrm{d}\xi^2_t &= -\gamma_2\xi^2_t\,\mathrm{d}t + \sigma'\,\mathrm{d}W^2_t
\end{aligned}$$

Cross section of $\mathcal{B}(h)$ is controlled by matrix

$$X^\star = \begin{pmatrix} \dfrac{1}{2(1+\gamma_1)} & \dfrac{1}{2(1+\gamma_1)} \\[2ex] \dfrac{1}{2(1+\gamma_1)} & \dfrac{1}{2\gamma_1} \end{pmatrix} + \mathcal{O}(\varepsilon)$$

Variance of $x-1 \simeq \sigma^2/(2(1+\gamma_1))$.

Reduced system for (y,ξ^2) is bistable (for appropriate μ).

Time-dependent freshwater flux

$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F(s) - Q(\Delta\rho)\Delta S$$

Possible causes:

1. Feedback: F or $\dot{F}$ depends on ΔT and ΔS.
2. External periodic forcing: Milankovich factors, . . .
3. Internal periodic forcing of ocean-atmosphere system.

1. $\Rightarrow$ relaxation oscillations, excitability
2., 3. $\Rightarrow$ stochastic resonance, hysteresis

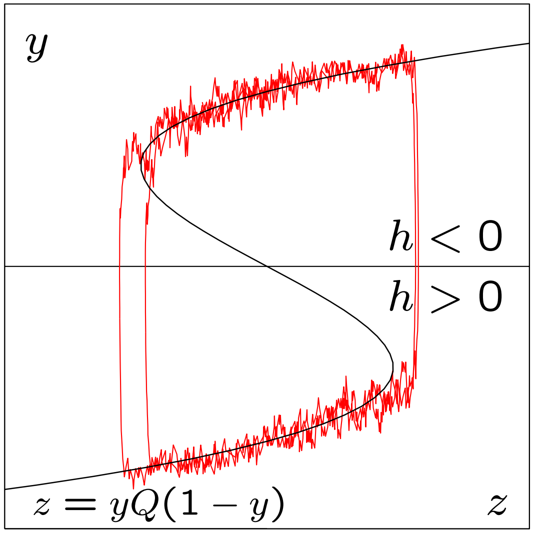
Case 1. Feedback

$$\mathrm{d}x_t = \frac{1}{\varepsilon_0}\Big[-(x_t-1) - \varepsilon_0 x_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon_0}}\,\mathrm{d}W_t^0$$
$$\mathrm{d}y_t = \Big[z_t - y_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_t^1$$
$$\mathrm{d}z_t = \varepsilon h(x_t,y_t,z_t)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma_2\,\mathrm{d}W_t^2$$

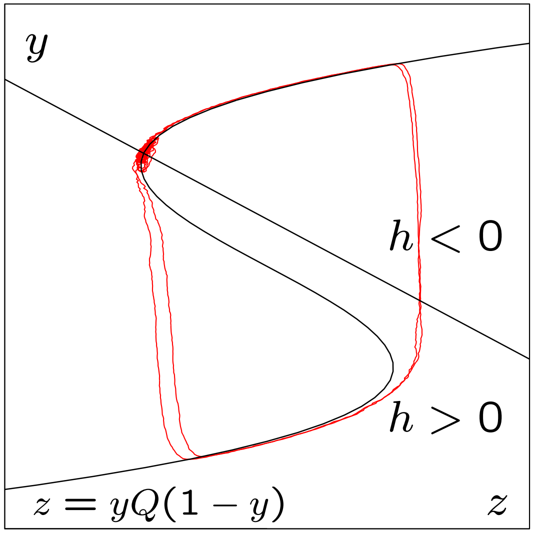
Reduced equation, $t \mapsto \varepsilon t$:

$$\mathrm{d}y_t = \frac{1}{\varepsilon}\Big[z_t - y_t Q(1-y_t)\Big]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^1$$
$$\mathrm{d}z_t = h(1,y_t,z_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^2$$

Relaxation oscillations

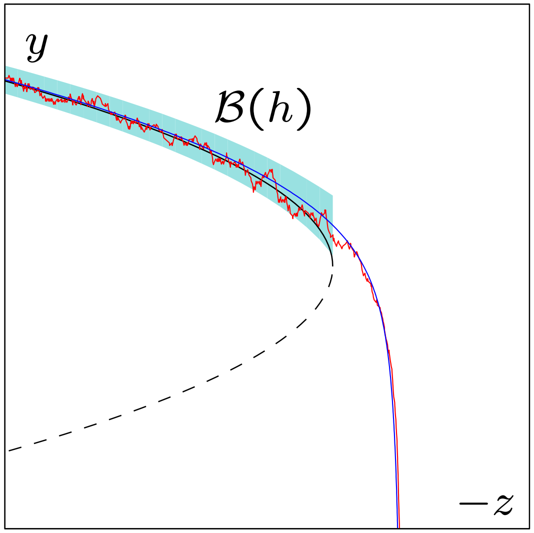


Excitability

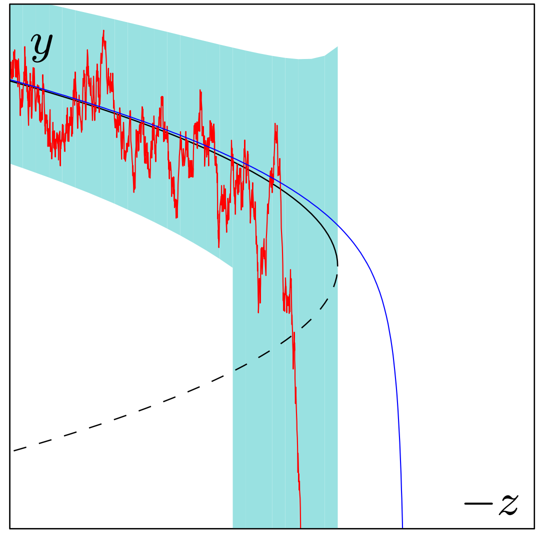


Saddle–node bifurcation

$\sigma \ll \sqrt{\varepsilon}$



$\sigma \gg \sqrt{\varepsilon}$



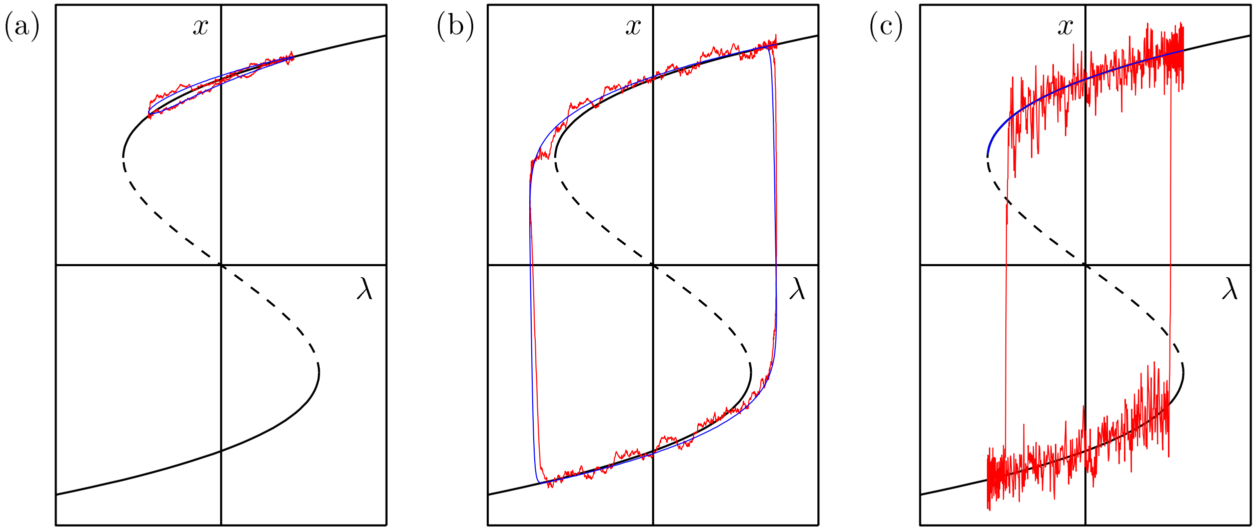
Deterministic case $\sigma = 0$: Solutions stay at distance $\varepsilon^{1/3}$ above bifurcation point until time $\varepsilon^{2/3}$ after bifurcation.

Theorem: [B. & G., Nonlinearity 2002]

1. If $\sigma \ll \sqrt{\varepsilon}$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation, maximal spreading $\sigma/\varepsilon^{1/6}$.
2. If $\sigma \gg \sqrt{\varepsilon}$: Paths likely to escape at time $\sigma^{4/3}$ *before* bifurcation.

Case 2. Periodic forcing

Assume $F(t)$ periodic (and centred w.r.t. bifurcation diagram).



Theorem: [B. & G., Nonlinearity 2002]

- Small amplitude, small noise: transitions unlikely during one cycle (However: see talk by Barbara Gentz)
- Large amplitude, small noise: hysteresis cycles
  Area = static area + $\mathcal{O}(\varepsilon^{2/3})$
- Large noise: stochastic resonance
  Area = static area − $\mathcal{O}(\sigma^{4/3})$

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Noise-Induced Phenomena
in Slow–Fast Dynamical Systems

A Sample-Paths Approach

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