Sample-path behaviour of noisy systems near critical transitions

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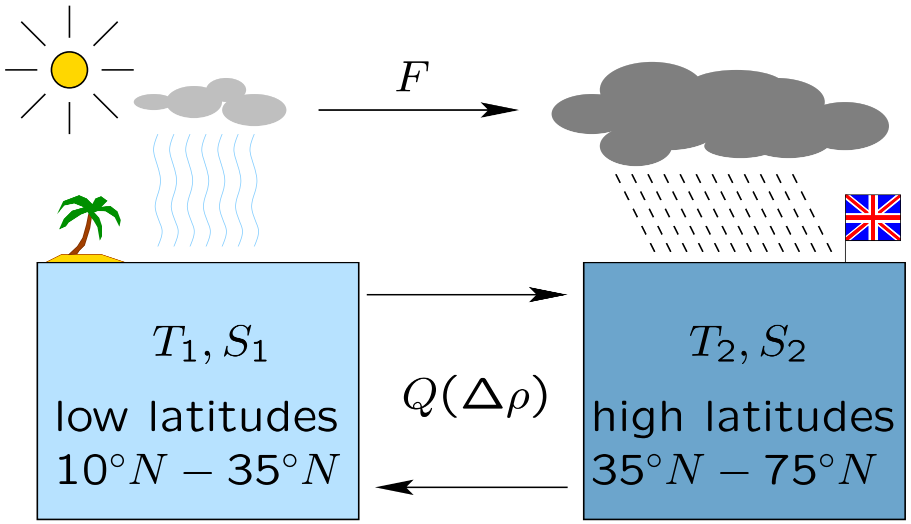
North-Atlantic THC: Stommel's Box Model ('61)

- T_i: temperatures
- S_i: salinities
- F: freshwater flux
- $Q(\Delta\rho)$: mass exchange
- $\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$



$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S$$

Model for Q [Cessi]: $Q(\Delta\rho) = \frac{1}{\tau_d} + \frac{q}{V}\Delta\rho^2$.

North-Atlantic THC: Stommel's Box Model ('61)

Scaling: $x = \Delta T/\theta$, $y = \Delta S \alpha_S/(\alpha_T \theta)$, $s = \tau_d t$
Separation of time scales: $\tau_r \ll \tau_d$, $\varepsilon = \tau_r/\tau_d \ll 1$

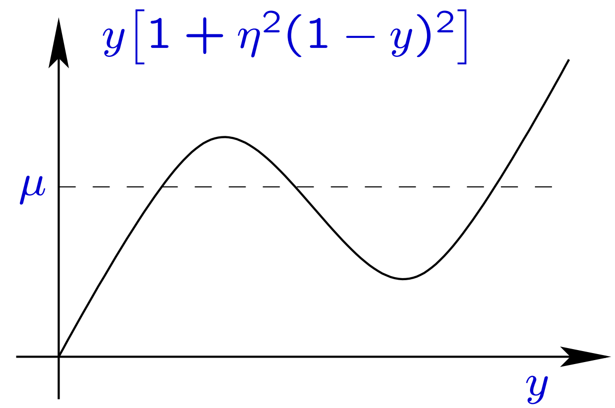
$$\varepsilon \dot{x} = -(x-1) - \varepsilon x \Big[1 + \eta^2 (x-y)^2\Big]$$
$$\dot{y} = \mu - y\Big[1 + \eta^2 (x-y)^2\Big]$$

Slow manifold [Fenichel '79]: $x = 1 + \mathcal{O}(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0$.

Reduced equation on slow manifold:

$$\dot{y} = \mu - y\Big[1 + \eta^2 (1-y)^2 + \mathcal{O}(\varepsilon)\Big]$$

One or two stable equilibria,
depending on μ (and η).



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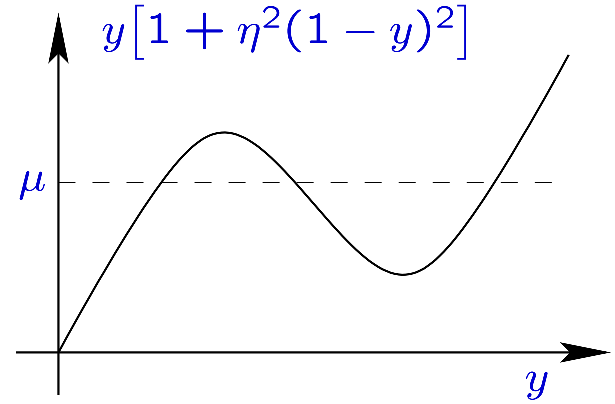

Questions:

- ▷ What happens when $\mu = \mu(\varepsilon' t)$ changes slowly?
- ▷ What is the effect of noise?

Deterministic slowly time-dependent systems

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f(x, \varepsilon s) \qquad x \in \mathbb{R}$$

On the slow time scale $t = \varepsilon s$:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)$$

- ▷ Equilibrium branch: $\{x = x^\star(t)\}$ where $f(x^\star(t), t) = 0$ for all t
- ▷ Stable if $a^*(t) = \partial_x f(x^\star(t), t) \leqslant -a_0 < 0$ for all t

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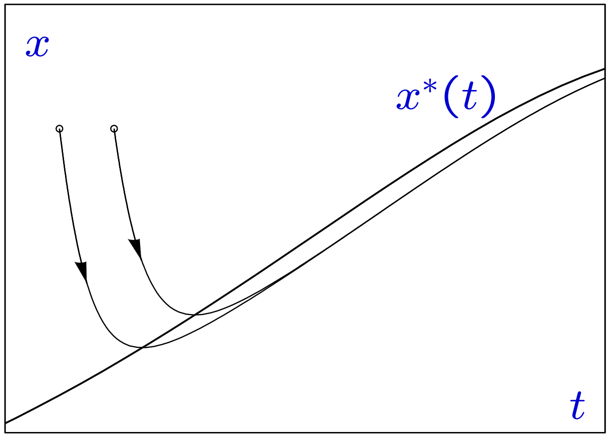
Then [Tikhonov '52, Fenichel '79]:

▷ There exists particular solution

$$\bar{x}(t) = x^*(t) + \mathcal{O}(\varepsilon)$$

▷ $\bar{x}$ attracts nearby orbits exp. fast
▷ $\bar{x}$ admits asymptotic series in ε



Theory generalises to higher-dimensional slow–fast systems

Noisy slowly time-dependent systems

$$\mathrm{d}x_s = f(x_s, \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

where W_s is a Brownian motion. On slow time scale

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Assume $x^\star(t)$ stable equilibrium branch

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Observation: Consider linearised equation at $\bar{x}(t)$:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\bar{a}(t)\xi_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

ξ_t: Gaussian process with variance $\sigma^2 v(t)$, s.t. $\varepsilon\dot{v} = 2\bar{a}(t)v + 1$

Asymptotically, $v(t) \simeq v^\star(t) = 1/2|\bar{a}(t)|$

$\mathcal{B}(h)$: confidence strip of width $\simeq h\sqrt{v^\star(t)}$ around $\bar{x}(t)$

Noisy slowly time-dependent systems

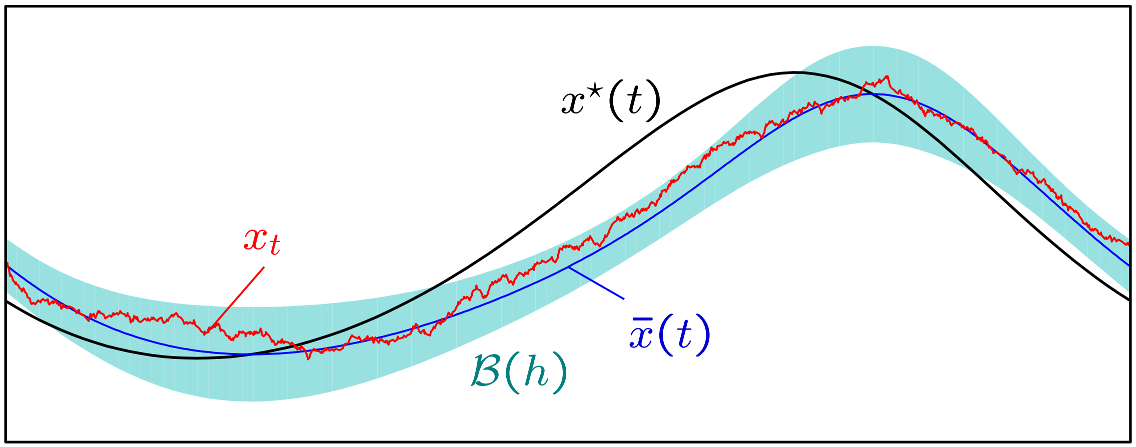
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Theorem: [B. & Gentz, PTRF 2002] For nonlinear equation

$$C(t,\varepsilon)\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

$$\kappa_\pm = 1 \mp \mathcal{O}(h)$$

$$C(t,\varepsilon) = \sqrt{\frac{2}{\pi}\frac{1}{\varepsilon}}\left|\int_0^t \bar{a}(s)\,\mathrm{d}s\right|\frac{h}{\sigma}\left[1 + \text{error of order } \mathrm{e}^{-h^2/\sigma^2}\, t/\varepsilon\right]$$



Deterministic fold or saddle–node bifurcation

Normal form:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = -x^2 - t$$

▷ Stable branch: $x_+^\star(t) = \sqrt{-t}$, $t \leqslant 0$

▷ Unstable branch: $x_-^\star(t) = -\sqrt{-t}$, $t \leqslant 0$

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- ▷ Unstable branch: $x^\star_-(t) = -\sqrt{-t}$, $t \leqslant 0$
- ▷ Outer region: there is a solution with asymptotic expansion

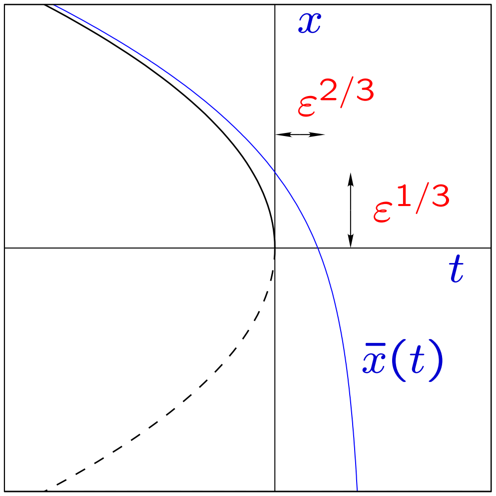
$$\bar{x}(t) = \sqrt{-t} + \frac{\varepsilon}{-4t} - \frac{5}{32}\frac{\varepsilon^2}{(-t)^{5/2}} + \dots$$

which becomes disordered at $t \asymp -\varepsilon^{2/3}$

- ▷ Inner region: $t = \mathcal{O}(\varepsilon^{2/3})$
  Scaling $x = \varepsilon^{1/3} u$, $t = \varepsilon^{2/3} s$

$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}s} = -u^2 - s$$

$\bar{x}(t)$ stays of order $\varepsilon^{1/3}$ up to time of order $\varepsilon^{2/3}$ then makes fast transition



Noisy fold or saddle–node bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[-x_t^2 - t\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Linearisation at $\bar{x}(t)$:

$$\bar{a}(t) \simeq \begin{cases} -\sqrt{t} & t \leqslant -\varepsilon^{2/3} \\ -\varepsilon^{1/3} & -\varepsilon^{2/3} \leqslant t \leqslant \varepsilon^{2/3} \end{cases}$$

Define as before confidence strip $\mathcal{B}(h)$ of width $\asymp h/\sqrt{|\bar{a}(t)|}$

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Define as before confidence strip $\mathcal{B}(h)$ of width $\asymp h/\sqrt{|\bar{a}(t)|}$

Theorem: [B. & Gentz, Nonlinearity 2002]

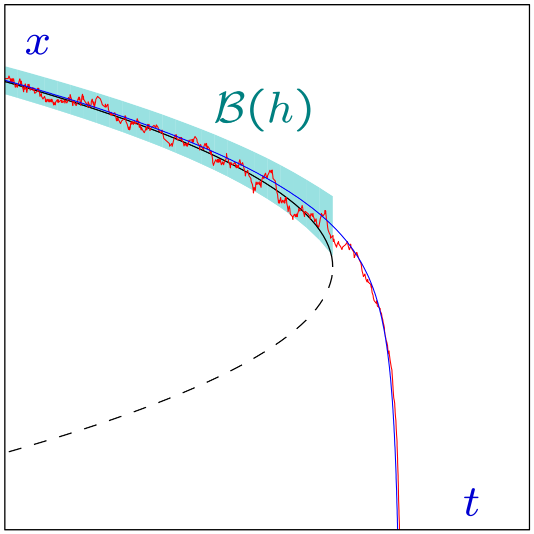
$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

as long as $h \leqslant |\bar{a}(t)|^{3/2}$ (linear part dominates nonlinear part)

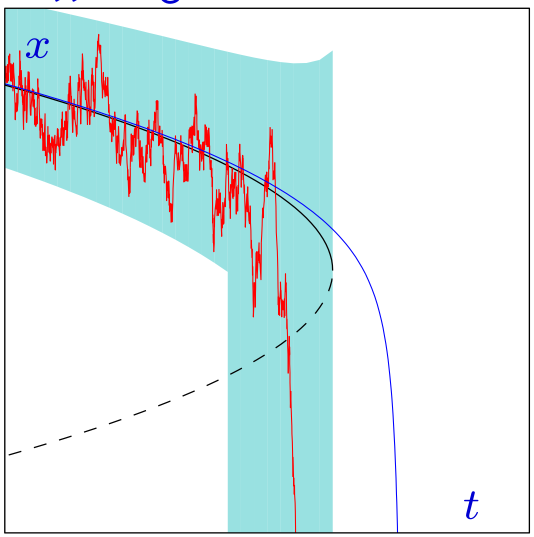
- ▷ Weak noise: $\sigma < \varepsilon^{1/2}$, apply thm at $t = \varepsilon^{2/3}$ with $h = \varepsilon^{1/2}$
  $\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C\,\mathrm{e}^{-\kappa\varepsilon/2\sigma^2}$
- ▷ Strong noise: $\sigma > \varepsilon^{1/2}$, thm applicable only for $t \ll -\sigma^{4/3}$

Noisy fold or saddle–node bifurcation

$\sigma \ll \sigma_c = \varepsilon^{1/2}$



$\sigma \gg \sigma_c = \varepsilon^{1/2}$



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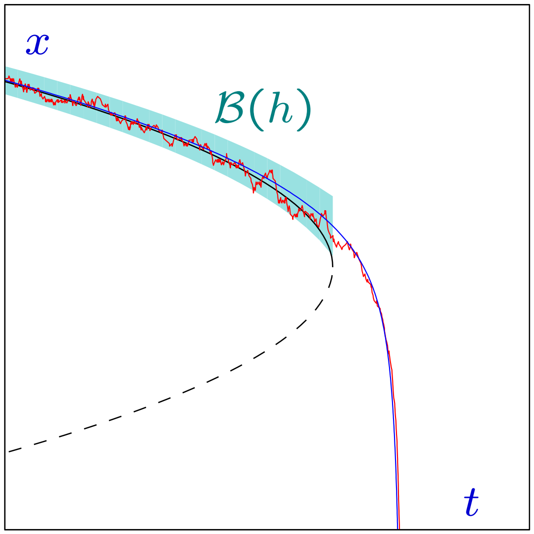

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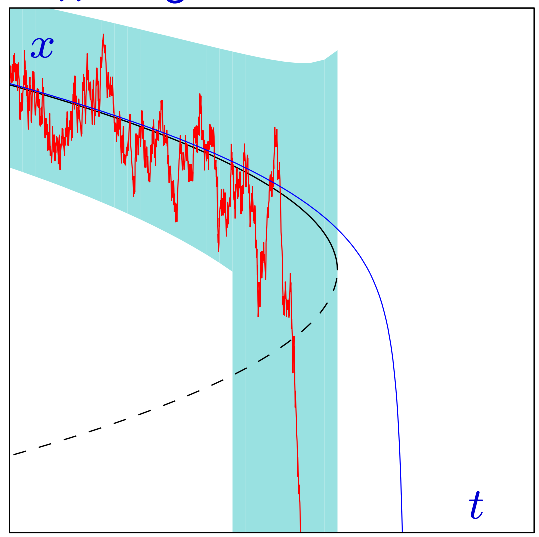


▷ Fluctuations grow like $\dfrac{\sigma}{|\bar{a}(t)|^{1/2}} \asymp \dfrac{\sigma}{\max\{(-t)^{1/4}, \varepsilon^{1/6}\}}$

▷ Early transitions occur if $\sigma \gg \varepsilon^{1/2}$ at time $\asymp -\sigma^{4/3}$

Theorem: [B. & Gentz, Nonlinearity 2002]

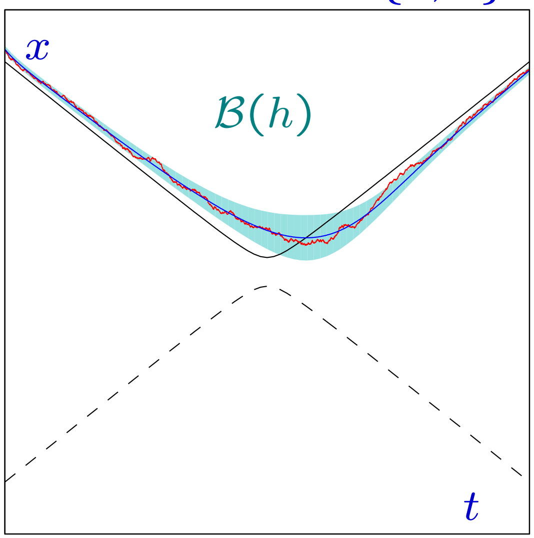
Probability of early transition $\geqslant 1 - \mathrm{e}^{-\kappa\sigma^2/\varepsilon|\log\sigma|}$

Proof uses idea of repeated attempts to escape

Avoided transcritical bifurcation

$$dx_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2\Big]\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t$$

$\sigma \ll \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$



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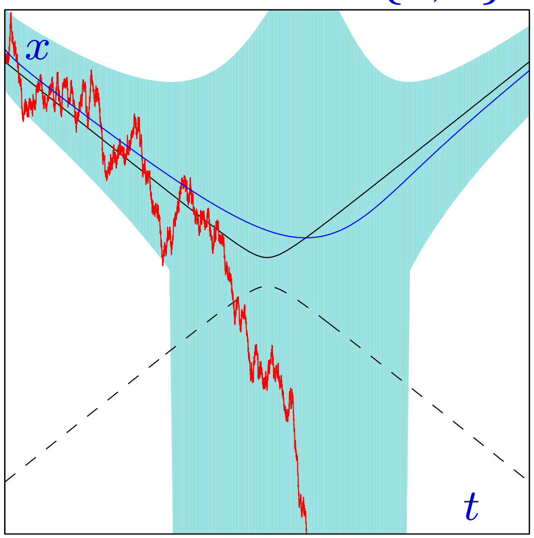


Minimal distance between branches $= \delta^{1/2}$
Det. case $\sigma = 0$: Solutions stay $\max\{\delta, \varepsilon\}^{1/2}$ above bif. point

Theorem: [B. & Gentz, Annals Applied Probab. 2002]

- ▷ Weak noise: $\sigma \ll \sigma_c$, transition probability $\leqslant e^{-c\sigma_c^2/\sigma^2}$
- ▷ Strong noise: $\sigma \gg \sigma_c$, Early transitions at $t \asymp -\sigma^{2/3}$, transition probability $\geqslant 1 - e^{-c\sigma^{4/3}/\varepsilon|\log \sigma|}$

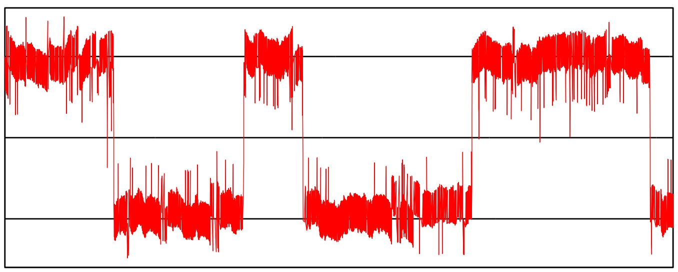
Stochastic resonance

$$\mathrm{d}x_s = [-x^3 + x + A\cos\varepsilon s]\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

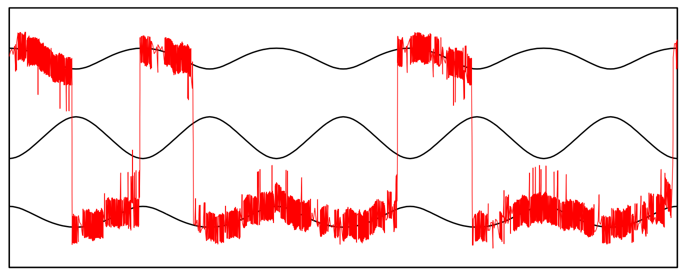
▷ deterministically bistable climate [Croll, Milankovitch]

▷ random perturbations due to weather [Benzi/Sutera/Vulpiani, Nicolis/Nicolis]

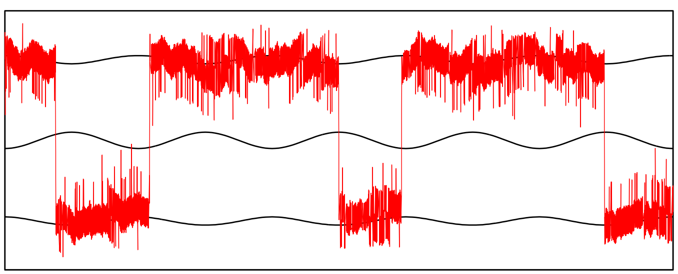
Sample paths $\{x_s\}_s$ for $\varepsilon = 0.001$:



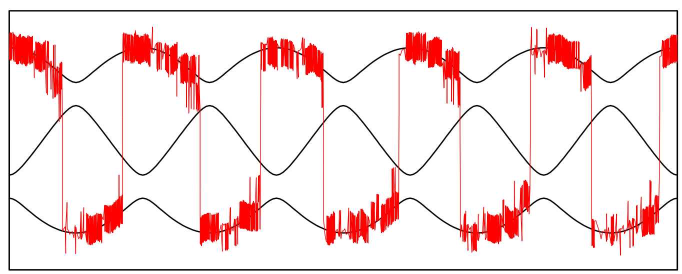
$A = 0$, $\sigma = 0.3$



$A = 0.24$, $\sigma = 0.2$



$A = 0.1$, $\sigma = 0.27$

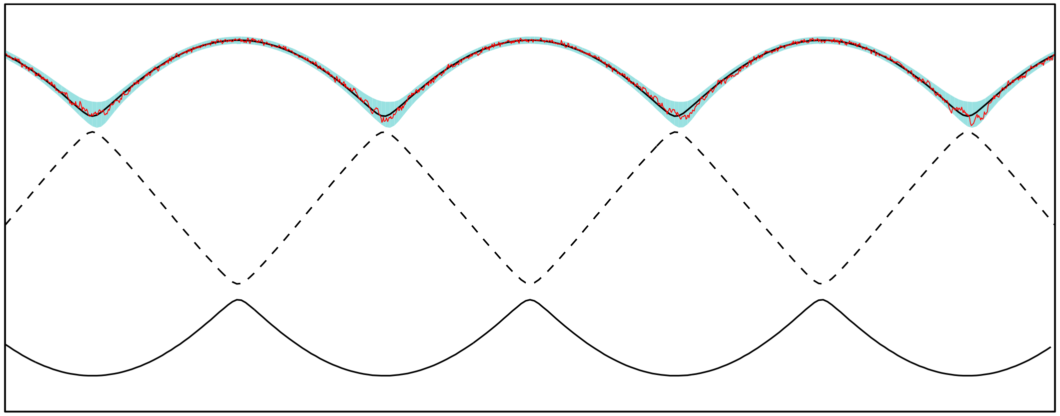


$A = 0.35$, $\sigma = 0.2$

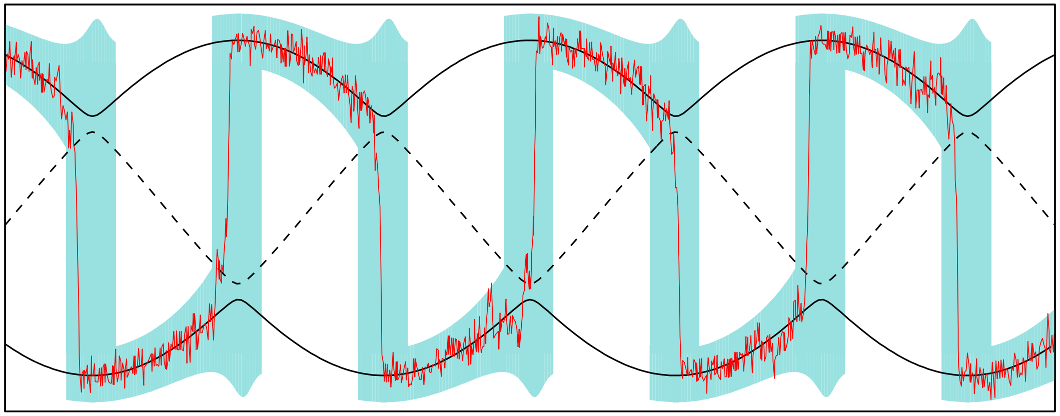
Stochastic resonance

Critical noise intensity: $\sigma_{\mathrm{c}} = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_{\mathrm{c}} - A$

$\sigma \ll \sigma_{\mathrm{c}}$: transitions unlikely



$\sigma \gg \sigma_{\mathrm{c}}$: synchronisation



Excitability

FitzHugh–Nagumo equations

$$\varepsilon \dot{x} = x - x^3 + y$$
$$\dot{y} = a - x$$

- ▷ $x \propto$ membrane potential of neuron
- ▷ $y \propto$ proportion of open ion channels (recovery variable)
- ▷ $\varepsilon \ll 1 \Rightarrow$ fast–slow system

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Stationary point $P = (a, a^3 - a)$

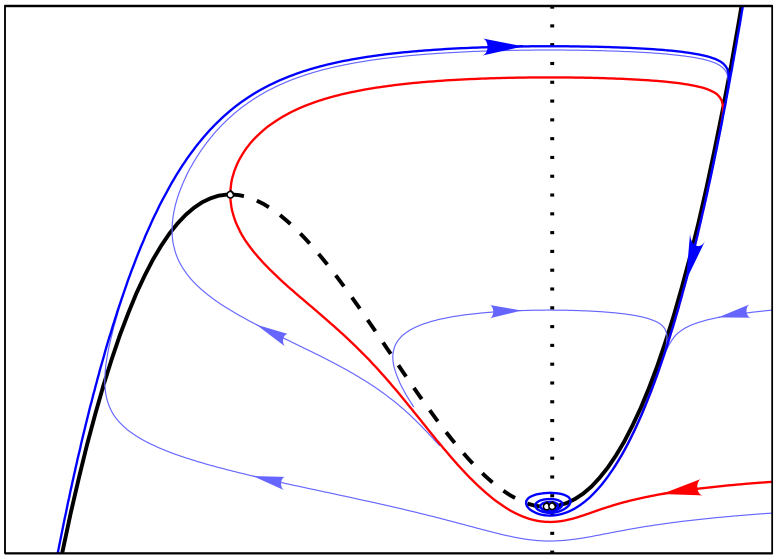
Linearisation has eigenvalues $\frac{-\delta\pm\sqrt{\delta^2-\varepsilon}}{\varepsilon}$ where $\delta = \frac{3a^2-1}{2}$

- ▷ $\delta > 0$: stable node ($\delta > \sqrt{\varepsilon}$) or focus ($0 < \delta < \sqrt{\varepsilon}$)
- ▷ $\delta = 0$: singular Hopf bifurcation [Erneux & Mandel '86]
- ▷ $\delta < 0$: unstable focus ($-\sqrt{\varepsilon} < \delta < 0$) or node ($\delta < -\sqrt{\varepsilon}$)

Excitability

$\delta > 0$:

- ▷ P is asymptotically stable
- ▷ the system is excitable
- ▷ one can define a separatrix



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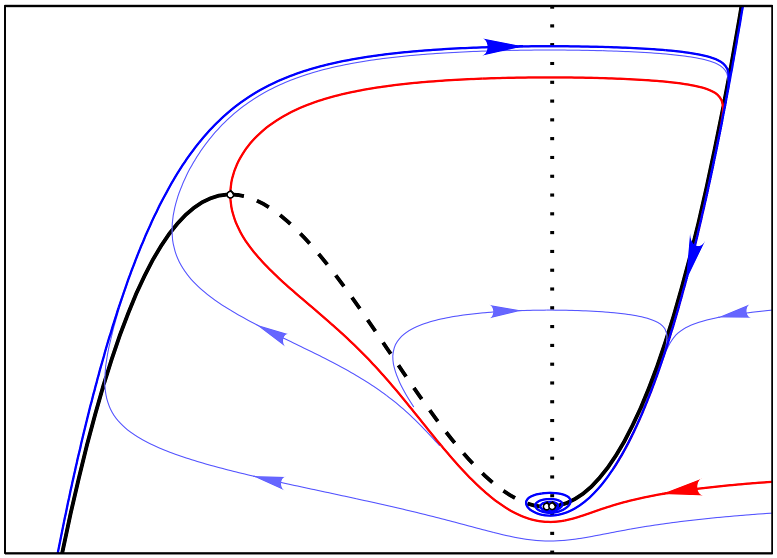
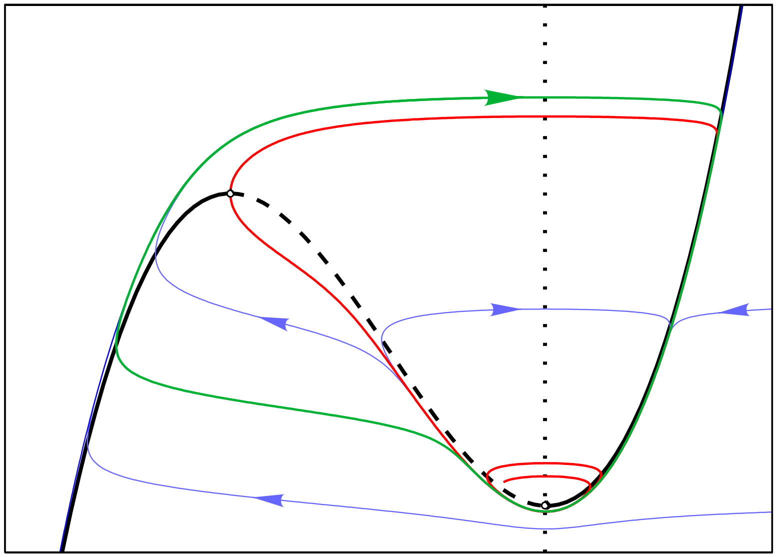


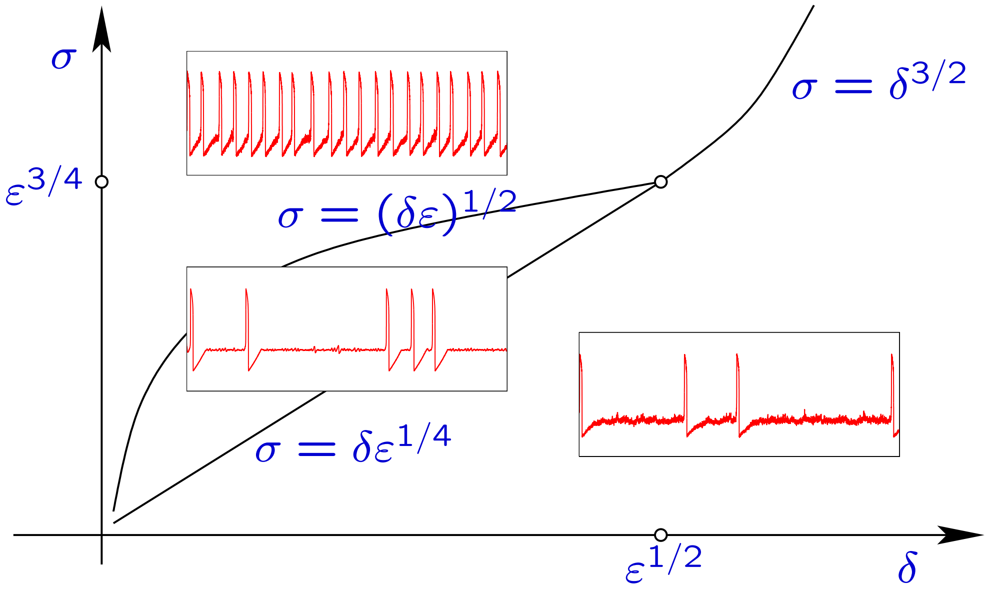
$\delta < 0$:

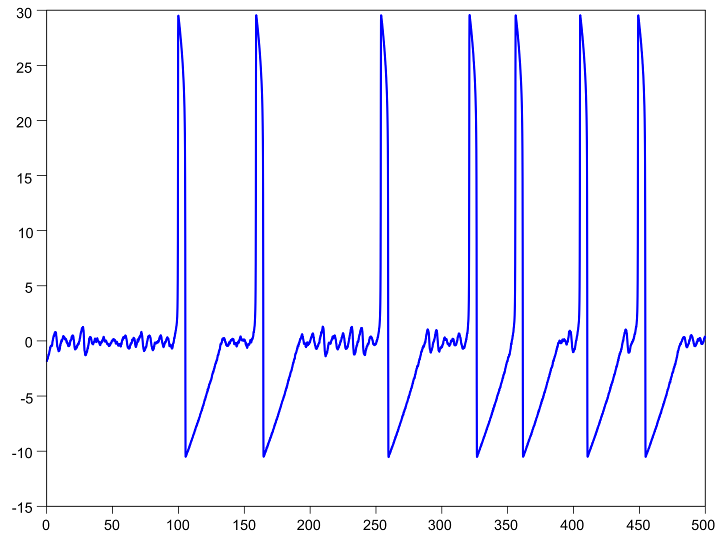
- ▷ P is unstable
- ▷ $\exists$ asympt. stable periodic orbit
- ▷ sensitive dependence on δ: canard (duck) phenomenon [Callot, Diener, Diener '78, Benoît '81, . . .]



Excitability

Proposed "phase diagram" [Muratov & Vanden Eijnden '08]





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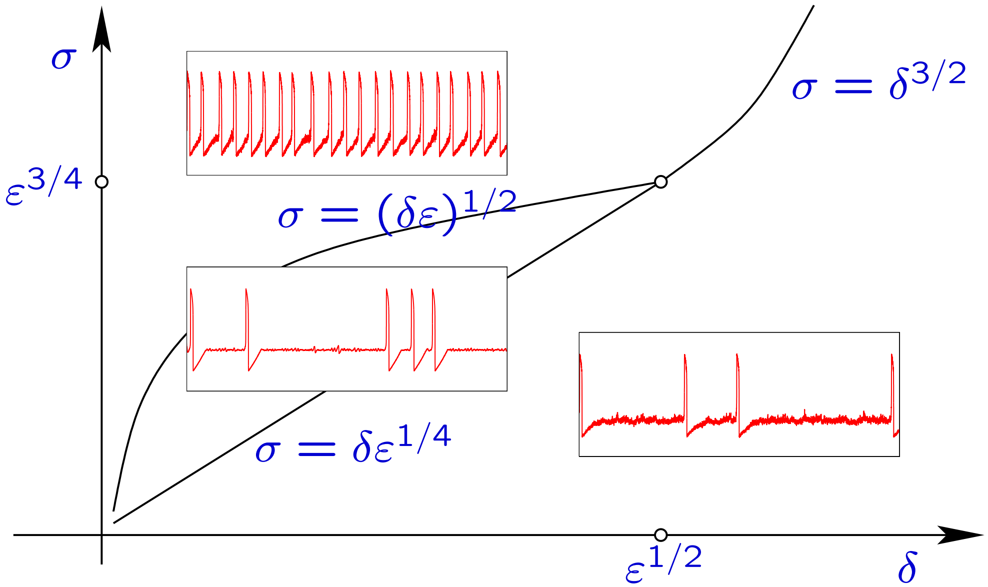
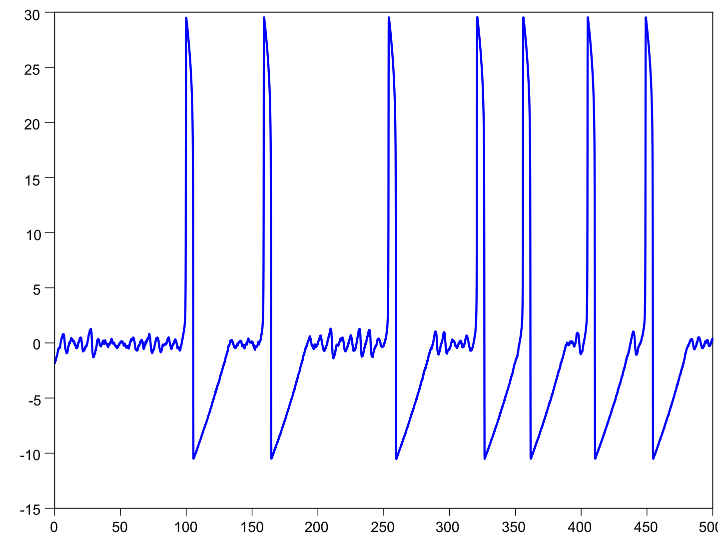




Theorem: [B. & Landon, 2011]

- ▷ # N of small oscillations between spikes asympt. geometric
- ▷ Weak noise: If $\delta < \sqrt{\varepsilon}$, $\sigma^2 \leqslant (\delta\varepsilon^{1/4})^2/\log(\sqrt{\varepsilon}/\delta)$

$$\frac{1}{\mathbb{E}[N]} \leqslant C\,\mathrm{e}^{-\kappa(\delta\varepsilon^{1/4})^2/\sigma^2}$$

- ▷ Intermediate noise: $\frac{1}{\mathbb{E}[N]} \simeq \Phi\left(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\right)$

Summary

When approaching a critical transition

- ▷ Fluctuations increase in a way caracteristic for the bifurcation
- ▷ Early transitions occur above a threshold noise intensity

Well understood:

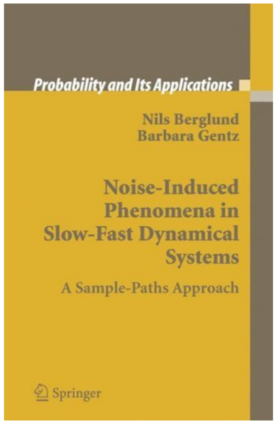
- ▷ Codimension-1 bifurcations (fold, (avoided) transcritical, pitchfork, Hopf)
- ▷ Higher codimension: case studies (cf. Kuehn)
- ▷ Extension to infinite dim may be possible for discrete spectrum

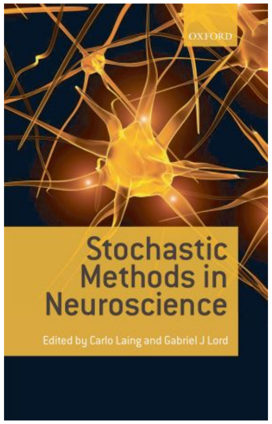
Essentially still open:

- ▷ Other types of noise (except Ornstein–Uhlenbeck)
- ▷ Equations with delay
- ▷ Infinite dimensions with continuous spectrum

Further reading

- N. B. & Barbara Gentz, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)
- ________, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Applied Probab. **12**, 1419-1470 (2002)
- ________, *The effect of additive noise on dynamical hysteresis*, Nonlinearity **15**, 605–632 (2002)
- ________, *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)
- ________, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)
- ________, *Stochastic dynamic bifurcations and excitability*, in C. Laing and G. Lord, (Eds.), *Stochastic methods in Neuroscience*, p. 65-93, Oxford University Press (2009)





- N.B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, `arXiv:1105.1278`, submitted (2011)