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Classical Billiards in a Magnetic Field and a Potential*

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Abstract

We study billiards in plane domains, with a perpendicular magnetic field and a potential. We give some results on periodic orbits, KAM tori and adiabatic invariants. We also prove the existence of bound states in a related scattering problem.

Classical billiards are popular models for various physical systems, in fields ranging from mechanics of systems with impacts and ergodic theory [1] to semiclassical methods in quantum chaos. In particular, billiards in a magnetic field appear to be relevant for the study of transport properties in mesoscopic systems, diamagnetism and the quantum Hall effect (see for instance [2]).

In this work, we consider the classical motion of a charged particle in a plane domain, with a perpendicular magnetic field of intensity B and a potential $V(\mathbf{x})$. We first discuss a method for proving existence of periodic and quasiperiodic orbits. Then we give some results on KAM tori and adiabatic invariants for billiards in a magnetic field. Finally, we consider the scattering on a hard disc in crossed electromagnetic fields, where we prove the existence of bound states. Details on the present results can be found in [3, 4, 5].

1 Billiards and Periodic Orbits

Bouncing Map: We consider the classical motion of a particle in a connected domain Q (which may be unbounded or not simply connected). The boundary ∂Q is parametrized by its arclength, $\mathbf{x}(s) = (X(s), Y(s))$, with $X'(s)^2 + Y'(s)^2 = 1$; the unit tangent vector and the curvature are given respectively by $\mathbf{t}(s) = (X'(s), Y'(s))$ and $\kappa(s) = X'(s)Y''(s) - X''(s)Y'(s)$.

Inside Q, the billiard flow is defined by the Lagrangian

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) - V(\mathbf{x}), \tag{1}$$

where $\mathbf{A}(\mathbf{x}) = \frac{1}{2}B(-y, x)$ is the vector potential in symmetric gauge (we will adopt the sign convention $qB < 0$).

The dynamics is defined in the following way. Assume that the billiard particle starts on the boundary at $\mathbf{x}(s_0)$, with a velocity $\dot{\mathbf{x}}_0$ making an angle θ_0 with $\mathbf{t}(s_0)$. It then evolves in Q according to the Lagrange equations. If the particle returns to the boundary, at a point $\mathbf{x}(s_1)$, with a velocity $\dot{\mathbf{x}}_1$ making an angle $-\theta_1$ with $\mathbf{t}(s_1)$, it is reflected elastically, meaning that it leaves the boundary again with an angle θ_1 (Fig. 1). The i–th collision may thus be parametrized by the coordinates (s_i, θ_i), or, alternatively, by the *Birkhoff variables* $(s_i, u_i) \equiv z_i$, where $u_i = \dot{\mathbf{x}}_i \cdot \mathbf{t}(s_i) = |\dot{\mathbf{x}}_i| \cos\theta_i$ denotes the tangent velocity.

As long as the particle returns to the boundary, we may thus describe the dynamics by the *bouncing map* $T : z_0 \mapsto z_1$.

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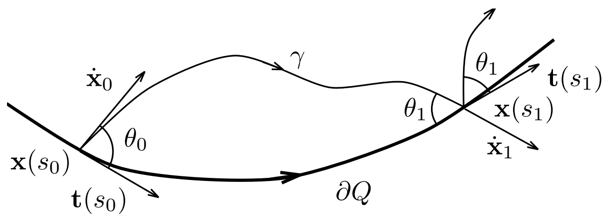

Figure 1: The abscissa s_0 of the starting point and the angle θ_0 uniquely define the trajectory γ, and, if the boundary ∂Q is reached again, the coordinates (s_1, θ_1) of the next collision.

Generating function: A very useful tool to study the bouncing map is its *generating function* $G(s_0, s_1)$ [6], which satisfies

$$\frac{\partial G}{\partial s_0} = -u_0, \qquad \frac{\partial G}{\partial s_1} = u_1. \tag{2}$$

It can be constructed in the following way: let γ be a trajectory connecting the points $\mathbf{x}(s_0)$ and $\mathbf{x}(s_1)$, and $F(s_0, s_1) = \int_\gamma \mathbf{p} \cdot \mathrm{d}\mathbf{x}$ the action along γ, where $\mathbf{p} = \partial_{\dot{\mathbf{x}}}\mathcal{L} = m\dot{\mathbf{x}} + q\mathbf{A}(\mathbf{x})$ is the momentum. We know from analytical mechanics that for infinitesimal variations of the end points $\mathrm{d}\mathbf{x}_i = \mathbf{t}(s_i)\mathrm{d}s_i$, the change of action is $\mathrm{d}F = -\mathbf{p}_0 \cdot \mathrm{d}\mathbf{x}_0 + \mathbf{p}_1 \cdot \mathrm{d}\mathbf{x}_1$. It is then easy to check that

$$G(s_0, s_1) = \frac{1}{m}F(s_0, s_1) + \frac{qB}{2m}\int_{s_0}^{s_1} Y(s)X'(s) - X(s)Y'(s)\,\mathrm{d}s \tag{3}$$

satisfies the relations (2).

Here we have assumed that there is a unique trajectory connecting the two points on the boundary. In fact, there may be several or no solution of the Lagrange equations for given end points, so that the generating function may be multiply defined on some domain, and not exist on another one. This gives rise to new complications, but the main arguments presented in the following can be transposed to this more difficult situation.

Periodic orbits: The relation (2) is useful to compute periodic orbits. If $s_0, s_1, \dots s_{n-1}$ is a sequence of arclengths on the boundary, we define the *n–point generating function*

$$G^{(n)}(s_0, s_1, \dots s_{n-1}) = G(s_0, s_1) + G(s_1, s_2) + \cdots + G(s_{n-1}, s_0). \tag{4}$$

The law of specular reflection implies that there is a periodic orbit of period n, hitting the boundary at $\mathbf{x}(s_0), \dots, \mathbf{x}(s_{n-1})$, if and only if $\partial G^{(n)}/\partial s_i = 0$ for each i (assuming $G^{(n)}$ is defined and sufficiently differentiable); in other words, periodic orbits correspond to stationary points of $G^{(n)}$.

The linear stability of the orbit is determined by noting that

$$\begin{aligned} \mathrm{d}u_0 &= -G_{20}\mathrm{d}s_0 - G_{11}\mathrm{d}s_1 \\ \mathrm{d}u_1 &= G_{11}\mathrm{d}s_0 + G_{02}\mathrm{d}s_1 \end{aligned} \qquad G_{jk} = \frac{\partial^{j+k}G}{\partial s_0^j\,\partial s_1^k}(s_0, s_1), \tag{5}$$

implies $\mathrm{d}z_1 = T'(s_0, s_1)\mathrm{d}z_0$, where

$$T'(s_0, s_1) = -\frac{1}{G_{11}}\begin{pmatrix} G_{20} & 1 \\ G_{20}G_{02} - G_{11}^2 & G_{02} \end{pmatrix} \tag{6}$$

is the Jacobian matrix of the bouncing map. Since T' has unit determinant, T is area–preserving. For n iterations, $\mathrm{d}z_n = S_n\mathrm{d}z_0$, where $S_n(s_0, \dots, s_{n-1}) = T'(s_0, s_{n-1})T'(s_{n-1}, s_{n-2}) \cdots T'(s_1, s_0)$. The eigenvalues of S_n are $\lambda_\pm = t \pm \sqrt{t^2 - 1}$, where $t = \frac{1}{2}\mathrm{Tr}S_n$. Hence, the orbit is hyperbolic if $|t| > 1$, elliptic if $|t| < 1$ (under certain conditions, t may be related to second derivatives of $G^{(n)}$ [7]).

The center manifold theorem implies that hyperbolic orbits are unstable, even when nonlinear terms are taken into account. Elliptic orbits are generically nonlinearly stable (in the sense of Lyapunov), as a consequence of the KAM theorem. A standard result is

Theorem 1 *Let T be measure–preserving and $\mathcal{C}^5$ in a neighborhood of the periodic orbit. Assume that the eigenvalues are such that $(\lambda_\pm)^3 \neq 1$ and $(\lambda_\pm)^4 \neq 1$. There exists C, depending only on the second and third derivatives of T along the orbit (and which can be explicitly computed), such that if the non–degeneracy condition $C \neq 0$ is satisfied, any point of the orbit has a neighborhood which is invariant under the map T^n.*

This result is proved by applying Moser's theorem [8] to the Birkhoff normal form of T^n.

2 Billiards in a Magnetic Field

The particular case when there is only the magnetic field ($V(\mathbf{x}) = 0$) was first considered by Robnik and Berry [9]. For a given energy E, the trajectories are arcs of Larmor radius $\mu = \sqrt{2mE}/|qB|$. The generating function and the Jacobian matrix can be explicitly expressed in terms of geometric properties of the boundary [3].

We first consider the billiard in a convex domain, i.e., we assume that the radius of curvature $\rho(s) = 1/\kappa(s)$ is smooth and bounded by positive constants $\rho_{\min}$ and $\rho_{\max}$. An important class of orbits are the "whispering gallery modes": they correspond physically to quasiperiodic trajectories skipping along the boundary; in phase space, these solutions live on invariant curves which are close to $\theta = 0, \pi$. In the zero field case, existence of such orbits was proved by Lazutkin [10]. As remarked in [9], when a magnetic field is added, the dynamics near the boundary depend on the relative value of μ, $\rho_{\min}$ and $\rho_{\max}$. This is confirmed by the following result:

Theorem 2 [3] *Assume the boundary is $\mathcal{C}^5$ and satisfies $0 < \rho_{\min} \leq \rho(s) \leq \rho_{\max} \leq \infty$. There exists a Cantor set of invariant curves with positive measure in the three following situations:*

1. *For $0 < \mu < \infty$, near $\theta = \pi$. It corresponds to backward skipping trajectories, which are curved towards the boundary.*
2. *For $\mu > \rho_{\max}$, near $\theta = 0$. It corresponds to forward skipping trajectories which are curved away from the boundary.*
3. *For $\mu < \rho_{\min}$, near $\theta = 0$. It corresponds to backward skipping trajectories starting with a forward glancing velocity, and performing almost complete Larmor circles.*

The proof relies on a perturbative expression of the bouncing map for small $\sin\theta$, which can by analyzed by Moser's theorem [8]. The difference between the regimes $\theta \sim 0$ and $\theta \sim \pi$ is due to the symmetry breaking effect of the magnetic field. When $\rho_{\min} < \mu < \rho_{\max}$, invariant curves near $\theta = 0$ are absent due to discontinuities by tangency. Note that in contrast with a theorem by Mather [11], in a magnetic field invariant curves still exist when the curvature is allowed to vanish. In this respect, the magnetic field has a stabilyzing effect.

This result can be extended to more general billiard domains. Consider for instance the billiard *outside* a given convex curve. One can try to construct a trajectory of the outside billiard by completing every arc of an inside trajectory to a full circle (Fig. 2). There is a one–to–one correspondence between inside and outside orbits if the following property is satisfied:

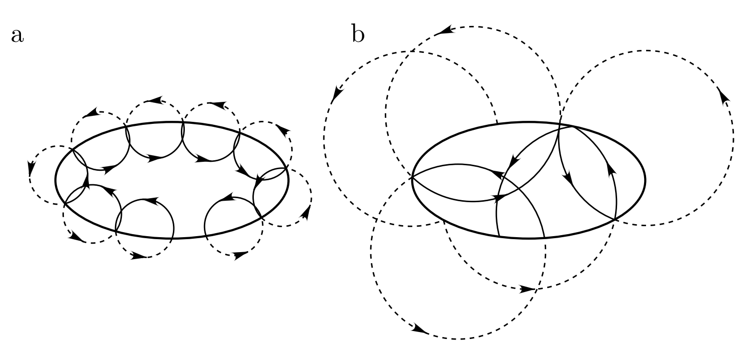

Figure 2: Inside–outside duality. In (a), the μ–intersection property is satisfied, there is a one–to–one correspondence between inside and outside trajectories. If this property is not satisfied, the duality may be destroyed (b).

Definition 1 *A closed plane $\mathcal{C}^2$ curve is said to have the μ–*intersection property *for some $\mu > 0$ if any circle of radius μ intersects it at most twice.*

Lemma 1 [3] *A closed plane convex curve with extremal radii of curvature $\rho_{\min}$ and $\rho_{\max}$ satisfies the μ–intersection property if $\mu < \rho_{\min}$ or $\mu > \rho_{\max}$.*

We conlude that the inside and outside billiards are equivalent in low or high magnetic field. In fact, it is possible to show that the inside–outside duality remains true for backward skipping trajectories, even for intermediate magnetic field. Thus, Theorem 2 is valid for outside as well as for inside billiards, implying that this large class of billiards is not ergodic.

More generally, one can consider domains which are not convex, but whose boundary has a bounded curvature: $|\kappa(s)| \leq 1/\rho_{\min}$. It is possible to show that point 1. of Theorem 2 remains true if $\mu < \rho_{\min}$, which is once again a manifestation of the stabilyzing effect of the magnetic field. In this case, it is of particular interest to consider the strong magnetic field limit.

Proposition 1 [3] *Assume the boundary is $\mathcal{C}^k$, $k \geq 3$, and has a bounded curvature. For sufficiently small μ, the bouncing map is $\mathcal{C}^{k-1}$ and takes the form*

$$\begin{aligned} s_1 &= s_0 - 2\mu \sin\theta_0 + \mu^2 \sin\theta_0 \, a(s_0, \theta_0, \mu) \pmod{|\partial Q|}, \\ \theta_1 &= \theta_0 + \mu^2 \sin^2\theta_0 \, b(s_0, \theta_0, \mu). \end{aligned} \tag{7}$$

The functions $a \in \mathcal{C}^{k-2}$ and $b \in \mathcal{C}^{k-3}$ are uniformly bounded for $s \in \mathbb{R}, 0 \leq \theta \leq \pi$, $|\partial Q|$-periodic in s, and admit expansions in μ which can be explicitly computed.

The bouncing map (7) has the structure of a perturbed integrable map, where the factors $\sin\theta_0$ ensure that the boundaries $\theta = 0, \pi$ are fixed. One can again study invariant curves, of the form $I(s, \theta) = \text{const}$, using Moser's theorem, although the analysis is complicated by the fact that the frequency $\Omega(\theta) = -2\mu \sin\theta$ is multiplied by the small parameter μ and is not monotonic.

An alternative approach to KAM theory is to construct *adiabatic invariants* $J(s, \theta)$, such that $J(s_1, \theta_1) - J(s_0, \theta_0)$ is as small as possible.

Theorem 3 [3] *If the boundary ∂Q is $\mathcal{C}^k$, $k \geq 3$, and has bounded curvature, there exists a function $J(s, \theta)$ such that $J(s_1, \theta_1) = J(s_0, \theta_0) + \mathcal{O}(\mu^{k+1})$. If ∂Q is analytic, there exists a function $J(s, \theta)$ such that $J(s_1, \theta_1) = J(s_0, \theta_0) + \mathcal{O}(\mathrm{e}^{-1/C|\mu|})$.*

In fact, Theorem 3 is true for a large class of maps, including (7). In the case of a billiard,

$$J(s, \theta; \mu) = \theta + \mu \sin\theta \left[\frac{1}{3}\kappa(s) + \frac{2}{9}\mu \cos\theta \kappa(s)^2 + \mathcal{O}(\mu^2) \right]. \tag{8}$$

If the boundary is analytic, Theorem 3 implies that for *any* initial condition (s_0, θ_0), (s_n, θ_n) remains at a distance of order μ from the level curve $J(s, \theta) = J(s_0, \theta_0)$, during a time of order e^B.

3 A Scattering System

We now consider the case where the potential is given by a uniform electric field, $V(\mathbf{x}) = -q\mathcal{E} \cdot \mathbf{x}$, $\mathcal{E} = (0, |\mathcal{E}|)$. One can introduce dimensionless variables such that the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{x}}^2 + \frac{1}{2}(y\dot{x} - x\dot{y}) - \varepsilon y, \tag{9}$$

where $\varepsilon = \left|\mathcal{E}m/qB^2\right|$ measures the strength of the electric field.

The trajectories are cycloids of the form

$$x(\psi) = a + \varepsilon\psi + \rho\cos(\psi - \bar{\psi}), \qquad y(\psi) = b + \rho\sin(\psi - \bar{\psi}), \tag{10}$$

where ρ reduces to the Larmor radius when $\varepsilon = 0$. To account for the conservation of energy $E = \frac{1}{2}(\varepsilon^2 + 2\varepsilon b + \rho^2)$, we introduce a second parameter $\mu = \sqrt{2E - \varepsilon^2}$, so that $\rho = \sqrt{\mu^2 - 2\varepsilon b}$. The range of the tangent velocity is $|u| \leq \mu(1 + \mathcal{O}(\varepsilon))$.

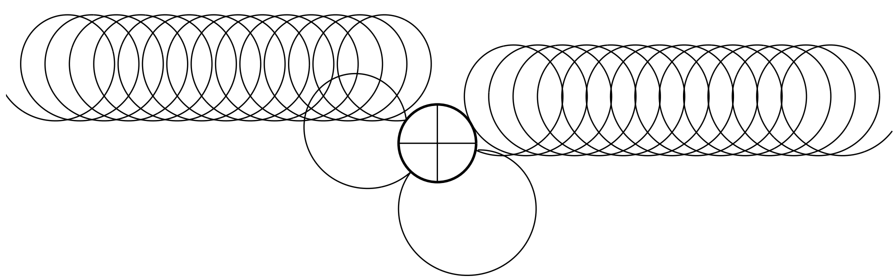

Figure 3: A trajectory scattered off the hard disc. Trajectories coming in from infinity leave the scatterer again with probability one. However, some orbits may form "bound states" which are indefinitely bouncing on the scatterer.

In [4], the following problem was studied: consider the billiard outside a circular scatterer, parametrized by $\mathbf{x}(s) = (\cos s, \sin s)$. Are there trajectories which remain "trapped" in the vicinity of the scatterer, in spite of the drift due to the electric field? In fact, one can easily show that a particle drifting in from infinity will leave the scatterer again with probability one (Fig. 3). However, it is possible that some trajectories bounce on the scatterer indefinitely in the past and in the future, forming a classical "bound state". To show this, we first need to construct the generating function. Given two end points $\mathbf{x}(s_0)$ and $\mathbf{x}(s_1)$, we have to determine the equation of the corresponding cycloid (10), and use the general formula (3). The perturbative result is:

Proposition 2 [5] *Let $s_\pm = (s_1 \pm s_0)/2$. There are positive constants c_1, c_2 and ε_0, such that for $c_1\varepsilon < s_- < \pi - c_1\varepsilon$, $\mu > 1 + c_2\varepsilon$ and $0 \leq \varepsilon < \varepsilon_0$, the generating function of the bouncing map is unique, an analytic function of $s_\pm$, μ and ε, and admits the expansion*

$$\begin{aligned} G(s_-, s_+) &= s_- + \frac{1}{2}\mu^2\Delta\psi - (C+R)S - \varepsilon\,[2S + (C+R)\Delta\psi]\sin s_+ \\ &\quad + \varepsilon^2\left[\left(\frac{(C+R)^2 S}{\mu^2 R} + \frac{C+R}{R}\Delta\psi + \frac{\mu^2}{4RS}\Delta\psi^2\right)\sin^2 s_+ - \frac{R}{4S}\Delta\psi^2\right] + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{11}$$

where C, S, R, $\Delta\psi$ denote functions of s_- alone:

$$C = \cos s_-, \quad S = \sin s_-, \quad R = \sqrt{\mu^2 - S^2}, \quad \Delta\psi = 2\pi - \mathrm{Arccos}[1 - 2S^2/\mu^2]. \tag{12}$$

The point is that the generating function has the form $G(s_0, s_1, \varepsilon) = G_0(s_1 - s_0) + \varepsilon G_1(s_0, s_1, \varepsilon)$. If we substitute this expression in (2), we obtain that the bouncing map has the structure of a perturbed integrable map:

Corollary 1 *There is a positive c_3 such that for $|u| < \mu(1 - c_3\varepsilon)$, $\mu > 1 + c_2\varepsilon$ and $0 \leq \varepsilon < \varepsilon_0$, the trajectory returns to the boundary, the bouncing map is analytic and of the form*

$$\begin{aligned} s_1 &= s_0 + \Omega(u_0) + \varepsilon f(s_0, u_0, \varepsilon) \pmod{2\pi}, \\ u_1 &= u_0 + \varepsilon g(s_0, u_0, \varepsilon). \end{aligned} \tag{13}$$

This result asserts that the particle will return to the scatterer if it starts with a sufficiently large normal velocity. The problem is now to show that some of these trajectories indefinitely return to the scatterer. This can be achieved once again by using Moser's theorem: indeed the existence of two distinct invariant curves in phase space implies the region between them to be invariant under the map.

Theorem 4 [5] *There is a positive ε_1 such that for $\varepsilon < \varepsilon_1$ and $\mu > 1 + c_2\varepsilon$, the scattering system has a set of bound states with positive measure.*

The problem with KAM theory is that one has in general very bad estimates on the bound ε_1. One can improve them by studying periodic orbits, which can be surrounded by islands of stability for much higher values of the electric field. As described in Section 1, we can compute the two–point function $G^{(2)}$ and analyze its stationnary points (for this purpose, we needed to know (11) at second order in ε). In this way we can prove the existence of two orbits of period 2:

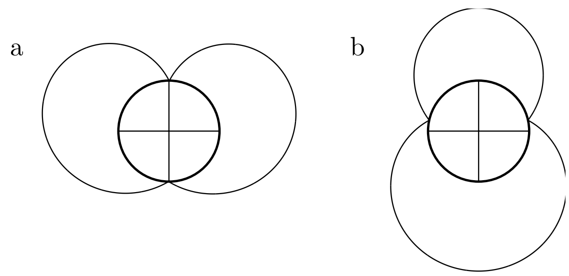

Figure 4: Orbits of period 2, for $\varepsilon = 0.1$ and $\mu = 1.5$: (a) hyperbolic orbit, (b) elliptic orbit. The elliptic orbit is generically surrounded by a set of bound states.

1. An orbit hitting the scatterer at $s = \pi/2, 3\pi/2$, which is hyperbolic for small ε (Fig. 4a).
2. An orbit hitting the scatterer at $s = \delta, \pi - \delta$, where $\delta = \varepsilon\Delta\psi(\pi/2)/R(\pi/2) + \mathcal{O}(\varepsilon^3)$, which is elliptic for $0 < \varepsilon < \varepsilon_2$ (Fig. 4b). Using Theorem 1, we can show that if the orbit is elliptic, then it is stable with probability 1 with respect to $\mathrm{d}\mu\mathrm{d}\varepsilon$. We have the estimation

$$\varepsilon_2 \simeq \left[\frac{4}{\mu^2} + \frac{4\Delta\psi(\pi/2)}{R(\pi/2)} + \Delta\psi(\pi/2)^2\right]^{-1/2}, \tag{14}$$

   which is in good agreement with numerics (roughly, bound states exist as long as the drift per cycle is smaller than the radius of the scatterer).

This method of searching elliptic orbits can be used to study existence of bound states for more general scatterers. In fact, if the billiard posseses elliptic orbits in zero electric field, they will generically survive small perturbations. Interesting open problems include (1) the existence of a critical electric field beyond which there are *no* bound states, and (2) an understanding of transport in phase space, and its influence on the transit time of a particle drifting in from infinity.

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