Multivariate Hawkes processes on inhomogeneous random graphs

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May 23rd 2022

Outlines

Modelling of N neurons in interaction on a graph

- The neuron i is located in $x_i \in I \subset \mathbb{R}^d$

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- $\left(Z_i^{(N)}(t)\right)_{t\geq 0}$ counting process :
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- conditional intensity at time t :

$$\lambda_i^{(N)}(t) = f\left(u_0(t,x_i) + \frac{1}{N}\sum_{j=1}^{N} w_{ij}^{(N)} \int_0^{t-} h(t-s)dZ_j^{(N)}(s)\right). \tag{1}$$

$\Rightarrow \left(Z_1^{(N)}(t), \cdots, Z_N^{(N)}(t)\right)_{t>0}$ multivariate Hawkes process.

Firing intensity

$$\lambda_i^{(N)}(t) = f\left(u_0\left(t-, x_i\right) + \frac{1}{N}\sum_{j=1}^{N} w_{ij}^{(N)} \int_{]0,t[} h\left(t-s\right) dZ_j^{(N)}\left(s\right)\right)$$

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- $h \ : \ \mathbb{R}_+ \longrightarrow \mathbb{R}$: memory function which models how a past jump of the system affects the present intensity (ex : decreasing exponential $h(t) = e^{-\alpha t}$, compact support $h(t) = 1_{0 \leq t \leq \theta}$)

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- $w_{ij}^{(N)}$: interaction between the neurons i and j

Graph of interaction

Interaction between neurons i and j : $w_{ij}^{(N)} = \kappa_i^{(N)} \xi_{ij}^{(N)}$ where

- $\kappa_i^{(N)} \geq 0$ dilution parameter so that the interaction term remains of order 1 as $N \to \infty$
- $\forall i, j,\ \xi_{ij}^{(N)} \in \{0, 1\} \sim \mathcal{B}\left(W_N\left(x_i, x_j\right)\right)$ where $W_N \ : \ I \times I \longrightarrow [0, 1]$ kernel of the microscopic structure

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Definition

Graph of interaction : $\mathcal{G}^{(N)} = \left(\{1, \cdots, N\}, \left(\xi_{ij}^{(N)}\right)_{1 \leq i, j \leq N}\right)$

Annealed graph :

$\mathcal{G}_m^{(N)} = \left(\{1, \cdots, N\}, \text{edge } j \to i \text{ with weight } \kappa_i^{(N)} W_N(x_i, x_j)\right)$

Some dense graphs, $I = [0,1], x_i = \frac{i}{N}, \kappa_i^{(N)} = 1, i = 1 \dots N$

Erdös-Rényi graph

$W_N(x,y) = \rho_N$ with $\rho_N \to \rho > 0$

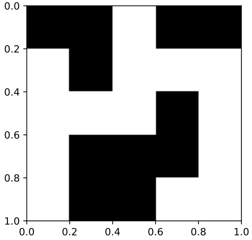

Figure – $\mathcal{G}^{(N)}$ with $\rho_N = 0.5$, $N = 5$

P-nearest neighbor

$W_N(x,y) = 1_{\min(|x-y|, 1-|x-y|) < r}$

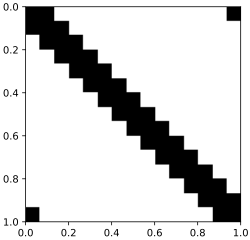

Figure – $\mathcal{G}^{(N)}$ with $r = 0.1$, $N = 15$

Inhomogeneous graph, $I = [0,1], x_i = \frac{i}{N}, \kappa_i^{(N)} = 1, i = 1...N$

Expected Degree Distribution (EDD)

$W_N(x,y) = g(x)k(y)$

Here : $W_N(x,y) = xy$

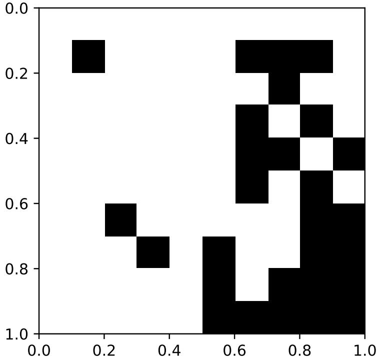

Figure – $\mathcal{G}^{(N)}$ with $N = 10$

$(\pi_i\,(ds,dz))_{1\leq i\leq N}$: i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dsdz$. For all $t \geq 0, i \in [\![1,N]\!]$:

$$Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds,dz).$$

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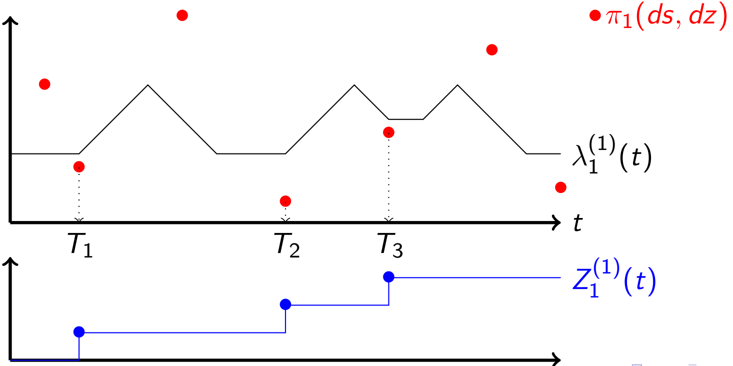

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Hypotheses

f Lipschitz continuous (L_f), h locally integrable, u_0 time continuous, Lipschitz continuous in space and uniformly bounded.

Proposition [Delattre et al., 2016, Chevallier et al., 2019]

For a fixed realisation of $(\pi_i)_{1 \leq i \leq N}$, there exists a pathwise unique multivariate Hawkes process $\left(Z_1^{(N)}(t), \cdots, Z_N^{(N)}(t)\right)_{t>0}$ such that $\left(\sup_{1 \leq i \leq N} \mathrm{E}[Z_i^{(N)}(t)]\right)_{t>0}$ is locally bounded.

Outlines

$$\lambda_i^{(N)}(t) = f\left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^{N} \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s)\right) \xrightarrow[N\to\infty]{} ?$$

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About the positions

- $\nu^{(N)} := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(dx) \xrightarrow[N\to\infty]{} \nu$, a macroscopical distribution

Scenario (1) : $I = [0,1]$ and $x_i = \frac{i}{N} \Rightarrow \nu(dx) = dx$
Scenario (2) : (x_i) i.i.d. of distribution ν

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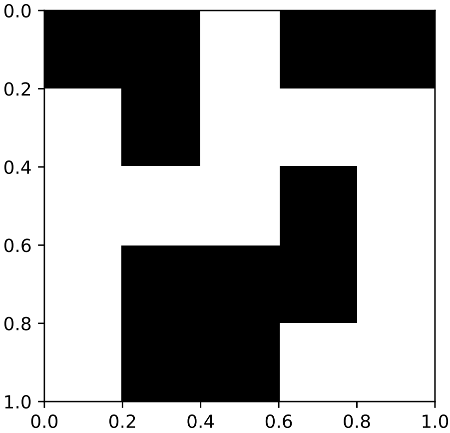
About the graph

$\mathcal{G}^{(N)} \xrightarrow[N \to \infty]{} ?$

Example : Erdös-Rényi graph



$$\mathcal{G}^{(N)}, \quad N = 5$$

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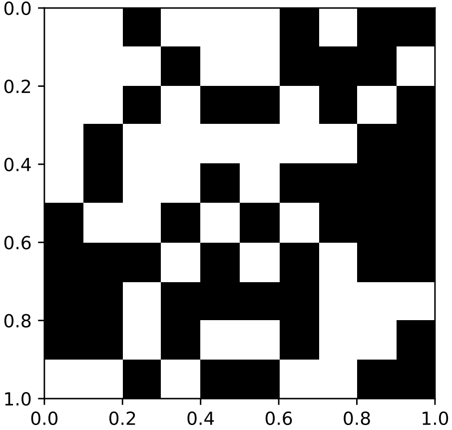
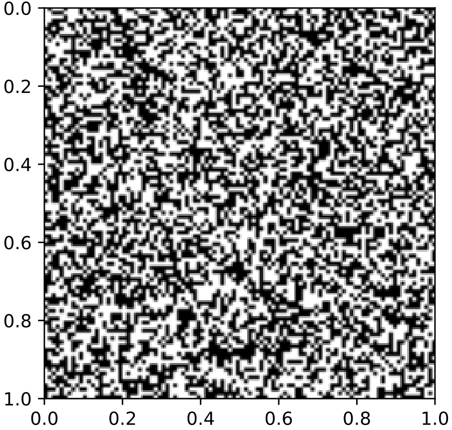


$$\mathcal{G}^{(N)}, \quad N = 10$$

Example : Erdös-Rényi graph



$$\mathcal{G}^{(N)}, \quad N = 100$$

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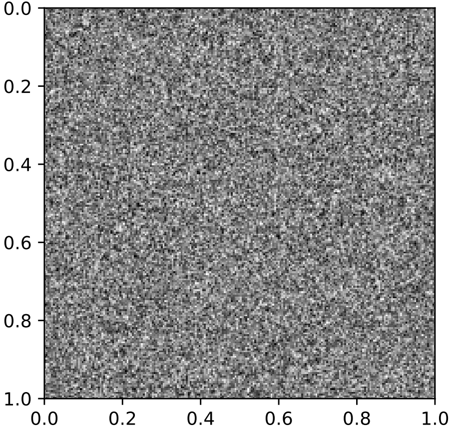
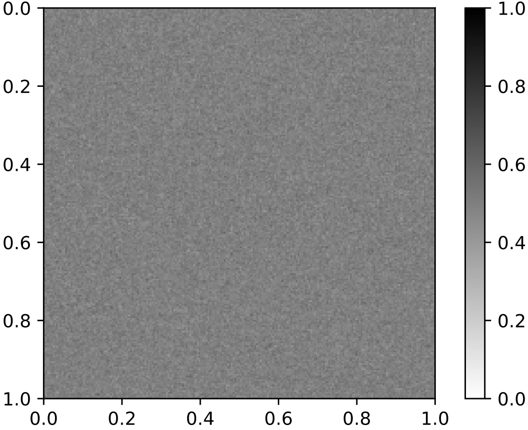


$$\mathcal{G}^{(N)}, \quad N = 500$$

Example : Erdös-Rényi graph



$$\mathcal{G}^{(N)}, \quad N = 3000$$

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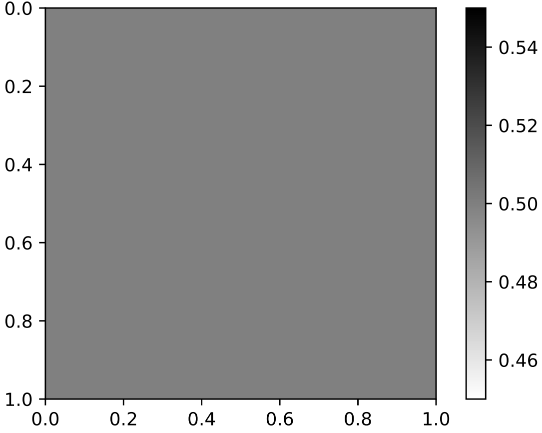
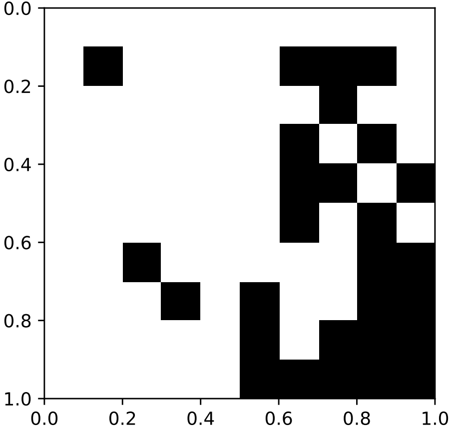


$$\mathcal{G}^{(N)}, \quad N \to \infty$$

Example : EDD with $W(x, y) = xy$



$\mathcal{G}^{(N)}, \quad N = 10$

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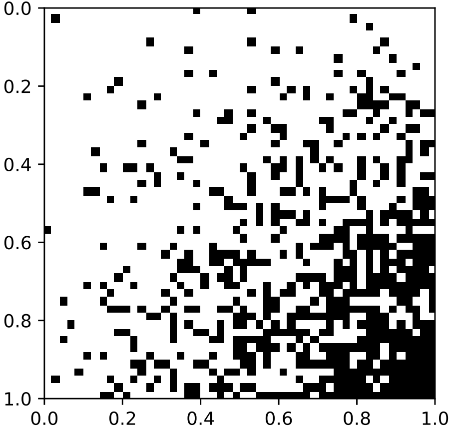
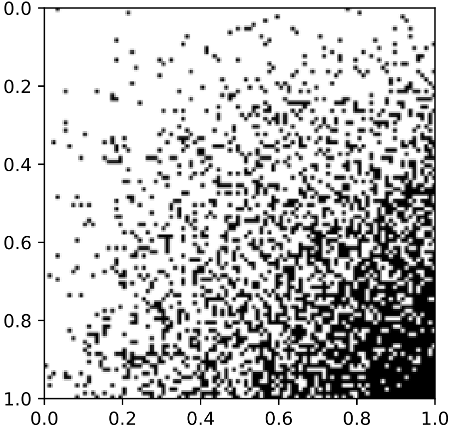


$\mathcal{G}^{(N)}, \quad N = 50$

Example : EDD with $W(x, y) = xy$



$\mathcal{G}^{(N)}, \quad N = 100$

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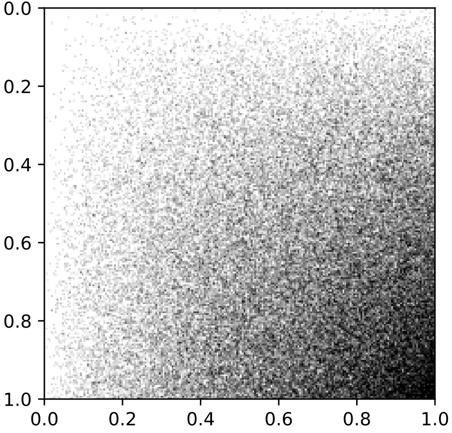

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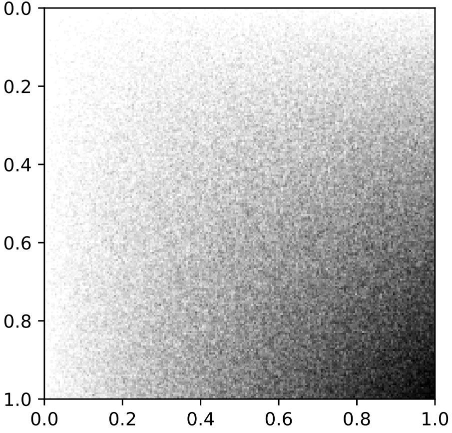

$\mathcal{G}^{(N)}, \quad N = 1000$

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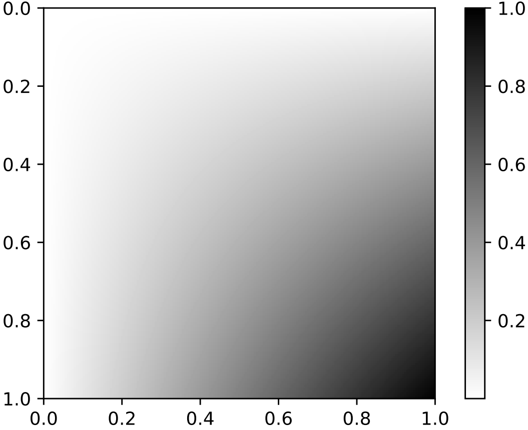

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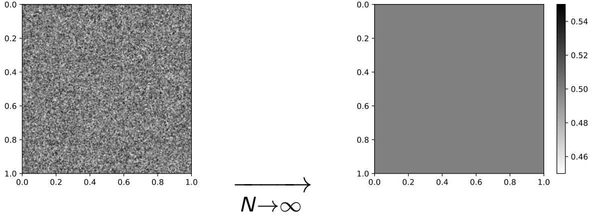

Figure – Erdös-Rényi

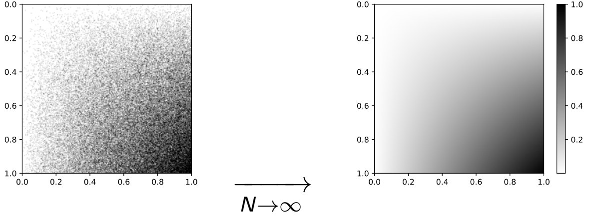

Figure – EDD - $N = 500$ à gauche, graphon $W(x, y) = xy$ à droite

Macroscopic interaction graphon $W : I^2 \to \mathbb{R}_+$

Assume : $\forall N, \exists\ I = \bigsqcup_{i=1}^{N} B_i^{(N)} / \quad \nu\left(B_i^{(N)}\right) = \frac{1}{N}.$

- Graph $\mathcal{G}^{(N)}$ $\Rightarrow$ Connectivity matrix $\Rightarrow$ step-function $W^{\mathcal{G}^{(N)}}$ (graphon) [Lovász, 2012]

$$W^{\mathcal{G}_m^{(N)}}(u, v) := \sum_{i,j=1}^{N} 1_{B_i^{(N)} \times B_j^{(N)}}(u, v) \kappa_i^{(N)} W_N(x_i, x_j)$$

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- $W^{\mathcal{G}_m^{(N)}} \xrightarrow[N\to\infty]{d_{\square,\nu}} W$ a macroscopic interaction graphon with

$$d_{\square,\nu}(W_1, W_2) := \sup_{S,T\subset I} \left| \int_{S\times T} (W_1 - W_2)(x, y)\, \nu(dx)\nu(dy) \right|$$

Heuristics of the limiting intensity

$$\lambda_i^{(N)}(t) = f\left(u_0(t,x_i) + \frac{\kappa_i^{(N)}}{N}\sum_{j=1}^{N}\xi_{ij}^{(N)}\int_0^{t-} h(t-s)dZ_j^{(N)}(s)\right)$$

$$\text{"}\xrightarrow[N\to\infty]{}\text{"}$$

$$\lambda(t,x) = f\left(u_0(t,x) + \int_I W(x,y)\int_0^t h(t-s)\lambda(s,y)ds\nu(dy)\right) \quad (2)$$

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Proposition - Let $T > 0$. Assume

- $D(x) := \int_I W(x, y)\nu(dy)$, $\sup_{x\in I} D(x) < \infty$
- $\int_I |W(x, y) - W(x', y)|\nu(dy) \leq C\|x - x'\|^\iota$, $\iota \in (0, 1]$

There exists a unique solution λ of (2) continuous and bounded on $[0, T] \times I$.

Coupling [Delattre et al., 2016, Chevallier et al., 2019]

For $t \in [0,T], i \in [\![1,N]\!]$ and the same (π_i) :

- $$Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds,dz) \text{ with}$$

  $$\lambda_i^{(N)}(t) = f\left(u_0(t,x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s)\right)$$

- $$\overline{Z}_i(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda(s,x_i)\}} \pi_i(ds,dz) \text{ with}$$

  $$\lambda(t,x) = f\left(u_0(t,x) + \int_{\mathbb{R}^d} W(x,y) \int_0^{t-} h(t-s)\lambda(s,y) ds \nu(dy)\right).$$

Hypotheses

- Control of the dilution of the graph $\mathcal{G}^{(N)}$
- Control of indegrees and outdegress :
$\sup_{i\in[\![1,N]\!]} \frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W$,
$\sup_{j\in[\![1,N]\!]} \frac{1}{N} \sum_{i=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W$

Theorem - Let $T > 0$, **for $\mathbb{P}$-almost realisations of the connectivity sequence** $\left(\xi^{(N)}\right)_{N\geq 1}$ and positions $\left(\underline{x}_N\right)_{N\geq 1}$:

$$\frac{1}{N} \sum_{i=1}^N \mathrm{E}\left[\sup_{t\in[0,T]} \left|Z_i^{(N)}(t) - \overline{Z}_i(t)\right|\right] \xrightarrow[N\to\infty]{} 0.$$

Spatial profile, with $I = [0,1]$ and $x_i = \frac{i}{N}$

- $X_t^{(N)}(x) :=$
$$\sum_{i=1}^{N} \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^{N} \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right) 1_{x \in (\frac{i-1}{N}, \frac{i}{N}]}$$
- $X_t(x) := u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds\, \nu(dy)$

Proposition - For $\mathbb{P}$-almost realisations of the connectivity sequence $\left(\xi^{(N)}\right)_{N\geq 1}$,

$$\mathrm{E}\left[\int_0^T \int_0^1 \left| X_t^{(N)}(x) - X_t(x) \right| dx\ dt\right] \xrightarrow[N\to\infty]{} 0.$$

Outlines

Linear case : $f = Id$

$$\lambda(t,x) = u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds \, \nu(dy)$$

Without spatial interaction [Delattre et al., 2016]

$$\lambda(t) = u_0 + \int_0^t h(t-s)\lambda(s)ds$$

Phase transition

- Subcritical case ($\|h\|_1 < 1$) : $\lambda(t) \xrightarrow[t\to\infty]{} \frac{u_0}{1-\|h\|_1}$
- Supercritical case ($\|h\|_1 > 1$) : $\lambda(t) \sim \alpha e^{\beta t} \to \infty$ for some $\alpha, \beta > 0$

$$\lambda(t,x) = u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds \, \nu(dy)$$
$$= u_0(t,x) + \int_0^t h(t-s) T_W \lambda(s,\cdot)(x) ds$$

$$\lambda(t,x) = u_0(t,x) + \int_I W(x,y) \int_0^t h(t-s)\lambda(s,y)ds\ \nu(dy)$$
$$= u_0(t,x) + \int_0^t h(t-s)T_W\lambda(s,\cdot)(x)ds$$

Integral operator T_W

$$T_W : \begin{array}{ccc} L^\infty(I) & \longrightarrow & L^\infty(I) \\ g & \longmapsto & \left(T_W g : x \longmapsto \int_I W(x,y)g(y)\nu(dy)\right). \end{array}$$

Spectral radius

$$r_\infty := r_\infty(T_W) = \sup_{\sigma \in Sp(T_W)} |\sigma| = \lim_{n\to\infty} \|T_W^n\|^{\frac{1}{n}}.$$

Subcritical case : $\|h\|_1 r_\infty < 1$

Theorem

In the subcritical case $\|h\|_1 r_\infty < 1$, if $\sup_{x\in I} |u_0(t,x) - u(x)| \xrightarrow[t\to\infty]{} 0$, then for any $x \in I$,

$$\lambda(t,x) \xrightarrow[t\to\infty]{} \ell(x)$$

where ℓ is the unique continuous and bounded function solving

$$\ell(x) = u(x) + \|h\|_1 \int_I W(x,y)\ell(y)\nu(dy).$$

Proposition

If u_0 is constant, ℓ is uniform if and only if the indegree is uniform ($\int_I W(x,y)\nu(dy) = D$ for every $x \in I$). In such case, $r_\infty = D$.

Subcritical case - Erdös-Rényi graph

$$W_N(x,y) = \rho = r_\infty = 0.5$$
$$N = 1000$$

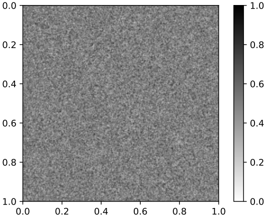
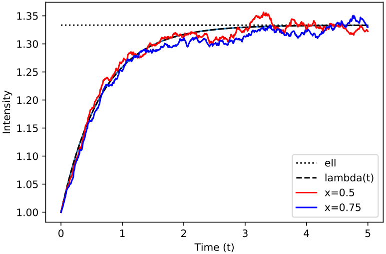


Figure – $\mathcal{G}^{(N)}$

Subcritical condition : $\|h\|_1 \rho < 1$



$$h(t) = e^{-2t}, \quad u_0(t,x) = 1$$
$$\lambda(t) = \frac{4}{3} - \frac{1}{3} e^{-\frac{3}{2}t}, \quad \ell = \frac{4}{3}$$

Subcritical case - P-nearest neighbor

$$W_N(x,y) = 1_{\min(|x-y|,1-|x-y|)<r}, \quad r_\infty = 2r$$
$$h(t) = e^{-2t}, \quad u_0(t,x) = 1, \quad r = 0.1$$

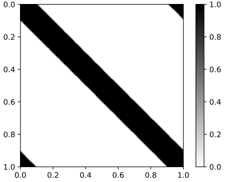
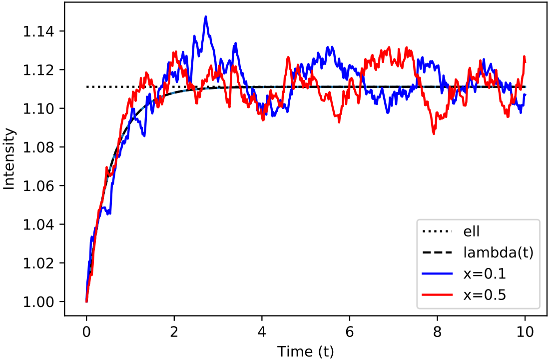


Figure – $\mathcal{G}^{(N)}$



$N = 500$
$\lambda(t) = \frac{10}{9} - \frac{1}{9} e^{-\frac{9}{5}t}$

Subcritical case - EDD

f, g densities on I, bounded

$$W_N(x,y) = f(x)g(y), \quad D(x) = f(x), \quad r_\infty = \langle f, g \rangle$$

When $\|h\|_1 \langle f, g \rangle < 1$:

$$\lambda(t,x) \xrightarrow[t\to\infty]{} \ell(x) = u(x) + \|h\|_1 \frac{f(x)\langle u, g\rangle}{1 - \|h\|_1 \langle f, g\rangle}.$$

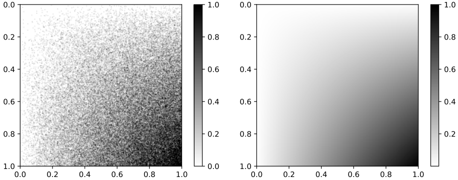
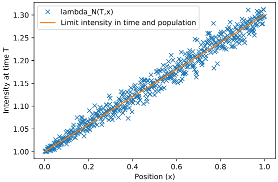
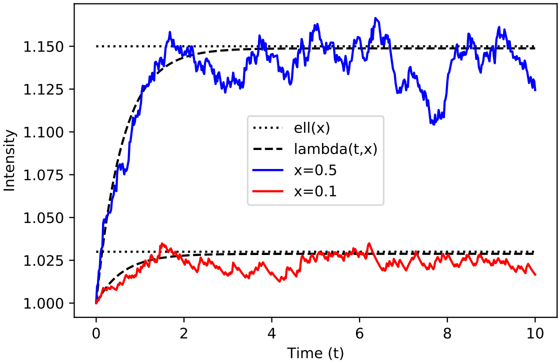


Figure – $\mathcal{G}^{(N)}$ with $N = 500$ and the graphon $W(x,y) = xy$

Subcritical case - EDD $W(x, y) = xy$

$$h(t) = e^{-2t}$$
$$u_0(t, x) = 1$$
$$\ell(x) = 1 + \frac{3}{10}x$$



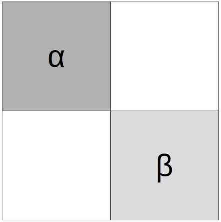


Supercritical case : $\|h\|_1 r_\infty > 1$

$$\|h\|_1 r_\infty > 1 \Rightarrow \lambda(t,x) \xrightarrow[t\to\infty]{} \infty \ ?$$

Example

W with 2 disconnected mean-field components, $\alpha > \beta$, $r_\infty = \alpha/2$:
A population can be in the subcritical case and the other in the supercritical case.



(critical parameters : $\alpha_c = \frac{2}{\|h\|_1}$ and $\beta_c = \frac{2}{\|h\|_1}$)

Supercritical case : $\|h\|_1 r_\infty > 1$

Hypotheses

- $\sup_x \int_I W(x,y)^2 \nu(dy) =: C_{W_2} < \infty$,
- $\forall (x,y) \in I^2, \quad W(x,y) = W(y,x)$
- there exists k such that $W^{(k)} > 0$ where $W^{(k)}(x,y) := \int_{I\times\cdots\times I} W(x,x_1)\cdots W(x_{k-1},y)dx_1\cdots dx_{k-1}$.

Proposition

$$\int_I \lambda(t,x)^2 \nu(dx) \xrightarrow[t\to\infty]{} \infty$$

Outlines

Model

When $N \to \infty$

Large time behaviour of λ

Current work
- Hypotheses
- Goal

- Exponential decay : $h(t) = e^{-\alpha t}$

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Then the large time limit of the spatial profile verifies

$$X_t(x) = \int_I W(x, y) \int_0^t e^{-\alpha(t-s)} F(X_s(y), u(s, y)) ds\ dy$$

(X is then the solution of the scalar neural field equation [Amari, 1977, Wilson - Cowan, 1972, Chevallier et al., 2019])

Work in progress

Let T_N be polynomial in N -i.e., there exists m such that $T_N = \left(\frac{N}{\kappa_N}\right)^m$. Then for any $\varepsilon > 0$,

$$\mathrm{P}\left(\sup_{t\in[0,T_N]} \left\|X_t^{(N)} - X_t\right\|_2 > \varepsilon\right) \xrightarrow[N\to\infty]{} 0.$$

Tool : the large time limit of X_t satisfying

$$\alpha X_\infty = T_W F(X_\infty, u_\infty)$$

- Finite time dynamic :

$$\mathrm{P}\left(\sup_{t\in[0,T]}\left\|X_t^{(N)} - X_t\right\|_2 > \varepsilon\right) \xrightarrow[N\to\infty]{} 0.$$

- Stability around X_∞ for (X_t) :

$$\partial_t\left(X_t - X_\infty\right) = -\alpha\left(X_t - X_\infty\right)dt + T_W\left(F(X_t, u_t) - F(X_\infty, u_\infty)\right)dt$$

- By an iteration argument in large time $X_t^{(N)}$ remains close to X_∞ as

$$\partial_t\left(X_t^{(N)} - X_\infty\right) = \mathcal{L}\left(X_t^{(N)} - X_\infty\right)dt + dM_N(t) + r_N(t)dt,$$

with $\mathcal{L}$ an accretive operator, M_N some Poissonian noise and r_N a drift term (from the graph).

Thanks !

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Long time behavior for Hawkes processes on random graphs
Work in progess.

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