



Exact simulation of the first time a diffusion process overcomes a given threshold

S. Herrmann

University of Burgundy, Dijon, France

**joint work with Cristina ZUCCA (University of Turin)
and Nicolas MASSIN (University of Valenciennes)**

May 24, 2022

Introduction

Simulation of random variables depending on the paths of a one-dimensional diffusion process: a challenging task.

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x < L.$$

Aim: Generation of different variates can be considered:

- the value X_t at a fixed time $t > 0$.
- the first passage time (FPT) through a given threshold

$$\tau_L := \inf\{t \geq 0 : X_t = L\}, \quad x < L.$$

- the exit time (ET) of an interval

$$\tau_I := \inf\{t \geq 0 : X_t \notin I\}, \quad x \in]a, b[.$$

Applications in different fields: breaking times (reliability), times of ruin (insurance), neuroscience, barrier options (finance),...

Different tools: explicit expression of the pdf, approximation of the density, approximation of the stochastic process, rejection sampling...

Explicit expressions for the FPT τ_L or for the ET τ_I .

Standard Brownian case ($B_0 = 0$):

1. The optional stopping thm applied to $M_t = \exp\{\lambda B_t - \frac{1}{2}\lambda^2 t\}$ leads to

$$\mathbb{E}[e^{-\lambda\tau_L}] = e^{-\sqrt{2\lambda}L}, \quad \lambda \geq 0.$$

Inversion of the Laplace transform:

$$\mathbb{P}(\tau_L \in dt) = \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{L^2}{2t}} dt, \quad t > 0.$$

Hence $\tau_L \sim L^2/G^2$

where $G \sim \mathcal{N}(0, 1)$.

Easy and exact simulation !

2. Concerning τ_I with $I = [-1, 1]$, we know that:

$$p_\tau(t) = \sum_{n=0}^{+\infty} (-1)^n R_1(2n+1, t) \quad \text{with } R_1(n, t) := \frac{2n}{t^{3/2}} \phi\left(\frac{n}{\sqrt{t}}\right).$$

The following expansion also holds:

$$p_\tau(t) = \sum_{n=0}^{+\infty} (-1)^n R_2(2n+1, t) \quad \text{with } R_2(n, t) := \frac{\pi n}{2} \exp\left(-\frac{n^2 \pi^2}{8} t\right).$$

- When the transition probability of (X_t) has an explicit expression...

Volterra-type integral equation (see Buonocore, Nobile, Ricciardi)

The pdf $f_L(t)$ of the FPT τ_L satisfies a Volterra-type equation depending on the probability current of the diffusion process.

Closed form results for the Brownian motion and for the O-U process.

In general: numerical approximation of the integral...

- **General method: time discretization (Euler scheme).**

$$X_{(n+1)\Delta} = X_{n\Delta} + \Delta b(X_{n\Delta}) + \sqrt{\Delta} \sigma(X_{n\Delta}) G_n, \quad n \geq 0.$$

τ_L^Δ the FPT of the **discrete-time process**: we often observe an overestimation of the FPT.

- 1 a shift of the boundary (Broadie-Glasserman-Kou, Gobet-Menozzi)
- 2 computation of the probability for a Brownian bridge to hit the boundary during a small time interval (Giraudo-Saccerdote-Zucca)

Acceptance-rejection sampling: an exact simulation of the FPT

Principal idea: Let f and g two probability distribution functions, such that $h(x) := f(x)/g(x)$ is upper-bounded by a constant $c > 0$.

Aim: simulation of X with pdf f .

- 1 Generate a rv Y with pdf g .
- 2 Set $X = Y$ with conditional probab. $h(Y)/c$ otherwise go back to 1.

For any positive measure function ψ :

$$\mathbb{E}[\psi(X)] = \int_{\mathbb{R}} \psi(x)f(x) dx = \int_{\mathbb{R}} \psi(x)h(x)g(x) dx = \mathbb{E}[\psi(Y)h(Y)]$$

Important: h should be bounded and explicit!

Not quite so simple: h is related to a series in particular situations.

The aim is to use this general procedure for specific variables:

- the diffusion value X_t at time t (Beskos & Roberts, 2005)
- the stopping times τ_L (FPT) and the exit time τ_I

Remark: Lamperti's transform \Rightarrow simpler diffusion process

$$dX_t = dB_t + b(X_t) dt, \quad X_0 = x.$$

Regular drift b . Set $\beta(x) = \int_0^x b(y) dy$ and $\gamma := \frac{b^2 + b'}{2}$.

1st Case: simulation of X_t for a given time t (Beskos & Roberts).

Using Girsanov's transformation and Itô's lemma:

$$\mathbb{E}_x[\psi(X_t)] = \mathbb{E}\left[\psi(x + B_t) e^{\beta(x+B_t) - \int_0^t \gamma(x+B_s) ds}\right] = \mathbb{E}[\psi(Y) h(Y)]$$

It permits to use a rejection sampling for Y whose distribution satisfies

$$g(y) := \frac{1}{\sqrt{2\pi t} \cdot g(\mathbb{R})} e^{\beta(y) - \frac{(y-x)^2}{2t}}, \quad \text{if } g(\mathbb{R}) < \infty,$$

associated with the weight of acceptance given by

$$h(y) := g(\mathbb{R}) \cdot \mathbb{E}\left[e^{-\int_0^t \gamma(x+B_s) ds} \middle| x + B_t = y\right] = g(\mathbb{R}) \cdot \mathbb{E}\left[e^{-\int_0^t \gamma(b_s^{x \rightarrow y}) ds}\right].$$

Here $(b_s^{x \rightarrow y}, 0 \leq s \leq t)$ stands for a Brownian bridge starting with the value x and ending in y at time t .

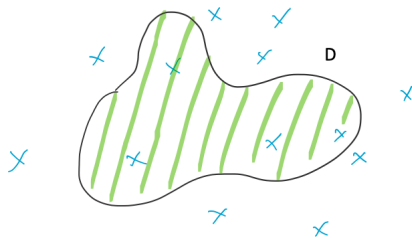
Proposal random variable: Y with p.d.f. $g(y) := \frac{1}{\sqrt{2\pi t} \cdot g(\mathbb{R})} e^{\beta(y) - \frac{(y-x)^2}{2t}}$
 accepted with the weight proportional to $\mathbb{E} \left[e^{-\int_0^t \gamma(b_s^{x \rightarrow y}) ds} \right]$.

Intuitive algorithm:

- 1 Generate Y with density $g \rightarrow y$
- 2 Generate a path of a Brownian bridge
 $\rightarrow b^{x \rightarrow y}$.
- 3 Accept y with probability weight
 proportional to $e^{-\int_0^t \gamma(b_s^{x \rightarrow y}) ds}$.
 \rightarrow area under the curve

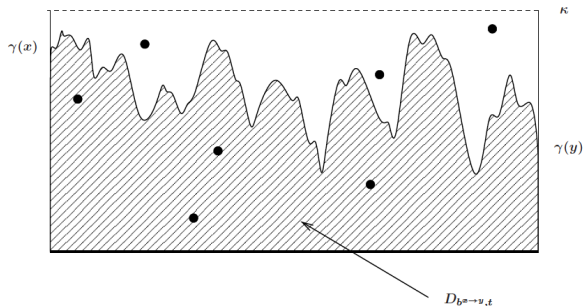
From now on, hyp: $0 \leq \gamma(\cdot) \leq \kappa$.

Poisson Process N with intensity λ , the Lebesgue measure on \mathbb{R}^2 .

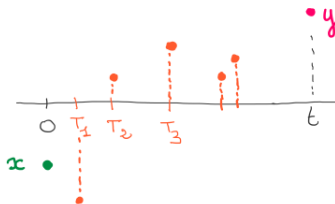
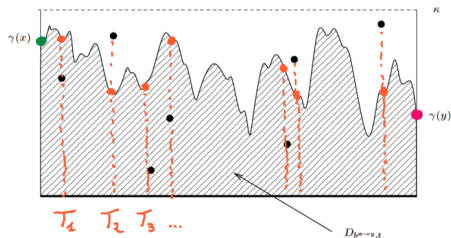


$$\mathbb{P}[N(D) = 0] = e^{-\lambda(D)}.$$

How to accept y with probability $\propto e^{-\int_0^t \gamma(b_s^{x \rightarrow y}) ds}$? (BR)



- Generate N a P.P. on $[0, t] \times \mathbb{R}_+$, independent of the Brownian bridge.
- If $N(D_{b^{x \rightarrow y}, t}) = 0$ then accept y .



Exact simulation of X_t – Algorithm $(BR)_t$

$(Y_n)_{n \geq 1}$ i.i.d. with density g , $(G_n)_{n \geq 1}$ i.i.d. $\mathcal{N}(0, 1)$, $(E_n)_{n \geq 1}$ i.i.d. $\mathcal{E}(\kappa)$,
 $(U_n)_{n \geq 1}$ i.i.d. $\mathcal{U}([0, 1])$. All sequences are independent.

Initialization: $k = 0$, $n = 0$.

Step 1. Set $k \leftarrow k + 1$ then $Z = x$, $W = Y_k$ and $\mathcal{T} = 0$.

Step 2. While $\mathcal{T} < t$ do:

- set $n \leftarrow n + 1$
- $Z \leftarrow Z + \frac{E_n}{t - \mathcal{T}} W + \sqrt{\frac{E_n(t - \mathcal{T} - E_n)_+}{t - \mathcal{T}}} G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T} + E_n, t)$
- If $(\mathcal{T} < t \text{ and } \kappa U_n < \gamma(Z))$ then go to Step 1.

Outcome: the random variable W .

Theorem (Beskos-Roberts) Under suitable hyp., the outcome W of Algorithm $(BR)_t$ and the diffusion value X_t are identically distributed.

2nd Case: simulation of τ_L with L fixed (H. & Zucca, 2019).

Let us recall that $X_0 = x$,

$$dX_t = dB_t + b(X_t) dt, \beta(x) = \int_0^x b(y) dy \text{ and } \gamma := \frac{b^2 + b'}{2}.$$

Combining the Girsanov transform and Itô's lemma permits to obtain:

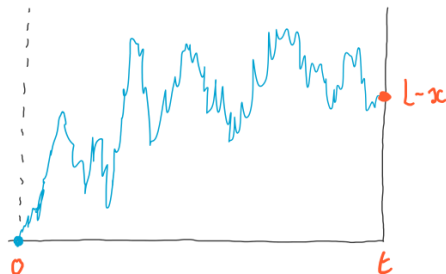
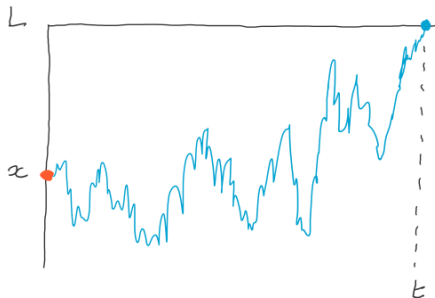
$$\begin{aligned} \mathbb{E}[\psi(\tau_L^X) 1_{\{\tau_L^X < \infty\}}] &= \mathbb{E}\left[\psi(\tau_L^B) \exp\left(\int_0^{\tau_L^B} b(B_s) dB_s - \frac{1}{2} \int_0^{\tau_L^B} b^2(B_s) ds\right)\right] \\ &= \mathbb{E}\left[\psi(\tau_L^B) e^{\beta(L) - \beta(x)} e^{-\int_0^{\tau_L^B} \gamma(B_s) ds}\right] = \mathbb{E}[\psi(\tau_L^B) h(\tau_L^B)] \end{aligned}$$

with

$$h(t) \propto \mathbb{E}\left[e^{-\int_0^t \gamma(B_s) ds} \mid B_0 = x, \tau_L^B = t\right] = \mathbb{E}\left[\exp - \int_0^t \gamma(L - R_s) ds\right].$$

Here $(R_t, t \geq 0)$ stands for a Bessel bridge of dimension 3 starting in 0 and ending with the value $L - x$ at time t .

Why is a Bessel process appearing in the computations ?



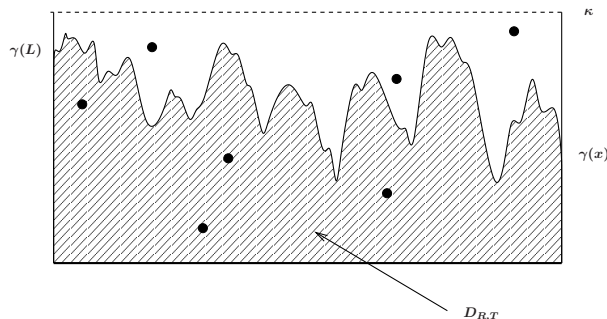
After a rotation of 180° ...

Heuristic algorithm for the generation of τ_L under the condition $0 \leq \gamma(x) \leq \kappa$.

Step 1: Generate $T = (L - x)^2 / G^2$ with $G \sim \mathcal{N}(0, 1)$.

Step 2: Generate a Bessel bridge of dim 3.

$$D_{R,T} := \left\{ (t, v) \in [0, T] \times \mathbb{R}_+ : v \leq \gamma(L - R_t) \right\}.$$



Step 3: Generate a P.P. N on $[0, T] \times \mathbb{R}_+$, indep. of the Bessel process.

Step 4: If $N(D_{R,T}) = 0$ then accept T otherwise go to Step 1.

Exact simulation of τ_L – Algorithm (HZ)

$(G_n)_{n \geq 1}$ i.i.d. $\mathcal{N}_3(0, \text{Id})$, $(e_n)_{n \geq 0}$ i.i.d. $\mathcal{E}(\kappa)$, $(V_n)_{n \geq 1}$ i.i.d. $\mathcal{U}([0, 1])$,
 $(g_n)_{n \geq 1}$ i.i.d. $\mathcal{N}(0, 1)$. All sequences are independent.

Initialization: $k = 0$, $n = 0$.

Step 1. $k \leftarrow k + 1$, $\delta = (0, 0, 0)$, $\mathcal{W} = 0$, $\mathcal{T}_k \leftarrow (L - x)^2 / g_k^2$, $\mathcal{E}_0 = 0$
 and $\mathcal{E}_1 = e_n$.

Step 2. While $\mathcal{E}_1 \leq \mathcal{T}_k$ do:

- set $n \leftarrow n + 1$
- $\delta \leftarrow \frac{\mathcal{T}_k - \mathcal{E}_1}{\mathcal{T}_k - \mathcal{E}_0} \delta + \sqrt{\frac{(\mathcal{T}_k - \mathcal{E}_1)(\mathcal{T}_k - \mathcal{E}_0)}{\mathcal{T}_k - \mathcal{E}_0}} G_n$
- If $\kappa V_n \leq \gamma(L - \| \mathcal{E}_1(L - x)(1, 0, 0) / \mathcal{T}_k + \delta \|)$ then $\mathcal{W} \leftarrow 1$ else
 $\mathcal{W} \leftarrow 0$
- $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ and $\mathcal{E}_1 \leftarrow \mathcal{E}_1 + e_n$

Step 3. If $\mathcal{W} = 0$ then $\mathcal{Y} \leftarrow \mathcal{T}_k$ otherwise go to Step 1.

Outcome: the random variable \mathcal{Y} .

Theorem

Under suitable conditions, the outcome \mathcal{Y} of Algorithm (HZ) and τ_L are identically distributed.

Efficiency of the algorithm.

Rem.: Be carefull with the generation of the PP: if you sample all points, their averaged number is $\mathbb{E}[\kappa T] = \infty$: efficiency to be improved!

Number of iterations (step 1): $\mathbb{E}[I] \leq \exp((L - x)\sqrt{2\kappa})$.

- Concerning $(L - x)$, linearization using space splitting.
- Concerning κ : if $0 < \gamma_0 \leq \gamma(x) \leq \kappa$ for all $x \in \mathbb{R}$,
then replace $\gamma(\cdot) \leftarrow \gamma(\cdot) - \gamma_0$, $\kappa \leftarrow \kappa - \gamma_0$ & introduce the
generation of $IG\left(\frac{L-x}{\sqrt{2\gamma_0}}, (L-x)^2\right)$ (Michael-Schucany-Haas).

Hyp. on γ , the average number of points used during the first iteration:

$$\mathbb{E}[\mathcal{N}_1] \leq M_{\gamma,1} + \kappa M_{\gamma,2}(x^2 + (L-x)^{(1+r)/2}), \quad \text{for } x < L.$$

Examples of generalization and numerics

Example. $dX_t = (2 + \sin(X_t)) dt + dB_t$, $X_0 = 0$. We have $0 \leq \gamma \leq 5$.

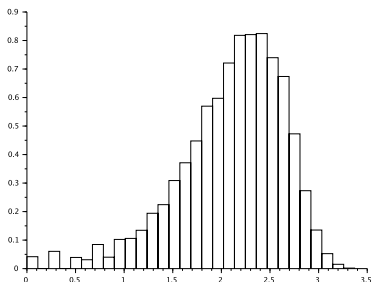
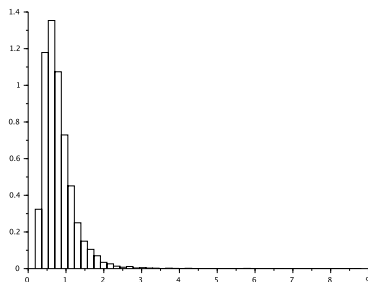


Figure: Histogram of the hitting time distribution for 10 000 simulations corresponding to the level $L = 2$ and starting position $X_0 = 0$ (left), histogram of the number of iterations in Algorithm (A1) in the \log_{10} -scale (right).

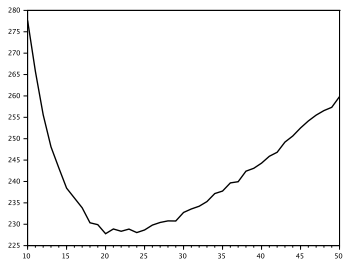
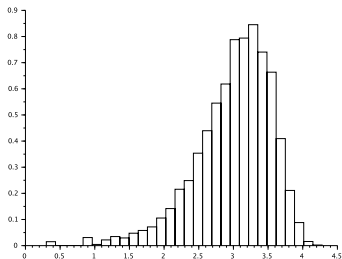


Figure: Number of random variables used in Algorithm (HZ-1) for 10 000 simulations with $L = 2$, $X_0 = 0$ in the \log_{10} -scale (left) and averaged number of random variables used in Algorithm (HZ-1) versus the number of slices k with $X_0 = 0$ and $L = 5$. The averaging uses 10 000 simulations.

Stopped diffusion processes:

- The algorithm (*HZ*) presented so far permits to observe τ_L and consequently the event $\tau_L < \mathbb{T}$ for \mathbb{T} any fixed time.
- Another algorithm (*A*) permits to generate the conditional distribution of

$$X_{\mathbb{T}} \quad \text{given} \quad \tau_L > \mathbb{T}.$$

Algo (*A*) based on:

- 1 Exact generation of the Brownian motion $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$. Pdf:

$$f_{\mathbb{T}}(x) = \frac{1}{\sqrt{\mathbb{T}}} \frac{\phi(x/\sqrt{\mathbb{T}}) - \phi((x-2L)/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}, \quad x < L.$$

- 2 Rejection sampling: Girsanov's transform in a similar way as (*HZ*).

Combining Algo (HZ) (generation of τ_L) and (A) (conditional distribution of $X_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$) permits to generate the first time a jump diffusion overcomes a given threshold L .

$$dX_t = b(X_{t-}) dt + \sigma(X_{t-}) dB_t + \int_{\mathcal{E}} j(t, X_{t-}, v) p_{\lambda}(dv \times dt), \quad t \geq 0.$$

- $p_{\lambda}(dv \times dt)$ is a Poisson measure on $\mathcal{E} \times [0, T]$ whose intensity measure is given by $\lambda(dv)dt$, λ being non negative finite.
- the jump rate corresponds to $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \rightarrow \mathbb{R}$

We build a new algorithm which generates $(\tau_L \wedge \mathbb{T}, X_{\tau_L \wedge \mathbb{T}})$

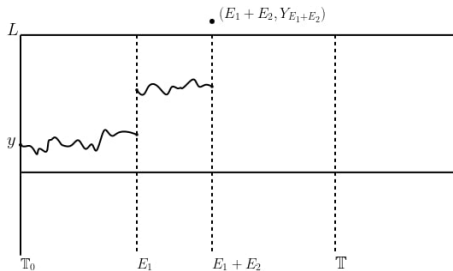
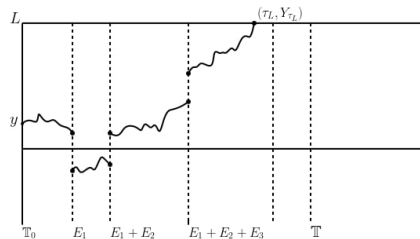
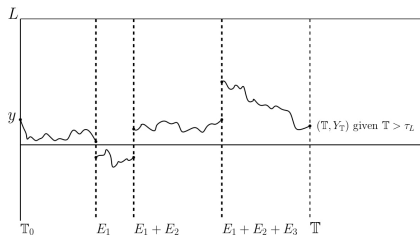


Figure: Three typical paths representing different scenarios

3rd Case: simulation of τ_I for a given $I = [a, b]$ (H. & Zucca, 2020).

We recall that $X_0 = x$,

$$dX_t = dB_t + b(X_t) dt, \beta(x) = \int_0^x b(y) dy \text{ and } \gamma := \frac{b^2 + b'}{2}.$$

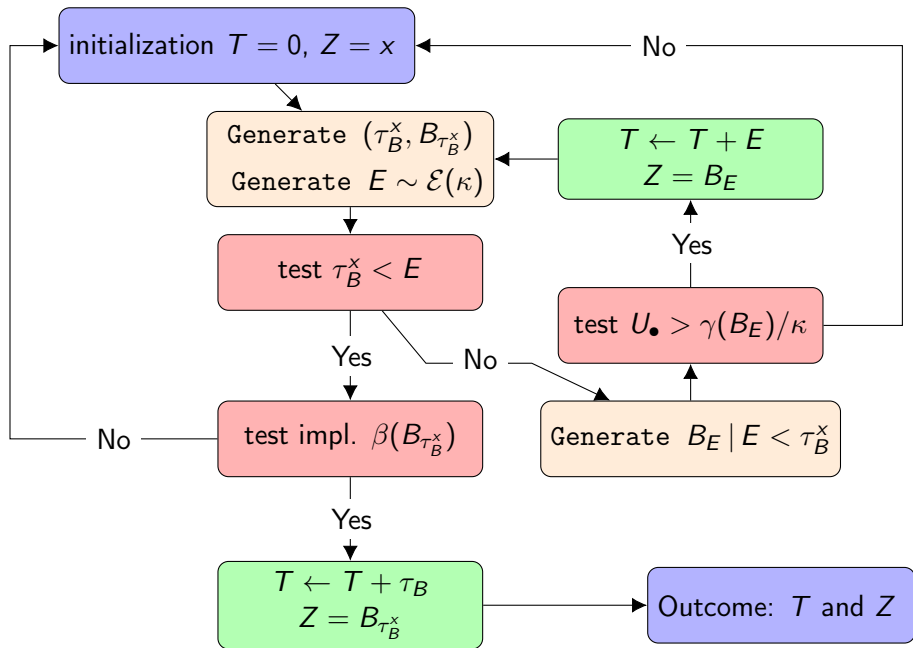
Using Girsanov's transformation and Itô's lemma:

$$\begin{aligned} \mathbb{E}[\psi(\tau_I^X, X_{\tau_I^X})] &= \mathbb{E}\left[\psi(\tau_I^B, X_{\tau_I^B}) e^{\beta(B_{\tau_I^B}) - \beta(x)} e^{-\int_0^{\tau_I^B} \gamma(B_s) ds}\right] \\ &= \mathbb{E}[\psi(\tau_I^B, B_{\tau_I^B}) h(\tau_I^B, B_{\tau_I^B})] \end{aligned}$$

with

$$\begin{aligned} h(t, y) &\propto e^{\beta(y)} \mathbb{E}\left[e^{-\int_0^t \gamma(B_s) ds} \middle| B_0 = x, \tau_I^B = t, B_{\tau_I^B} = y\right] \\ &= e^{\beta(y)} \mathbb{E}\left[e^{-\int_0^t \gamma(\xi_s) ds}\right] \end{aligned}$$

where $(\xi_s, 0 \leq s \leq t)$ is a constrained Brownian motion.



$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x \in (a, b),$$

Theorem

- If $\gamma(\cdot)$ is a non negative function on $[a, b]$ and upper bounded by κ , then the outcome of the algorithm (Z, T) has the same distribution as (X_{τ_I}, τ_I) .
- Moreover the global cost is given by $\mathcal{N}_{\text{tot}}^x$ satisfying::

$$\mathbb{E}[\mathcal{N}_{\text{tot}}^x] \leq C(t_c, t_e) \cosh \left(\sqrt{\frac{\kappa}{2}}(b - a) \right), \quad \forall x \in]a, b[.$$

Generalization:

- 1 to any drift term $b \in \mathcal{C}^1([a, b])$. The modified algorithm is based on an iterative procédure.
- 2 to any diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ using the Lamperti transform.

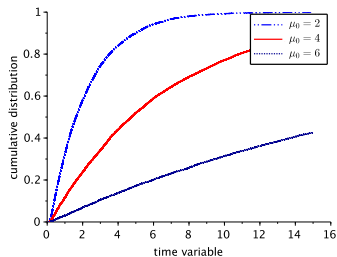


Figure: Empirical cumulative distribution function on $[0, 15]$ for the OU exit time of $[-1, 1]$ (10 000 simulations)

$$dX_t = -\mu_0 X_t dt + dB_t$$

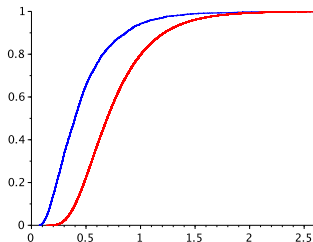


Figure: FET from $[a, b] = [-1, 2]$ for the diffusion

$$dX_t = (2 + \sin(X_t)) dt + dB_t$$

(sample size: 100 000).

To sum up...

- 1 Exact simulation of the first passage time for a continuous diffusion.
- 2 Exact simulation of the first time a jump diffusion overcomes L
- 3 Exact simulation of the first exit time for continuous diffusion

Related questions:

- Exit time from a domain in \mathbb{R}^d with $d \geq 2$.
- Exit time for nonlinear diffusions ???

S. H. and C. Zucca, *Exact simulation of the first-passage time of diffusions*

J. Sci. Comput. 79 (2019), no. 3, 1477-1504.

S. H. and C. Zucca, *Exact simulation of first exit times for one-dimensional diffusion processes*, ESAIM M2NA 54 (2020), no.3, 811–844

S. H. and C. Zucca, *Exact simulation of diffusion first exit times: algorithm acceleration*. J. Mach. Learn. Res. 23 (2022)

S. H. and N. Massin, *Exact simulation of the first passage time through a given level for jump diffusions* (2021) arXiv:2106.05560