Periodicity and large time behavior for excitable mean-field systems.

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Joint works with Christophe Poquet (Lyon 1).

Outline

A general model of mean-field particles

Consider N interacting diffusions $X_t^i \in \mathbb{R}^d$, $i = 1, \ldots, N$, $t \geqslant 0$ solving

$$\mathrm{d}X_t^i = \left(\delta F(X_t^i) - K\left(X_t^i - \frac{1}{N}\sum_{j=1}^N X_t^j\right)\right)\mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_t^i$$

The above dynamics is decomposed into :

- Local dynamics : $\delta F(X_t^i)\mathrm{d}t$, ($\delta > 0$: scaling parameter),
- Linear interaction with the mean value : $-K\left(X_t^i - \frac{1}{N}\sum_{j=1}^N X_t^j\right)$, ($K = diag\,(k_1, \ldots, k_d)$, $k_i > 0$: matrix of interaction),
- Noise : $B^1, \ldots, B^N$: standard i.i.d. Brownian motions, ($\sigma = diag\,(\sigma_1, \ldots, \sigma_d)$, $\sigma_i > 0$: diffusion coefficients).

Motivation : excitable systems

The local dynamics of each particle is governed by

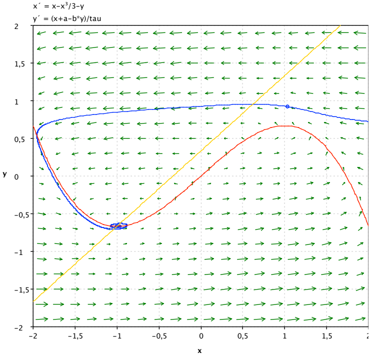
$$\mathrm{d}X_t = F(X_t)\mathrm{d}t$$

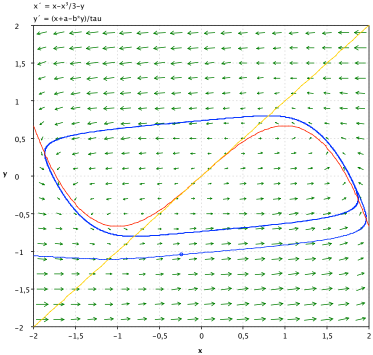
Excitable system [Lindner 2004] :

- without perturbation : rests in a stable state,
- with a sufficiently strong perturbation : the system leaves this resting state, resulting in a large excursion in the phase space.

An example : FitzHugh-Nagumo model : $X_t = (v_t, w_t) \in \mathbb{R}^2$, ($v_t$: membrane potential) and

$$F(X) = F(v, w) = \left(v - \frac{v^3}{3} - w, \frac{1}{c}(v + a - bw)\right), \ a, b, c \in \mathbb{R}.$$





One is particularly interested in phenomena like

- Persistence of oscillatory behaviors (e.g. when individual oscillations persist for the whole macroscopic population)
- Emergence of macroscopic structured dynamics due to noise and interaction (when individual dynamics is not oscillatory)

Kinetic or not kinetic

Two frameworks for FHN :

•The case with full coupling and noise :

$$\begin{cases} \mathrm{d}V_t^i = \left(\delta\left(V_t^i - \frac{\left(V_t^i\right)^3}{3} - W_t^i\right) - k_1\left(V_t^i - \frac{1}{N}\sum_{j=1}^N V_t^j\right)\right)\mathrm{d}t + \sqrt{2}\sigma_1 \mathrm{d}B_t^{i,V}, \\ \mathrm{d}W_t^i = \left(\frac{\delta}{c}\left(V_t^i + a - bW_t^i\right) - k_2\left(W_t^i - \frac{1}{N}\sum_{j=1}^N W_t^j\right)\right)\mathrm{d}t + \sqrt{2}\sigma_2 \mathrm{d}B_t^{i,W} \end{cases}$$

•The kinetic case : interaction and noise only on the V variable :

$$\begin{cases} \mathrm{d}V_t^i = \left(\delta\left(V_t^i - \frac{\left(V_t^i\right)^3}{3} - W_t^i\right) - k_1\left(V_t^i - \frac{1}{N}\sum_{j=1}^N V_t^j\right)\right)\mathrm{d}t + \sqrt{2}\sigma_1 \mathrm{d}B_t^{i,V}, \\ \mathrm{d}W_t^i = \frac{\delta}{c}\left(V_t^i + a - bW_t^i\right)\mathrm{d}t \end{cases}$$

Emergence of periodicity : 1) not enough noise

Emergence of periodicity : 2) some more noise

Behavior of the system as $N \to \infty$ on $[0, T]$

$$\mathrm{d}X_{i,t} = \left(\delta F(X_{i,t}) - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)\right)\mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_{i,t}$$

Let μ_N be the empirical measure

$$\mu_{N,t} = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_{j,t}}.$$

Then, under mild assumptions on F, [Sznitman, McKean] $\mu_{N,t} \xrightarrow[N\to\infty]{} \mu_t$ weak solution to the nonlinear Fokker-Planck (NFP) equation

$$\partial_t \mu_t = -\delta\nabla\cdot(F(x)\mu_t) + \nabla\cdot\left(K\left(x - \int y\mu_t(\mathrm{d}y)\right)\right) + \nabla\cdot\left(\sigma^2\nabla\mu_t\right).$$

More precisely, take e.g. the bounded-Lipschitz distance $d_{BL}(\mu,\nu) := \sup_{\|f\|_{Lip} \leqslant 1} \left|\int f\mathrm{d}\mu - \int f\mathrm{d}\nu\right|$,

$$\mathbf{E}\left[\sup_{0\leqslant t\leqslant T} d_{BL}\left(\mu_{N,t}, \mu_t\right)\right] \leqslant \frac{e^{CT}}{\sqrt{N}}.$$

Outline

Question 1 : emergence of periodic solutions for the NFP

$$\partial_t \mu_t = -\delta \nabla \cdot (F(x)\mu_t) + \nabla \cdot \left(K \left(x - \int y \mu_t(\mathrm{d}y) \right) \right) + \nabla \cdot \left(\sigma^2 \nabla \mu_t \right).$$

μ is the law of the nonlinear process

$$\mathrm{d}X_t = \delta F(x_t)\mathrm{d}t - K\left(X_t - \mathbf{E}\left[X_t\right]\right)\mathrm{d}t + \mathrm{d}B_t$$

Previous works

- [Mischler, Quiñinao, Touboul 2016] : long-time analysis for the FHN model in the kinetic case, in the regime $k_1 \to 0$.
- [Quiñinao, Touboul 2018] : long-time analysis for the kinetic FHN model when $\sigma \to 0$.

Here, we are looking at the regime of strong noise and interaction w.r.t. the individual dynamics, namely

K and σ are fixed and $\delta \ll 1$.

Taking $\delta \to 0$: a slow-fast analysis

$$\partial_t \mu_t = \nabla\cdot\left(\sigma^2 \nabla \mu_t\right) + \nabla\cdot\left(K\mu_t(x)\left(x - \int_{\mathbb{R}^d} z\mu_t(\mathrm{d}z)\right) - \delta\nabla\cdot\left(\mu_t(x)F(x)\right)\right).$$

The main point of the analysis is to decompose the process μ_t in terms of

- its mean value $m_t = \int_{\mathbb{R}^d} z\mu_t(\mathrm{d}z)$
- the centered process p_t defined by

$$\int_{\mathbb{R}^d} \varphi(x) p_t(\mathrm{d}x) := \int_{\mathbb{R}^d} \varphi(x - m_t)\mu_t(\mathrm{d}x).$$

Rewriting everything in terms of (p_t, m_t), we obtain

$$\begin{cases} \partial_t p_t(x) &= \nabla\cdot\left(\sigma^2 \nabla p_t(x)\right) + \nabla\cdot\left(p_t(x)(Kx + \dot{m}_t - \delta F(x + m_t))\right) \\ \dot{m}_t &= \delta \int_{\mathbb{R}^d} F(x + m_t) p_t(\mathrm{d}x) \end{cases}$$

This is a slow-fast dynamics !

The case $\delta = 0$: a simple Ornstein-Uhlenbeck process

In the case $\delta = 0$, the non-linear process reads

$$\mathrm{d}\bar{X}_t = -K\left(\bar{X}_t - \mathbb{E}\left(\bar{X}_t\right)\right)\mathrm{d}t + \sqrt{2}\sigma\mathrm{d}B_t.$$

Hence, the mean-value $\mathbb{E}\left(\bar{X}_t\right) = m_0$ is constant and the dynamics of the process μ_t is simply given by

$$\partial_t \mu_t = \nabla \cdot (\sigma^2 \nabla \mu_t(x)) + \nabla \cdot (\mu_t(x) K x).$$

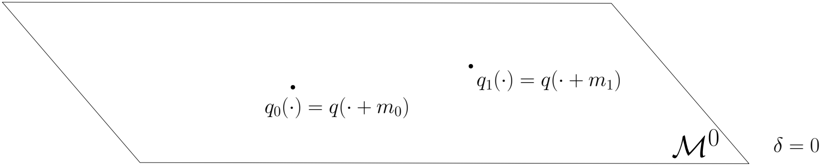
This is nothing else than the law of the Ornstein-Uhlenbeck process ! In this case, the dynamics is reversible and we have exponential convergence to a Gaussian invariant measure :

$$\left\|\mu_t - q\right\|_{L^2(w)} \leqslant e^{-k_0 t}\left\|\mu_0 - q\right\|_{L^2(w)},$$

for

$$q(x) = \frac{1}{\left((2\pi)^d \det(\sigma^2 K^{-1})\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} x \cdot K\sigma^{-2} x\right),\ x \in \mathbb{R}^d.$$

Perturbing normally hyperbolic invariant manifolds



Claim

- For $\delta = 0$, there is a trivial normally hyperbolic invariant (Gaussian) manifold for the evolution.
- Such structure is stable by small perturbations (Geometric singular perturbation theory : [Fenichel 1972], [Giacomin, Pakdaman, Pellegrin, Poquet 2012], [Wiggins 2013])

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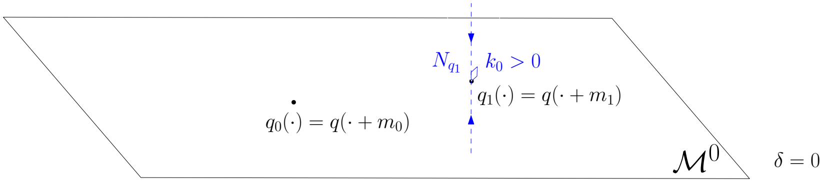

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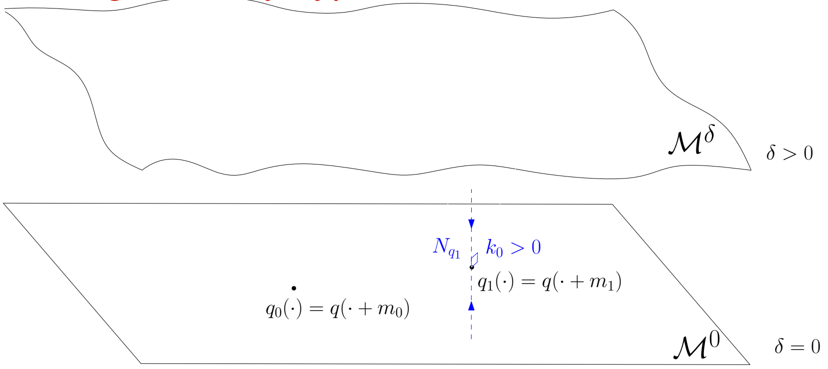

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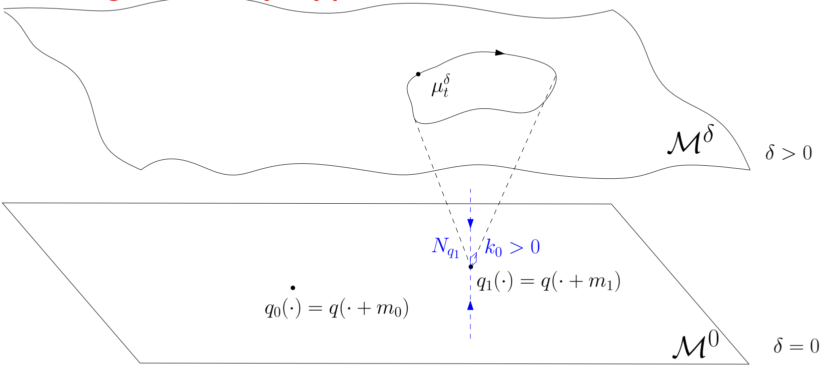

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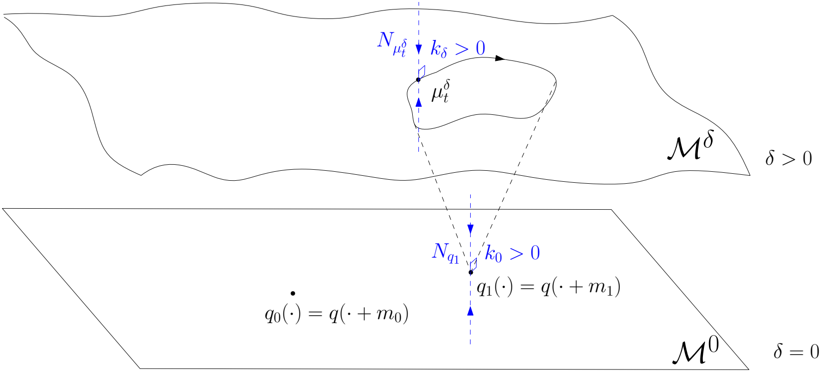

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What is, at first order in δ, the dynamics of the mean-value m along the perturbed manifold ?

$$\begin{cases} \partial_t p_t(x) & = \nabla \cdot (\sigma^2 \nabla p_t(x)) + \nabla \cdot (p_t(x)(Kx + \dot{m}_t - \delta F(x + m_t))) \\ \dot{m}_t & = \delta \int_{\mathbb{R}^d} F(x + m_t) p_t(\mathrm{d}x) \end{cases}$$

Intuitively,

- The fast component $\mu_t^\delta \approx \mu_t^{\delta=0}$ which converges exponentially fast to its invariant measure q.
- Hence, it is natural to replace μ_t^δ by the Gaussian measure q in the slow component m.
- Thus, it is natural to expect

$$\dot{m}_t = \delta \int_{\mathbb{R}^d} F(x + m_t) q(x) \mathrm{d}x + O(\delta^2).$$

Theorem (L., Poquet, 2019)

There exists $\bar{\delta} > 0$ such that for all $0 \leqslant \delta \leqslant \bar{\delta}$, there exists a positively invariant manifold $\mathcal{M}^\delta = \left\{(p_m^\delta, m)\right\}_m$ for the nonlinear Fokker-Planck PDE, where μ_m^δ is a probability measure on $\mathbb{R}^d$.

- *$\mathcal{M}^\delta$ is a perturbation of size δ of the Gaussian manifold $\mathcal{M}^0$*

$$\sup_m \|p_m^\delta - q\|_{L^2(w)} \leqslant C\delta.$$

- *$\mathcal{M}^\delta$ is stable in the following sense : if $\|p_0 - p_{m_0}^\delta\|_{L^2(w)} \leqslant c\delta$, then*

$$\|p_t - p_{m_t}^\delta\|_{L^2(w')} \leqslant ce^{-\lambda t}\|p_0 - p_{m_0}^\delta\|_{L^2(w')}$$

- *The trajectory $t \mapsto m_t^\delta$ of the mean-value of $\mu_{m_t}^\delta$ has the following expansion :*

$$\dot{m}_t^\delta = \delta \int_{\mathbb{R}^d} F(u + m_t) q(u) \mathrm{d}u + \delta^2 g^\delta(m_t^\delta),$$

with $\|g^\delta\| \leqslant C$.

Idea of proof : balance between dynamics and equilibrium

Define the Ornstein-Uhlenbeck operator

$$\mathcal{L}f \;=\; \nabla \cdot (\sigma^2 \nabla f) + \nabla \cdot (Kxf) \,.$$

Then,

$$\partial_t(p_t - q_0) \;=\; \mathcal{L}(p_t - q_0) + \nabla \cdot (p_t \dot{m}_t) - \delta \nabla \cdot \big(p_t F_t\big).$$

The key estimate is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_t - q_0\|^2_{L^2(w)} \;\leqslant\; -(k - c'\delta)\|p_t - q_0\|^2_{L^2(w)} + C\delta\|p_t - q_0\|_{L^2(w)} \,,$$

which gives

$$\|p_t - q_0\|_{L^2(w)} \;\leqslant\; \max\big(\|p_0 - q_0\|_{L^2(w)}, C\delta\big)$$

To sum up

With this result, the dynamics of the infinite dimensional measure μ_t along the manifold $\mathcal{M}^\delta$ can be parameterized by the finite dimensional dynamics of its mean-value m_t^δ.
Namely, everything boils down to comparing now

- the dynamics of an isolated unit

$$\dot{m}_t = F(m_t) \tag{1}$$

- with the dynamics of the mean-value of connected units, approximated at first order in δ by

$$\dot{m}_t = \delta \int_{\mathbb{R}^d} F(x + m_t) q(x) \mathrm{d}x. \tag{2}$$

Claim

In the FHN case, for a suitable choice of parameters, Eq. (1) is in a resting state, whereas Eq. (2) oscillates : there is emergence of structured dynamics due to noise and interaction.

The FitzHugh-Nagumo model

In the case of a FHN dynamics

$$F(v,w) = \left(v - \frac{v^3}{3} - w, \frac{1}{c}(v + a - bw) \right) ,$$

with diagonal interaction $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and noise $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$.

Averaging F by the Gaussian kernel q gives

$$\int_{\mathbb{R}^2} F(z + (v,w))q(z)\mathrm{d}z = \left(\left(1 - \frac{\sigma_1^2}{K_1} \right) v - \frac{v^3}{3} - w, \frac{1}{c}(v + a - bw) \right)$$

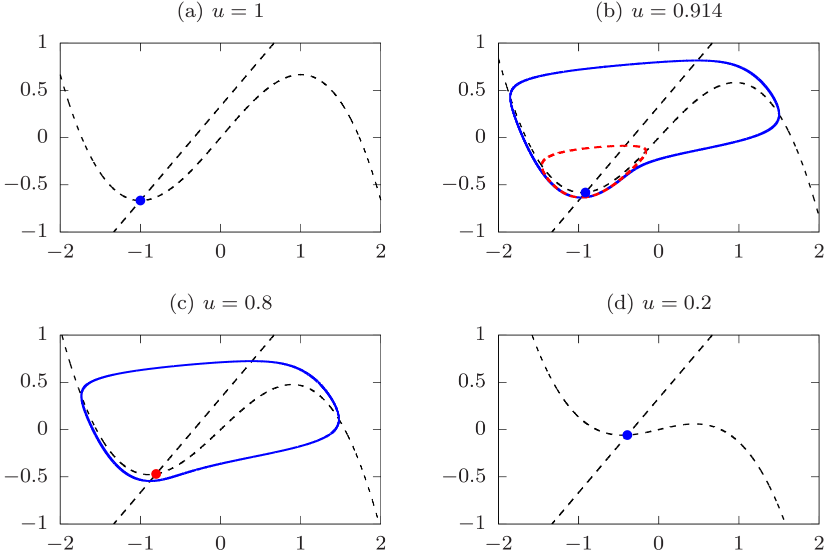
which is again a FitzHugh Nagumo system, where the parameter $u = 1$ has been changed into $u = 1 - \frac{\sigma_1^2}{K_1}$.

Summary

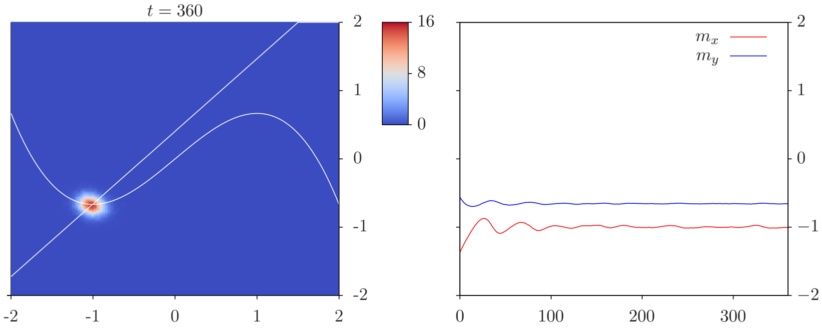
Adding noise and interaction simply means here decreasing the parameter u from 1 to $u = 1 - \frac{\sigma_1^2}{K_1} < 1$.

Phase transition as $u = 1 - \frac{\sigma_1^2}{K_1} \searrow 0$ for

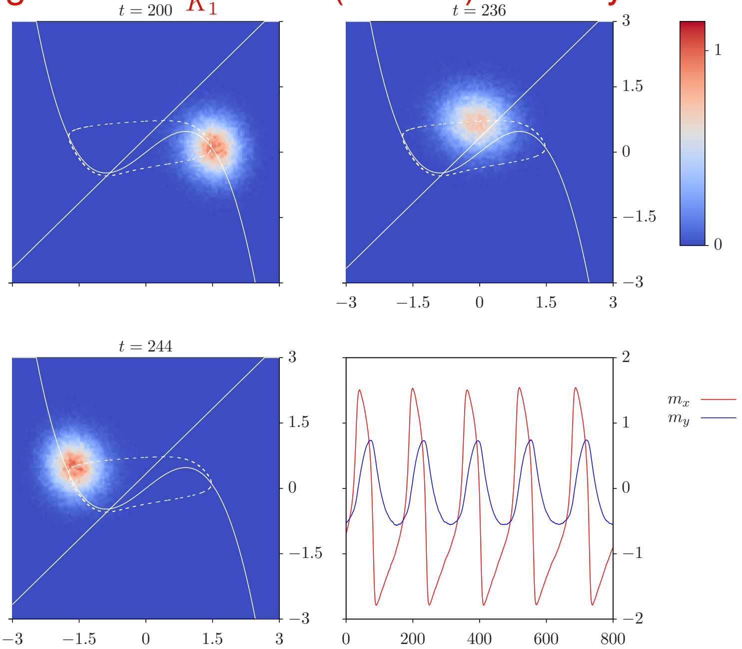
$$\dot{v} = uv - \frac{v^3}{3} - w, \dot{w} = \frac{1}{c}(v + a - bw)$$



No noise : $\frac{\sigma_1^2}{K_1} = 0$ ($u = 1$)



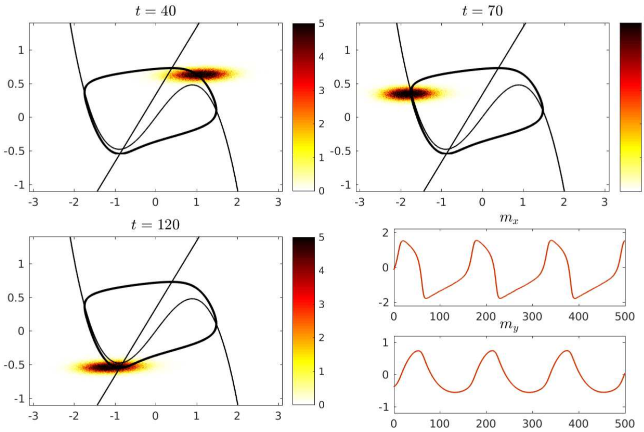
Large noise : $\frac{\sigma_1^2}{K_1} = 0.2$ $(u = 0.8)$: limit cycle



The kinetic case

[L., Poquet 2020]

$$\begin{cases} \mathrm{d}V_{i,t} = \delta\left(V_{i,t} - \frac{V_{i,t}^3}{3} - w_{i,t}\right)\mathrm{d}t - K_1\left(V_{i,t} - \frac{1}{n}\sum_{j=1}^n V_{j,t}\right)\mathrm{d}t + \sqrt{2}\sigma_1 \mathrm{d}B_{i,t}^{(1)} \\ \mathrm{d}w_{i,t} = \frac{\delta}{c}\left(V_{i,t} + a - bw_{i,t}\right)\mathrm{d}t \end{cases}$$



Outline

1. Mean-field excitable systems
2. Emergence of periodicity when $N = \infty$
3. The case $N < \infty$: the empirical measure on long time scales

Question 2 : the empirical measure on large time scales

$$\mathbf{E}\left[\sup_{0 \leqslant t \leqslant T} d_{BL}\left(\mu_{N,t}, \mu_t\right)\right] \leqslant \frac{e^{CT}}{\sqrt{N}}.$$

The previous estimate is only relevant for bounded time intervals $T = O(1)$ (or $T \sim c\log(N)$, c small). The question we ask is the following :

1. Can we transpose the existence of a limit cycle for μ_t to a similar periodic behavior for the empirical measure $\mu_{N,t}$?

The answer is NO : $\mu_{N,t}$ is Markovian and for well-confining dynamics F, μ_N has a unique invariant measure. On long time-scales, the system diffuses and cannot have a periodic behavior. So it should be better to rephrase the question into

1'. Is there a time scale α_N on which the behavior of $\mu_{N,\alpha_N t}$ remains close to the periodic solution of μ_t ?

This issue has a long story, especially in the reversible case where $F = -\nabla V$ for some sufficiently convex potential V ([Bolley, Guillin, Malrieu]).

A “simpler model” : phase oscillators

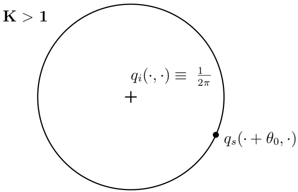
Here the state space is the circle $\mathbb{S}^1$ and $X_{i,t}$ solves

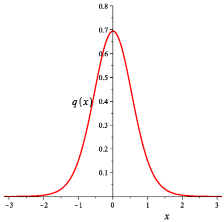
$$\mathrm{d}X_{i,t} = \frac{K}{N}\sum_{j=1}^{N}\sin(X_{j,t}-X_{i,t})\mathrm{d}t + \sqrt{2}\sigma\mathrm{d}B_i(t), \quad i=1,\ldots,N,$$

and the nonlinear Fokker-Planck equation reduces to

$$\partial_t\mu_t(x) = \sigma^2\partial_x^2\mu_t(x) - K\partial_x\left[\mu_t(x)\Big(\sin*\mu_t(x)\Big)\right].$$

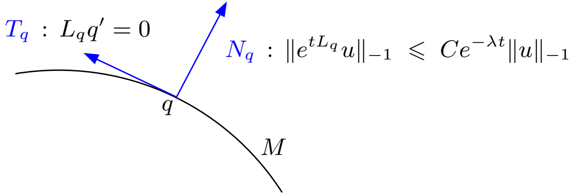
The steady states are given as a manifold of stationary steady states M





Local stability of the invariant manifold

Let $L_q h := \frac{1}{2}\partial_x^2 h + K\partial_x \left(h \sin * q + q \sin * h\right)$ be the linearization around some stationary solution $q \in M$. L_q can be decomposed in $H^{-1}_{1/q}$ as follows :



This decomposition implies the local stability of M. [Dahms 2002, Bertini, Giacomin, Pakdaman, 2010], but we also have global stability [Giacomin, Pakdaman, Pellegrin, Poquet, 2012].

Long-time diffusion for the Kuramoto model :

Theorem ([Bertini, Giacomin, Poquet, 2014])

For $T > 0$ *and* $\varepsilon > 0$, *if*

$$\lim_{N \to \infty} \mathbb{P}\left(\mathrm{d}\left(\mu_{N,0}, M\right) \leqslant \varepsilon\right) = 1$$

then

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{t \in [\log(N)/N, T]} \left\| \mu_{N,tN} - q_{\psi_0 + DW_{N,t}} \right\|_{-1} \leqslant \varepsilon\right) = 1$$

where $W_{N,t}$ *converges weakly to a standard Brownian motion as* $N \to \infty$.

Further references

- [Brassesco, De Masi, Presutti 1998, Funaki 1995] : Allen-Cahn model.
- [L., Poquet, 2017]] : extension to the case of the Kuramoto model with inhomogeneities.
- [Coppini, 2019] : long time stability results for the Kuramoto model on Erdös-Renyi graphs.
- [Giacomin, Poquet, Shapira, 2018]] : long-time diffusion for SDEs with small noise in finite dimension : specific use of the isochron map to derive the motion along the limit cycle.

Going back to limit cycles in $\mathbb{R}^d$

Main issues concerning the previous results :

1. The existence of limit cycle is stated in weighted L^2 norm, too strong to accommodate for empirical measures
2. The existence result is not sufficiently complemented with regularity results (in particular, regularity of the isochron, a C^2-regularity being required, as we will need to apply Ito formula)

Solutions :

1. We now work weighted Sobolev spaces H_θ^{-r} (recall that $\mu^n \in H_\theta^{-r}$ for $r > d/2$), for the H^{-r} space weighted by

$$w_\theta(x) = \exp\left(-\frac{\theta}{2}\,|x|^2_{K\sigma^{-2}}\right).$$

2. Main assumption : F and all its derivatives regular and bounded.

We use here an abstract result of [Bate, Lu, Zeng 2008] concerning Approximately Invariant Manifolds.

Regularity of limit cycles for the F-P equation

Recall that

$$\begin{cases} \partial_t p_t(x) & = \nabla \cdot (\sigma^2 \nabla p_t(x)) + \nabla \cdot (p_t(x)(Kx + \dot{m}_t - \delta F(x + m_t))) \\ \dot{m}_t & = \delta \int_{\mathbb{R}^d} F(x + m_t) p_t(\mathrm{d}x) \end{cases} \tag{NFP}$$

Theorem ([L., Poquet, 2021])

For some small $\delta > 0$ and large $r \geqslant 1$, for all $\theta \in (0, 1]$, (NFP) admits a periodic solution $\left(\Gamma_t^\delta\right)_t := (q_t^\delta, \gamma_t^\delta)_t$ in $\mathbf{H}_\theta^{-r}$ with period $T_\delta > 0$. Moreover q_t^δ is a probability distribution for all $t \geqslant 0$, and $t \mapsto \partial_t \Gamma_t^\delta$ and $t \mapsto \partial_t^2 \Gamma_t^\delta$ are in $C([0, T_\delta), \mathbf{H}_\theta^{-r})$.
Secondly, there exists a neighborhood $\mathcal{W}^\delta \in \mathbf{H}_\theta^{-r}$ of Γ^δ and a C^2 mapping $\Theta^\delta : \mathcal{W}^\delta \to \mathbb{R}/T_\delta\mathbb{Z}$ that satisfies, for all $\mu \in \mathcal{W}^\delta$, denoting $\mu_t = T^t \mu$,

$$\Theta^\delta(\mu_t) = \Theta^\delta(\mu) + t \quad \textit{mod}\ T_\delta,$$

and for $C_{\Theta,\delta} > 0$, for all $\mu \in \mathcal{W}^\delta$ with $\mu_t = T^t \mu$,

$$\left\| \mu_t - \Gamma^\delta_{\Theta^\delta(\mu)+t} \right\|_{\mathbf{H}_\theta^{-r}} \leqslant C_{\Theta,\delta} e^{-\lambda_\delta t} \left\| \mu - \Gamma^\delta_{\Theta^\delta(\mu)} \right\|_{\mathbf{H}_\theta^{-r}} .$$

Now turn to the dynamics of the empirical measure

- F-P equation :

$$\partial_t \mu_t = -\delta \nabla \cdot (F(x)\mu_t) + \nabla \cdot \left(K \left(x - \int y \mu_t(\mathrm{d}y) \right) \mu_t \right) + \nabla \cdot (\sigma^2 \nabla \mu_t)$$

$\mu_t \leftrightarrow (p_t, m_t)$ where $\dot{m}_t := \delta \int F_{m_t} \mathrm{d}p_t$ and

$$\partial_t p_t = \nabla \cdot (\sigma^2 \nabla p_t) + \nabla(p_t K x) - \delta \nabla \cdot \left(p_t \left(F_{m_t} - \int F_{m_t} \mathrm{d}p_t \right) \right)$$

- Particle system :

$$\mathrm{d}X_{i,t} = \left(\delta F(X_{i,t}) - K \left(X_{i,t} - \frac{1}{N} \sum_{j=1}^{N} X_{j,t} \right) \right) \mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_{i,t}.$$

$\mu_{N,t} \leftrightarrow (p_{N,t}, m_{N,t})$ where $m_{N,t} := \frac{1}{N} \sum_{j=1}^{N} X_{j,t}$ and $p_{N,t} := \frac{1}{N} \sum_{j=1}^{N} \delta_{X_{j,t} - m_{N,t}}$.

The main result for the particle system

Theorem ([L., Poquet, 2021])

Under the previous hypotheses, if

$$\mathbb{P}\left(\|\mu_{N,0} - \Gamma_{u_0}\|_{-r} \leqslant \varepsilon \right) \xrightarrow[N\to\infty]{} 1$$

then for all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left\|\mu_{N,Nt} - \Gamma_{u_0+Nt+v_{N,t}}\right\|_{-r} \leqslant \varepsilon \right) \xrightarrow[N\to\infty]{} 1$$

where the random process converges weakly to $v_t = gt + a^2 w_t$*, where* w *is a standard BM and* a, b *depend “explicitly” on the two first derivatives of* Θ*.*

Step 1 : staying close to the manifold Γ

Proposition

Suppose that $\mathbb{P}\left(\|\mu_{N,0} - \Gamma_{u_0}\|_{-r} \leqslant \varepsilon\right) \xrightarrow[N\to\infty]{} 1$. *Then*

$$\mathbb{P}\left(\sup_{t\in[\log(N)/N,T]} \mathrm{dist}\left(\mu_{N,Nt},\Gamma\right) \leqslant N^{-(1/2-)}\right) \xrightarrow[N\to\infty]{} 1.$$

Key argument : mild formulation for the dynamics of $\nu_{N,t} := \mu_{N,t} - proj_\Gamma(\mu_{N,t})$ on $[0,T]$:

$$\nu_{N,t} = \Phi_{t,0}\nu_{N,0} + \int_0^t \Phi_{t,s}R_s(\nu_{N,s})\mathrm{d}s + Z_{N,t},$$

- $\Phi_{t,s}$ is the semigroup of the linearized dynamics in a neighborhood of Γ, which has good contracting properties,
- $R(\nu)$ is quadratic in ν
- Z_N is a noise term of order $N^{-(1/2-)}$.

Step 2 : deriving the dynamics along the manifold Γ

Key argument : apply Ito formula to the the isochron map $\Theta(\mu_{N,t})$:

$$\Theta(\mu_{N,t}) = \Theta(\mu_{N,0}) + t - \frac{1}{N}\int_0^t D_1\Theta_{\mu_{N,s}} \nabla \cdot \left(\sigma^2 \nabla p_{N,s}\right) \mathrm{d}s + \frac{1}{2}\int_0^t D^2\Theta_{\mu_{N,s}} \mathrm{d}\left[\!\left[M_N \right]\!\right]_s + W_{N,t}.$$

It remains to prove that the remaining terms in this formula give a nontrivial contribution on a time scale of order N.

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Merci de votre attention !