Project PERISTOCH's meeting

# Stochastic resonance in stochastic PDEs

Rita Nader

Institut Denis Poisson, University of Orléans, France

23-24 May 2022

based on joint work with Nils Berglund

project PERISTOCH

# A simple model of stochastic resonance

**Particle subjected to two perturbations:**

- A deterministic periodic driving force
- An additive noise

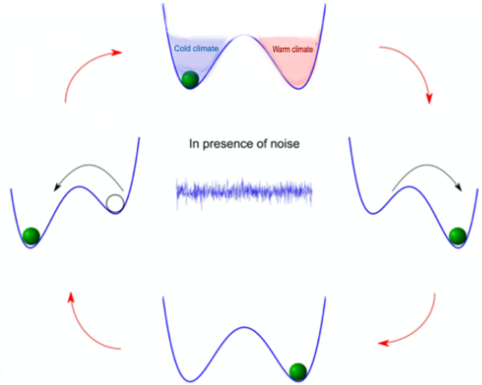


Figure: Overdamped motion of a particle in a double-well potential.

# Stochastic resonance in one dimensional SDEs

$$\mathrm{d}X_t = f(\varepsilon t, X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- $W_t$ is a standard Wiener process
- Example: "A double-well potential"

$$f(\varepsilon t, x) = x - x^3 + A\cos(\varepsilon t) = -\frac{\partial}{\partial x} U(\varepsilon t, x)$$

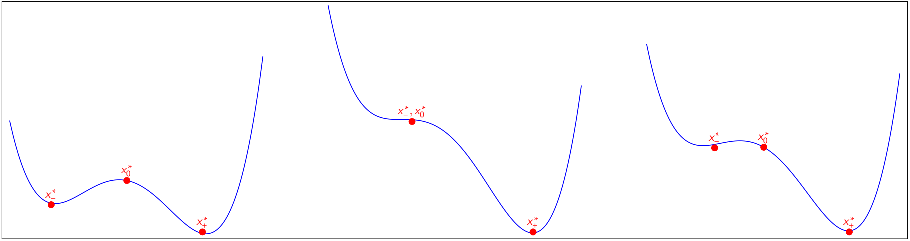


Figure: An asymmetric potential $U(\varepsilon t, x)$.For $A\cos(\varepsilon t) < A_c$ (left, right), a double-well potential . For $A\cos(\varepsilon t) = A_c$ (middle), a single well potential.

# Stochastic resonance in one dimensional SDEs

Stationary solutions:

- If $A = 0$, $x - x^3 = 0$ has exactly three solutions: $\pm 1$ and $0$
- If $A \neq 0$ and $A < A_c = \frac{2}{3\sqrt{3}}$, $x - x^3 + A\cos(t) = 0$ has exactly three solutions: $x_-^*(t) < x_0^*(t) < x_+^*(t)$

Added noise: Mechanism of Stochastic Resonance

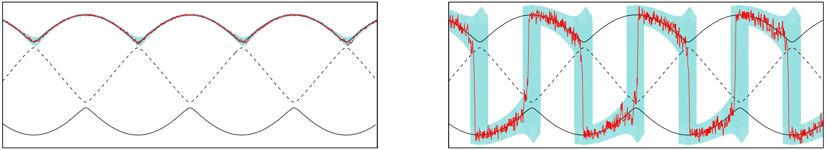

Figure: Sample paths of the SDE $\mathrm{d}X_t = \frac{1}{\varepsilon}\left[X_t - X_t^3 + A\cos(t)\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$ for $\varepsilon = 0.005$, and $\sigma = 0.02$ (left picture) and $\sigma = 0.14$ (right picture).

# Infinite-dimensional stochastic PDEs

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\left[\Delta\phi(t,x) + f(t,\phi(t,x))\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $\phi(t,x) \in \mathbb{R}$, $t \geqslant 0$, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$;
- $0 \leqslant \varepsilon, \sigma \ll 1$;
- $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $f(t,\phi) = -\partial_\phi U(t,\phi)$.
  Example: "Periodically forced Allen-Cahn equation"
  $$f(t,\phi(t,x)) = \phi(t,x) - \phi(t,x)^3 + A\cos(t);$$
- $\mathrm{d}W(t,x) = \xi(t,x)\,\mathrm{d}t$ where
  - $\xi$ space-time white noise: centered, Gaussian, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$;
  - $\xi$ distribution, $\langle \xi, \varphi \rangle \sim \mathcal{N}(0, \|\varphi\|^2_{L^2})$, $\mathbb{E}[\langle \xi, \varphi_1\rangle\langle \xi, \varphi_2\rangle] = \langle \varphi_1, \varphi_2\rangle$.

# Stable case: Deterministic dynamics

- Define the fractional Sobolev norm of $\phi$ by $\|\phi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}}(1+k^2)^s\phi_k^2$
- Assume that

$$\exists\phi^* : I \to \mathbb{R}; f(t,\phi^*(t)) = 0 \text{ and } \partial_\phi f(t,\phi^*(t)) < 0 \quad \forall t \in I = [0,T]$$

Proposition 1:

$\exists\ C, \varepsilon_0 > 0$; for $0 < \varepsilon < \varepsilon_0$,

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}[\Delta\phi(t,x) + f(t,\phi(t,x))]\,\mathrm{d}t$$

admits a solution $\bar{\phi}(t,x)$ satisfying

$$\|\bar{\phi}(t,\cdot) - \phi^*(t)e_0\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I,$$

where $e_0(x) = \frac{1}{\sqrt{L}}$.

Hint: Lyapounov function

$\psi(t,\cdot) = \phi(t,\cdot) - \phi^*(t)e_0$;

$V(\psi) = \frac{1}{2}\|\psi\|_{H^1}^2$

# Stable case: Stochastic dynamics

Define for any $h > 0$,

- $\mathcal{B}(h) = \Big\{(t,\phi) \colon t \in I, \|\phi - \bar\phi(t,\cdot)\|_{H^s} < h\Big\}$
- $\tau_{\mathcal{B}(h)} = \inf\Big\{t > 0 \colon \|\phi - \bar\phi(t,\cdot)\|_{H^s} \geqslant h\Big\}$

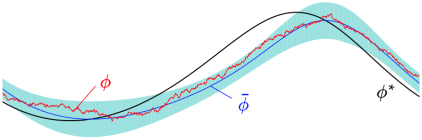


Figure: Concentration of sample paths.

Theorem 1: For any $s \in (0, \frac{1}{2})$ and $\nu > 0$, $\exists\ \varepsilon_0, h_0, \kappa(s), C(\kappa, T, \varepsilon, s) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h \leqslant h_0 \varepsilon^\nu$ one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < t\Big\} \leqslant C(\kappa, T, \varepsilon, s) \exp\bigg\{-\kappa \frac{h^2}{\sigma^2}\bigg[1 - \mathcal{O}\bigg(\frac{h}{\varepsilon^\nu}\bigg)\bigg]\bigg\} \ \forall t \in I \ .$$

# Proof

- ▷ $\psi(t,x) = \phi(t,x) - \overline{\phi}(t,x)$
- ▷ $h = h_0 + h_1$ with $h_0, h_1 > 0$,

$$\mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < t\Big\} = \mathbb{P}\Bigg\{\sup_{0\leqslant t\leqslant T\wedge\tau_{\mathcal{B}(h)}} \|\psi(t,\cdot)\|_{H^s} > h\Bigg\}$$

$$\leqslant \mathbb{P}\Bigg\{\sup_{0\leqslant t\leqslant T} \|\psi^0(t,\cdot)\|_{H^s} > h_0\Bigg\}$$

$$+ \mathbb{P}\Bigg\{\sup_{0\leqslant t\leqslant T\wedge\tau_{\mathcal{B}(h)}} \|\psi^1(t,\cdot)\|_{H^s} > h_1, \sup_{0\leqslant t\leqslant T} \|\psi^0(t,\cdot)\|_{H^s} \leqslant h_0\Bigg\}$$

# Proof

- $$\psi^0(t,\cdot) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar\alpha(t,t_1)/\varepsilon}\,\mathrm{e}^{[(t-t_1)/\varepsilon]\Delta}\,\mathrm{d}W(t_1,\cdot),$$

  where $\bar\alpha(t,t_1) = \int_{t_1}^t \bar a(u)\,\mathrm{d}u$ and $\bar a(t) = \partial_\phi f(t,\overline{\phi}(t,x))$,
- $h_0^2 = \sum_{k\in\mathbb{Z}} h_k^2$,

  $$\mathbb{P}\Big\{\sup_{t\in I}\|\psi^0(t,\cdot)\|_{H^s} > h_0\Big\} \leqslant \sum_{k\in\mathbb{Z}} \mathbb{P}\bigg\{\sup_{t\in I}|\psi_k^0(t)|^2 > h_k^2(1+k^2)^{-s}\bigg\}$$
  $$\leqslant \sum_{k\in\mathbb{Z}} C_T(1+k^2)\exp\bigg\{-\kappa(1+k^2)\frac{h_k^2(1+k^2)^{-s}}{\sigma^2}\bigg\}$$

  Choosing $h_k^2 = C(\eta,s)h^2(1+k^2)^{-1+s+\eta/2}$, $0<\eta<2\rho$.

# Proof

- $$\psi^1(t,\cdot) = \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\bar\alpha(t,t_1)/\varepsilon}\,\mathrm{e}^{[(t-t_1)/\varepsilon]\Delta}\, b(t_1,\psi(t_1,\cdot))\,\mathrm{d}t_1,$$

  where $|b(t,\psi(t,x))| \leqslant M\psi^2$, for all $t \in I$,
- $\mathbb{P}\big\{\sup_{0\leqslant t\leqslant T\wedge\tau_{\mathcal{B}(h)}} \|\psi^1(t,\cdot)\|_{H^s} > h_1, \sup_{0\leqslant t\leqslant T}\|\psi^0(t,\cdot)\|_{H^s} \leqslant h_0\big\} = 0$
  1. Assume $\psi(t,\cdot) \in H^s$ for all $0 < s < \frac{1}{2}$ then $b(t,\psi(t,\cdot)) \in H^r$ for all $r < \frac{1}{2}$.
  2. $\forall q < r+2$, $\exists M'(q,r) < \infty$; $\forall t \in I$, $\psi^1(t,\cdot) \in H^q$ and

     $$\|\psi^1(t,\cdot)\|_{H^q} \leqslant M'(q,r)\varepsilon^{\frac{q-r}{2}-1} \sup_{0\leqslant t_1\leqslant t} \|b(t_1,\psi(t_1,\cdot))\|_{H^r}\ .$$

  3. Choosing $h_1 = M'(q,r)\varepsilon^{\frac{q-r}{2}-1} Mh^2$.

# Unstable case: Near (0,0)

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + g(t) - \phi(t,x)^2 - b(t,\phi(t,x))\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x),$$

- $g(t) = \delta + t^2 + \mathcal{O}(t^3), \delta \geqslant 0$ and $b(t,\phi) = \mathcal{O}(\phi^3) + \mathcal{O}(t\phi^2) + \mathcal{O}(t^2\phi)$
- $\phi^*_\pm(t)$ such that $f(t,\phi^*_\pm(t)) = 0$, $\partial_\phi f(t,\phi^*_\pm(t)) \asymp \mp\big(\sqrt{\delta} + |t|\big)$

$\Updownarrow \phi(t,x) = \phi_0(t)e_0(x) + \phi_\perp(t,x)$

Coupled SDE-SPDE system:

$$\mathrm{d}\phi_0 = \tfrac{1}{\varepsilon}\Big[g(t) - \phi_0^2 - b(t,\phi_0(t)e_0) + b_0(t,\phi_0(t),\phi_\perp(t))\Big]\,\mathrm{d}t + \tfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0$$

$$\mathrm{d}\phi_\perp = \tfrac{1}{\varepsilon}\Big[\Delta\phi_\perp + a(t,\phi_0)\phi_\perp + b_\perp(t,\phi_0(t),\phi_\perp(t))\Big]\,\mathrm{d}t + \tfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_\perp$$

- $a(t,\phi_0) = \mathcal{O}(-2\phi_0(t)) < 0$ and $b_0, b_\perp$ are non-local remainders.

# Unstable case: Deterministic dynamics

**Proposition 2:**

The deterministic equation with $\sigma = 0$ admits $\phi_\perp(t,x) = 0$ and $\phi_0$ obeys

$$\varepsilon\dot{\phi}_0(t) = g(t) - \phi_0(t)^2 - b(t, \phi_0(t)e_0).$$

[N.B. & Barbara Gentz 2002]:
$\exists\ \bar{\phi}_0(t)$ tracking $\phi_+^\star(t)$:

$$\exists T_0, c_0 > 0;\ \bar{\phi}_0(t) - \phi_+^*(t) \asymp \begin{cases} \dfrac{\varepsilon}{|t|} & \text{for } -T_0 \leqslant t \leqslant -c_0\max\{\sqrt{\delta}, \sqrt{\varepsilon}\} \\ -\dfrac{\varepsilon}{|t|} & \text{for } c_0\max\{\sqrt{\delta}, \sqrt{\varepsilon}\} \leqslant t \leqslant T_0. \end{cases}$$

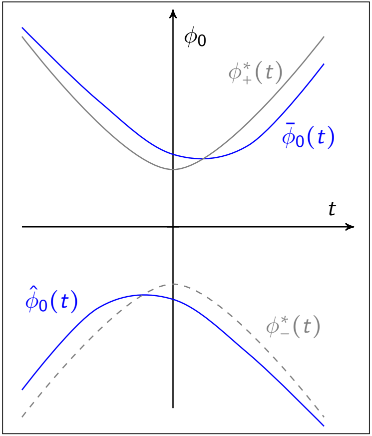


Figure: Equilibrium branches and associated adiabatic solutions near $(0,0)$.

# Transverse stochastic dynamics for $\phi_\perp$

Given $h_\perp > 0$, we define the set

$$\mathcal{B}_\perp(h_\perp) = \Big\{(t,\phi)\colon t \in [-T_0, T_0], \|\phi_\perp\|_{H^s} < h_\perp\Big\}\,.$$

Theorem 2: "Exit from $\mathcal{B}_\perp(h_\perp)$"

If $T_0$ is sufficiently small, then for any $s \in (0, \frac{1}{2})$ and $\nu > 0$, $\exists \varepsilon_0, h_\perp^0, \kappa(s)$ and $C(\kappa, T, \varepsilon, s) > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$ and $0 < h_\perp \leqslant h_{0\perp}\varepsilon^\nu$ one has

$$\mathbb{P}\Big\{\tau_{\mathcal{B}_\perp(h_\perp)} < t \wedge \tau_{\mathcal{B}_0(h)}\Big\} \leqslant C(\kappa, T, \varepsilon, s)\exp\left\{-\kappa\frac{h_\perp^2}{\sigma^2}\left[1 - \mathcal{O}\left(\frac{h_\perp}{\varepsilon^\nu}\right)\right]\right\}.$$

# Stochastic dynamics near $\bar{\phi}_0(t)$

- $\zeta(t) \asymp \frac{1}{|\bar{a}(t,\bar{\phi}_0)|}$ and $\hat{\zeta}(t) = \sup_{-T_0 \leqslant s \leqslant t} \zeta(s) \quad \forall t \in [-T_0, T_0]$
- $\mathcal{B}_0(h) = \left\{(t, \phi_0) : t \in [-T_0, T_0], |\phi_0 - \bar{\phi}_0(t)| < h\sqrt{\zeta(t)}\right\}$

**Theorem 3: "Exit from $\mathcal{B}_0(h)$"**

There exist $\varepsilon_0, h_0, \kappa, c_\perp > 0$; whenever $0 < \varepsilon \leqslant \varepsilon_0$, $0 < h \leqslant h_0 \hat{\zeta}(t)^{-3/2}$ and $0 < h_\perp \leqslant c_\perp h \hat{\zeta}(t)^{1/2}$ one has

$$\mathbb{P}\left\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\right\} \leqslant C(t,\varepsilon) \exp\left\{-\kappa \frac{h^2}{2\sigma^2}\right\},$$

where $\kappa = 1 - \mathcal{O}(h\hat{\zeta}(t)^{3/2})$.

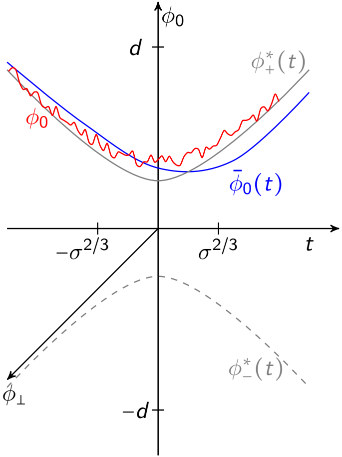


Figure: Weak noise regime

# Noise regime

- Weak-noise regime: if $\sigma \ll \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can be applied for any $t \in [-T_0, T_0]$, and shows that $\phi_0(t)$ remains close to $\bar{\phi}_0(t)$ with high probability during the whole time interval.

# Noise regime

- Weak-noise regime: if $\sigma \ll \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can be applied for any $t \in [-T_0, T_0]$, and shows that $\phi_0(t)$ remains close to $\bar{\phi}_0(t)$ with high probability during the whole time interval.
- Strong-noise regime: if $\sigma \geqslant \max\{\varepsilon, \delta\}^{3/4}$, Theorem 3 can only be applied up to times $t$ of order $-\sigma^{2/3}$, showing that $\phi_0(t)$ is unlikely to become negative up to times of that order.

# Stong-noise regime

**Theorem 4:**

Assume $\sigma \geqslant \max\{\varepsilon, \delta\}^{3/4}$. Fix $d, c_1 > 0$.
Let $h > 0$;

$$\bar{\phi}_0(t) + h\sqrt{\zeta(t)} \leqslant d \quad \forall t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}].$$

Then there exist $\kappa, \bar{c}_\perp > 0$; for $0 < h_\perp < \bar{c}_\perp[\sigma^{2/3} \wedge \sqrt{h}\hat{\zeta}(t)^{-1/4}]$, one has

$$\mathbb{P}\Big\{\phi_0(t_1) > -d \ \forall t_1 \in [-c_1\sigma^{2/3}, t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}]\Big\}$$

$$\leqslant \frac{3}{2}\exp\Big\{-\kappa\frac{\sigma^{4/3}}{\varepsilon\log(\sigma^{-1})}\Big\} + C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2},$$

for all $t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$.

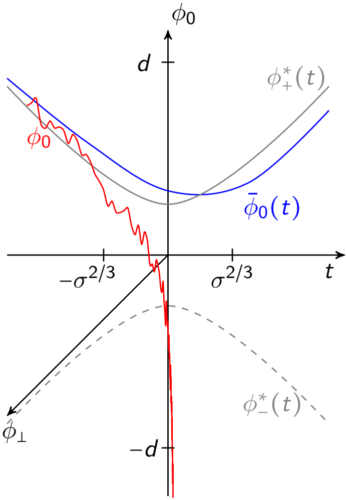


Figure: Strong noise regime.

# Stong-noise regime

Proposition 3:

Assume $\sigma \geqslant \max\{\varepsilon, \delta\}^{3/4}$. There exist $\tilde{c}, \tilde{\kappa} > 0$; for all $t_0 \in [-T_0, T_0 - \tilde{c}\varepsilon]$, the solution with initial condition $\phi_0(t_0) = -d$ satisfies

$$\mathbb{P}\Big\{\phi_0(t_1) > -d_0 \ \forall t_1 \in [t_0, (t_0 + \tilde{c}\varepsilon) \wedge \tau_{\mathcal{B}_\perp}(h_\perp)]\Big\} \leqslant e^{-\tilde{\kappa}/\sigma^2} .$$

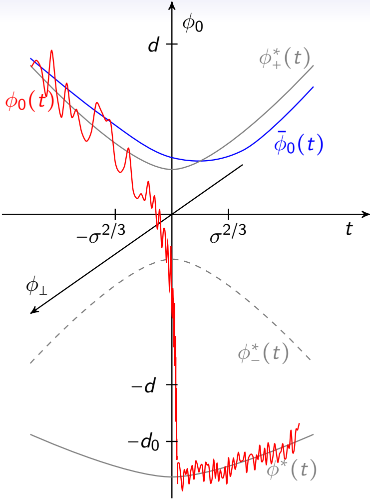


Figure: Strong noise regime, behaviour after reaching level $-d$.

# Stochastic PDEs on $\mathbb{T}^2$

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + f(t,\phi(t,x))\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

▷ $\sigma = 0$: PDE is well-defined.

# Stochastic PDEs on $\mathbb{T}^2$

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + f(t,\phi(t,x))\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $\sigma = 0$: PDE is well-defined.
- $\sigma \neq 0$: SPDE is not well-posed and $\phi^0(t,\cdot) \in \mathcal{C}^\alpha$ for any $\alpha < 0$.

# Stochastic PDEs on $\mathbb{T}^2$

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}[\Delta\phi(t,x) + f(t,\phi(t,x))]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $\sigma = 0$: PDE is well-defined.
- $\sigma \neq 0$: SPDE is not well-posed and $\phi^0(t,\cdot) \in \mathcal{C}^\alpha$ for any $\alpha < 0$.

A finite-dimensional spectral Galerkin approximation of the two-dimensional stochastic Allen-Cahn renormalised equation:

$$\mathrm{d}\phi_N(t,x) = \frac{1}{\varepsilon}[\Delta\phi_N(t,x) + \phi_N(t,x) - :\phi_N(t,x)^3:_{c_N}]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_N(t,x)$$

- $\phi_N(t,x) = \sum_{\substack{k\in\mathbb{Z}^2\\|k|\leqslant N}} \phi_k(t)e_k(x)$ and $c_N \sim \log(N) \xrightarrow[N\to+\infty]{} +\infty$ ;
- $:\phi_N(t,x)^3:_{c_N} = \phi_N(t,x)^3 - 3c_N\phi_N(t,x)$ is the wick power.

Argument of Da Prato-Debussche: $\phi = \phi^0 + \varphi$, where $\varphi$ is expected to be more regular than $\phi$.

# References


- N. B. & Nader Rita, *Stochastic resonance in stochastic PDEs*, Stoch PDE: Anal Comp, 2022
- N. B. & Barbara Gentz, *Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond*, Electron. J. Probab., **18**(24):1–58, 2013
- N. B. & Barbara Gentz, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields, **122**(3):341–388, 2002
- N. B. & Barbara Gentz., *A sample-paths approach to noise-induced synchronization: stochastic resonance in a double-well potential*, Ann. Appl. Probab., **12**(4):1419–1470, 2002
- Bourdaud Gérard & Meyer Yves, *Fonctions qui opèrent sur les espaces de Sobolev*, J. Funct. Anal., **97**(2):351–360, 1991


Thanks for listening!