[theorem]Lemma

Scaling limit for a stochastic Lokta-Volterra process

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Motivation

Stochastic cyclic Lokta-Volterra system with N particles distributed over 3 species : hen-fox-viper, which are prey/predators of one another.

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Stochastic cyclic Lokta-Volterra system with N particles distributed over 3 species : hen-fox-viper, which are prey/predators of one another.

$$N \to +\infty \;:\; \text{what happens on} \begin{cases} \text{finite time scales ?} \\ \text{long time scales ?} \end{cases}$$

Modelling

N indiscernable particles moving over a network of 3 sites in a circular way. Let x_i =proportion of particules on the site i, and $x = (x_1; x_2; x_3)$. So x 2 Σ (simplex of R^3).

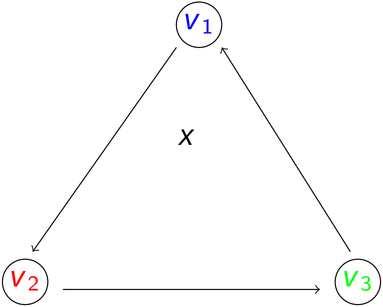

Figure – Representation of the composition of the system

Jump rate of the process

$(X_t^N)_{t \geq 0}$ is a Markov jump process on a finite set.

Jump of a particle from i to $i+1$ with the following jump rate :

$$^N_{\{i \to i+1\}} = Nx_i^N \;\; a + Nx_{i+1}^N \quad , \text{ where } a \geq 0.$$

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Case $a = 0$ **:** Interaction between particles

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If $x_j(t_0) = 0$: then for $t \ \ t_0$ the site j is no longer filled.

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If $x_j(t_0) = 0$: then for $t \quad t_0$ the site j is no longer filled.

Consequence : $X_t^N \ \underset{t\ !\ +1}{!} \ v_i$ *a.s.* for a certain i.

Simulation for $a = 0$

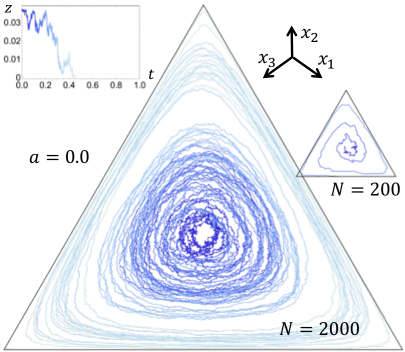

Figure – Trajectory (X_t^N) for $a = 0$.

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Large numbers law : $X_t^N \underset{t \to +1}{\to} \quad \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \quad a.s.$

Simulation for $a > 0$

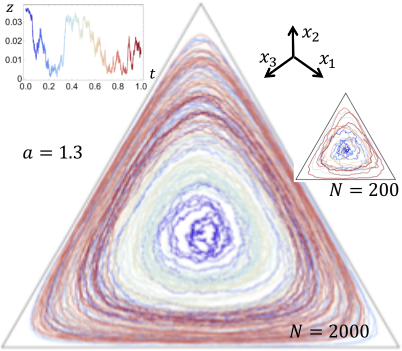

Figure – Trajectory (X_t^N) for $a = 1{:}3$.

Asymptotic behaviour

L_N is the Markov generator of the process with N particles. For $g \in C^2(S)$:

$$L_N g \underset{N \to +\infty}{\sim} N L_0 g + L_1 g.$$

$L_0 = v_0 \cdot \nabla$, with $v_0 \in C^1(\Sigma; \mathbb{R}^3)$.

L_1 : 2-order elliptic operator (drift+diffusion).

Fast dynamic L_0

For x_s satisfying $\dot{x}_s = v_0(x_s)$: $z(x) = 27(x_1 x_2 x_3)$ is constant.

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$z(x) \in [0;1]$: parameter of homogeneity.

Figure – Level lines of z on .

4 stationnary points : $\{v_1; v_2; v_3\}$, and $\left(\frac{1}{3}; \frac{1}{3}; \frac{1}{3}\right)$.

For $T(z_0)$ the period of rotation of (x_t) over $\{z(x) = z_0\}$

$$(\bar{L}_1 f) \circ z(x_0) = \frac{1}{T(z_0)} \int_0^{T(z_0)} L_1(f \circ z)(x_s)\, ds.$$

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We obtain $\overline{L}_1 = b\,\partial_z + \frac{\sigma^2}{2}\partial_z^2$, where $b, \sigma \in C([0,1])$.

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Our aim : prove, in a certain meaning, that for $f \in C^2[0;1]$

$$L_N(f \circ z) \underset{N \to \infty}{\to} (\overline{L}_1 f) \circ z;$$

Consider the process $Z_t^N = z \; X_t^N$ for $t \in [0; \;]$.

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Recall $\overline{L}_1 = b\,\partial_z + \frac{\;^2}{2}\,\partial_z^2$.

Theorem

If $Z_0^N \underset{N \to \infty}{\Rightarrow} z_0 \in [0; 1]$, then $P_{Z^N} \underset{N \to \infty}{\Rightarrow} P_Z$, satisfying $Z_0 = z_0$ a.s and the SDE

$$dZ_t = b(Z_t)\,dt + \;(Z_t)\,dW_t;$$

where W is a Wiener process.

$$Z_t^N = A_t^N + M_t^N$$

where (A^N) is a finite variation process and (M^N) is a martingale.

Theorem : $(A^N)/(M^N)$ is relatively compact if $(A^N)/(\langle M^N \rangle)$ satisfy the Aldous criterion : $\forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0, \exists N_0 \in \mathbb{N}$,

$$\sup_{N \geq N_0} \; \sup_{S, S' \text{ stopping times} ;\, S \leq S'} P\left(|A^N_{S'} - A^N_S| > \varepsilon\right) :$$

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$\Longrightarrow$ For P_{Z^N} : existence of a limit point P_Z.

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$\Rightarrow$ For P_{Z^N} : existence of a limit point P_Z.

Question : How to identify P_Z ?

For g 2 $C^2(\ )$, the following is a martingale for P_{X^N} :

$$M_t^{N;g} : x \; 2 \; D \; [0; \ ]; \quad 7! \quad g(x_t) \qquad \int_0^t L_N g(x_s) \, ds:$$

For $g \in C^2(\;)$, the following is a martingale for P_{X^N} :

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Question : For $f \in C^2([0;1])$, is the following a martingale for P_Z ?

$$M_t^{f} : z \in D([0;\;];[0;1]) \mapsto \quad f(z_t) \quad \int_0^t L_1 f(z_s)\,ds.$$

Consider $(\Omega; T; P)$ on which X^N and Z are defined and

$$Z^N \underset{N\to\infty}{\longrightarrow} Z_t \text{ a.s.}$$

If we prove that $\forall t \in [0; \;]$,

$$E \; M_t^{N;(f \;\; z)} \; X^N \quad M_t^f \; Z \quad \underset{N\to+\infty}{\longrightarrow} 0;$$

then (M^f) is a martingale for P_Z.

$(x_t)_{t \quad 0}$ trajectory over satisfying $\underline{x}_t = v_0 \; x_t$ for $t \qquad 0$.

$T_N = \ln \quad \ln N$, satisfying : $\frac{1}{N} \qquad \frac{T_N}{N} \qquad 1$.

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Lemma

For $(X^N_t)_{t \quad 0}$ with generator L_N, satisfying $X^N_0 = x_0$ a.s

$$\sup_{x_0 2} \quad \sup_{t \, 2 \, [0;T_N]} E \;\; X^N_{t=N} \qquad x_t \quad \underset{N!1}{!} \quad 0:$$

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For $(X_t^N)_{t \quad 0}$ with generator L_N, satisfying $X_0^N = x_0$ a.s

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Lemma

$$\sup_{x_0 2} \frac{1}{T_N} \int_0^{T_N} L_1(f \quad z)(x_s)\, ds \quad \overline{L}_1 f \quad z(x_0) \quad \underset{N!1}{!} \quad 0:$$

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In particular

$$\frac{1}{T(Z_0)} \int_0^{T(z_0)} h(x_s)\, ds \underset{z_0 \to 0^+}{\longrightarrow} \frac{1}{3} \sum_{i=1;2;3} h(e_i):$$

For $h = L_1(f - z)$: $h(e_i) = 0$: does not depend on i.

T_i^{i+1} corresponds to the mixing time for z_0 small.

(M_t^f) being a martingale for $f\ 2\ C^2\ [0;1]$,

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Further questions :

How do (Z_t) behaves at the points f 0; 1g ?

Is (Z_t) a Feller process ?

Operator $Lf = b\partial_x f + \frac{\sigma^2}{2}\partial_x^2 f$, where $b, \sigma \in C(]0;1[)$ and $\sigma > 0$, defined over

$$D(L) = \left\{ f \in C([0;1]) \cap C^2(]0;1[) ;\ Lf \in C([0;1]) \right\} .$$

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Then

$$L = \frac{d}{dm}\frac{d}{dp};$$

where $m, p \in C(]0;1[)$.

For $r \in \left]0;1\right[$, $x \in \{0;1\}$, let

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Then the boundary x is said to be

regular	if	${}_{\text{in}} < 1$	and	${}_{\text{out}} < 1$	
exit	if	${}_{\text{in}} < 1$	and	${}_{\text{out}} = 1$	
entrance	if	${}_{\text{in}} = 1$	and	${}_{\text{out}} < 1$	
(natural	if	${}_{\text{in}} = 1$	and	${}_{\text{out}} = 1$	:)

If x is regular, for q 2 [0;1] :

$$D_x(L) = \quad f \;\; 2\, D(L);\; q\, L f(x) = (\quad 1)^x (1 \quad q) \frac{df}{dp}(x) \quad :$$

If x is exit :

$$D_x(L) = \quad f \;\; 2\, D(L);\; L f(x) = 0 \quad :$$

If x is entrance/(natural) :

$$D_x(L) = \; D(L):$$

Theorem : $L; D_0(L) \backslash D \;\; _1(L)$ generates Feller process over ;1]

For the SDE satisfied by (Z_t) :

$z = 1$ is entrance for all $a \geq 0$.

$$z = 0 \text{ is } \begin{cases} \text{exit for } a = 0 \\ \text{regular for } a \in]0;1[\ (*) \\ \text{entrance for } a \geq 1 \ (*). \end{cases}$$

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Problem : If $a > 0$, then for $f \in C^2([0;1])$:

$$L_1 f(z = 0) = 0, \text{ and } \frac{df}{dp}(z = 0) = 0 ;$$

so $C^2([0;1])$ is not a core for L_1.

For 1 a_N N, averaged operator :

$$L_\infty f = v(z)\partial_z f ;$$

where $v \in C^\infty]0;1]$.

For $1 \ll a_N \ll N$, averaged operator :

$$L_1 f = v(z)\partial_z f \,;$$

where $v \in C^1(]0;1])$.

Then $P_{Z^N} \Longrightarrow P_Z$ which satisfies $\dot{Z}_t = v(Z_t)$:

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Problem : Solution of this ODE not unique.

Interpretation : If $L_N g(v_i) := 0$, this modification is not detected for $g = f$ z with $f \in C^2$ [0; 1] :

For $g \in C^2(\ )$, the sequence of laws of $M^{N;g}(X^N)$ is relatively compact.

On $(\ ;T;P)$ where $(Z_t^N) \underset{N\to\infty}{\to} (Z_t)$ a.s, does $M^{N;g}(X^N)$ converge in L^1 ?

$$M_t^{N;g}(X^N) = \quad g(X_t^N) \quad \int_0^t N L_0 g(X_s^N)\, ds \quad \int_0^t L_N \quad N L_0 g \ (X_s^N)\, ds$$

$$g(X_t^N) \quad \int_0^t N L_0 g(X_s^N)\, ds \underset{N\to\infty}{\to} g(Z_t) \text{ in } L^1 \, ?$$

$$\int_0^t L_N \quad N L_0 g \ (X_s^N)\, ds \underset{N\to\infty}{\to} \int_0^t \overline{L_1 g}\,(Z_s)\, ds \text{ in } L^1.$$

If the previous is justified : $g(Z_t) \quad \int_0^t \overline{L_1 g}(Z_s)\, ds$ is a martingale.

Question : for $f \in D(\overline{L_1})$ existence of $g \in C^2(\ )$ satisfying

$$g = f$$
$$\overline{L_1 g} = \overline{L_1} f \ ?$$

Advantage : $\overline{L_1 g}(z = 0) \neq 0$.

Conclusion

Scaling limit = slow/fast dynamics.

Other scalings : a_N 1 and a_N 1.

Dynamics generalized : D 1 sites, and jump rate

$$ {}^{N}_{i\,!\,j}(x) = c_{ij} \quad (x_i) \; a_j + N \quad (x_j) \; ; $$

where $\;\; 2\; C^1(\,[0;1]\,)$ such that $\;(0) = 0$ and $\;^{0} > 0$, and
$c_{ij} \quad 0;\; a_j \quad 0.$