Scaling limit for a stochastic Lokta-Volterra process

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Motivation

• Stochastic cyclic Lokta-Volterra system with N particles distributed over 3 species : hen-fox-viper, which are prey/predators of one another.

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• $N \longrightarrow +\infty$: what happens on $\begin{cases} \text{finite time scales ?} \\ \text{long time scales ?} \end{cases}$

Modelling

N indiscernable particles moving over a network of 3 sites in a circular way. Let x_i =proportion of particules on the site i, and $\boldsymbol{x} = (x_1, x_2, x_3)$. So $\boldsymbol{x} \in \Sigma$ (simplex of $\mathbb{R}^3$).

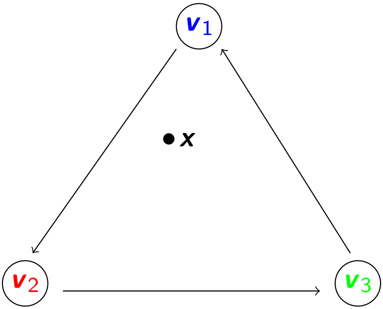

Figure – Representation of the composition of the system

Jump rate of the process

- $(\boldsymbol{X}_t^N)_{t\geq 0}$ is a Markov jump process on a finite set.
- Jump of a particle from i to $i+1$ with the following jump rate :

$$\tau^N_{\{i\rightarrow i+1\}} = Nx_i^N\left(a + Nx_{i+1}^N\right), \text{ where } a \geq 0.$$

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Consequence : $\boldsymbol{X}_t^N \underset{t\to+\infty}{\longrightarrow} \boldsymbol{v}_i$ *a.s.* for a certain i.

Simulation for $a = 0$

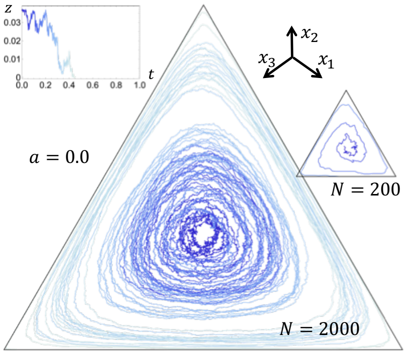

FIGURE – Trajectory $(\boldsymbol{X}_t^N)$ for $a = 0$.

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- Jump of a particle from i to $i+1$ with the following jump rate :

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Large numbers law : $\boldsymbol{X}^N_t \underset{t\to+\infty}{\longrightarrow} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ *a.s.*

Simulation for $a > 0$

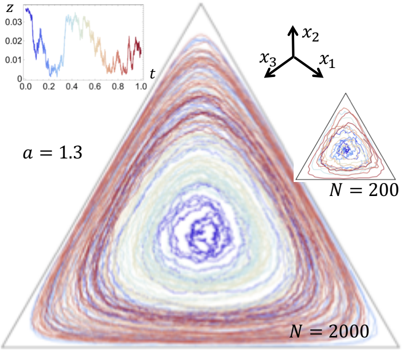

FIGURE – Trajectory $(\boldsymbol{X}_t^N)$ for $a = 1.3$.

Asymptotic behaviour

L_N is the Markov generator of the process with N particles. For $g \in \mathcal{C}^2(S)$:

$$L_N g \underset{N \to +\infty}{\approx} N \mathcal{L}_0 g + \mathcal{L}_1 g.$$

- $\mathcal{L}_0 = \boldsymbol{v}_0 \cdot \nabla$, with $\boldsymbol{v}_0 \in \mathcal{C}^1(\Sigma, \mathbb{R}^3)$.
- $\mathcal{L}_1$: 2-order elliptic operator (drift+diffusion).

Fast dynamic $\mathcal{L}_0$

- For $(\boldsymbol{x}_s)$ satisfying $\dot{\boldsymbol{x}}_s = \boldsymbol{v}_0(\boldsymbol{x}_s)$: $z(\boldsymbol{x}) = 27(x_1 x_2 x_3)$ is constant.

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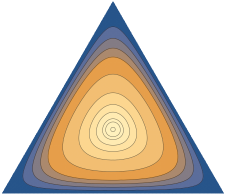

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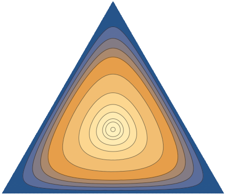

FIGURE – Level lines of z on Σ.

- 4 stationnary points : $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3\}$, and $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The averaged operator $\overline{\mathcal{L}}_1$

For $T(z_0)$ the period of rotation of $(\boldsymbol{x}_t)$ over $\{z(\boldsymbol{x}) = z_0\}$

$$(\overline{\mathcal{L}}_1 f) \circ z(\boldsymbol{x}_0) = \frac{1}{T(z_0)} \int_0^{T(z_0)} \mathcal{L}_1(f \circ z)(\boldsymbol{x}_s)\, \mathrm{d}s.$$

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We obtain $\overline{\mathcal{L}}_1 = b \, \partial_z + \frac{\sigma^2}{2} \partial_z^2$, where $b, \sigma \in \mathcal{C}(\,[0,1]\,)$.

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We obtain $\overline{\mathcal{L}}_1 = b\,\partial_z + \frac{\sigma^2}{2}\,\partial_z^2$, where $b, \sigma \in \mathcal{C}(\,[0,1]\,)$.

Our aim : prove, in a certain meaning, that for $f \in \mathcal{C}^2\big([0,1]\big)$

$$L_N(f \circ z) \underset{N \to \infty}{\approx} (\overline{\mathcal{L}}_1 f) \circ z,$$

Slow variable : theorem of convergence

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Theorem

If $Z^N_0 \underset{N\to\infty}{\Longrightarrow} z_0 \in [0,1]$, then $\mathbb{P}_{Z^N} \underset{N\to\infty}{\Longrightarrow} \mathbb{P}_Z$, satisfying $Z_0 = z_0$ a.s and the SDE

$$\mathrm{d}Z_t = b(Z_t)\,\mathrm{d}t + \sigma(Z_t)\,\mathrm{d}W_t,$$

where W is a Wiener process.

Relative compactness of (Z^N)

$$Z_t^N = A_t^N + M_t^N$$

where (A^N) is a finite variation process and (M^N) is a martingale.

Theorem : $(A^N)/(M^N)$ is relatively compact if $(A^N)/(\langle M^N\rangle)$ satisfy the Aldous criterion : $\forall\,\varepsilon > 0, \forall\,\eta > 0, \exists\,\delta > 0,\ N_0 \in \mathbb{N}$,

$$\sup_{N\geq N_0}\ \sup_{S,S' \text{ stopping times}\,;\ S\leq S'\leq\delta} \mathbb{P}\big(|A_{S'}^N - A_S^N| > \varepsilon\big) \leq \eta.$$

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$\Longrightarrow$ For $\big(\mathbb{P}_{Z^N}\big)$: existence of a limit point $\mathbb{P}_Z$.

Question : How to identify $\mathbb{P}_Z$?

Identification of the limit

For $g \in \mathcal{C}^2(\Sigma)$, the following is a martingale for $\mathbb{P}_{\boldsymbol{X}^N}$:

$$M_t^{N,g} : \boldsymbol{x} \in \mathbb{D}\big([0,\tau],\Sigma\big) \longmapsto g(\boldsymbol{x}_t) - \int_0^t L_N g(\boldsymbol{x}_s)\,\mathrm{d}s.$$

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Question : For $f \in \mathcal{C}^2([0,1])$, is the following a martingale for $\mathbb{P}_Z$?

$$M_t^{f} : z \in \mathbb{D}([0,\tau],[0,1]) \longmapsto f(z_t) - \int_0^t \bar{\mathcal{L}}_1 f(z_s)\,\mathrm{d}s.$$

Convergence

- Consider $(\Omega, \mathcal{T}, \mathbb{P})$ on which $\boldsymbol{X}^N$ and Z are defined and

$$Z^N \underset{N\to\infty}{\longrightarrow} Z_t \ a.s.$$

If we prove that $\forall\, t \in [0, \tau]$,

$$\mathbb{E}\left| M_t^{N,(f\circ z)}(\boldsymbol{X}^N) - M_t^f(Z) \right| \underset{N\to+\infty}{\longrightarrow} 0,$$

then (M^f) is a martingale for $\mathbb{P}_Z$.

Two lemmas

- $(\boldsymbol{x}_t)_{t\geq 0}$ trajectory over Σ satisfying $\dot{\boldsymbol{x}}_t = \boldsymbol{v}_0\big(\boldsymbol{x}_t\big)$ for $t \geq 0$.
- $T_N = \ln \circ \ln N$, satisfying : $\frac{1}{N} \ll \frac{T_N}{N} \ll 1$.

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Lemma

For $(\boldsymbol{X}_t^N)_{t\geq 0}$ with generator L_N, satisfying $\boldsymbol{X}_0^N = \boldsymbol{x}_0$ a.s

$$\sup_{\boldsymbol{x}_0 \in \Sigma} \sup_{t\in[0,T_N]} \mathbb{E}\Big\|\boldsymbol{X}_{t/N}^N - \boldsymbol{x}_t\Big\| \underset{N\to\infty}{\longrightarrow} 0.$$

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Lemma

$$\sup_{\boldsymbol{x}_0\in\Sigma} \left| \frac{1}{T_N}\int_0^{T_N} \mathcal{L}_1(f\circ z)(\boldsymbol{x}_s)\,\mathrm{d}s - \big(\overline{\mathcal{L}}_1 f\big)\circ z(\boldsymbol{x}_0)\right| \underset{N\to\infty}{\longrightarrow} 0.$$

Second lemma : idea of the proof

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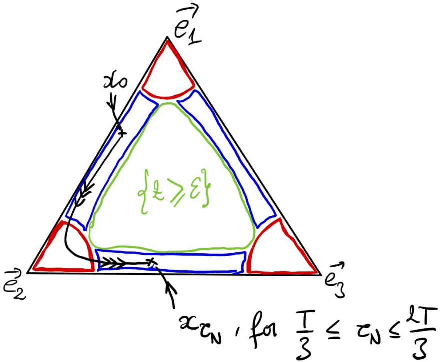

- Time in blue area : bounded
- Time in red area : $\to +\infty$ as $\varepsilon \to 0$

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- In particular

$$\frac{1}{T(Z_0)} \int_0^{T(z_0)} h(\boldsymbol{x}_s)\,\mathrm{d}s \underset{z_0 \to 0^+}{\longrightarrow} \frac{1}{3} \sum_{i=1,2,3} h(\boldsymbol{e}_i).$$

For $h = \mathcal{L}_1(f \circ z)$: $h(\boldsymbol{e}_i) = 0$: does not depend on i.

T_i^{i+1} corresponds to the mixing time for z_0 small.

SDE and boundary points

(M_t^f) being a martingale for $f \in \mathcal{C}^2([0,1])$,

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Further questions :

- How do (Z_t) behaves at the points $\{0,1\}$?
- Is (Z_t) a Feller process ?

Boundary points for 1D diffusion

Operator $\mathcal{L}f = b\,\partial_z f + \frac{\sigma^2}{2}\,\partial_z^2 f$, where $b, \sigma \in \mathcal{C}(]0,1[)$ and $\sigma > 0$, defined over

$$\mathcal{D}(\mathcal{L}) = \big\{ f \in \mathcal{C}(\,[0,1]\,) \cap \mathcal{C}^2(\,]0,1[\,),\ \mathcal{L}f \in \mathcal{C}(\,[0,1]\,) \big\}.$$

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Then

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}p},$$

where $m, p \in \mathcal{C}(\,]0,1[\,)$.

Boundary points for 1D diffusion

For $r \in]0,1[$, $x \in \{0,1\}$, let

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Then the boundary x is said to be

$$\begin{aligned}
\textbf{regular} \quad & \text{if } \tau_{\mathrm{in}} < \infty \text{ and } \tau_{\mathrm{out}} < \infty \\
\textbf{exit} \quad & \text{if } \tau_{\mathrm{in}} < \infty \text{ and } \tau_{\mathrm{out}} = \infty \\
\textbf{entrance} \quad & \text{if } \tau_{\mathrm{in}} = \infty \text{ and } \tau_{\mathrm{out}} < \infty \\
(\textbf{natural} \quad & \text{if } \tau_{\mathrm{in}} = \infty \text{ and } \tau_{\mathrm{out}} = \infty.)
\end{aligned}$$

Boundary points for 1D diffusion

- If x is **regular**, for $q \in [0,1]$:

$$\mathcal{D}_x(\mathcal{L}) = \left\{ f \in \mathcal{D}(\mathcal{L}),\ q\,\mathcal{L}f(x) = (-1)^x(1-q)\frac{\mathrm{d}f}{\mathrm{d}p}(x) \right\}.$$

- If x is **exit** :

$$\mathcal{D}_x(\mathcal{L}) = \left\{ f \in \mathcal{D}(\mathcal{L}),\ \mathcal{L}f(x) = 0 \right\}.$$

- If x is **entrance/(natural)** :

$$\mathcal{D}_x(\mathcal{L}) = \mathcal{D}(\mathcal{L}).$$

Theorem : $\left(\mathcal{L}, \mathcal{D}_0(\mathcal{L}) \cap \mathcal{D}_1(\mathcal{L})\right)$ generates Feller process over $[0,1]$.

Boundary points for the SDE associated with $\bar{\mathcal{L}}_1$

For the SDE satisfied by (Z_t) :

- $z = 1$ is **entrance** for all $a \geq 0$.
- $z = 0$ is $\begin{cases} \textbf{exit} \text{ for } a = 0 \\ \textbf{regular} \text{ for } a \in]0,1[\ (*) \\ \textbf{entrance} \text{ for } a \geq 1 \ (*). \end{cases}$

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Problem : If $a > 0$, then for $f \in \mathcal{C}^2([0,1])$:

$$\bar{\mathcal{L}}_1 f(z=0) = 0, \text{ and } \frac{\mathrm{d}f}{\mathrm{d}p}(z=0) = 0,$$

so $\mathcal{C}^2([0,1])$ is not a core for $\bar{\mathcal{L}}_1$.

Another scaling

- For $1 \ll a_N \ll N$, averaged operator :

$$\bar{\mathcal{L}}_1 f = v(z)\partial_z f,$$

where $v \in \mathcal{C}^1\big(\,]0,1]\,\big)$.

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Problem : Solution of this ODE not unique.

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Problem : Solution of this ODE not unique.

Interpretation : If $L_N g(\boldsymbol{v}_i) := 0$, this modification is not detected for $g = f \circ z$ with $f \in \mathcal{C}^2\big(\,[0,1]\,\big)$.

An attempt

- For $g \in \mathcal{C}^2(\Sigma)$, the sequence of laws of $M^{N,g}(\boldsymbol{X}^N)$ is relatively compact.

- On $(\Omega, \mathcal{T}, \mathbb{P})$ where $(Z_t^N) \underset{N\to\infty}{\longrightarrow} (Z_t)$ *a.s*, does $M^{N,g}(\boldsymbol{X}^N)$ converge in L^1 ?

$$M_t^{N,g}(\boldsymbol{X}^N) = \left(g(\boldsymbol{X}_t^N) - \int_0^t N\mathcal{L}_0 g(\boldsymbol{X}_s^N)\,\mathrm{d}s \right) - \int_0^t (L_N - N\mathcal{L}_0 g)(\boldsymbol{X}_s^N)\,\mathrm{d}s$$

$$g(\boldsymbol{X}_t^N) - \int_0^t N\mathcal{L}_0 g(\boldsymbol{X}_s^N)\,\mathrm{d}s \underset{N\to\infty}{\longrightarrow} \bar{g}(Z_t) \text{ in } L^1\,?$$

$$\int_0^t (L_N - N\mathcal{L}_0 g)(\boldsymbol{X}_s^N)\,\mathrm{d}s \underset{N\to\infty}{\longrightarrow} \int_0^t \overline{\mathcal{L}_1 g}\,(Z_s)\,\mathrm{d}s \text{ in } L^1.$$

An attempt

If the previous is justified : $\bar{g}(Z_t) - \int_0^t \overline{\mathcal{L}_1 g}(Z_s)\,\mathrm{d}s$ is a martingale.

Question : for $f \in \mathcal{D}(\overline{\mathcal{L}}_1)$ existence of $g \in \mathcal{C}^2(\Sigma)$ satisfying

$$\bar{g} = f$$
$$\overline{\mathcal{L}_1 g} = \overline{\mathcal{L}}_1 f?$$

Advantage : $\overline{\mathcal{L}_1 g}(z = 0) \neq 0$.

Conclusion

- Scaling limit = slow/fast dynamics.
- Other scalings : $a_N \ll 1$ and $a_N \gg 1$.
- Dynamics generalized : $D \geq 1$ sites, and jump rate

$$\tau^N_{i \to j}(\boldsymbol{x}) = c_{ij}\, \eta(x_i)\big(a_j + N\eta(x_j)\big),$$

where $\eta \in \mathcal{C}^1(\,[0,1]\,)$ such that $\eta(0) = 0$ and $\eta' > 0$, and $c_{ij} \geq 0$, $a_j \geq 0$.