Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras

Nils Berglund, Tom Klose

Abstract

We give a relatively short, almost self-contained proof of the fact that the partition function of the suitably renormalised Φ^4_3 measure admits an asymptotic expansion, the coefficients of which converge as the ultraviolet cut-off is removed. We also examine the question of Borel summability of the asymptotic series. The proofs are based on Wiener chaos expansions, Hopf-algebraic methods, and bounds on the value of Feynman diagrams obtained through BPHZ renormalisation.

Date. July 16, 2022.
2010 *Mathematical Subject Classification.* 60H15, 35R11 (primary), 81T17, 82C28 (secondary).
Keywords and phrases. Phi-four-three model, BPHZ renormalisation, Hopf algebras, Wiener chaos expansion, cumulants.

1 Introduction

The Φ^4_d model, defined on the d-dimensional torus with $d \in \{1,2,3\}$, is probably one of the simplest non-trivial models in Euclidean quantum field theory. Here non-trivial means that the model can be proven to behave differently from a Gaussian field. In dimension $d = 4$, it has been shown that the Φ^4_d model is indeed trivial [1].

Despite it being simpler than other models, the analysis of the Φ^4 model is by no means easy. The earliest works by Glimm and Jaffe and by Feldman approached the problem via a detailed combinatorial analysis of Feynman diagrams [20, 21, 19, 22], entailing very long and technical proofs. Over the years, the analysis of the model has been gradually simplified. The works [3, 4] introduced the idea of using a renormalisation group approach, consisting in a decomposition of the covariance of the underlying Gaussian reference field into scales, which then allows to integrate sucessively over one scale after the other. This method was further perfected in [11], using polymers to control error terms, an approach based on ideas from statistical physics [23].

In another direction, the approach provided in [12, 13] allows to bound correlation functions (or n-point functions) without having to compute the partition function explicitly, by using it as a generating function. This involves the derivation of skeleton inequalities, which were obtained up to third order in [12], and later extended to all orders in [9]. A relatively compact derivation of bounds on the partition function based on the Boué–Dupuis formula was recently obtained in [2].

In this review, we argue that there is still room for improvement in the analysis of the Φ^4_3 model, taking advantage of quite recent developments in more algebraic approaches. We will present a rather compact argument showing that the partition function of the suitably renormalised Φ^4_3 model admits a well-defined asymptotic expansion, which can furthermore be resummed using the theory of Borel transforms. Important sources of inspiration for our approach are the monograph [29] by Peccati and Taqqu on Wiener chaos and cumulant expansions, the article [17] by Ebrahimi–Fard *et al* on deformations of Hopf algebras, and Hairer's overview [24] of BPHZ renormalisation.

This article is organized as follows. In Section 2, we introduce the set-up, including a definition of the renormalised Φ^4_3 measure with cut-off N. In Section 3, we give a relatively concise proof of the fact that the partition function of the model admits an asymptotic expansion, all terms of which converge as the cut-off N is sent to infinity. Finally, in Section 4, we examine the question of Borel resummation of the asymptotic expansion.

Acknowledgements: Both authors thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) of the University of Vienna for its kind hospitality and financial support during the Masterclass and Workshop of the Graduate School on "Higher Structures Emerging from Renormalisation" (8 to 19 November 2021). NB was partly supported by the ANR project PERISTOCH, grant ANR–19–CE40–0023. TK additionally thanks the Institut Denis Poisson in Orléans for its generous financial support and the warm hospitality during his visit in June 2022.

2 Set-up

We are interested in the invariant measure of the massive Φ^4_3 model on the torus $\Lambda = \mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$, which can be formally defined as

$$\mu_{\Phi^4_3}(\mathrm{d}\phi) = \frac{1}{Z(\varepsilon)} \exp\biggl\{-\int_\Lambda \biggl(\frac12 \|\nabla\phi(x)\|^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\varepsilon}{4}\phi(x)^4\biggr)\mathrm{d}x\biggr\}\mathrm{d}\phi\,, \tag{2.1}$$

where the partition function $Z(\varepsilon)$ is the normalisation making $\mu_{\Phi^4_3}$ a probability measure. In what follows, we will consider for convenience the case $m^2 = 1$. However, there is no difficulty in extending the results to any $m^2 \geqslant 0$ by a Gaussian change of measure. In fact, even negative values of m^2 can be considered: they appear in the stochastic Allen–Cahn equation, see for instance [7, 5].

As such, the measure (2.1) is ill-defined, because there is no Lebesgue measure $\mathrm{d}\phi$ on $L^2(\Lambda)$. This issue can be solved in several steps, the first of which consists in considering a regularised version of the problem. Here it will be convenient to use a spectral Galerkin approximation with ultra-violet cut-off N. For $k \in \mathbb{Z}^3$, we write $e_k(x) = \exp(2\pi\,\mathrm{i}\,k\cdot x)$ for the Fourier basis functions of $L^2(\Lambda)$, and set

$$\mathscr{H}_N := \operatorname{span}\bigl\{e_k \colon k \in \mathscr{K}_N\bigr\}\,, \qquad \mathscr{K}_N := \bigl\{k \in \mathbb{Z}^3 \colon |k| := |k_1| + |k_2| + |k_3| \leqslant N\bigr\}\,.$$

For any finite N, (2.1) defines a probability measure on $\mathscr{H}_N$. In particular, the partition function can be written as

$$Z_N(\varepsilon) = \mathbb{E}^{\mu_N}\biggl[\exp\biggl\{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4\,\mathrm{d}x\biggr\}\biggr]\,,$$

where μ_N is the Gaussian measure on $\mathscr{H}_N$ with covariance function $[-\Delta+1]^{-1}$.

The limit $N \to \infty$ of this sequence of measures is not well-defined, which is why a renormalisation procedure is required. The first step of this procedure is called Wick renormalisation. It consist in replacing (2.1) for finite N by

$$\mu^{\text{Wick}}_{\Phi^4_3,N}(\mathrm{d}\phi) = \frac{1}{Z_N(\varepsilon)} \exp\biggl\{-\int_\Lambda \biggl(\frac12 \|\nabla\phi(x)\|^2 + \frac12\phi(x)^2 + \frac{\varepsilon}{4}{:}\phi(x)^4{:}\biggr)\mathrm{d}x\biggr\}\mathrm{d}\phi\,, \tag{2.2}$$

where we write

$$:\phi(x)^n: := H_n\big(\phi(x), C_N^{(1)}\big)\,.$$

Here $H_n(\cdot, C)$ denotes the nth Hermite polynomial with variance C, and

$$C_N^{(1)} := \frac{1}{|\Lambda|}\,\mathrm{Tr}\big((-\Delta+1)^{-1}\big)$$

is a counterterm which diverges like N.

In the case of the two-dimensional torus, the analogue of the measure (2.2) is known to converge to a well-defined limit. However, in the three-dimensional case, additional counterterms are required. The highly no-trivial result is that three additional such terms are sufficient. The correctly renormalised measure takes the form (see for instance [3, p. 145])

$$\mu^{\mathrm{BPHZ}}_{\Phi^4_3,N}(\mathrm{d}\phi) = \frac{1}{Z_N(\varepsilon)}\exp\biggl\{-\int_\Lambda\biggl(\frac12\,\|\nabla\phi(x)\|^2 + \frac12\big[1-\varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \frac{\varepsilon}{4}:\phi(x)^4: + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}\biggr)\mathrm{d}x\biggr\}\,\mathrm{d}\phi\,,$$

where the new counterterms are defined as follows. We write G_N for the truncated Green function given by

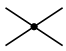
$$G_N(x,y) = G_N(x-y) := \sum_{k\in\mathcal{K}_N}\frac{1}{\lambda_k+1}\underbrace{e_k(x)\overline{e_k(y)}}_{=e_k(x-y)}\,, \tag{2.3}$$

where $-\lambda_k = (2\pi)^2\,\|k\|^2$ are the eigenvalues of the Laplacian on Λ. Then we have

$$\begin{aligned}
C_N^{(2)} &:= 3!\int_\Lambda G_N(x)^3\,\mathrm{d}x = \mathcal{O}(\log N)\,,\\
C_N^{(3)} &:= \frac{4!}{2!4^2}\int_\Lambda G_N(x)^4\,\mathrm{d}x = \mathcal{O}(N)\,,\\
C_N^{(4)} &:= \frac{2^3}{3!4^3}\binom{4}{2}^3\int_\Lambda\int_\Lambda G_N(x)^2 G_N(y)^2 G_N(x-y)^2\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\log N)\,.
\end{aligned} \tag{2.4}$$

Our aim in the following is to provide a compact partial proof of this result, based on recent developments in combinatorics in the Wiener chaos, on Hopf-algebraic methods, and on analytic bounds for BPHZ renormalisation. More precisely, we are going to address the question of convergence of the truncated partition function Z_N in the limit $N\to\infty$.

It will be useful to introduce some additional notation. We will use the symbols

$X =$  $:= \int_{\mathbb{T}^3} :\phi(x)^4:\,\mathrm{d}x\,,\qquad Y =$ $:= \int_{\mathbb{T}^3} :\phi(x)^2:\,\mathrm{d}x$ (2.5)

for Wick powers, as well as the shorthands

$$\alpha := \frac{\varepsilon}{4}\,,\qquad \beta := \frac12\varepsilon^2 C_N^{(2)}\,,\qquad \gamma := \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}\,. \tag{2.6}$$

In this way, the partition function can be written as

$$Z_N(\varepsilon) = \mathbb{E}\big[\mathrm{e}^{-\alpha X-\beta Y-\gamma}\big] = \mathrm{e}^{-\gamma}\,\mathbb{E}\big[\mathrm{e}^{-\alpha X-\beta Y}\big]\,.$$

Integrals as in (2.4) can be conveniently expressed as Feynman diagrams (more precisely, vacuum diagrams). If $\Gamma = (\mathscr{V}, \mathscr{E})$ is a multigraph with vertex set $\mathscr{V}$ and edge set $\mathscr{E}$ (multiple edges between vertices are allowed), then the *valuation* Π_N is the map defined by

$$\Pi_N(\Gamma) := \int_{\Lambda^{\mathscr{V}}} \prod_{e\in\mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x\,,$$

where $e_\pm$ denote the vertices connected by the edge e (G_N being even, their order does not matter here). In particular, we have the expressions

$C_N^{(1)} = \Pi_N$  , $\quad C_N^{(2)} = 3!\Pi_N$  , $\quad C_N^{(3)} = \frac{4!}{2!4^2}\Pi_N$  , $\quad C_N^{(4)} = \frac{2^3}{3!4^3}\binom{4}{2}^3\Pi_N$ 

for the counterterms. The graphical notation emphasizes how these expressions are a consequence of Wick calculus, which states in particular that the expectation of a product of Wick powers can be written as a sum over all pairings of their "legs", also called contractions, see for instance [29] as well as Example 3.4 below.

There are two different questions that one may want to address:

1. Show that the partition function (or its logarithm) admits an asymptotic expansion in powers of ε, with coefficients that converge to finite limits as $N \to \infty$.

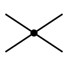
2. Analyse the Borel summability of the asymptotic series. Indeed, it is known that the asymptotic expansion of the partition function will remain divergent, even after removing all divergences in terms of N. Nonetheless, Borel summation allows to recover information on the partition function from its Borel transform.

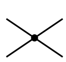
We will address the first question in Section 3, and the second one in Section 4.

3 Asymptotic expansion

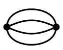
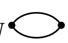
3.1 Cumulant expansion

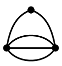
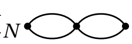
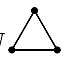
Define the centred moments

$\mu_n := (-1)^n \mathbb{E}\Big[\Big(\alpha$  $+\beta$ $\Big)^n\Big]$

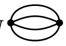
$= (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}\,, \qquad A_{nm} := \mathbb{E}\Big[$  ^{m} $^{n-m}\Big]\,.$

As already alluded to, the coefficients A_{nm} can be computed using the properties of Wick calculus, by summing over all contractions, that is, all pairings of legs of different diagrams (see Example 3.4 for more details). For instance, we have

$\mu_2 = \alpha^2 4!\Pi_N$  $+ \beta^2 2!\Pi_N$  ,

$\mu_3 = -\alpha^3\binom{4}{2}^3 2^3\Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$  .

We see that each A_{nm} has the form of a combinatorial numerical constant times the value of a Feynman diagram, obtained by performing all possible contractions. In general, A_{nm} may be a linear combination of Feynman diagrams, and these diagrams need not all be connected. For instance, A_{44} contains the term

$$3 \cdot (4!)^2 \Big(\Pi_N$$  $$\Big)^2 , \tag{3.1}$$

where the combinatorial factor 3 counts the number of pairings of the 4 four-vertex diagrams, and each factor 4! counts the number of pairwise matchings of the legs within each pair.

The cumulant expansion reads

$$-\log \mathbb{E}[\mathrm{e}^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} , \tag{3.2}$$

where the coefficients κ_n can be computed recursively with the Leonov–Shiraev relation

$$\kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m} .$$

It will be useful to write

$$\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} B_{nm} ,$$

where the coefficients B_{nm} are again linear combinations of Feynman diagrams. The first few cumulants are

$$\begin{aligned}
\kappa_2 &= \mu_2 , \\
\kappa_3 &= \mu_3 , \\
\kappa_4 &= \mu_4 - 3\mu_2^2 , \\
\kappa_5 &= \mu_5 - 10\mu_2\mu_3 .
\end{aligned}$$

An important observation is that $\kappa_n - \mu_n$ is always either zero, or a sum of products of at least two factors. In terms of Feynman diagrams, this means that $\kappa_n - \mu_n$ is a linear combination of non-connected graphs.

In particular, we see that the term $-3\mu_2^2$ kills exactly the non-connected term (3.1) of μ_4. Therefore, κ_4 is represented by a linear combination of *connected* Feynman diagrams. The fact that this generalises to all cumulants is well known in the quantum field theory literature, though full proofs seem not easy to find. See however [10, Sect. 3], or [30, Sect. 4.2].

Proposition 3.1. *Every κ_n is a linear combination of* connected *Feynman diagrams.*

The next subsection is dedicated to a proof of this statement.

3.2 A combinatorial proof of Proposition 3.1

With X and Y as introduced in (2.5), we set

$$\begin{aligned}
X &\equiv \times \equiv \int_{\mathbb{T}^3} {:}\phi(x)^4{:}\,\mathrm{d}x =: \int_{\mathbb{T}^3} X(x)\,\mathrm{d}x , \quad X(x) := \overset{x}{\times} , \\
Y &\equiv -\!\!\bullet\!\!- \equiv \int_{\mathbb{T}^3} {:}\phi(x)^2{:}\,\mathrm{d}x =: \int_{\mathbb{T}^3} Y(x)\,\mathrm{d}x , \quad Y(x) := -\!\!\overset{x}{\bullet}\!\!- ,
\end{aligned}$$

and by μ_N we always denote the Gaussian Free Field (GFF) at cut-off level N, i.e. the centered Gaussian measure on $L^2(\Lambda)$ with covariance kernel $G_N(x,y) = G_N(x-y)$ given in (2.3).

We also define

$$\tilde{G}_N(x) := \sum_{k\in\mathcal{K}_N} \frac{1}{\sqrt{\lambda_k+1}} e_k(x)$$

so that $(\tilde{G}_N * \tilde{G}_N)(x) = G_N(x)$ when $*$ denotes convolution, as can be verified by a straightforward calculation.

Whenever no explicit measure is mentioned, the reference measure is always that of spatial white noise on $L^2(\Lambda)$, i.e., the centred Gaussian measure with covariance given by the Dirac kernel $\delta(x-y)$. Accordingly, we write $\phi \sim \mu$ as

$$\phi(x) = I_1\big(\tilde{G}_N(x-\cdot)\big) =: \int_\Lambda \tilde{G}_N(x-z)\xi(\mathrm{d}z)\,, \tag{3.3}$$

where I_1 is the first Wiener-Itô isometry with respect to spatial white noise, see for example the textbook by Nualart [28]. Note that this is consistent with the calculation

$$\begin{aligned}\mathbb{E}^{\mu_N}\Big(\phi(x)\phi(y)\Big) &= G_N(x-y) = (\tilde{G}_N * \tilde{G}_N)(x-y) = \int_\Lambda \tilde{G}_N(z)\tilde{G}_N(z-(x-y))\,\mathrm{d}z \\ &= \int_\Lambda \tilde{G}_N(x-z)\tilde{G}_N(y-z)\,\mathrm{d}z = \mathbb{E}\Big(I_1\big(\tilde{G}_N(x-\cdot)\big)I_1\big(\tilde{G}_N(y-\cdot)\big)\Big)\,,\end{aligned}$$

where we have used that G_N is even, i.e. $G_N(x) = G_N(-x)$, and translation invariance in the penultimate step.

It is well-known that products of stochastic integrals such as ϕ in (3.3) produce correction terms in lower order Wiener-Itô chaoses (see e.g. [28, Prop. 1.1.3]) — but the Wick product is the projection onto the highest component, so we have

$$X(x) = :\phi(x)^4: = I_4(\tilde{G}_N(x-\cdot)^{\otimes 4})\,, \quad Y(x) = :\phi(x)^2: = I_2(\tilde{G}_N(x-\cdot)^{\otimes 2})\,.$$

Recall that

$$\kappa(X_1,\dots,X_n) := \frac{\partial^n}{\partial t_1\dots\partial t_n} \log\mathbb{E}\Big(\exp\Big(\sum_{\ell=1}^n t_\ell X_\ell\Big)\Big)\Big|_{t_1=\dots t_n=0}$$

denotes the *cumulant functional.* With $\kappa_n(X) = \kappa(X,\dots,X)$ where κ has n entries, we will use the well-known binomial-type formula

$$\kappa_n(\alpha X+\beta Y) = \sum_{k=0}^n \binom{n}{k}\alpha^k\beta^{n-k}\kappa(\underbrace{X,\dots,X}_{k\text{ times}},\underbrace{Y,\dots,Y}_{(n-k)\text{ times}})\,.$$

Multi-linearity of κ also gives

$$\kappa(\underbrace{X,\dots,X}_{k\text{ times}},\underbrace{Y,\dots,Y}_{(n-k)\text{ times}}) = \int_{\Lambda^n} \kappa\big(X(x_1),\dots,X(x_k),Y(x_{k+1}),\dots,Y(x_n)\big)\,\mathrm{d}x_{1:n}$$

where $\mathrm{d}x_{1:n} := \mathrm{d}x_1\dots\mathrm{d}x_n$. The following theorem implies the validity of Proposition 3.1.

Theorem 3.2. *The identity*

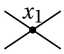
$$\kappa(\underbrace{X,\dots,X}_{k\text{ times}},\underbrace{Y,\dots,Y}_{(n-k)\text{ times}}) = \sum_{\Gamma\in\mathcal{G}} \Pi_N\Gamma$$

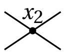
holds, where $\mathcal{G} = \mathcal{G}(k,n)$ denotes the set of connected multigraphs without self-loops that correspond to pairwise matchings.

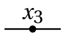
We will not introduce all the terminology in the previous statement abstractly, but rather illustrate it in a specific case. As the reader will see, the arguments easily generalise to all combinations of k and n and thus lead to a “proof by example.”

Remark 3.3. Essentially, the previous theorem is a direct consequence of [29, Coro. 7.3.1], a generalisation of Wick's theorem followed by a projection onto “connected diagrams” to account for the cumulant. Since the whole book [29] is written in the language of set-partition combinatorics, we hope our example aids the reader in seeing the connections clearly. $\diamondsuit$

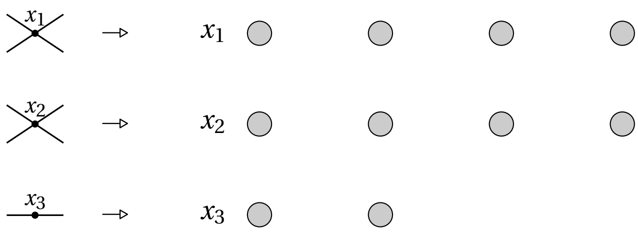
Example 3.4 ($k = 2, n = 3$: “Proof” of Theorem 3.2)**.** We consider

$X(x_1) =$  $= I_4(\underbrace{\tilde{G}_N(x_1 - \cdot)^{\otimes 4}}_{=: f_1(x_1;\,\cdot)}) \quad \rightarrow \quad n_1 = 4\,,$

$X(x_2) =$  $= I_4(\underbrace{\tilde{G}_N(x_2 - \cdot)^{\otimes 4}}_{=: f_2(x_2;\,\cdot)}) \quad \rightarrow \quad n_2 = 4\,,$

$Y(x_3) =$  $= I_2(\underbrace{\tilde{G}_N(x_3 - \cdot)^{\otimes 2}}_{=: f_3(x_3;\,\cdot)}) \quad \rightarrow \quad n_3 = 2\,.$

(1) In a first step, we convert the legs of the diagrams $X(x_i)$, $i = 1, 2$, and $Y(x_3)$ into nodes and keep the label x_i on the left side of the row. Accordingly, we have $n = \sum_{i=1}^{3} n_i = 10$ nodes in the following diagram:



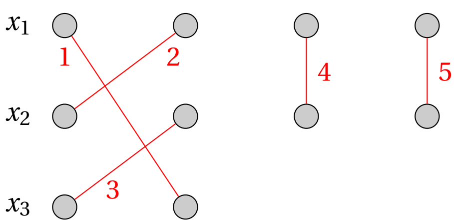
(2) We form pairwise matchings of these nodes, signified by lines between two nodes, abiding by the following rules:

 (i) One must not match two nodes that are in the same row. This would correspond to self-loops in the associated graphs. Peccati and Taqqu call these matchings “non-flat.”

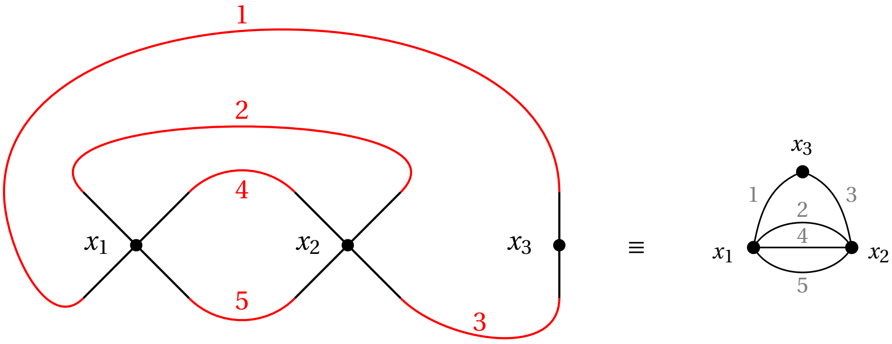
 (ii) The resulting matchings must be such that one cannot divide the rows without intersecting one line that symbolises a matching of two nodes. Otherwise, one could partition the rows into two or more subsets of rows and form pairwise matchings within each subset. This would correspond to disconnected graphs.

The set that contains all of these matchings is called $\mathcal{M}_2([n], \pi^\star)$ by Peccati and Taqqu, see point (4) below for the definition of $\pi^\star$ in our context.

We denote the specific matching in the following diagram by σ:



(3) One converts back these matchings into pairings of the legs of the variables $X(x_1)$, $X(x_2)$, and $Y(x_3)$. For the specific matching σ above, this leads to the following multigraph (or *vacuum Feynman diagram*):



(4) There is an analytical expression corresponding to this multi-graph respectively the matching that leads to it. Let us illustrate how this can be obtained using the result of Peccati and Taqqu [29, Coro. 7.3.1].

(4.1) Note that

$$\Big(\bigotimes_{k=1}^{3} f_k(x_k,\cdot)\Big)(z_{1:n}) = f_1(x_1,z_{1:4})f_2(x_2,z_{5:8})f_3(x_3,z_{9:10})\,. \tag{3.4}$$

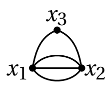
The partition $\pi^\star$ in Peccati's and Taqqu's book corresponds exactly to the partitioning of the indices of the z_i's:

$$\begin{aligned}\pi^\star &= \big\{\{1,\dots,n_1\},\{n_1+1,\dots,n_1+n_2\},\{n_1+n_2+1,\dots,n_1+n_2+n_3\}\big\}\\ &= \big\{\{1,\dots,4\},\{5,\dots,8\},\{9,10\}\big\}\,.\end{aligned}$$

(4.2) The function $f_{\sigma,\ell} \equiv f_{\sigma,3}$ built from (3.4) is obtained by simply identifying the variables that are matched via σ. In precise terms, this means that

$$\begin{aligned}f_{\sigma,3}(z_1,\dots,z_5) = \tilde G_N(x_1-z_1)&\tilde G_N(x_1-z_2)\tilde G_N(x_1-z_4)\tilde G_N(x_1-z_5)\\ \times\, &\tilde G_N(x_2-z_2)\tilde G_N(x_2-z_3)\tilde G_N(x_2-z_4)\tilde G_N(x_2-z_5)\\ \times\, &\tilde G_N(x_3-z_1)\tilde G_N(x_3-z_3)\,.\end{aligned}$$

(4.3) We then find

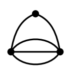
$$\int_{\Lambda^5} f_{\sigma,3}(z_{1:5})\,\mathrm{d}z_{1:5} = G_N(x_2-x_1)^3 G_N(x_3-x_1)G_N(x_3-x_2) = $$ 

by repeatedly using the identity $(\tilde G_N * \tilde G_N)(x) = G_N(x)$.

(4.4) Integrating over all the x_i's (above called the *valuation* Π_N) we then find

$$\begin{aligned}\int_{\Lambda^3}\int_{\Lambda^5} f_{\sigma,3}(z_{1:5})\,\mathrm{d}z_{1:5}\,\mathrm{d}x_{1:3} &= \int_{\Lambda^3} G_N(x_2-x_1)^3 G_N(x_3-x_1)G_N(x_3-x_2)\,\mathrm{d}x_{1:3}\\ &= \int_{\Lambda^{|\mathcal V|}} \prod_{e\in\mathcal E} G_N(x_{e_+}-x_{e_-})\,\mathrm{d}x = \Pi_N\Gamma\end{aligned}$$

for

$\Gamma =$  .

In the literature, the previous drawing sometimes symbolises $\Gamma \in \mathcal G$ and sometimes denotes $\Pi_N\Gamma \in \mathbb R$. We follow the former convention.

The above procedure clearly generalises to arbitrary values of n and k and to different matchings that produce $\Gamma \in \mathscr{G} = \mathscr{G}(k,n)$. Therefore, Theorem 3.2 follows by the same route. ⧫

3.3 BPHZ renormalisation

We now examine the cumulant expansion (3.2) in more detail. Each coefficient A_{nm} and B_{nm} is a linear combination of Feynman diagrams $\Gamma^{(k)}_{nm}$ having each

- m vertices of degree 4,
- $n-m$ vertices of degree 2,
- $n+m$ edges.

We associate with a multigraph $\Gamma = (\mathscr{V}, \mathscr{E})$ a degree given by

$$\deg(\Gamma) = 3(|\mathscr{V}|-1) - |\mathscr{E}|\,,$$

so that $\deg(\Gamma^{(k)}_{nm}) = 2n - m - 3$ for all k. We call a diagram Γ *divergent* if $\deg\Gamma \leqslant 0$.

For small divergent diagrams, one can check that their value diverges like $N^{-\deg(\Gamma)}$, possibly with logarithmic corrections. This is however not true for many larger diagrams, because of the presence of divergent subdiagrams. In fact, there is only one possible divergent subdiagram in our situation, namely the "bubble"

 ,

the value of which diverges like $\log N$.

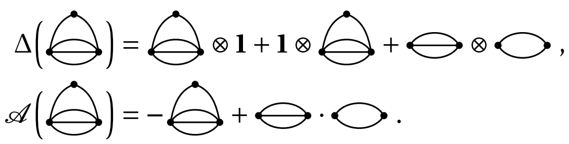
BPHZ renormalisation, named after Bogoliubov, Parasiuk, Hepp and Zimmermann [8, 25, 33] provides a way of dealing with these divergent subdiagrams. It can be formulated in a convenient way by using the Connes–Kreimer extraction-contraction coproduct on graphs [15, 16], given by

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\mathbf{1} \neq \overline{\Gamma} \subsetneq \Gamma} \overline{\Gamma} \otimes (\Gamma/\overline{\Gamma})\,, \tag{3.5}$$

where $\mathbf{1}$ denotes the empty graph, the sum ranges over all divergent subdiagrams, and $\Gamma/\overline{\Gamma}$ is obtained by contracting all edges in $\overline{\Gamma}$ to one vertex. We further define a (twisted) antipode $\mathscr{A}$ as the linear map satisfying $\mathscr{A}(\mathbf{1}) = \mathbf{1}$, and extended inductively by

$$\begin{aligned}\mathscr{A}(\Gamma) &= -\mathscr{M}(\mathscr{A} \otimes \mathrm{id})(\Delta\Gamma - \Gamma \otimes \mathbf{1}) \\ &= -\Gamma - \sum_{\mathbf{1} \neq \overline{\Gamma} \subsetneq \Gamma} \mathscr{A}(\overline{\Gamma}) \cdot (\Gamma/\overline{\Gamma})\,,\end{aligned} \tag{3.6}$$

where the product $\cdot$ denotes the disjoint union of graphs, and $\mathscr{M}(\Gamma_1 \otimes \Gamma_2) = \Gamma_1 \cdot \Gamma_2$. For instance,



(3.7)

A *character* is by definition a linear form $g : \mathscr{G} \to \mathbb{R}$, $\Gamma \mapsto \langle g, \Gamma\rangle$ which is also multiplicative, in the sense that

$$\langle g, \Gamma_1 \cdot \Gamma_2\rangle = \langle g, \Gamma_1\rangle\langle g, \Gamma_2\rangle \qquad \forall \Gamma_1, \Gamma_2 \in \mathscr{G}\,.$$

To such a character, we can associate a renormalisation transformation, given by the linear map $M^g : \mathscr{G} \to \mathscr{G}$ defined by

$$M^g(\Gamma) := (g \otimes \mathrm{id})\Delta\Gamma\,.$$

The *BPHZ character* is given by

$$\langle g^{\mathrm{BPHZ}}, \Gamma \rangle := \begin{cases} \Pi_N \mathscr{A}(\Gamma) & \text{if } \deg\Gamma \leqslant 0 \;, \\ 0 & \text{otherwise} \;. \end{cases}$$

We then define

$$\begin{aligned} \Pi_N^{\mathrm{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) \\ &= (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma \;. \end{aligned}$$

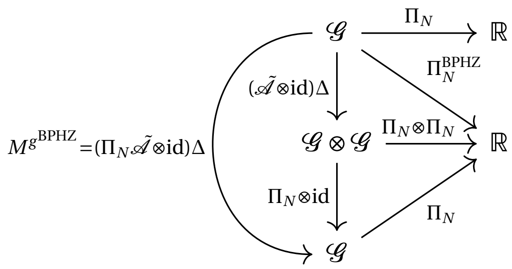
A compact way of writing this is to introduce the map $\tilde{\mathscr{A}}$ defined by

$$\tilde{\mathscr{A}}(\Gamma) := \mathscr{A}(\Gamma) 1_{\deg\Gamma \leqslant 0} \;, \tag{3.8}$$

which implies

$$\begin{aligned} \Pi_N^{\mathrm{BPHZ}}(\Gamma) &= (\Pi_N \otimes \Pi_N)(\tilde{\mathscr{A}} \otimes \mathrm{id})\Delta\Gamma \\ &= (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma \;. \end{aligned}$$

The following commutative diagram summarises the situation:



The interest of this construction is that one can show that $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ is bounded uniformly in N if Γ is non-divergent, and otherwise diverges like $N^{-\deg(\Gamma)}$, possibly with logarithmic corrections [24, 6].

The aim of the remainder of this section is to give a mostly algebraic proof of the following combinatorial result.

Theorem 3.5. *For* $0 \leqslant m \leqslant n$*, write*

$$B_{nm} = \sum_k b_{nm}^{(k)} \Pi_N\big(\Gamma_{nm}^{(k)}\big) \;,$$

where each sum runs over finitely many k*, the* $b_{nm}^{(k)}$ *are combinatorial coefficients, and each* $\Gamma_{nm}^{(k)}$ *is a connected Feynman diagram with* m *vertices of degree* 4*,* $n-m$ *vertices of degree* 2*, and* $n+m$ *edges. Then*

$$\sum_{n=2}^{\infty} \frac{\kappa_n}{n!} = -\sum_{p=2}^{\infty} \frac{1}{p!}(-\alpha)^p \sum_k b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}\big(\Gamma_{pp}^{(k)}\big) \;,$$

where equality is in the sense of formal power series.

Corollary 3.6. *All terms in the asymptotic cumulant expansion* (3.2) *are bounded uniformly in the cut-off* N*.*

PROOF: Since $\deg(\Gamma_{pp}^{(k)}) = p-3$, the only divergent $\Gamma_{pp}^{(k)}$ are those with $p \in \{2,3\}$. The choice of γ in (2.6) precisely compensates these two terms. The result follows at once. □

3.4 Zimmermann's forest formula

A first step in proving Theorem 3.5 is to obtain a simpler expression than (3.6) for the twisted antipode. This is provided by Zimmermann's forest formula [33, 24], which reads

$$\mathscr{A}(\Gamma) = -\sum_{\mathscr{F}} (-1)^{|\mathscr{F}|} \mathscr{C}_{\mathscr{F}} \Gamma \,.$$

Here the sum ranges over all forests $\mathscr{F}$ not containing Γ, where a *forest* is a set of subgraphs of Γ which are pairwise either included in one another, or vertex-disjoint. The operator $\mathscr{C}_{\mathscr{F}}$ extracts all subgraphs in $\mathscr{F}$ from Γ (for a forest, this operation is independent of the order of the elements of $\mathscr{F}$).

In our simple situation, forests are just unions of disjoint bubbles. The forest formula thus takes the following form: if Γ contains g bubbles, then

$$\mathscr{A}(\Gamma) = -\sum_{S\subset\{1,\dots,g\}} (-1)^{|S|}$$  $$^{|S|} \mathscr{C}_S \Gamma \,,$$

where we write $\mathscr{C}_S$ for the operation consisting in contracting all bubbles labelled by an element of S (for an arbitrary fixed labelling of the bubbles).

As a result, using the fact that

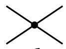
$$\Pi_N\Big($$  $$\Big) = \frac{\beta}{3\varepsilon^2} = \frac{\beta}{48\alpha^2} \,, \tag{3.9}$$

we obtain

$$\Pi_N^{\mathrm{BPHZ}}\big(\Gamma_{pp}^{(k)}\big) = -\sum_{S\subset\{1,\dots,g\}} \Big(-\frac{\beta}{48\alpha^2}\Big)^{|S|} \Pi_N\big(\mathscr{C}_S\Gamma_{pp}^{(k)}\big) \,,$$

where $\mathscr{C}_S\Gamma_{pp}^{(k)}$ is a diagram with $p-|S|$ vertices, $|S|$ of which are of degree 2.

3.5 A Hopf-algebra-flavoured proof

As we have already alluded to in the last sections, it is well-known that the triple $(\mathscr{G},\cdot,\Delta)$ can canonically be turned into a Hopf algebra, itself isomorphic to the *Connes–Kreimer Hopf algebra* [14] of rooted trees. Therefore, it is natural to expect that a particularly elegant proof of Theorem 3.5 may be achieved if one were to interpret $X =$  and $Y =$ as *monomials* in a polynomial Hopf algebra. Indeed, such a construction has been performed by Ebrahimi-Fard et al. [17]: In this section, we describe how to adapt it for our purposes.[1]

We let $\mathbf{X} := (X,Y)$ and for each $\boldsymbol{n} = (n_1,n_2) \in \mathbb{N}^2$ then define

$$\mathbf{X}^{\boldsymbol{n}} := X^{n_1} Y^{n_2} \,, \quad \mathbf{X}^{(0,0)} := \mathbf{1} \,, \quad H := \operatorname{span}\{\mathbf{X}^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\} \,.$$

Remark 3.7. In [17], the authors allow for $\mathbf{X} = (X_a : a \in I)$ where I is some (possibly infinite) index set and then consider the set

$$M(I) = \{B : I \to \mathbb{N} : B(a) \neq 0 \text{ for finitely many } a \in \mathscr{A}\} \,.$$

In our setting $|I| = 2$, so we can identify $M(I) \simeq \mathbb{N}^2$, i.e. we just consider $B = \boldsymbol{n} \in \mathbb{N}^2$. In particular, we still consider polynomials $\mathbf{X}^{\boldsymbol{n}}$ as above and, contrary to [17], do not base our analysis on the (isomorphic) vector space freely generated from $M(I)$.

[1] Note that X and Y do satisfy the assumptions in that article: They have moments of all orders because probabilistic L^p-norms coincide in all homogeneous Wiener-Itô chaoses [26, Thm. 3.50].

We repeat some of the constructions in [17], albeit in a way slightly adapted to our setting. For $\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k} \in \mathbb{N}_0^2$, we define

$$\mathscr{M}(\mathbf{X}^{\boldsymbol{n}} \otimes \mathbf{X}^{\boldsymbol{m}}) := \mathbf{X}^{\boldsymbol{n}} \cdot \mathbf{X}^{\boldsymbol{m}} := \mathbf{X}^{\boldsymbol{n}\cdot\boldsymbol{m}}, \quad \boldsymbol{k}\cdot\boldsymbol{m} := (k_1 + m_1, k_2 + m_2),$$
$$\hat{\Delta}\mathbf{X}^{\boldsymbol{n}} := \sum_{\substack{\boldsymbol{k},\boldsymbol{m}\in\mathbb{N}_0^2 \\ \boldsymbol{k}\cdot\boldsymbol{m}=\boldsymbol{n}}} \binom{\boldsymbol{n}}{\boldsymbol{m},\boldsymbol{k}} \mathbf{X}^{\boldsymbol{k}} \otimes \mathbf{X}^{\boldsymbol{m}}, \quad \binom{\boldsymbol{n}}{\boldsymbol{m},\boldsymbol{k}} := \prod_{i=1}^{2} \frac{n_i!}{m_i! k_i!},$$

with a slight abuse of notation for the product $\cdot$. It is proved in [17] that Δ defines a coproduct on H. ♢

In the previous sections, we have seen that the (twisted) antipode $\mathscr{A}$ (resp. $\tilde{\mathscr{A}}$) plays a crucial role in the renormalisation of multigraphs $\Gamma \in \mathscr{G}$. In this section, we want to define a corresponding map $\hat{\mathscr{A}}_\eta$ acting on H, as well as a map $\mathscr{P}$ that sends H to $\mathscr{G}$, such that the following two diagrams between spaces respectively objects commute:

$$\begin{array}{ccccc}
H & \xrightarrow{\mathscr{P}} & \mathscr{G} & \xrightarrow{\Pi_N} & \mathbb{R} \\
\downarrow{\scriptstyle \chi_\eta=(\hat{\mathscr{A}}_\eta\otimes\mathrm{id})\hat{\Delta}} & & \downarrow{\scriptstyle (\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta} & \searrow{\scriptstyle \Pi_N^{\mathrm{BPHZ}}} & \\
H\otimes H & & \mathscr{G}\otimes\mathscr{G} & \xrightarrow{\Pi_N\otimes\Pi_N} & \mathbb{R} \\
\downarrow{\scriptstyle \mathscr{M}} & & \downarrow{\scriptstyle \Pi_N\otimes\mathrm{id}} & \nearrow{\scriptstyle \Pi_N} & \\
H & \xrightarrow{\mathscr{P}} & \mathscr{G} & &
\end{array} \tag{3.10}$$

$$\begin{array}{ccccc}
e^{-\alpha X} & \xmapsto{\mathscr{P}} & \sum_{n,k} \frac{(-\alpha)^n}{n!} b_{nn}^{(k)} \Gamma_{nn}^{(k)} & \xmapsto{\Pi_N} & \sum_n \frac{(-\alpha)^n}{n!} B_{nn} \\
\downarrow{\scriptstyle \chi_\eta=(\hat{\mathscr{A}}_\eta\otimes\mathrm{id})\hat{\Delta}} & & \downarrow{\scriptstyle (\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta} & \searrow{\scriptstyle \Pi_N^{\mathrm{BPHZ}}} & \\
\sum_{n,m} \frac{(-\alpha)^n(-\eta)^m}{(n-2m)!m!} Y^m \otimes X^{n-2m} & & \sum_{n,k,S} \frac{(-\alpha)^n}{n!} b_{nn}^{(k)} \tilde{\mathscr{A}}\left(\ominus^{|S|}\right) \otimes \mathscr{C}_S \Gamma_{nn}^{(k)} & \xmapsto{\Pi_N\otimes\Pi_N} & \sum_n \frac{(-\alpha)^{n-m}(-\beta)^m}{(n-m)!m!} B_{nm} \\
\downarrow{\scriptstyle \mathscr{M}} & & \downarrow{\scriptstyle \Pi_N\otimes\mathrm{id}} & \nearrow{\scriptstyle \Pi_N} & \\
e^{-\alpha X-\beta Y} & \xmapsto{\mathscr{P}} & \sum_{n,m,k} \frac{(-\alpha)^{n-m}(-\beta)^m}{m!(n-m)!} b_{nm}^{(k)} \Gamma_{nm}^{(k)} & &
\end{array} \tag{3.11}$$

The following definition introduces the desired maps $\hat{\mathscr{A}}_\eta$ and $\mathscr{P}$. Recall that

$$(2\ell-1)!! := \prod_{i=1}^{\ell}(2i-1) = \frac{(2\ell)!}{2^\ell \ell!}. \tag{3.12}$$

Definition 3.8. *We define the linear map $\hat{\mathscr{A}}_\eta : H \to H$ by*

$$\hat{\mathscr{A}}_\eta \mathbf{X}^{\boldsymbol{n}} := \begin{cases} (2\ell-1)!!(-2\eta Y)^\ell & \text{if } n_1 = 2\ell,\ n_2 = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.13}$$

Note that the condition on the first line is satisfied precisely if $\mathbf{X}^{\boldsymbol{n}} = X^{2\ell}$ for some $\ell \in \mathbb{N}_0$ and, in particular, implies that $\hat{\mathscr{A}}_\eta \mathbf{1} = \mathbf{1}$. We also decree that $\hat{\mathscr{A}}_\eta$ is linear w.r.t. infinite sums and define

$$\chi_\eta : H \to H \otimes H, \quad \chi_\eta(\mathbf{X}^{\boldsymbol{n}}) := (\hat{\mathscr{A}}_\eta \otimes \mathrm{id})\hat{\Delta}\mathbf{X}^{\boldsymbol{n}},$$

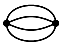
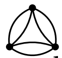
as well as the operation $\mathscr{P} : H \to \mathscr{G}$ *by*

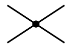
$$\mathscr{P}\mathbf{X}^{\boldsymbol{n}} := \sum_{\Gamma \in \mathscr{G}(n_1, n_1+n_2)} \Gamma\,,$$

for $\mathscr{G}(n_1, n_1+n_2)$ *as given in Theorem 3.2. Note that this corresponds exactly to* $\mathbf{X}^{\boldsymbol{n}} = X^{n_1} Y^{n_2}$. *In other words,* $\mathscr{P}$ *coincides with* κ *up to taking the valuation* Π_N.

Remark 3.9. The previous definition might seem somewhat “ad-hoc”, so let us explain the motivation behind it. Essentially, we want to mirror the down-facing arrows in the middle of (3.10) and (3.11), respectively. We neglect combinatorial factors which are accounted for by the binomial coefficient.

(1) We know that the extraction-contraction co-product Δ given in (3.5) extracts divergent subgraphs of some *connected* graph Γ.

The only divergent diagrams in our setting are , , and the bubble  — but all the vertices of the first two graphs have valence 4, so they cannot be subgraphs of Γ because it is connected, see also Section 3.4.

(2) The bubble, however, is produced by pairing three of the legs of $X =$  with three legs of another instance of X; extracting a bubble then locally produces  · , i.e. ηY for $\eta \simeq \beta/\alpha^2$. Because “two instances of X produce Y”, the latter should count double for the argument to be correct; that is the reason for the action of $\hat{\mathscr{A}_\eta}$ on even powers of X.

(3) Encoding the factor η in $\hat{\mathscr{A}_\eta}$ (and not $\hat{\Delta}$) seemed more convenient in computations and resembles the action of $\mathscr{A}$ in (3.7) more closely.

(4) The factor $(2\ell-1)!!$ counts all possible ways to pair the 2ℓ four-vertex diagrams.

(5) The map $\hat{\mathscr{A}_\eta}$ should act trivially on non-divergent diagrams, so it sends all monomials that are not even powers of X to 0. ◊

The following proposition proves that the two leftmost down-facing arrows in the diagrams (3.10) and (3.11) correspond to well-defined operations:

Proposition 3.10. *The identity* $(\mathscr{M} \circ \chi_\eta)\, \mathrm{e}^{-\alpha X} = \mathrm{e}^{-\alpha X - \beta Y}$ *holds.*

PROOF: Observe that $X^n = \mathbf{X}^{\boldsymbol{n}}$ for $\boldsymbol{n} = (n_1, n_2) = (n, 0)$. By definition, we thus have

$$(\mathscr{M} \circ \chi_\eta) X^n = \sum_{\substack{\boldsymbol{m},\boldsymbol{k} \in \mathbb{N}^2 \\ \boldsymbol{m}\cdot\boldsymbol{k} = \boldsymbol{n}}} \binom{\boldsymbol{n}}{\boldsymbol{m},\boldsymbol{k}} \hat{\mathscr{A}_\eta}(\mathbf{X}^{\boldsymbol{m}})\mathbf{X}^{\boldsymbol{k}}\,,$$

and since $n_2 = 0$, the condition $\boldsymbol{m}\cdot\boldsymbol{k} = \boldsymbol{n}$ implies that $m_2 + k_2 = 0$. As $m_2, k_2 \in \mathbb{N}_0$, we then find $m_2 = k_2 = 0$ and the preceding formula simplifies to

$$(\mathscr{M} \circ \chi_\eta) X^n = \sum_{\substack{m,k \in \mathbb{N} \\ m+k=n}} \frac{n!}{m!k!} \hat{\mathscr{A}_\eta}(X^m) X^k = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \hat{\mathscr{A}_\eta}(X^m) X^{n-m}.$$

Accounting for the definition of the antipode in (3.13), we arrive at the identity

$$\begin{aligned}(\mathscr{M}\circ\chi_\eta)X^n &= \sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!}{(2\ell)!(n-2\ell)!}\hat{\mathscr{A}}_\eta(X^{2\ell})X^{n-2\ell}\\ &= \sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!}{(2\ell)!(n-2\ell)!}(2\ell-1)!!(-2\eta Y)^\ell X^{n-2\ell}\\ &= \sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!}{\ell!(n-2\ell)!}(-\eta Y)^\ell X^{n-2\ell}.\end{aligned}$$

Next, we expand the exponential function

$$\mathrm{e}^{-\alpha X} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\alpha^n X^n$$

so that the previous computation and $\eta = \beta/\alpha^2$ implies that

$$(\mathscr{M}\circ\chi_\eta)\,\mathrm{e}^{-\alpha X} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{\ell=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2\ell)!\ell!}(-\beta Y)^\ell(\alpha X)^{n-2\ell}\ .$$

We now set $p := n-\ell$ and $q := n-2\ell$ so that

$$\ell = p-q\,,\quad n = 2p-q\,,\quad (-1)^n = (-1)^{n-2\ell}(-1)^{2\ell} = (-1)^q \quad \text{for} \quad \ell = 0,\ldots,\lfloor n/2\rfloor\,,$$

and then reorganise the series accordingly to get

$$\begin{aligned}(\mathscr{M}\circ\chi_\eta)\,\mathrm{e}^{-\alpha X} &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{1}{q!(p-q)!}(-\beta Y)^{p-q}(-\alpha X)^q\\ &= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}(-\beta Y)^{p-q}(-\alpha X)^q = \mathrm{e}^{-\alpha X+\beta Y}\ .\end{aligned}$$

The proof is complete. □

Remark 3.11. For $n \in \{2,3\}$, we have

$$(\mathscr{M}\circ\chi_\eta)X^2 = X^2 - 2\eta Y,\quad (\mathscr{M}\circ\chi_\eta)X^3 = X^3 - 6\eta Y X.$$

From these examples, it is interesting to note that, upon identifying

$$X \longleftrightarrow Z \sim \mathscr{N}(0,1),\quad E[X^2] \longleftrightarrow 2\eta Y,$$

the map $\mathscr{M}\circ\chi_\eta$ behaves like a Wick product. We leave further investigations of this observation for future work. ◊

It remains to check that the diagrams are indeed commutative. This is the content of the next section.

3.6 Combinatoric proof of the diagram's commutativity

To complete the proof of Theorem 3.5, we need to show that the identity

$$\begin{aligned}\mathscr{P}\circ\mathscr{M}\circ\chi_\eta &= (\Pi_N\otimes\mathrm{id})\circ(\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta\circ\mathscr{P} \\ &= (\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta\circ\mathscr{P}\end{aligned} \tag{3.14}$$

holds on the space spanned by all monomials X^n. This is equivalent to showing that the following diagram commutes:

$$\begin{array}{ccc} X^n & \xmapsto{\quad\mathscr{P}\quad} & \sum_k b_{nn}^{(k)}\Gamma_{nn}^{(k)} \\ \Big\downarrow{\scriptstyle \mathscr{M}\circ\chi_\eta} & & \Big\downarrow{\scriptstyle (\Pi_N\otimes\mathrm{id})(\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta} \\ \sum_{n,m}\frac{n!(-\eta)^m}{(n-2m)!m!}Y^mX^{n-2m} & \xmapsto{\mathscr{P}} & \sum_{k,S} b_{nn}^{(k)}\left(\frac{-\eta}{48}\right)^{|S|}\mathscr{C}_S\Gamma_{nm}^{(k)} \end{array} \tag{3.15}$$

We have already obtained in the proof of Proposition 3.10 the expression

$$(\mathscr{M}\circ\chi_\eta)(X^n) = \sum_{m=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2m)!m!}(-\eta)^mY^mX^{n-2m}\,.$$

On the other hand, recalling that $\eta = \beta/\alpha^2$, we get

$$(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta\Gamma = -\sum_{m=0}^{\lfloor n/2\rfloor}\left(\frac{\eta}{48}\right)^m\sum_{S\colon |S|=m}\mathscr{C}_S\Gamma\,,$$

where the second sum runs over all sets of m bubbles (if Γ has fewer than m bubbles, the last sum is zero by definition). Comparing the last two expressions, we see that (3.14) holds if

$$\mathscr{P}(Y^mX^{n-2m}) = \frac{(n-2m)!m!}{48^m n!}\sum_{S\colon |S|=m}\mathscr{C}_S(\mathscr{P}X^n) \tag{3.16}$$

for any $n \geqslant 1$. Recall that both sides of this relation involve in general a sum over several Feynman diagrams. There is, however, a natural identification between these diagrams, so that we may lighten the notation by pretending that there is only one diagram in each sum.

We now observe that $\mathscr{P}(Y^mX^{n-2m})$ is a (sum of) diagram(s) having m vertices of degree 2, $n-m$ vertices of degree 4, and $2n-m$ edges. To produce the corresponding term on the right-hand side, we argue that instead of extracting bubbles on the right-hand side, we can also insert bubbles on the left-hand side. This amounts to inserting $2m$ four-vertex diagrams at the vertices of degree 2 of $\mathscr{P}(Y^mX^{n-2m})$. To do that, there are

- $\binom{n}{2m}$ ways of selecting the $2m$ four-vertex diagrams;
- $(2m-1)!!$ ways to pair the $2m$ four-vertex diagrams;
- $4^2\cdot 3! = 96$ ways of matching six pairs of legs of each set of two four-vertex diagrams, amounting to 96^m matchings;
- and finally, $m!$ ways of inserting the resulting bubbles at the m vertices of index 2.

Multiplying all the above combinatorial factors, by (3.12) we arrive at

$$\frac{48^m n!}{m!(n-2m)!}$$

ways of inserting the four-vertex diagrams. This is indeed compatible with the desired relation (3.16). The proof of Theorem 3.5 is complete.

We close this section with an example that deals with mixed monomials.

Example 3.12. As we have seen, it suffices for the purposes of this article that the diagram in (3.15) commutes for the base monomial X^n. However, the following example gives us hope that the commutativity might still be true for monomials containing non-trivial powers of Y.

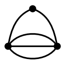
Let $\mathbf{X}^{\boldsymbol{n}} = X^2 Y$, i.e. $\boldsymbol{n} = (n_1, n_2) = (2, 1)$. We find

$$\hat{\Delta}(X^2Y) = X^2Y \otimes \mathbf{1} + \mathbf{1} \otimes X^2Y + X^2 \otimes Y + Y \otimes X^2 + 2XY \otimes X + 2X \otimes XY ,$$
$$\chi_\eta(X^2Y) = \mathbf{1} \otimes X^2Y + \hat{\mathscr{A}}_\eta(X^2) \otimes Y ,$$

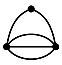
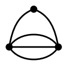
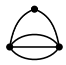
and then

$$(\mathscr{M} \circ \chi_\eta)(X^2Y) = X^2Y - 2\eta Y^2 .$$

Next, note that

$\mathscr{P}(X^2Y) = 4^2 \cdot 2! \cdot 3!$  , $\qquad \mathscr{P}(Y^2) = 2!$  , (3.17)

where the combinatorial factor in the first expression counts the 4^2 ways of choosing one leg in each four-vertex diagram, the 2! ways of matching these with the legs of the two-vertex diagram, and the 3! pairwise matchings of the remaining legs. From the equalities in (3.7), (3.8), as well as (3.9), we obtain

$(\Pi_N \otimes \mathrm{id})(\tilde{\mathscr{A}} \otimes \mathrm{id})\Delta$  $=$  $- \Pi_N$  $\cdot$  $=$  $- \frac{\eta}{48}$  .

Applying the expressions (3.17) of $\mathscr{P}$ to X^2Y and $X^2Y - 2\eta Y^2$, we see that the commutativity relation (3.14) is indeed satisfied. We expect that the same conclusion holds for other mixed monomials, modulo a suitable encoding of the combinatorics in the definition of the map $\hat{\mathscr{A}}_\eta$ from Definition 3.8. ♦

4 Borel resummation

In this section, we examine the question of whether the asymptotic expansion (3.2), though not convergent, can nevertheless be related to a convergent quantity. A positive answer to this question can be given thanks to the theory of Borel summation. This fact has been known in quantum field theory for quite a while, though first proofs of Borel summability for the Φ^4_2 and Φ^4_3 model [18, 27] were quite difficult. Here we show that with modern analytical tools, which combine Hopf-algebraic methods, an improved Borel summation result by Sokal [31], and a decomposition originally obtained by Hepp in [25], the proof of Borel summability can be simplified considerably. The analytical arguments are strongly based on the presentation in [24], made more quantitative in [6].

We start in Section 4.1 by presenting the main ideas of Borel summation, and the result by Sokal, in the case of the zero-dimensional Φ^4 model, whose partition function is simply an integral over $\mathbb{R}$. See also [30] for a more detailed account of various resummation techniques for that model. Then we show in Section 4.2 how these methods can be extended to the three-dimensional case, using in particular methods introduced by Hepp.

4.1 The case of the Φ^4_0 model

The Φ^4_0 model is simply the Φ^4 model for a field ϕ defined at a single point. Its potential is

$$V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4 ,$$

and its partition function is given by

$$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi .$$

The integral is clearly well-defined for $\varepsilon \geqslant 0$. It can also be extended to complex values of ε, at least if $\operatorname{Re}\varepsilon \geqslant 0$, and possibly to other complex values. However, the integral is clearly not convergent for real $\varepsilon < 0$. Therefore, Z is not analytic in a neighbourhood of $\varepsilon = 0$, and does not admit a convergent expansion in powers of ε.

If we nevertheless expand the exponential, we obtain

$$Z(\varepsilon) \asymp \sum_{n\geqslant 0} \frac{1}{n!}\Bigl(-\frac{\varepsilon}{4}\Bigr)^n \int_{-\infty}^{\infty} \phi^{4n}\,\mathrm{e}^{-\phi^2/2}\,\mathrm{d}\phi ,$$

where the symbol $\asymp$ denotes an asymptotic expansion. We can interpret $\mathrm{e}^{-\phi^2/2}$ as the density of a Gaussian measure (up to normalisation), which yields

$$\begin{aligned} Z(\varepsilon) &\asymp \sqrt{2\pi}\sum_{n\geqslant 0} \frac{1}{n!}\Bigl(-\frac{\varepsilon}{4}\Bigr)^n \mathbb{E}^{\mu}\bigl[\phi^{4n}\bigr] \\ &= \sqrt{2\pi}\sum_{n\geqslant 0} \Bigl(-\frac{\varepsilon}{4}\Bigr)^n \frac{(4n-1)!!}{n!} , \end{aligned}$$

where $(4n-1)!!$ is defined in (3.12), and we have used the Isserlis–Wick theorem to compute the moments of the normal law. Recalling that $n! = \Gamma(n+1)$ and using (3.12), we get

$$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0} \Bigl(-\frac{\varepsilon}{16}\Bigr)^n \frac{\Gamma(4n+1)}{\Gamma(2n+1)\Gamma(n+1)} .$$

The general term of this formal series can be analysed by using Legendre's duplication formula for the Gamma function

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) ,$$

which yields

$$\Gamma(4n+1) = 4n\Gamma(4n) = \frac{2}{\sqrt{\pi}} n 2^{4n}\Gamma(2n)\Gamma(2n+\tfrac{1}{2}) .$$

Therefore

$$\begin{aligned} Z(\varepsilon) &\asymp 2\sqrt{2}\sum_{n\geqslant 0} (-\varepsilon)^n \frac{n\Gamma(2n)\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)\Gamma(2n+1)} \\ &= \sqrt{2}\sum_{n\geqslant 0} (-\varepsilon)^n \frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)} \\ &= \frac{1}{\sqrt{\pi}}\sum_{n\geqslant 0} (-4\varepsilon)^n \frac{\Gamma(n+\frac{1}{4})\Gamma(n+\frac{3}{4})}{\Gamma(n+1)} , \end{aligned} \tag{4.1}$$

where we used $2n\Gamma(2n) = \Gamma(2n+1)$ to get the second line. Stirling's formula implies

$$\Gamma(z+\alpha) = z^\alpha \Gamma(z)\left[1 + \mathcal{O}\left(\frac{\alpha-1}{z}\right)\right],$$

showing that the general term in the series (4.1) diverges like $\Gamma(n)$. Therefore, the series is indeed divergent.

Remark 4.1. A more direct way of obtaining the asymptotic expansion (4.1) is to notice that

$$Z(\varepsilon) = 2\int_0^\infty e^{-\phi^2/2} e^{-\varepsilon\phi^4/4} \, d\phi = \sqrt{2}\int_0^\infty e^{-t} \frac{e^{-\varepsilon t^2}}{\sqrt{t}} \, dt,$$

where we have used the change of variables $\phi = \sqrt{2t}$. Expanding the exponential, we obtain

$$Z(\varepsilon) \asymp \sqrt{2} \sum_{n\geqslant 0} \frac{(-\varepsilon)^n}{n!} \int_0^\infty t^{2n-\frac{1}{2}} e^{-t} \, dt = \sqrt{2} \sum_{n\geqslant 0} (-\varepsilon)^n \frac{\Gamma(2n+\frac{1}{2})}{n!}$$

which agrees with (4.1). ♢

Certain divergent series can be resummed by a procedure known as Borel summation. Consider a formal power series

$$A(\varepsilon) = \sum_{n\geqslant 0} a_n \varepsilon^n .$$

We can rewrite it as

$$A(\varepsilon) = \sum_{n\geqslant 0} a_n \varepsilon^n \frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0} \frac{a_n \varepsilon^n}{n!} \int_0^\infty t^n e^{-t} \, dt .$$

Define the Borel-transformed power series by interchanging the sum and the integral, that is

$$A_{\text{Borel}}(\varepsilon) := \int_0^\infty e^{-t} \sum_{n\geqslant 0} \frac{a_n \varepsilon^n t^n}{n!} \, dt = \int_0^\infty e^{-t} \mathscr{B}A(\varepsilon t) \, dt,$$

which is the Laplace transform of the Borel sum

$$\mathscr{B}A(t) := \sum_{n\geqslant 0} \frac{a_n}{n!} t^n .$$

Watson's theorem [32] gives conditions under which $A_{\text{Borel}}(\varepsilon)$ admits the asymptotic series $A(\varepsilon)$.

In the case of the expansion (4.1) we find

$$\mathscr{B}Z(t) = \frac{1}{\sqrt{\pi}} \sum_{n\geqslant 0} b_n t^n , \tag{4.2}$$

where

$$b_n = (-4)^n \frac{\Gamma(n+\frac{1}{4})\Gamma(n+\frac{3}{4})}{\Gamma(n+1)^2} = \frac{(-4)^n}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

The series (4.2) has radius of convergence $\frac{1}{4}$, with a pole located at $-\frac{1}{4}$. One can thus expect that it admits an analytic continuation to a domain including all positive reals, so that its Laplace transform indeed converges.

In [31], Alan Sokal has proved the following improvement of Watson's theorem. Assume that A is analytic in a disc $D_R = \{\varepsilon\colon \operatorname{Re}\varepsilon^{-1} > R^{-1}\}$, which is tangent to the imaginary axis. Assume further that A admits the asymptotic expansion

$$A(\varepsilon) = \sum_{k=0}^{n-1} a_k \varepsilon^k + R_n(\varepsilon)\;, \tag{4.3}$$

where

$$|R_n(\varepsilon)| \leqslant C r^n n! |\varepsilon|^n \tag{4.4}$$

for some $C, r > 0$, uniformly in n and ε in D_R. Then $\mathscr{B}A(t)$ converges for $|t| < 1/r$, and has an analytic continuation to a $1/r$-neighbourhood of the positive real axis. Furthermore, A can be represented by the absolutely convergent integral

$$A(\varepsilon) = \frac{1}{\varepsilon}\int_0^\infty \mathrm{e}^{-t/\varepsilon}\,\mathscr{B}A(t)\,\mathrm{d}t$$

for any $\varepsilon \in D_R$.

To apply this to our situation, we write

$$\frac{Z(\varepsilon)}{\sqrt{2\pi}} = \sum_{k=0}^{n-1} \frac{1}{k!}\Bigl(-\frac{\varepsilon}{4}\Bigr)^n \mathbb{E}\bigl[\phi^{4k}\bigr] + \mathbb{E}\biggl[\mathrm{e}^{-\varepsilon\phi^4/4} - \sum_{k=0}^{n-1} \frac{1}{k!}\Bigl(-\frac{\varepsilon}{4}\Bigr)^k \phi^{4k}\biggr]\;.$$

We then use the fact that for any $n \in \mathbb{N}$, one has the Taylor expansion

$$\begin{aligned}
\mathrm{e}^{-z} &= \sum_{k=0}^{n-1} \frac{1}{k!}(-z)^k + (-z)^n \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathrm{e}^{-t_n z}\,\mathrm{d}t_n \ldots \mathrm{d}t_1 \\
&= \sum_{k=0}^{n-1} \frac{1}{k!}(-z)^k + (-z)^n \int_0^1 \int_0^1 \cdots \int_0^1 s_1 \ldots s_{n-1}\,\mathrm{e}^{-s_1\ldots s_n z}\,\mathrm{d}s_n \ldots \mathrm{d}s_1\;,
\end{aligned} \tag{4.5}$$

showing that for any z with positive real part,

$$\biggl|\mathrm{e}^{-z} - \sum_{k=0}^{n-1} \frac{1}{k!}(-z)^k\biggr| \leqslant \frac{1}{n!}|z|^n\;.$$

This implies that $Z(\varepsilon)/\sqrt{2\pi}$ satisfies (4.3) with a remainder R_n such that

$$|R_n(\varepsilon)| \leqslant \frac{1}{n!}\biggl(\frac{|\varepsilon|}{4}\biggr)^n \mathbb{E}\bigl[\phi^{4n}\bigr]\;.$$

By the above computations, the remainder indeed meets Sokal's conditions.

4.2 The case of the Φ^4_3 model

As mentioned above, Borel summability of the perturbation expansions of correlation functions (or Schwinger functions) of the Φ^4_d model has been proved in [18] in the case $d = 2$, and in [27] in the case $d = 3$. The proofs are based on cluster expansion techniques from statistical physics, and are quite technical. Here we outline a comparatively simple proof of Borel summability of the expansion of the partition function, based on the techniques described above. The essential (and non-trivial) analytical ingredient is a bound on the value of BPHZ-renormalised Feynman diagrams, explained in [24], and made more quantitative in [6].

We start by observing that for any $n \geqslant 1$, we have the Taylor expansion

$$\mathrm{e}^{-\alpha X} = S_n^0 + R_n^0\,, \qquad S_n^0 = \sum_{p=0}^{n-1} \frac{(-\alpha)^p}{p!} X^p\,, \qquad R_n^0 = \frac{(-\alpha)^n}{n!} X^n r_n(X)\,,$$

with a remainder satisfying $r_n(X) = \mathrm{e}^{-\theta(X)\alpha X}$ for some $\theta(X) \in [0,1]$. Disregarding for the moment any questions related to renormalisation, we note that

$$\log \mathbb{E}[\mathrm{e}^{-\alpha X}] - \log \mathbb{E}[S_n^0] = \log \frac{\mathbb{E}[\mathrm{e}^{-\alpha X}]}{\mathbb{E}[S_n^0]} = \log\biggl(1 + \frac{\mathbb{E}[R_n^0]}{\mathbb{E}[S_n^0]}\biggr)\,.$$

The Cauchy–Schwarz inequality yields

$$|\mathbb{E}[R_n^0]| \leqslant \frac{|\alpha|^n}{n!} \sqrt{\mathbb{E}[X^{2n}]\mathbb{E}[r_n(X)^2]}\,,$$

so that since $\log(1+|x|) \leqslant |x|$, we obtain the bound

$$\Bigl|\log \mathbb{E}[\mathrm{e}^{-\alpha X}] - \log \mathbb{E}[S_n^0]\Bigr| \leqslant \frac{|\alpha|^n}{n!} \sqrt{\mathbb{E}[X^{2n}]}\, \frac{\sqrt{\mathbb{E}[r_n(X)^2]}}{\mathbb{E}[S_n^0]}$$

on the remainder of the cumulant expansion. Note in particular that (4.5) implies

$$\begin{aligned} \frac{\mathbb{E}[r_n(X)^2]}{\mathbb{E}[S_n^0]^2} &= (n!)^2 \int_{[0,1]^{2n}} s_1 \dots s_{n-1} \tilde{s}_1 \dots \tilde{s}_{n-1} \frac{\mathbb{E}[\mathrm{e}^{-[s_1 \dots s_n + \tilde{s}_1 \dots \tilde{s}_n]\alpha X}]}{\mathbb{E}[S_n^0]^2}\, \mathrm{d}s\, \mathrm{d}\tilde{s} \\ &\leqslant \sup_{(s,\tilde{s}) \in [0,1]^{2n}} \frac{\mathbb{E}[\mathrm{e}^{-[s_1 \dots s_n + \tilde{s}_1 \dots \tilde{s}_n]\alpha X}]}{\mathbb{E}[S_n^0]^2}\,, \end{aligned} \tag{4.6}$$

which one would expect to have order 1 if no renormalisation were needed.

We now want to transpose this idea to the situation with renormalisation. Applying the linear map $\mathscr{M} \circ \chi_\eta$ to $\mathrm{e}^{-\alpha X}$, we obtain

$$\mathrm{e}^{-\alpha X - \beta Y} = (\mathscr{M} \circ \chi_\eta)(S_n^0) + (\mathscr{M} \circ \chi_\eta)(R_n^0) =: S_n + R_n\,.$$

By the same computation as above, we have

$$\log \mathbb{E}[\mathrm{e}^{-\alpha X - \beta Y}] = \log \mathbb{E}[S_n] + \log\biggl(1 + \frac{\mathbb{E}[R_n]}{\mathbb{E}[S_n]}\biggr)\,,$$

which implies that the partition function satisfies

$$\begin{aligned} -\log Z_N &= \gamma - \log \mathbb{E}[\mathrm{e}^{-\alpha X - \beta Y}] \\ &= \gamma - \log \mathbb{E}[S_n] - \log\biggl(1 + \frac{\mathbb{E}[R_n]}{\mathbb{E}[S_n]}\biggr)\,. \end{aligned}$$

Commutativity of the diagram (3.10) yields

$$\begin{aligned} -\log \mathbb{E}[S_n] &= (\Pi_N \circ \mathscr{P})(S_n) \\ &= (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P})(S_n^0) \\ &= \sum_{p=2}^{n-1} \sum_k \frac{(-\alpha)^p}{p!} b_{pp}^{(k)} \Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)})\,. \end{aligned}$$

As pointed out above, only the terms $p=2$ and $p=3$ of this expansion diverge in the limit $N\to\infty$, and are compensated exactly by the counterterm γ.

The remainder R_n can be written as

$$R_n=\sum_{q=0}^{\lfloor n/2\rfloor}R_{nq}\,,\qquad R_{nq}=\frac{(-\alpha)^{n-2q}(-\beta)^q}{(n-2q)!q!}X^{n-2q}Y^q r_{nq}(X,Y)\,,$$

where the functions r_{nq} admit integral representations similar to the remainder in (4.5). The Cauchy–Schwarz inequality yields

$$\begin{aligned}|\mathbb{E}[R_n]|&\leqslant\sum_{q=0}^{\lfloor n/2\rfloor}\frac{|\alpha|^{n-2q}|\beta|^q}{(n-2q)!q!}\sqrt{\mathbb{E}[X^{2(n-2q)}Y^{2q}]}\sqrt{\mathbb{E}[r_{nq}^2]}\\&=\Bigl(\frac{\varepsilon}{4}\Bigr)^n\sum_{q=0}^{\lfloor n/2\rfloor}\frac{\bigl(2C_N^{(2)}\bigr)^q}{(n-2q)!q!}\sqrt{\mathbb{E}[X^{2(n-2q)}Y^{2q}]}\sqrt{\mathbb{E}[r_{nq}^2]}\,,\end{aligned}\tag{4.7}$$

where we have used the expressions (2.6) for α and β. The last factor in this bound is comparable to $\mathbb{E}[S_n]$ by the same argument as in (4.6). It thus remains to bound the expectations of polynomial terms.

Let $\mathscr{P}_0$ denote the linear operator performing all pairwise matchings, but without projecting on connected diagrams. Recall that this amounts to taking expectations without taking the logarithm. Then we argue that

$$\begin{aligned}\bigl(\Pi_N^{\mathrm{BPHZ}}\circ\mathscr{P}_0\bigr)(X^{2n})&=\bigl(\Pi_N\circ\mathscr{P}_0\circ\mathscr{M}\circ\chi_\eta\bigr)(X^{2n})\\&=\sum_{m=0}^{n}\frac{(2n)!}{(2n-2m)!m!}\Bigl(-\frac{\beta}{\alpha^2}\Bigr)^m\bigl(\Pi_N\circ\mathscr{P}_0\bigr)\bigl(X^{2n-2m}Y^m\bigr)\\&=\sum_{m=0}^{n}\frac{(2n)!}{(2n-2m)!m!}\bigl(-8C_N^{(2)}\bigr)^m\bigl(\mathbb{E}[X^{2n-2m}Y^m]\bigr)\,.\end{aligned}$$

The point of this observation is that the quantity $\Pi_N^{\mathrm{BPHZ}}\circ\mathscr{P}_0(X^{2n})$ can be estimated with a procedure detailed in [6], based on the exposition in [24]. Indeed, $\mathscr{P}_0(X^{2n})$ can be written as a sum of Feynman diagrams, obtained by pairwise matching of the $8n$ legs of X^{2n}. The number of these matchings cannot exceed $(8n-1)!!$ — in fact, it is somewhat smaller, due to the condition of all matchings involving different vertices.

Now a straightforward extension of [6, Proposition 6.1] shows that whenever Γ has strictly positive degree, one has

$$\Bigl|(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta(\Gamma)\Bigr|\leqslant K^{|\mathscr{E}(\Gamma)|}$$

where $|\mathscr{E}(\Gamma)|$ is the number of edges of Γ, and K is a constant depending only on the Green function $G=\lim_{N\to\infty}G_N$. Since this number grows at most linearly with n, we obtain the bound

$$\bigl(C_N^{(2)}\bigr)^m\mathbb{E}[X^{2n-2m}Y^m]\lesssim(8n-1)!!K^{cn}\frac{(2n-2m)!m!}{(2n)!}\lesssim\frac{(8n)!(2n-2m)!m!}{(4n)!(2n)!}\widetilde{K}^{cn}$$

for constants $c,\widetilde{K}>0$. Inserting in (4.7) and using Stirling's formula yields

$$\mathbb{E}\bigl[|R_n|\bigr]\lesssim\varepsilon^n n!\widetilde{K}^{cn}\sqrt{\mathbb{E}[r_{nq}^2]}\,,$$

so that indeed Sokal's condition (4.4) is satisfied.

Remark 4.2. In the last computation, we have argued that the sum over q and the sum over m behave similarly, though there may be unexpected cancellations between terms. This can be circumvented by a applying the same argument as above to a whole family of monomials of the form $(\Pi_N^{\mathrm{BPHZ}}\circ\mathscr{P}_0)(X^{2n-m}Y^m)$, which would allow to estimate the expectations of the monomials $X^{2(n-2q)}Y^{2q}$ individually. $\Diamond$

References

[1] Michael Aizenman and Hugo Duminil-Copin. Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models. *Ann. Math. (2)*, 194(1):163–235, 2021.

[2] Nikolay Barashkov and Massimiliano Gubinelli. A variational method for ϕ_3^4. *Duke Mathematical Journal*, 169(17):3339 – 3415, 2020. `arXiv:1805.1081`.

[3] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, and E. Scacciatelli. Some probabilistic techniques in field theory. *Comm. Math. Phys.*, 59(2):143–166, 1978.

[4] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, and E. Scacciatelli. Ultraviolet stability in Euclidean scalar field theories. *Comm. Math. Phys.*, 71(2):95–130, 1980.

[5] Nils Berglund. *An Introduction to Singular Stochastic PDEs.* EMS Press, 2022.

[6] Nils Berglund and Yvain Bruned. BPHZ renormalisation and vanishing subcriticality limit of the fractional ϕ_d^3 model. `arXiv:1907.13028`, 2019.

[7] Nils Berglund, Giacomo Di Gesù, and Hendrik Weber. An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two. *Electron. J. Probab.*, 22:1–27, 2017.

[8] N. N. Bogoliubow and O. S. Parasiuk. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. *Acta Math.*, 97:227–266, 1957.

[9] Anton Bovier and Giovanni Felder. Skeleton inequalities and the asymptotic nature of perturbation theory for Φ^4-theories in two and three dimensions. *Comm. Math. Phys.*, 93(2):259–275, 1984.

[10] Christian Brouder. Quantum field theory meets Hopf algebra. *Math. Nachr.*, 282(12):1664–1690, 2009.

[11] D. Brydges, J. Dimock, and T. R. Hurd. The short distance behavior of $(\phi^4)_3$. *Comm. Math. Phys.*, 172(1):143–186, 1995.

[12] David C. Brydges, Jürg Fröhlich, and Alan D. Sokal. A new proof of the existence and nontriviality of the continuum φ_2^4 and φ_3^4 quantum field theories. *Comm. Math. Phys.*, 91(2):141–186, 1983.

[13] David C. Brydges, Jürg Fröhlich, and Alan D. Sokal. The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities. *Comm. Math. Phys.*, 91(1):117–139, 1983.

[14] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. In *Quantum field theory: perspective and prospective (Les Houches, 1998)*, volume 530 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 59–108. Kluwer Acad. Publ., Dordrecht, 1999.

[15] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.*, 210(1):249–273, 2000.

[16] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group. *Comm. Math. Phys.*, 216(1):215–241, 2001.

[17] Kurusch Ebrahimi-Fard, Frédéric Patras, Nikolas Tapia, and Lorenzo Zambotti. Hopf-algebraic deformations of products and Wick polynomials. *Int. Math. Res. Not.*, 2020(24):10064–10099, 2020.

[18] Jean-Pierre Eckmann, Jacques Magnen, and Roland Sénéor. Decat properties and Borel summability for the Schwinger functions in $p(\phi)_2$ theories. *Comm. Math. Phys.*, 39:251–271, 1975.

[19] Joel Feldman. The $\lambda\varphi_3^4$ field theory in a finite volume. *Comm. Math. Phys.*, 37:93–120, 1974.

[20] James Glimm and Arthur Jaffe. A $\lambda\phi^4$ quantum field without cutoffs. I. *Phys. Rev. (2)*, 176:1945–1951, 1968.

[21] James Glimm and Arthur Jaffe. Positivity of the ϕ_3^4 Hamiltonian. *Fortschr. Physik*, 21:327–376, 1973.

[22] James Glimm and Arthur Jaffe. *Quantum physics. A functional integral point of view.* Springer-Verlag, New York-Berlin, 1981.

[23] C. Gruber and H. Kunz. General properties of polymer systems. *Comm. Math. Phys.*, 22:133–161, 1971.

[24] M. Hairer. An analyst's take on the BPHZ theorem. In *Comput. Combin. Dyn. Stoch. Control*, pages 429–476, Cham, 2018. Springer International Publishing.

[25] K. Hepp. Proof of the Bogoliubov–Parasiuk theorem on renormalization. *Comm. Math. Phys.*, 2(4):301–326, 1966.

[26] Svante Janson. *Gaussian Hilbert spaces*, volume 129 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, reprint of the 1997 hardback ed. edition, 2008.

[27] J. Magnen and R. Sénéor. Phase space cell expansion and borel summability for the Euclidean ϕ_3^4 theory. *Comm. Math. Phys.*, 56:237–276, 1977.

[28] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[29] Giovanni Peccati and Murad S. Taqqu. *Wiener chaos: moments, cumulants and diagrams*, volume 1 of *Bocconi & Springer Series.* Springer, Milan; Bocconi University Press, Milan, 2011. A survey with computer implementation, Supplementary material available online.

[30] V. Rivasseau. Constructive field theory in zero dimension. *Adv. Math. Phys.*, 2009:12, 2009. Id/No 180159.

[31] Alan D. Sokal. An improvement of Watson's theorem on Borel summability. *J. Math. Phys.*, 21(2):261–263, 1980.

[32] G. N. Watson. A theory of asymptotic series. *Philos. Trans. Soc. London Ser. A*, 211:279, 1912.

[33] W. Zimmermann. Convergence of Bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.*, 15:208–234, 1969.

Nils Berglund
Institut Denis Poisson (IDP)
Université d'Orléans, Université de Tours, CNRS – UMR 7013
Bâtiment de Mathématiques, B.P. 6759
45067 Orléans Cedex 2, France
E-mail address: `nils.berglund@univ-orleans.fr`

Tom Klose
Institut für Mathematik
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin, Germany
E-mail address: `klose@math.tu-berlin.de`