Metastability and stochastic resonance in slow–fast systems with noise

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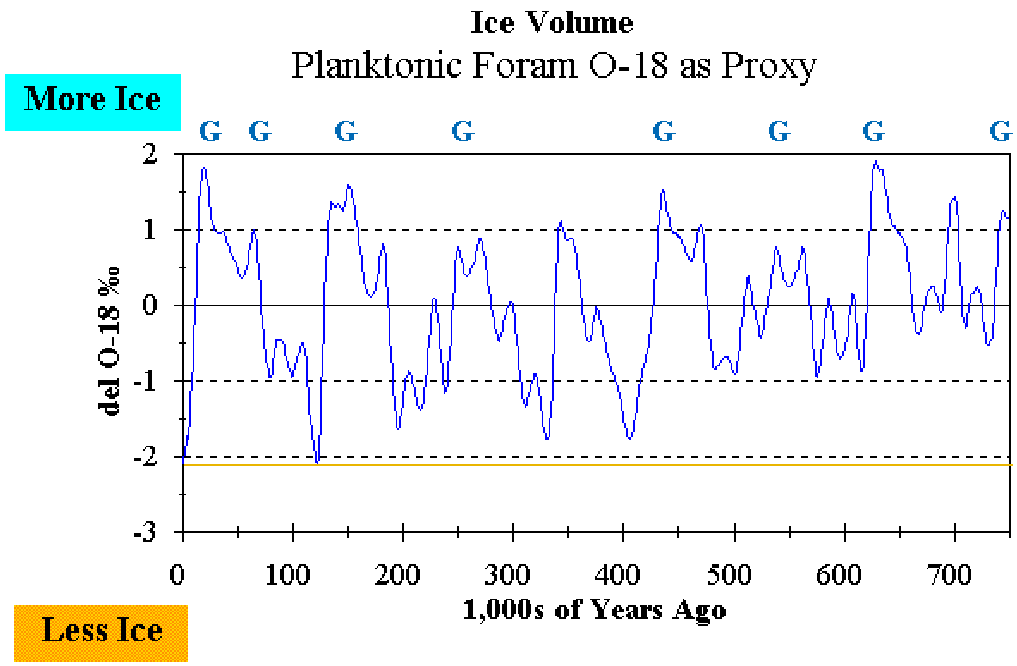
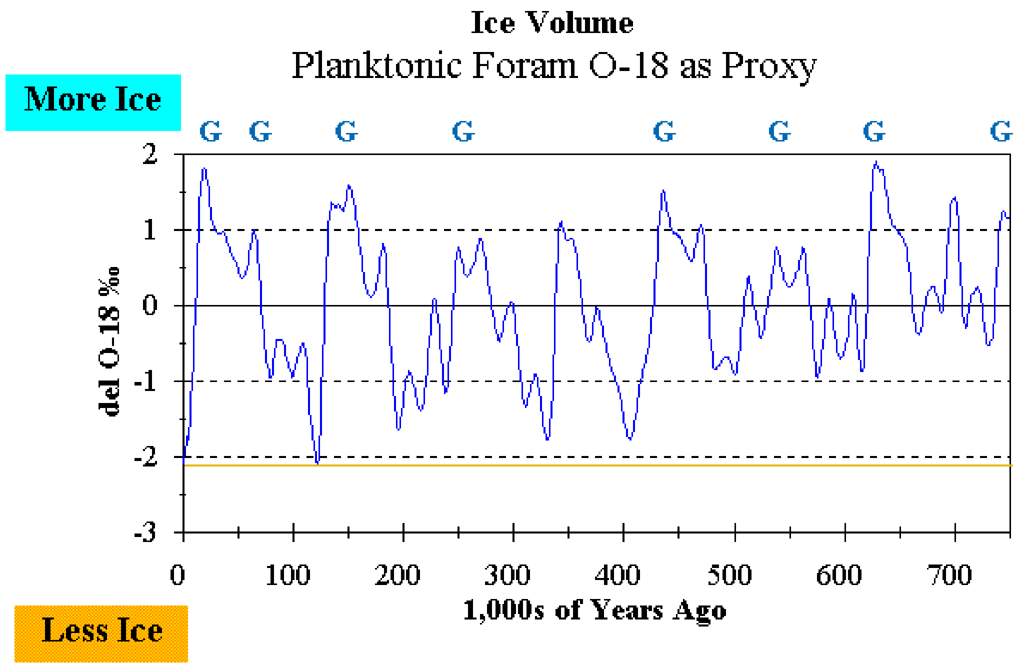
Ice Ages



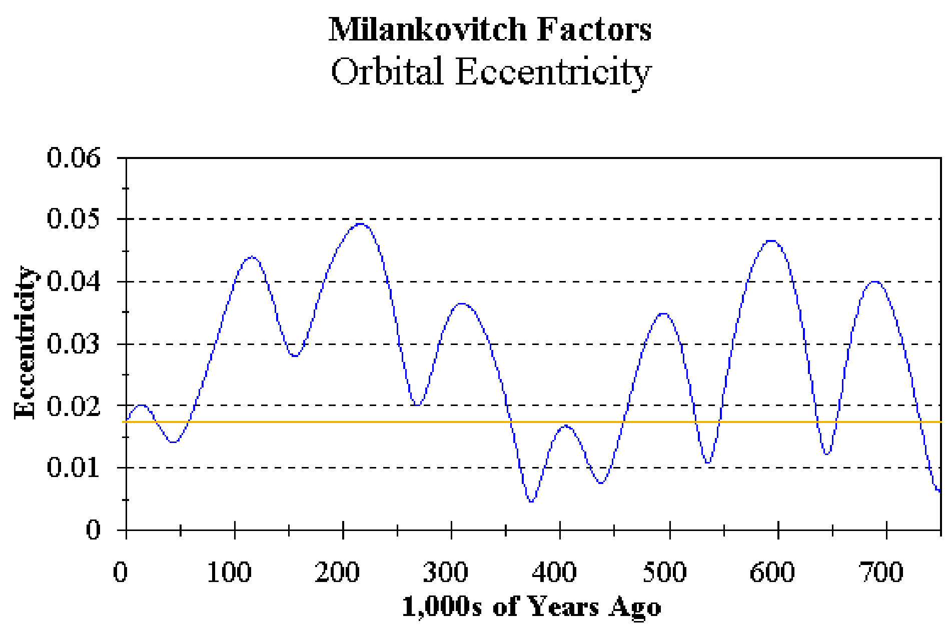
Ice Volume
Planktonic Foram O-18 as Proxy
More Ice
Less Ice
G G G G G G G G
2
1
0
-1
-2
-3
del O-18 ‰
0 100 200 300 400 500 600 700
1,000s of Years Ago

Ice Ages



Croll 1875

Milankovitch 1930s



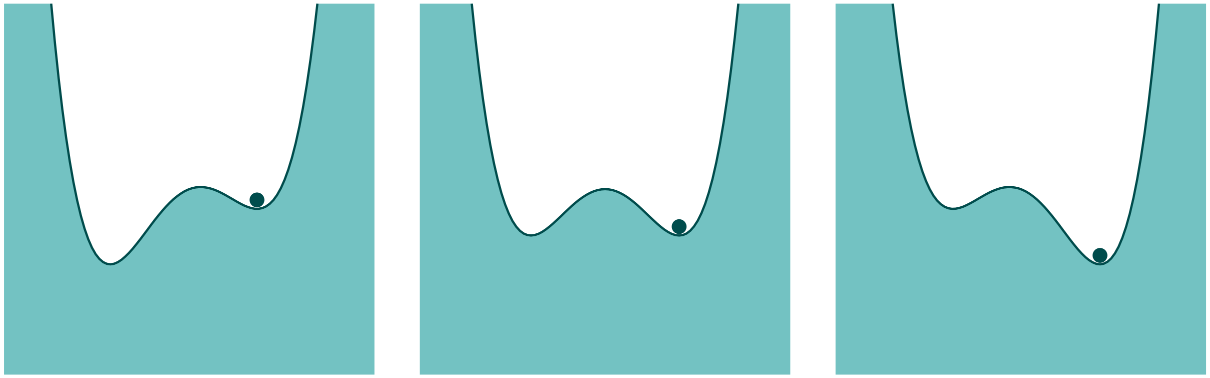
Stochastic resonance

Energy-balance model: $x \sim$ temperature

$$\dot{x} = -\frac{\partial}{\partial x} V(x)$$

$V(x)$ double-well potential, e.g. $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$

With periodic forcing: $V(x) \mapsto V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos\varepsilon t$



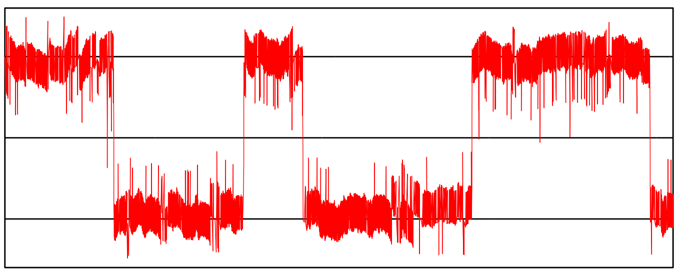
Random influence of weather
(Benzi/Sutera/Vulpiani and Nicolis/Nicolis 1981):

$$\mathrm{d}x_t = -\frac{\partial}{\partial x} V(x_t, t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

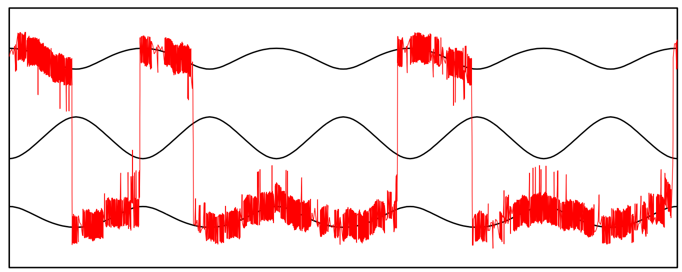
Stochastic resonance

$$\mathrm{d}x_t = -\frac{\partial}{\partial x}\Big[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 - Ax\cos\varepsilon t\Big]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$
$$= [-x^3 + x + A\cos\varepsilon t]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

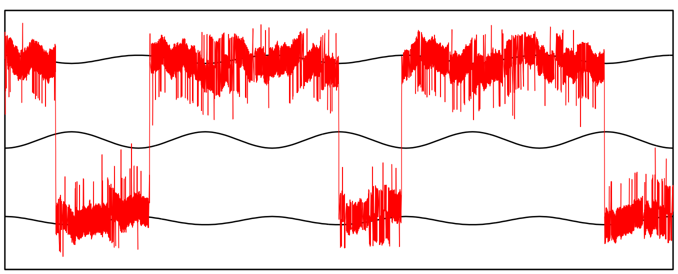
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



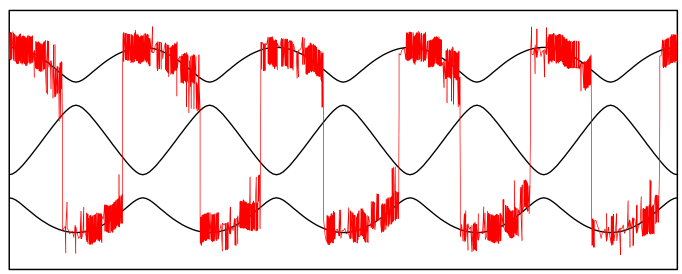
$A = 0$, $\sigma = 0.3$



$A = 0.24$, $\sigma = 0.2$



$A = 0.1$, $\sigma = 0.27$



$A = 0.35$, $\sigma = 0.2$

Equilibrium branches

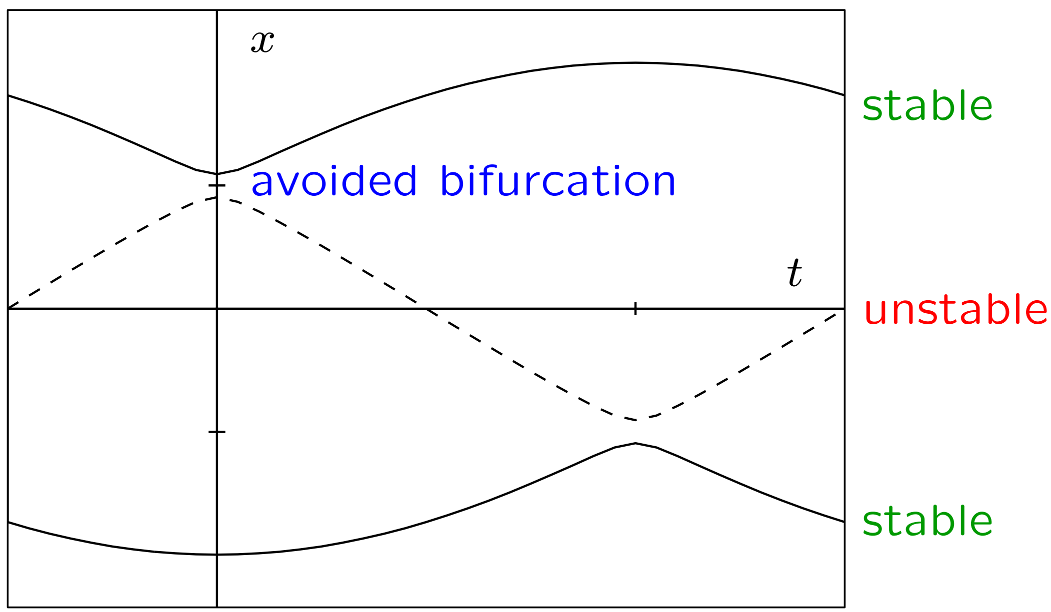
$$\mathrm{d}x_t = \underbrace{[-x^3 + x + A\cos\varepsilon t]}_{f(x,\varepsilon t)}\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Time change $\varepsilon t \mapsto t$

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x,t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equilibrium branches: $f(x^\star(t),t) = 0$

$A < A_\mathrm{c} = 2/3\sqrt{3} \cong 0.385$:



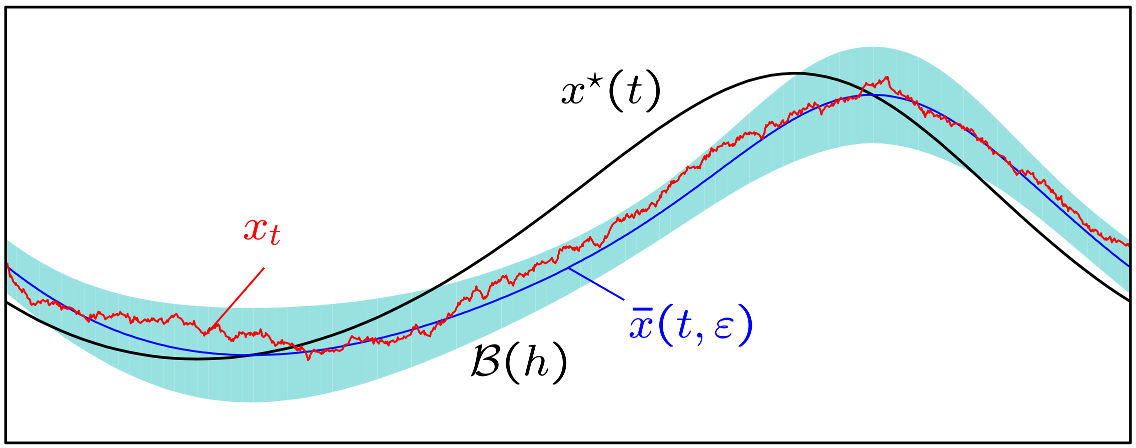
Dynamics near a stable branch

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Stable equil. branch: $f(x^\star(t), t) = 0$, $a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0$

Adiabatic solution: $\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$

$\mathcal{B}(h)$: strip of width $\simeq h/\sqrt{|a^\star(t)|}$ around $\bar{x}(t, \varepsilon)$.



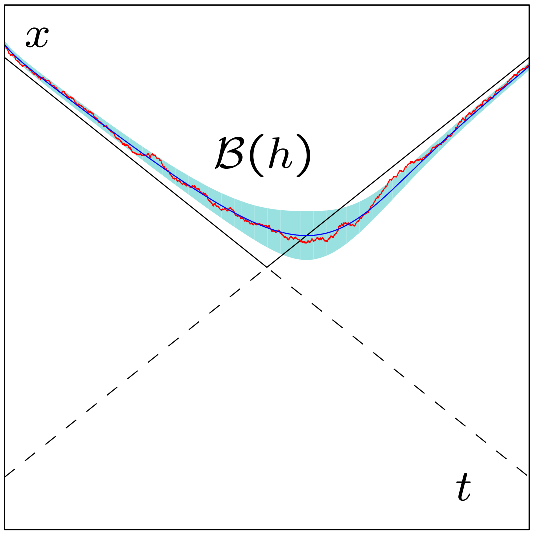
Theorem: [B. & G., PTRF 2002]

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \simeq \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon} \left| \int_0^t a^\star(s)\,\mathrm{d}s \right| \frac{h}{\sigma}\, \mathrm{e}^{-h^2/2\sigma^2}$$

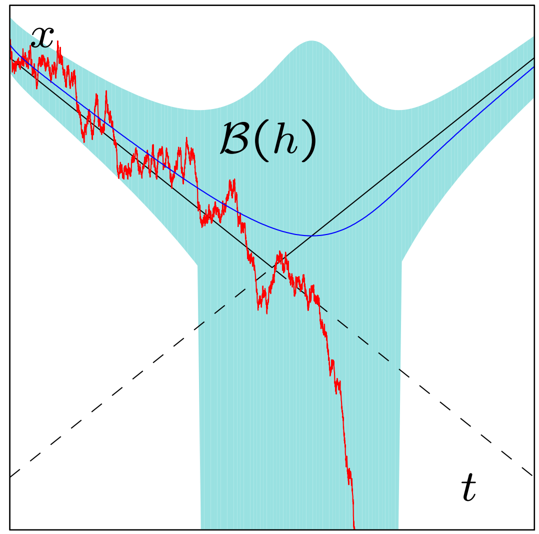
Case $A = A_\mathrm{c}$: Transcritical bifurcation

locally $\mathrm{d}x_t = \frac{1}{\varepsilon}(-x^2 + t^2 + \dots)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Det. case $\sigma = 0$: Solutions stay $\varepsilon^{1/2}$ above bif. point



$\sigma \ll \sigma_\mathrm{c} = \varepsilon^{3/4}$



$\sigma \gg \sigma_\mathrm{c} = \varepsilon^{3/4}$

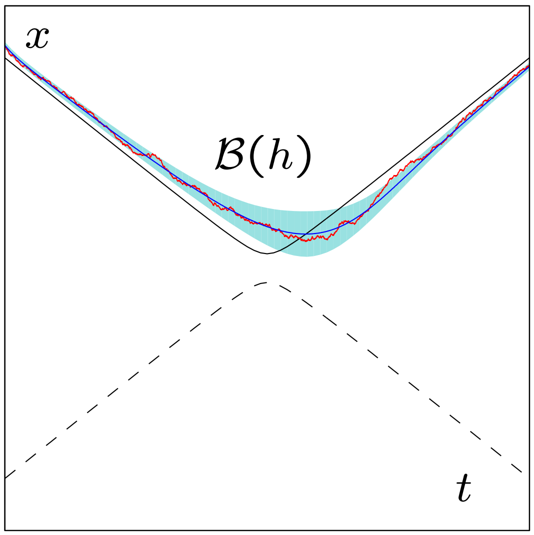
Theorem: [B. & G., Ann. App. Probab. 2002]

1. If $\sigma \ll \sigma_\mathrm{c}$: Paths likely to stay in $\mathcal{B}(h)$, transition probability $\leqslant \mathrm{e}^{-c\sigma_\mathrm{c}^2/\sigma^2}$.
2. If $\sigma \gg \sigma_\mathrm{c}$: Transition typically for $t \asymp -\sigma^{2/3}$ transition probability $\geqslant 1 - \mathrm{e}^{-c\sigma^{4/3}/\varepsilon|\log\sigma|}$

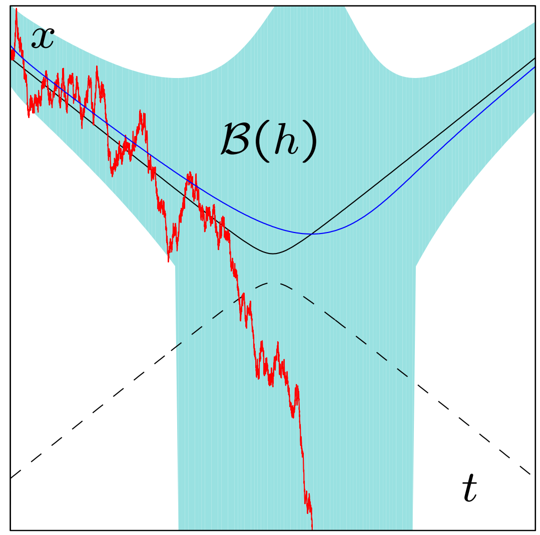
Case $A = A_\mathrm{c} - \delta$: Avoided transcritical bifurcation

locally $\mathrm{d}x_t = \frac{1}{\varepsilon}(-x^2 + \delta + t^2 + \dots)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Det. case $\sigma = 0$: Solutions stay $(\delta \vee \varepsilon)^{1/2}$ above bif. point



$\sigma \ll \sigma_\mathrm{c} = (\delta \vee \varepsilon)^{3/4}$



$\sigma \gg \sigma_\mathrm{c} = (\delta \vee \varepsilon)^{3/4}$

Theorem: [B. & G., Ann. App. Probab. 2002]

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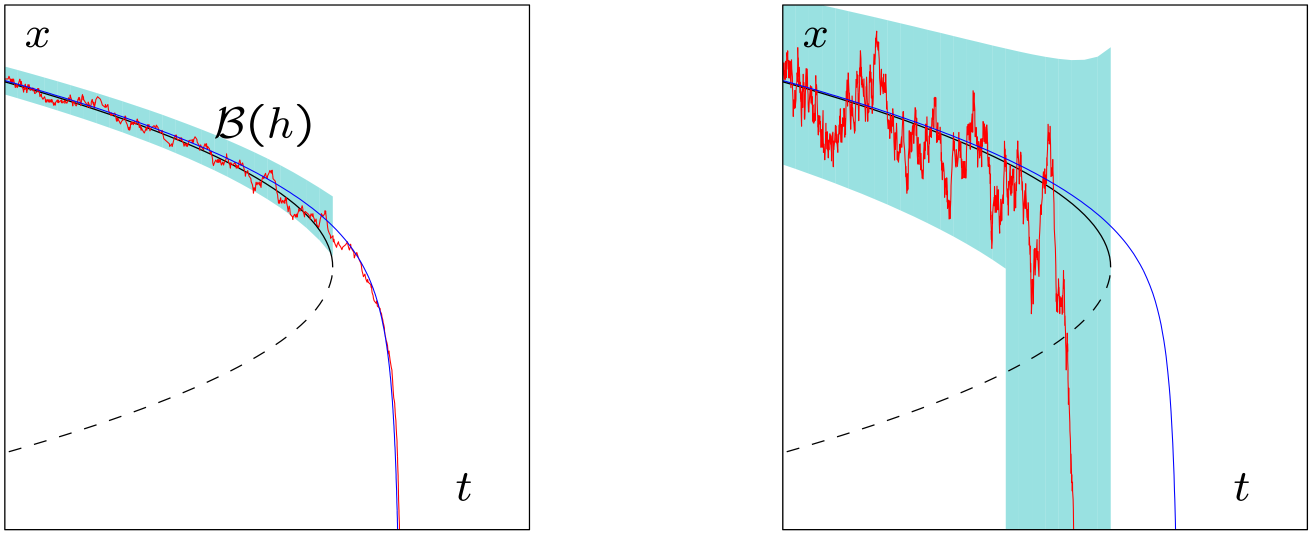
2. If $\sigma \gg \sigma_\mathrm{c}$: Transition typically for $t \asymp -\sigma^{2/3}$ transition probability $\geqslant 1 - \mathrm{e}^{-c\sigma^{4/3}/\varepsilon|\log\sigma|}$

Case $A > A_c$: Saddle–node bifurcation

locally $\mathrm{d}x_t = \frac{1}{\varepsilon}(-x^2 - t + \dots)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Det. case $\sigma = 0$: Solutions stay $\varepsilon^{1/3}$ above bif. point until $t \asymp \varepsilon^{2/3}$.



$\sigma \ll \sigma_c = \varepsilon^{1/2}$ $\qquad$ $\sigma \gg \sigma_c = \varepsilon^{1/2}$

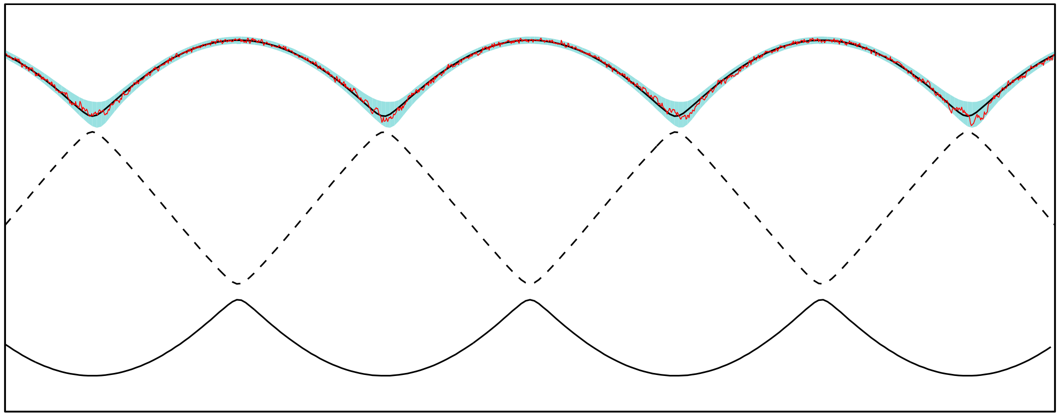
Theorem: [B. & G., Nonlinearity 2002]

1. If $\sigma \ll \sigma_c$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation, maximal spreading $\sigma/\varepsilon^{1/6}$.
2. If $\sigma \gg \sigma_c$: Transition typically for $t \asymp -\sigma^{4/3}$ transition probability $\geqslant 1 - \mathrm{e}^{-c\sigma^2/\varepsilon|\log\sigma|}$

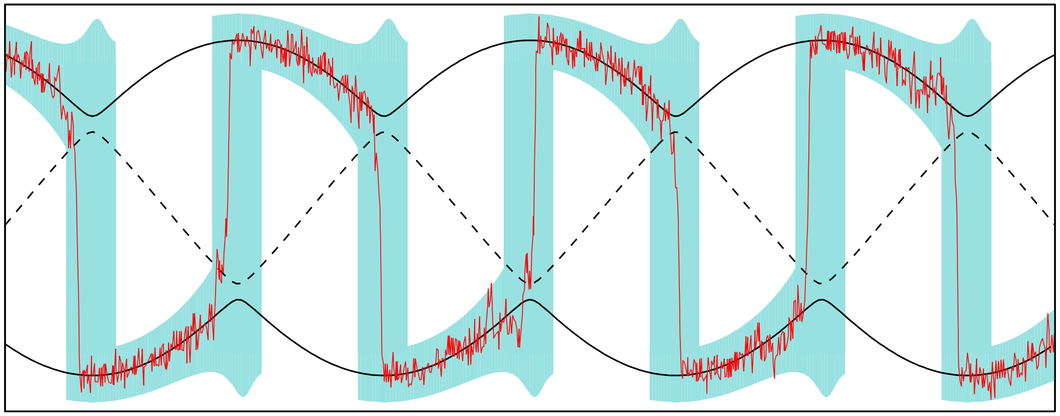
Global behaviour

Critical noise intensity: $\sigma_c = (\delta \vee \varepsilon)^{3/4}$, $\delta = A_c - A$

$\sigma \ll \sigma_c$: transitions unlikely

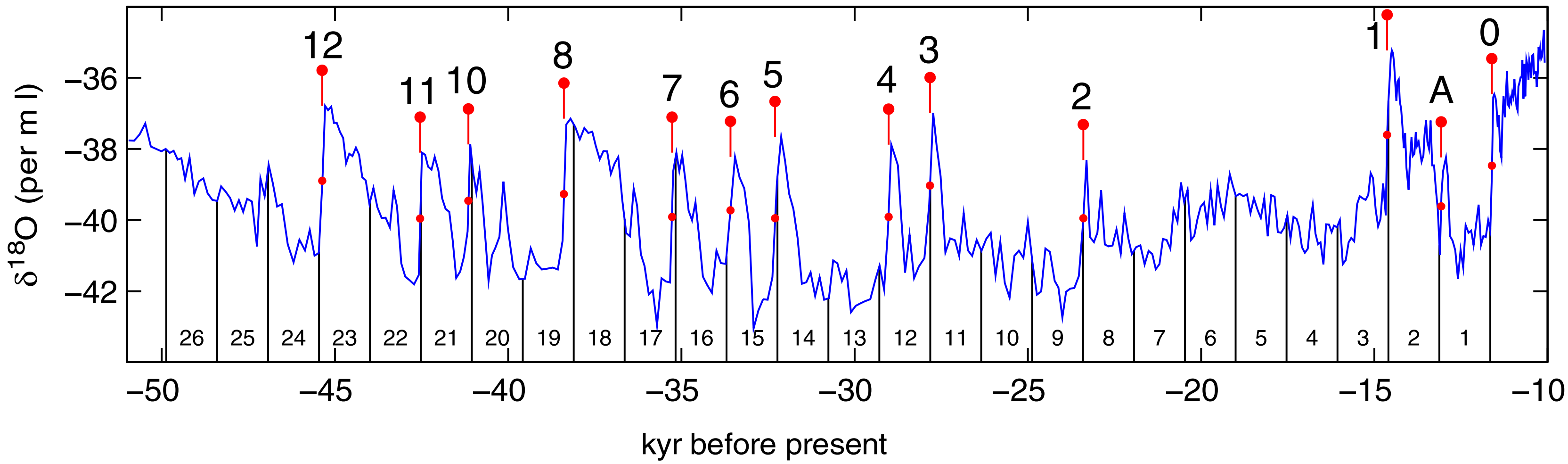


$\sigma \gg \sigma_c$: synchronisation

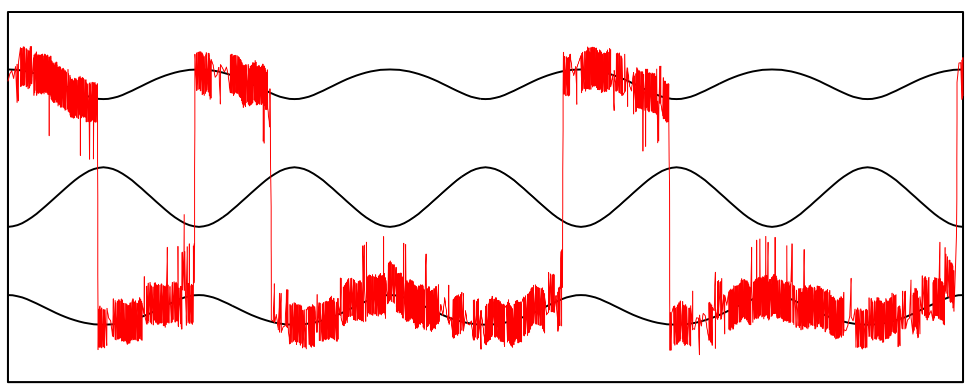


Residence-time distributions

Dansgaard–Oesheger events:



Model equation for $A = 0.24$, $\sigma = 0.2$:



Residence-time distribution

$q(t)$: probability density of time between transitions

Without forcing ($A = 0$): $q(t) \sim$ exponential.

With forcing ($A \gg \sigma^2$):

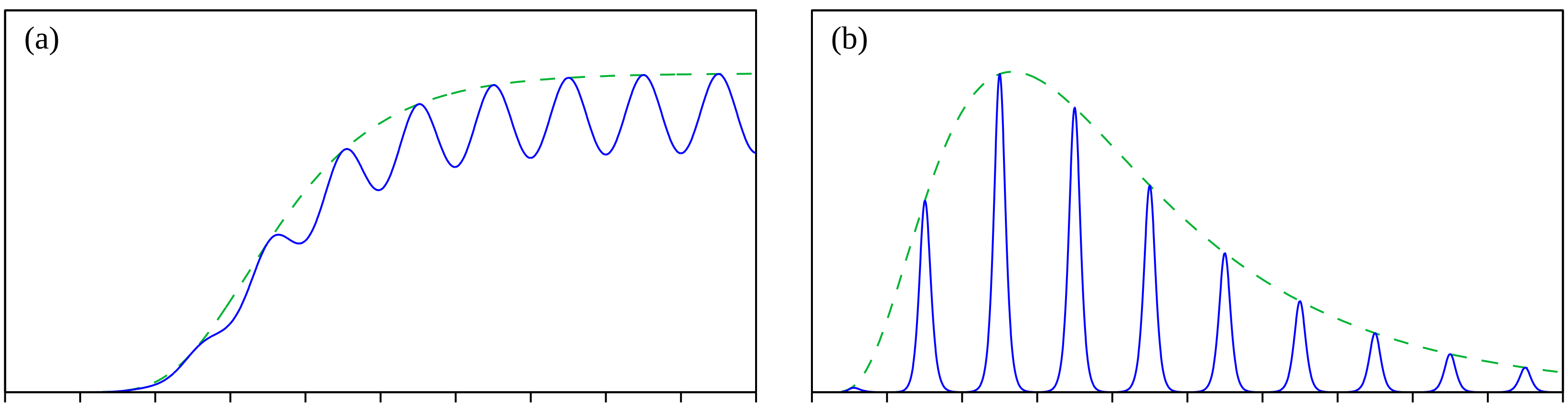
Theorem: [B. & G., Europhys Letters 2005]

$$q(t) \simeq f_{\text{trans}}(t) \frac{\mathrm{e}^{-t/T_{\text{K}}}}{T_{\text{K}}} \frac{\lambda T}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(\lambda(t + T/2 - kT))}$$

T: forcing period

T_{K}: Kramers' time, $T_{\text{K}} \simeq \frac{1}{\sigma} \mathrm{e}^{2H/\sigma^2}$

λ: Lyapunov exponent



$\sigma = 0.2$, $T = 2$ $\quad$ $\sigma = 0.4$, $T = 10$

References

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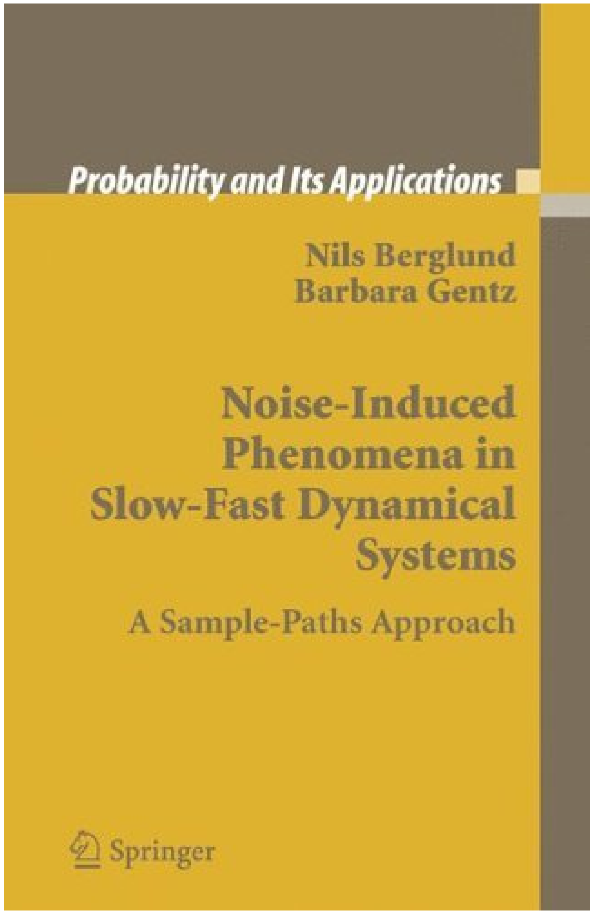
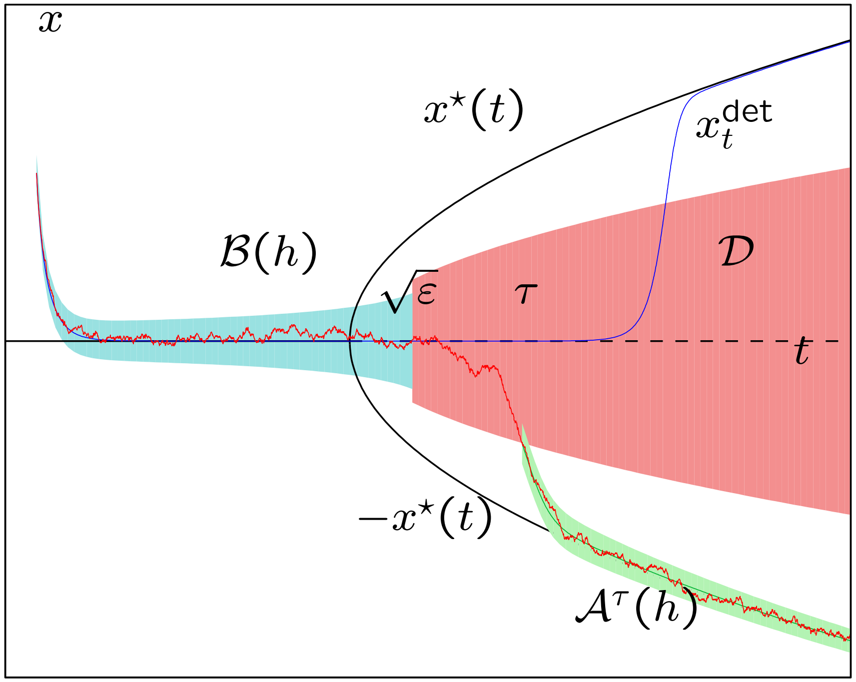


`http://berglund.univ-tln.fr/poitiers.pdf`

Appendix: Pitchfork bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[tx - x^3]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$



Theorem [B. & G., PTRF 2002]

- Paths concentrated in $\mathcal{B}(h)$ up to time $\sqrt{\varepsilon}$
  Typical spreading $\sigma\varepsilon^{-1/4}$
- Paths likely to leave $\mathcal{D}$ at time of order $\sqrt{\varepsilon|\log\sigma|}$
- Paths likely to stay in $\mathcal{A}^\tau(h)$ after leaving $\mathcal{D}$