The effect of additive noise on dynamical hysteresis

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Abstract

We investigate the properties of hysteresis cycles produced by a one-dimensional, periodically forced Langevin equation. We show that depending on amplitude and frequency of the forcing and on noise intensity, there are three qualitatively different types of hysteresis cycles. Below a critical noise intensity, the random area enclosed by hysteresis cycles is concentrated near the deterministic area, which is different for small and large driving amplitude. Above this threshold, the area of typical hysteresis cycles depends, to leading order, only on the noise intensity. In all three regimes, we derive mathematically rigorous estimates for expectation, variance, and the probability of deviations of the hysteresis area from its typical value.

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1 Introduction

For a long time, hysteresis was considered as a purely static phenomenon. As a consequence, it has been modeled by various integral operators relating the "output" of the system to its "input", for operators not depending on the speed of variation of the input (see for instance [May] and [MNZ] for reviews).

This situation changed drastically a decade ago, when Rao and coauthors published a numerical study of the effect of the input's frequency on shape and area of hysteresis cycles [RKP]. They proposed in particular that the area $\mathcal{A}$ of a hysteresis cycle, which measures the energy dissipation per period, should obey a scaling law of the form

$$\mathcal{A} \simeq A^{\alpha}\varepsilon^{\beta} \tag{1.1}$$

for small amplitude A and frequency ε of the periodic input (e. g. the magnetic field), and some model-dependent exponents α and β. This work triggered a substantial amount of numerical, experimental and theoretical studies, trying to establish the validity of the scaling law (1.1) for various systems, a problem which has become known as *dynamical hysteresis*.

The first model investigated in [RKP] is a Langevin partial differential equation for the spatially extended, N-component order parameter (e. g. the magnetization), in a $(\Phi^2)^2$-potential with $O(N)$-symmetry, in the limit $N \to \infty$. Their numerical experiments suggested that (1.1) holds with $\alpha \simeq 2/3$ and $\beta \simeq 1/3$. Various theoretical arguments [DT, SD, ZZ] indicate that the scaling law should be valid, but with $\alpha = \beta = 1/2$.

The second model considered in [RKP] is an Ising model with Monte-Carlo dynamics. Here the situation is not so clear. Different numerical simulations (for instance [LP, AC, ZZL]) suggested scaling laws with widely different exponents. More careful simulations [SRN], however, showed that the behaviour of hysteresis cycles depends in a complicated way on the mechanism of magnetization reversal, and no universal scaling law of the form (1.1) should be expected. Rigorous results on hysteresis in the Ising model are only available for discontinuous reversal (quenching) of the field [SS].

A third kind of models for which scaling laws of hysteresis cycles have been investigated belong to the mean field class, and include the Curie–Weiss model. A one-dimensional deterministic equation modeling a bistable laser, and being equivalent to the equation of motion of an overdamped particle in a periodically forced double-well potential, was considered in [JGRM]. The area of hysteresis cycles was shown to obey the scaling law

$$\mathcal{A} \simeq \mathcal{A}_0 + \varepsilon^{2/3} \tag{1.2}$$

for sufficiently large driving amplitude. A similar equation governs the dynamics of the magnetization in the Curie–Weiss model, in the limit of infinite system size. This equation was examined in [TO], where it was shown that the behaviour changes drastically when the amplitude of the forcing crosses a threshold, a phenomenon they termed "dynamic phase transition".

As pointed out in [Rao], the difference between the scaling laws (1.1) and (1.2) can be attributed to the existence of a potential barrier for the one-dimensional order parameter, which is absent in higher dimensions. The deterministic equation, however, neglects both thermal fluctuations and the finite system size, whose effects may be modeled by an additive white noise (see for instance [Mar]). Noise, however, may help to overcome the potential barrier and change the scaling law.

The aim of the present work is to give a rigorous characterization of the effect of additive white noise on scaling properties of hysteresis cycles. For definiteness, we shall consider the case of a Ginzburg–Landau potential, i. e., the stochastic differential equation

$$\mathrm{d}x_s = -\frac{\partial}{\partial x}\Bigl[\frac{1}{4}x_s^4 - \frac{1}{2}x_s^2 - \lambda(\varepsilon s)x_s\Bigr]\,\mathrm{d}s + \sigma\,\mathrm{d}W_s, \tag{1.3}$$

where W_s is a standard Brownian motion, and

$$\lambda(\varepsilon s) = -A\cos(2\pi\varepsilon s), \quad A > 0. \tag{1.4}$$

However, our results depend only on certain qualitative features of the bifurcation diagram and the proofs carry over to a more general setup as in [BG1, BG2].

In the deterministic case $\sigma = 0$, it is known [TO, JGRM, BK] that

- for $A < \lambda_{\rm c} + \mathcal{O}(\varepsilon)$ (where $\lambda_{\rm c} = 2/(3\sqrt{3})$ is such that the potential has two wells if and only if $|\lambda| < \lambda_{\rm c}$), solutions of (1.3) are attracted by hysteresis cycles (one for each potential well) enclosing an area of order ε, and with nonzero mean;
- for $A > \lambda_{\rm c} + \mathcal{O}(\varepsilon)$, solutions are attracted by a hysteresis cycle enclosing an area of order $\mathcal{A}_0 + \varepsilon^{2/3}(A-\lambda_{\rm c})^{1/3}$, where the static hysteresis area $\mathcal{A}_0$ is a constant, depending only on the geometry of the equilibrium branches.

For positive σ, the area $\mathcal{A}$ enclosed by a trajectory during one period is a random variable, depending on the realization $W_s(\omega)$ of the Brownian motion. Our aim is to characterize the distribution of $\mathcal{A}$ as a function of the parameters ε, σ and $a_0 = A - \lambda_{\rm c}$.

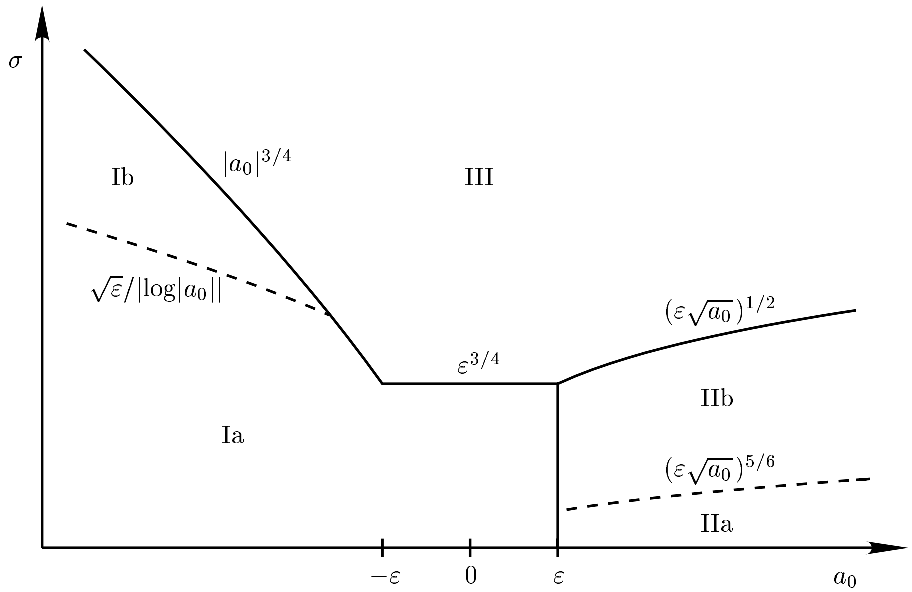

FIGURE 1. The different regimes as a function of amplitude $A = \lambda_c + a_0$ and noise intensity σ, for a given value of the frequency ε.

It turns out that the distribution is usually concentrated around a deterministic reference value. We determine the expectation and variance of $\mathcal{A}$. Furthermore, we estimate the behaviour of deviations of $\mathcal{A}$ from its reference value.

One of the main results is the existence of a threshold value for the noise intensity σ, depending on A and ε: Below this threshold, the area is concentrated near the corresponding deterministic value, while above the threshold, it depends, to leading order, only on the noise intensity and is slightly smaller than $\mathcal{A}_0$.

There are thus three parameter regimes, as shown in Figure 1, with qualitatively different behaviour of the area $\mathcal{A}$.

- In Case I, the *small amplitude regime*, the area is close to the deterministic value of order ε. There is a further subdivision into Case Ia, where the distribution of $\mathcal{A}$ is close to a Gaussian with standard deviation $\sigma\sqrt{\varepsilon}$ smaller than ε, and Case Ib, where the distribution is more spread out (see Theorem 2.3 and Figure 3).
- In Case II, the *large amplitude regime*, the area is concentrated near the deterministic value of order $\mathcal{A}_0 + (\varepsilon\sqrt{a_0})^{2/3}$. In Case IIa, the distribution is close to a Gaussian with standard deviation of order $\sigma(\varepsilon\sqrt{a_0})^{1/6}$. In Case IIb, we can only show that $\mathcal{A}$ is concentrated in an interval of width $(\varepsilon\sqrt{a_0})^{2/3}$ (see Theorem 2.4 and Figure 4).
- In Case III, the *large noise regime*, $\mathcal{A}$ is likely to be close to a reference area $\hat{\mathcal{A}}$ of order $\mathcal{A}_0 - \sigma^{4/3}$, which is smaller than the static hysteresis area. This is due to the noise driving x over the potential barrier before it becomes minimal or vanishes. The deviation $-\sigma^{4/3}$ does not depend on ε or A (see Theorem 2.5 and Figure 5).

Hysteresis does not only occur in ferromagnets and lasers, but also in mechanical systems displaying relaxation oscillations, such as the Van der Pol oscillator. Here additive noise can also have the effect of enabling jumps between stable states separated by a potential barrier [Fr]. Simple climate models can also display hysteresis, as has been observed for instance for the Atlantic thermohaline circulation [Rah, Mo]. In these systems, the effect of small scale degrees of freedom is represented by additive noise. Our results describe quantitatively how noise may cause the system to switch to another equilibrium state, at an earlier time than expected from the deterministic approximation.

We presented our results in detail in Section 2. Section 3 contains a short description of the deterministic dynamics, while the remaining sections present the proofs for the various parameter regimes.

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2 Results

We consider the non-autonomous SDE

$$\mathrm{d}x_s = F(x_s, \lambda(\varepsilon s))\,\mathrm{d}s + \sigma\,\mathrm{d}W_s, \tag{2.1}$$

where F derives from a periodically forced double-well potential and W_s is a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For definiteness, we shall consider the case

$$F(x,\lambda) = x - x^3 + \lambda = -\frac{\partial}{\partial x}\Big[\frac{1}{4}x^4 - \frac{1}{2}x^2 - \lambda x\Big] \tag{2.2}$$

$$\lambda(\varepsilon s) = -A\cos(2\pi\varepsilon s), \qquad A > 0. \tag{2.3}$$

We introduce the notation $\mathbb{P}^{t_0,x_0}$ for the law of the process $\{x_t\}_{t\geqslant t_0}$, starting in x_0 at time t_0, and use $\mathbb{E}^{t_0,x_0}$ to denote expectations with respect to $\mathbb{P}^{t_0,x_0}$. Note that the stochastic process $\{x_t\}_{t\geqslant t_0}$ is an inhomogeneous Markov process.

Before turning to the precise statements of our results, let us introduce some notations. We shall use

- $y \vee z$ and $y \wedge z$ to denote the maximum or minimum, respectively, of two real numbers y and z.
- If $\varphi(t,\varepsilon)$ and $\psi(t,\varepsilon)$ are defined for small ε and for t in a given interval I, we write $\psi(t,\varepsilon) \asymp \varphi(t,\varepsilon)$ if there exist strictly positive constants $c_\pm$ such that $c_-\varphi(t,\varepsilon) \leqslant \psi(t,\varepsilon) \leqslant c_+\varphi(t,\varepsilon)$ for all $t \in I$ and all sufficiently small ε. The constants $c_\pm$ are understood to be independent of t and ε (and hence also independent of small quantities like σ and, possibly, a_0, which we consider as functions of ε).
- By $g(u) = \mathcal{O}(u)$ we indicate that there exist $\delta > 0$ and $K > 0$ such that $g(u) \leqslant Ku$ for all $u \in [0,\delta]$, where δ and K of course do not depend on ε or on the other small parameters a_0 and σ.
- Let I be an interval. The notation $1_I(x)$ is used for the indicator function, taking value 1 if $x \in I$ and 0 otherwise.

Finally, let us point out that most estimates hold for small enough ε only, and often only for $\mathbb{P}$-almost all $\omega \in \Omega$. We will stress these facts only where confusion might arise.

Let us first consider the deterministic case $\sigma = 0$. It is convenient to introduce the slow time $t = \varepsilon s$, and rewrite (2.1) for $\sigma = 0$ as

$$\varepsilon\frac{\mathrm{d}x_t}{\mathrm{d}t} = F(x_t, \lambda(t)). \tag{2.4}$$

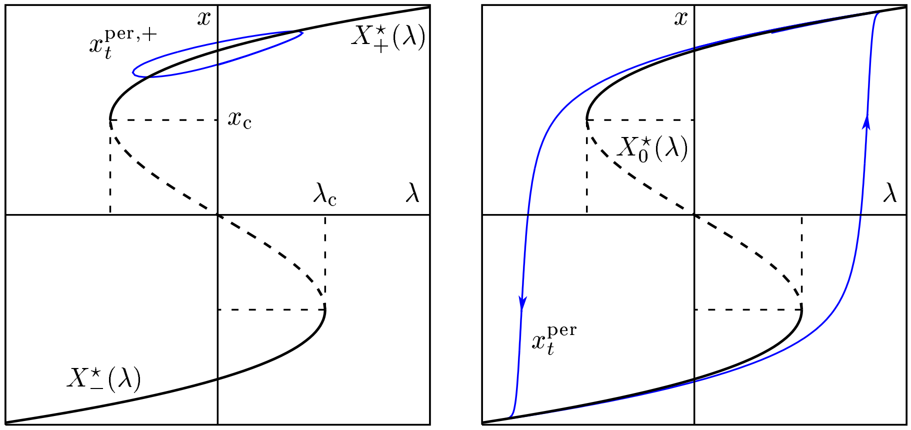

FIGURE 2. Equilibrium branches of F (heavy curves) and periodic solutions of the deterministic equation (light curves), for $A < \lambda_c$ (left) and $A > \lambda_c$ (right). For $A < \lambda_c + \mathcal{O}(\varepsilon)$, the enclosed area is of order ε while for $A > \lambda_c + \mathcal{O}(\varepsilon)$, it is of order $\mathcal{A}_0 + \varepsilon^{2/3}(A - \lambda_c)^{1/3}$.

We start by discussing some properties of this equation, which will be summarized in Theorem 2.2 below. As ε goes to zero, solutions of (2.4) are known to approach equilibrium branches of F, that is, solutions of $F(x, \lambda) = 0$ (see Figure 2). Let $\lambda_c = 2/(3\sqrt{3})$.

- For $|\lambda| < \lambda_c$, F has three equilibrium branches $X_-^\star(\lambda) < X_0^\star(\lambda) < X_+^\star(\lambda)$, where $X_\pm^\star(\lambda)$ are stable equilibria and $X_0^\star(\lambda)$ is an unstable equilibrium of the associated frozen system $\dot{x} = F(x, \lambda)$.
- At $\lambda = -\lambda_c$, the branches $X_+^\star(\lambda)$ and $X_0^\star(\lambda)$ undergo a saddle–node bifurcation, and $X_+^\star(-\lambda_c) = X_0^\star(-\lambda_c) = x_c := 1/\sqrt{3}$.
- For $\lambda < -\lambda_c$, $X_-^\star(\lambda)$ is the only equilibrium branch.
- A similar bifurcation occurs at $\lambda = +\lambda_c$, where $X_-^\star(\lambda_c) = X_0^\star(\lambda_c) = -x_c$.
- For $\lambda > \lambda_c$, $X_+^\star(\lambda)$ is the only equilibrium branch.

We can thus expect a qualitative difference, in the limit $\varepsilon \to 0$, between the regime $A < \lambda_c$, where F always derives from a double-well potential, and the regime $A > \lambda_c$, where F has only one equilibrium part of the time.

Definition 2.1. *Let $x_t^{\text{per},\varepsilon}$ be a periodic solution of* (2.4)*. We say that this solution* does not display hysteresis *if there exists a continuous function $\lambda \mapsto X^\star(\lambda)$ such that*

$$\lim_{\varepsilon \to 0} x_t^{\text{per},\varepsilon} = X^\star(\lambda(t)). \tag{2.5}$$

If no such function exists, we say that $x_t^{\text{per},\varepsilon}$ displays hysteresis*.*

If $A < \lambda_c$, solutions starting near a stable equilibrium branch $X_+^\star(\lambda)$ or $X_-^\star(\lambda)$ will remain close to that branch, and relation (2.5) holds with $X^\star(\lambda) = X_+^\star(\lambda)$ or $X_-^\star(\lambda)$, depending on the initial condition. If $A > \lambda_c$, however, it turns out that

$$\lim_{\varepsilon \to 0} x_t^{\text{per},\varepsilon} = \begin{cases} X_+^\star(\lambda(t)) & \text{if } \lambda(t) > \lambda_c \text{ or if } \lambda(t) > -\lambda_c \text{ and } \lambda'(t) < 0 \\ X_-^\star(\lambda(t)) & \text{otherwise.} \end{cases} \tag{2.6}$$

Thus the solution displays hysteresis since the instantaneous value of λ alone does not suffice to determine the state of the system in the adiabatic limit. This so-called *hysteresis* *cycle* can be characterized by its area, defined as

$$\mathcal{A}(\varepsilon) = -\int_{-1/2}^{1/2} x_t^{\text{per},\varepsilon} \lambda'(t)\,\mathrm{d}t. \tag{2.7}$$

If $A < \lambda_{\text{c}}$, we have $\lim_{\varepsilon\to 0} \mathcal{A}(\varepsilon) = 0$, while for $A > \lambda_{\text{c}}$,

$$\lim_{\varepsilon\to 0} \mathcal{A}(\varepsilon) = \mathcal{A}_0 := \int_{-\lambda_{\text{c}}}^{\lambda_{\text{c}}} (X_+^\star(\lambda) - X_-^\star(\lambda))\,\mathrm{d}\lambda = \frac{3}{2}. \tag{2.8}$$

The situation is thus relatively simple in the limit $\varepsilon \to 0$. Since in practice, however, the variation of λ will not be infinitely slow, it is important to understand what happens for small but positive values of ε. We summarize the necessary facts in the following theorem.

Theorem 2.2 (Deterministic Case)**.** *There exist constants $\gamma_1 > \gamma_0 > 0$ such that the following behaviour holds for sufficiently small ε.*

- *If $a_0 = A - \lambda_{\text{c}} \leqslant \gamma_0\varepsilon$, Equation* (2.4) *has exactly two stable periodic solutions $x_t^{\text{per},+}$ and $x_t^{\text{per},-}$, and one unstable periodic solution $x_t^{\text{per},0}$. These solutions track, respectively, the equilibrium branches $X_\pm^\star(\lambda(t))$ and $X_0^\star(\lambda(t))$ at a distance not larger than $\mathcal{O}(\varepsilon|a_0|^{-1/2} \wedge \sqrt{\varepsilon})$, and enclose an area*

$$\mathcal{A}(\varepsilon) \asymp \varepsilon A. \tag{2.9}$$

  All solutions which do not start on $x_t^{\text{per},0}$ are attracted either by $x_t^{\text{per},+}$ or by $x_t^{\text{per},-}$.
- *If $a_0 = A - \lambda_{\text{c}} \geqslant \gamma_1\varepsilon$, Equation* (2.4) *admits exactly one periodic solution x_t^{per}. This solution is stable, satisfies* (2.6) *in the adiabatic limit, and encloses an area $\mathcal{A}(\varepsilon)$ satisfying*

$$\mathcal{A}(\varepsilon) - \mathcal{A}_0 \asymp \varepsilon^{2/3} a_0^{1/3}. \tag{2.10}$$

In the case where a_0 is of order 1, the scaling law (2.10) was first obtained in [JGRM]. We outline the proof of Theorem 2.2 in Section 3. Note that in the transition zone $\gamma_0\varepsilon < a_0 < \gamma_1\varepsilon$, the situation is more complicated, since more than two stable periodic orbits can coexist [TO, BK].

Let us now return to the stochastic differential equation (2.1). In slow time $t = \varepsilon s$, it can be written as

$$\mathrm{d}x_t = \frac{1}{\varepsilon} F(x_t, \lambda(t))\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t. \tag{2.11}$$

Let us fix, say, $t_0 = -1/2$ as initial time, and some $x_0 > 0$ as initial condition, such that the solution x_t^{det} of the deterministic equation (2.4) with the same initial condition is attracted by $x_t^{\text{per},+}$ or x_t^{per}, respectively. We denote by $x_t(\omega)$ the solution of the SDE (2.11) with initial condition $x_{t_0} = x_0$ for a given realization ω of the Brownian motion, and associate with it the area

$$\mathcal{A}(\varepsilon, \sigma; \omega) = -\int_{-1/2}^{1/2} x_t(\omega) \lambda'(t)\,\mathrm{d}t. \tag{2.12}$$

Note that $\mathcal{A}(\varepsilon, \sigma; \omega)$ also depends on $a_0 = A - \lambda_{\text{c}}$. We do not stress this dependence here but consider a_0 as a (possibly constant) function of ε. Of course, since $x_t(\omega)$ is not periodic in general, the integral (2.12) does not represent the area of enclosed by a closed curve. However it is still physically meaningful since it describes the energy dissipation if

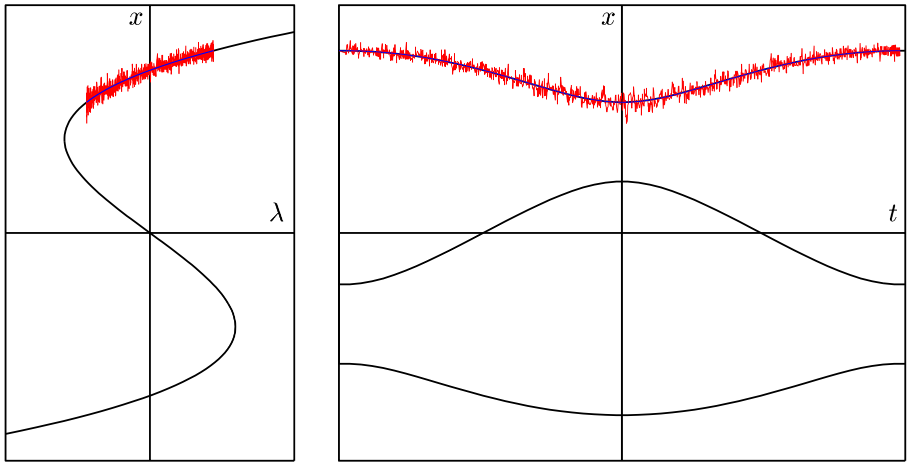

FIGURE 3. A sample path of Equation 2.1 for $\varepsilon = 0.001$, $\sigma = 0.05$ and $a_0 = -0.1$, corresponding to the small amplitude regime.

x and λ are thermodynamically conjugate variables. One can check that for $|x_0 - x_{t_0}^{\text{per},+}|$ sufficiently small (but still of order one), $|x_t^{\text{det}} - x_t^{\text{per},+}|$ decreases exponentially fast in $(t-t_0)/\varepsilon$ and thus $\mathcal{A}(\varepsilon,0)$ still behaves like (2.9). The same is true for x_t^{per} and the validity of (2.10).

Our main purpose is to characterize the distribution of the random variable $\mathcal{A}(\varepsilon,\sigma)$ as a function of the parameters ε, $a_0 = A - \lambda_{\text{c}}$ and σ. The following three theorems describe the situation in three different parameter regimes.

Theorem 2.3 (Case I – Small amplitude regime). *Assume that $a_0 = A - \lambda_{\text{c}} \leqslant \gamma_0\varepsilon$ and that $\sigma \leqslant (|a_0| \vee \varepsilon)^{3/4}$. Then there exist positive constants κ, h_1, h_2, c_0 and C such that the following properties hold for sufficiently small ε.*

- *The probability that a sample path starting near one potential well crosses the potential barrier during one period is smaller than*

$$\frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa(|a_0|\vee\varepsilon)^{3/2}/\sigma^2}\,. \tag{2.13}$$

- **Case Ia:** *Assume that either $a_0 \geqslant -\varepsilon$ or $\sigma \leqslant \sqrt{\varepsilon}/|\log|a_0||$. Then the deviation of the area $\mathcal{A}(\varepsilon,\sigma)$ from its deterministic value $\mathcal{A}(\varepsilon,0)$ satisfies*

$$\mathbb{P}\bigl\{|\mathcal{A}(\varepsilon,\sigma)-\mathcal{A}(\varepsilon,0)| \geqslant H\bigr\} \leqslant \begin{cases} \dfrac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} & \text{for } 0 \leqslant H \leqslant H_1(\varepsilon,a_0), \\ \dfrac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^4/\sigma^2} & \text{for } H \geqslant h_2, \end{cases} \tag{2.14}$$

where $H_1(\varepsilon,a_0) = h_1\sqrt{\varepsilon}(|a_0|\vee\varepsilon)^{3/4}\wedge(\varepsilon/|\log(|a_0|\vee\varepsilon)|)$. Furthermore, under the slightly stronger assumption $\sigma|\log\varepsilon| \leqslant c_0(|a_0|\vee\varepsilon)^{3/4}$,

$$\bigl|\mathbb{E}\bigl\{\mathcal{A}(\varepsilon,\sigma)-\mathcal{A}(\varepsilon,0)\bigr\}\bigr| \leqslant \mathcal{O}(\sigma^2|\log(|a_0|\vee\varepsilon)|) \tag{2.15}$$

$$\operatorname{Var}\bigl\{\mathcal{A}(\varepsilon,\sigma)-\mathcal{A}(\varepsilon,0)\bigr\} \asymp \sigma^2\varepsilon. \tag{2.16}$$

- **Case Ib:** *Assume now that $a_0 \leqslant -\varepsilon$ and $\sigma \geqslant \sqrt{\varepsilon}/|\log|a_0||$. Then (2.14) still holds, and in addition, we have*

$$\mathbb{P}\bigl\{|\mathcal{A}(\varepsilon,\sigma)-\mathcal{A}(\varepsilon,0)| \geqslant H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2|\log|a_0||)} \tag{2.17}$$

for $\varepsilon/|\log|a_0|| \leqslant H \leqslant H_2(\varepsilon, a_0) = h_1|a_0|^{3/2}|\log|a_0||$. *Moreover, under the slightly stronger assumption* $\sigma|\log\varepsilon| \leqslant c_0(|a_0| \vee \varepsilon)^{3/4}$,

$$\big|\mathbb{E}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)\big\}\big| \leqslant \mathcal{O}(\sigma^2|\log|a_0||) \tag{2.18}$$

$$\operatorname{Var}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)\big\} \leqslant \mathcal{O}(\sigma^4|\log|a_0||^2). \tag{2.19}$$

The proof follows as a particular case of more general results presented in Section 4.

In Case Ia, the distribution of $\mathcal{A}(\varepsilon,\sigma)$ is close to a Gaussian centred at $\mathcal{A}(\varepsilon,0)$. Both the expectation of $\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)$ and its standard deviation are smaller than the deterministic value $\mathcal{A}(\varepsilon,0) \asymp \varepsilon$. Thus one will still observe, with a high probability, an area of the same order as the deterministic one.

In Case Ib, the distribution of $\mathcal{A}(\varepsilon,\sigma)$ becomes more spread out, with a standard deviation possibly exceeding the deterministic value $\mathcal{A}(\varepsilon,0)$. Thus, although typical values of $\mathcal{A}(\varepsilon,\sigma)$ will still be small, the probability of negative values is no longer negligible, and the deterministic scaling law $\mathcal{A}(\varepsilon,0) \asymp \varepsilon$ can no longer be observed.

The quartic decay of the probability of deviations of order larger than 1 from the deterministic area is a consequence of the cubic growth of the drift term F for large $|x|$. In fact, this property holds in *all* parameter regimes, since it does not depend on the details of the dynamics near the origin. For the sake of brevity, we will not repeat this estimate in the other regimes.

Note that there is a gap between $H \leqslant H_1$ or H_2 and $H \geqslant h_2$ where we do not describe the deviations. In fact, the distribution of $\mathcal{A}(\varepsilon,\sigma)$ will not be unimodal. Sample paths are unlikely to jump from one potential well to the other one, but if they do so, then most likely near the instant of minimal barrier height, producing a small peak in the distribution for areas $\mathcal{A}(\varepsilon,\sigma)$ of order 1.

Theorem 2.4 (Case II – Large amplitude regime)**.** *Assume that* $a_0 = A - \lambda_{\rm c} \geqslant \gamma_1\varepsilon$ *and that* $\sigma \leqslant (\varepsilon\sqrt{a_0})^{1/2}$. *Then there exist positive constants* κ, h_1, h_2, c_0, L_0, L_1, L_2 *and* C *such that the following properties hold for sufficiently small* ε.

- *Let* λ^0 *denote the (random) value of* λ *when* x_t *changes sign for the first time. Then*

$$\mathbb{P}\big\{|\lambda^0| < \lambda_{\rm c} - L\big\} \leqslant \frac{C}{\varepsilon}\,{\rm e}^{-\kappa(L^{3/2}\vee\varepsilon\sqrt{a_0})/\sigma^2} \tag{2.20}$$

  for $-L_1(\varepsilon\sqrt{a_0})^{2/3} \leqslant L \leqslant L_0/|\log(\varepsilon\sqrt{a_0})|$, *and*

$$\mathbb{P}\big\{|\lambda^0| > \lambda_{\rm c} + L\big\} \leqslant 3\,1_{(0,a_0]}(L)\,\exp\bigg\{-\frac{\kappa}{\sigma^2}\frac{L}{(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})|}\bigg\} \tag{2.21}$$

  for $L \geqslant L_2(\varepsilon\sqrt{a_0})^{2/3}$.
- **Case IIa:** *Assume that* $\sigma \leqslant (\varepsilon\sqrt{a_0})^{5/6}$. *Then*

$$\mathbb{P}\big\{|\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)| \geqslant H\big\} \leqslant \frac{C}{\varepsilon}\exp\bigg\{-\kappa\frac{H^2}{\sigma^2(\varepsilon\sqrt{a_0})^{1/3}}\bigg\} \qquad \forall\, H \leqslant h_1\varepsilon\sqrt{a_0}. \tag{2.22}$$

  Furthermore, under the slightly stronger assumption $\sigma|\log\varepsilon| \leqslant c_0(\varepsilon\sqrt{a_0})^{5/6}$,

$$\big|\mathbb{E}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)\big\}\big| \leqslant \mathcal{O}\bigg(\frac{\sigma^2|\log\varepsilon|}{(\varepsilon\sqrt{a_0})^{2/3}}\bigg) \tag{2.23}$$

$$\operatorname{Var}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)\big\} \asymp \sigma^2(\varepsilon\sqrt{a_0})^{1/3}. \tag{2.24}$$

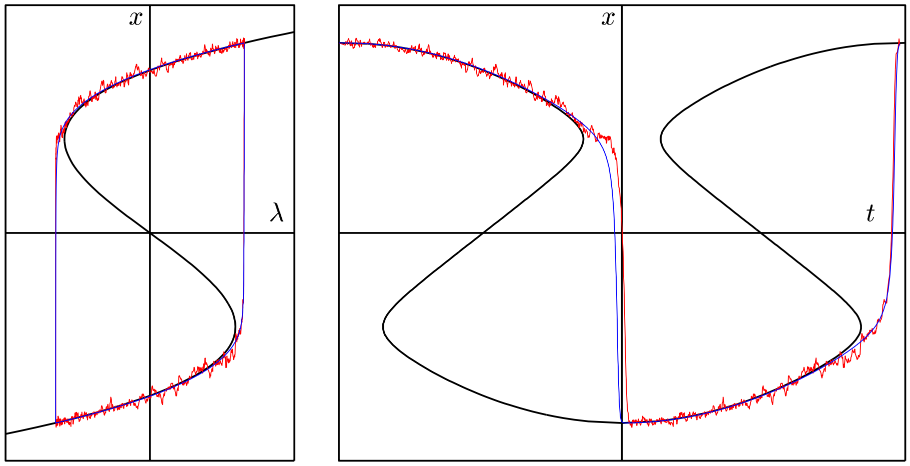

FIGURE 4. A sample path of Equation 2.1 for $\varepsilon = 0.005$, $\sigma = 0.04$ and $a_0 = 0.04$, corresponding to the large amplitude regime. A solution for $\sigma = 0$ is shown for comparison.

- **Case IIb:** *Assume now that* $(\varepsilon\sqrt{a_0})^{5/6} \leqslant \sigma \leqslant (\varepsilon\sqrt{a_0})^{1/2}$. *Then*

$$\mathbb{P}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0) \leqslant -H\big\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} \tag{2.25}$$

$$\mathbb{P}\big\{\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0) \geqslant +H\big\} \leqslant \frac{C}{\varepsilon}\exp\Big\{-\kappa\frac{(\varepsilon\sqrt{a_0})^{1/3}H}{\sigma^2|\log(\varepsilon\sqrt{a_0})|}\Big\} \tag{2.26}$$

for $h_1(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})| \leqslant H \leqslant h_2(\varepsilon\sqrt{a_0})^{1/3}|\log(\varepsilon^{2/3}a_0^{-1/6})|$. *As a consequence, if* $\sigma \leqslant c_0(\varepsilon\sqrt{a_0})/\sqrt{|\log\varepsilon|}$, *then expectation and standard deviation of* $\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}(\varepsilon,0)$ *are both at most of order* $(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})|$.

The proof is given in Section 6.

The estimates (2.20) and (2.21) show that the value of the parameter (e. g. the magnetic field) for which x_t changes sign is most likely $\pm(\lambda_c + \mathcal{O}((\varepsilon\sqrt{a_0})^{2/3}))$, which corresponds to the deterministic value.

In Case IIa, the distribution of $\mathcal{A}(\varepsilon,\sigma)$ is again close to a Gaussian in a neighbourhood of $\mathcal{A}(\varepsilon,0)$. Both the expectation of $\mathcal{A}(\varepsilon,\sigma)-\mathcal{A}(\varepsilon,0)$ and its standard deviation are smaller than the deterministic value of $\mathcal{A}(\varepsilon,0)-\mathcal{A}_0 \asymp (\varepsilon\sqrt{a_0})^{2/3}$. Thus one will still observe, with a high probability, an area of the same order as the deterministic one.

In Case IIb, we can only show that $\mathcal{A}(\varepsilon,\sigma)$ is likely to belong to an interval of size $(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})|$ centred at the deterministic value, so that $\mathcal{A}(\varepsilon,\sigma) - \mathcal{A}_0$ is not necessarily positive with probability close to 1. There is a gap between the estimates (2.25) and (2.26) outside this interval, and the trivial bound 1 inside the interval. This is due to the large spreading of paths during the jump. However, this result may conceivably fall short of being optimal.

Theorem 2.5 (Case III – Large noise regime)**.** *Assume that either* $a_0 \leqslant \varepsilon$ *and* $\sigma > (|a_0| \vee \varepsilon)^{3/4}$, *or* $a_0 \geqslant \varepsilon$ *and* $\sigma > (\varepsilon\sqrt{a_0})^{1/2}$. *Then there exists a (deterministic) reference area* $\hat{\mathcal{A}}$, *satisfying*

$$\hat{\mathcal{A}} - \mathcal{A}_0 \asymp -\sigma^{4/3}, \tag{2.27}$$

and positive constants κ, h_1, h_2, c_0, c_1, c_2 *and* C, *such that the following properties hold for sufficiently small* ε.

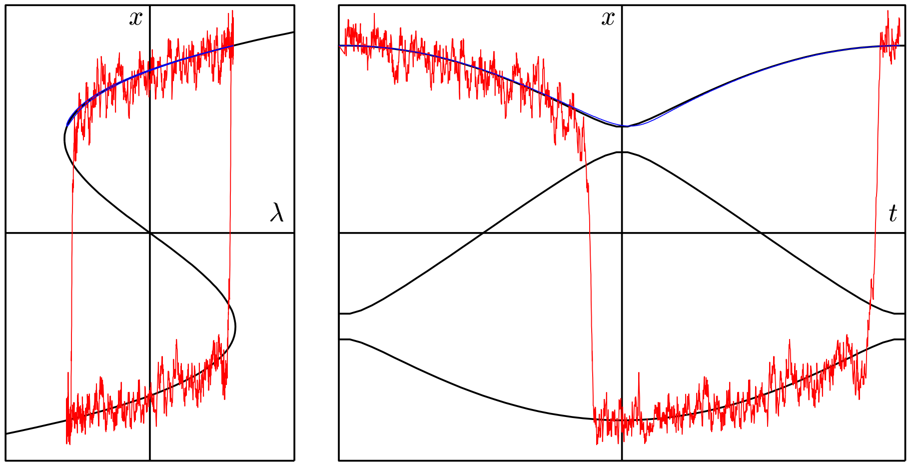

FIGURE 5. A sample path of Equation 2.1 for $\varepsilon = 0.005$, $\sigma = 0.16$ and $a_0 = -0.01$, corresponding to the large noise regime. A solution for $\sigma = 0$ is shown for comparison.

- **Case IIIa:** *Either* $a_0 \leqslant \varepsilon$ *or* $\sigma > a_0^{3/4}$. *Then the deviation of the area* $\mathcal{A}(\varepsilon,\sigma)$ *from the reference value* $\hat{\mathcal{A}}$ *satisfies*

$$\mathbb{P}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} + \frac{3}{2}\,\mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)} \tag{2.28}$$

$$\mathbb{P}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}} > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2|\log\sigma|)} + \frac{3}{2}1_{[0,h_2\sigma^{4/3})}(H)\,\mathrm{e}^{-\kappa H/(\varepsilon|\log\sigma|)} \tag{2.29}$$

for $0 \leqslant H \leqslant h_1\sigma^{2/3}|\log\sigma|$. *Moreover, if the noise intensity satisfies* $c_1\varepsilon < \sigma^{4/3}/|\log\sigma|^2$ *and* $\sigma^{2/3}|\log\sigma| \leqslant c_2/|\log\varepsilon|$, *then*

$$\mathbb{E}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \in [-C\sigma^{4/3}|\log\varepsilon|^{2/3}, C(\varepsilon \vee \sigma^2|\log\varepsilon|)|\log\sigma|] \tag{2.30}$$

$$\mathrm{Var}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \leqslant C\bigl(\sigma^{4/3}|\log\varepsilon|^{2/3}\bigr)^2. \tag{2.31}$$

- **Case IIIb:** $a_0 \geqslant \varepsilon$ *and* $\sigma \leqslant a_0^{3/4}$. *Let* $\ell_0 = |\log(\sigma^{4/3}/\sqrt{a_0})|$. *Then*

$$\mathbb{P}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} + \frac{3}{2}\,\mathrm{e}^{-\kappa\sigma^2/(\varepsilon\sqrt{a_0}|\log\sigma|)} \tag{2.32}$$

$$\mathbb{P}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}} > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2\ell_0)} + \frac{3}{2}1_{[0,h_2a_0)}(H)\,\mathrm{e}^{-\kappa\sigma^{2/3}H/(\varepsilon\sqrt{a_0}|\log\sigma|)} \tag{2.33}$$

for $0 \leqslant H \leqslant h_1\sigma^{2/3}\ell_0$. *In addition, if* $\sigma \geqslant c_1(\varepsilon\sqrt{a_0})^{1/2}|\log\varepsilon|$, *then*

$$\mathbb{E}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \in [-C\sigma^{4/3}|\log\varepsilon|^{2/3}, C(\sigma^2\ell_0|\log\varepsilon| \vee \varepsilon\sqrt{a_0}|\log\sigma|/\sigma^{2/3})] \tag{2.34}$$

$$\mathrm{Var}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \leqslant C\bigl(\sigma^{4/3}|\log\varepsilon|^{2/3}\bigr)^2. \tag{2.35}$$

The value λ^0 *of* λ *at the first time* x_t *reaches* 0 *behaves in a similar way as the area, when compared to a reference value* $\hat{\lambda}$ *equal to* $\lambda_{\mathrm{c}} - \mathcal{O}(\sigma^{4/3})$.

The proof is given in Section 5.

The main feature in this parameter regime is that the noise intensity is sufficiently large to drive x_t over the potential barrier before it reaches its minimal height or even vanishes. The barrier is typically crossed when $|\lambda|$ equals $\lambda_{\mathrm{c}} - \mathcal{O}(\sigma^{4/3})$.

The distribution of $\mathcal{A}(\varepsilon,\sigma)$ decays faster to the right of $\hat{\mathcal{A}}$ than to the left. The probability that $\mathcal{A}(\varepsilon,\sigma)$ exceeds $\mathcal{A}_0$ is very small (unless σ approaches its threshold value), so that it is indeed likely to observe an area that is smaller than $\mathcal{A}_0$, by an amount of order $\sigma^{4/3}$.

3 Deterministic case

In this section we discuss the deterministic equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = x - x^3 + \lambda(t) \tag{3.1}$$

with initial condition $x_{-1/2} = x_0$. Recall that we are interested in the case $\lambda(t) = -A\cos(2\pi t)$ with $A = \lambda_{\mathrm{c}} + a_0$. Since this equation has already been studied in [JGRM, TO, BK, BG2], we only outline the main properties without proofs.

3.1 The case $a_0 \leqslant \gamma_0\varepsilon$

The simplest situation occurs when a_0 is negative and of order 1. Then the three curves $x^\star_\pm(t) = X^\star_\pm(\lambda(t))$ and $x^\star_0(t) = X^\star_0(\lambda(t))$ are uniformly hyperbolic equilibrium curves of the associated family of frozen systems $\dot{x} = x - x^3 + \lambda$. Thus Tihonov's theorem [Ti, Gr] shows the existence of particular solutions $x^\pm_t$ and x^0_t tracking, respectively, $x^\star_\pm(t)$ and $x^\star_0(t)$ at a distance of order ε. These solutions are not necessarily periodic, but the curves $x^\pm_t$ attract a neighbourhood of order 1 exponentially fast. Thus the Poincaré map $P : x_{-1/2} \mapsto x_{1/2}$ maps neighbourhoods of order 1 of $x^\star_\pm(-1/2)$, respectively, to two exponentially small intervals containing $x^\pm_{1/2}$. This implies the existence of a unique fixed point (corresponding to a periodic orbit) in each interval. A similar statement is true for P^{-1} in a neighbourhood of $x^\star_0(1/2)$. The fact that x_t is monotonous between the equilibrium branches excludes the existence of other periodic orbits (see, e.g. [Ber, Proposition 4.8]).

If a_0 is a small parameter, Tihonov's theorem can also be applied outside a given interval $[-T, T]$, T a constant of order 1, while the dynamics in $[-T, T]$ has to be analysed separately. For $|t| \geqslant c_0(|a_0| \vee \varepsilon)^{1/2}$, c_0 a sufficiently large constant, one can consider the deviation $y_t = x_t - x^\star_+(t)$, which obeys the equation

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} = a^\star_+(t)y + b^\star_+(y,t) - \varepsilon\frac{\mathrm{d}x^\star_+}{\mathrm{d}t}, \tag{3.2}$$

where

$$\begin{aligned} a^\star_+(t) &= 1 - 3(x^\star_+(t))^2 \\ b^\star_+(y,t) &= -y^2\bigl[3x^\star_+(t) + y\bigr]. \end{aligned} \tag{3.3}$$

This equation is used to show (see [BG2, Section 4.1]) that

$$y_t \asymp \frac{\varepsilon}{|t|} \qquad \text{for } -T \leqslant t \leqslant -c_0(|a_0| \vee \varepsilon)^{1/2}. \tag{3.4}$$

In the case $a_0 \leqslant -\varepsilon/\gamma_0$ for some small enough $\gamma_0 > 0$, one also obtains from (3.2) that for $|t| \leqslant c_0\sqrt{|a_0|}$,

$$y_t = -tC_1(t) + C_2(t) \qquad \text{with} \qquad C_1(t) \asymp \frac{\varepsilon}{|a_0|}, \quad C_2(t) \asymp \frac{\varepsilon^2}{|a_0|^{3/2}}, \tag{3.5}$$

which implies that y_t becomes negative at a time of order $\varepsilon|a_0|^{-1/2}$. If $-\varepsilon/\gamma_0 \leqslant a_0 \leqslant \gamma_0\varepsilon$, the dynamics for $|t| \leqslant c_0\sqrt{\varepsilon}$ is analysed by the change of variables

$$t = c\sqrt{\varepsilon}s, \qquad x = \frac{1}{\sqrt{3}}\Big(1 + \frac{\sqrt{\varepsilon}}{c}z\Big), \tag{3.6}$$

where $2\pi^2 Ac^4 = 1$. Then z obeys a perturbation of order $\sqrt{\varepsilon}$ of the Riccati equation

$$\frac{\mathrm{d}z}{\mathrm{d}s} = s^2 - z^2 - \delta, \qquad \text{where } \delta = \sqrt{3}\,c^2\,\frac{a_0}{\varepsilon} \leqslant \sqrt{3}\,c^2\,\gamma_0, \tag{3.7}$$

which can be used to show that for γ_0 small enough, $z_s \asymp 1$ for s of order 1. It follows that for *all* $a_0 \leqslant \gamma_0\varepsilon$,

$$x_t - \frac{1}{\sqrt{3}} \asymp (|a_0| \vee \varepsilon)^{1/2} \qquad \text{for } |t| \leqslant c_0(|a_0| \vee \varepsilon)^{1/2}. \tag{3.8}$$

Finally, one obtains as before that

$$y_t \asymp -\frac{\varepsilon}{|t|} \qquad \text{for } c_0(|a_0| \vee \varepsilon)^{1/2} \leqslant t \leqslant T. \tag{3.9}$$

Hence there is a solution of (3.1) tracking $x^\star_+(t)$ at a distance of order $\varepsilon/(|t| \vee \sqrt{|a_0|} \vee \sqrt{\varepsilon})$ (if $a_0 > 0$, $x^\star_+(t)$ does not exist during a small time interval, but this gap is too small for x_t to slip through). Later we will use the fact that the linearization of F around this solution satisfies

$$a(t) := \frac{\partial}{\partial x}\big[x - x^3 - A\cos(2\pi t)\big]\Big|_{x=x_t} \asymp -(|t| \vee \sqrt{|a_0|} \vee \sqrt{\varepsilon}). \tag{3.10}$$

Furthermore, we will need that fact that $|a'(t)|$ is bounded above by a constant independent of ε and a_0. This can be shown by using the relation

$$a'(t) = -6x_t\frac{\mathrm{d}}{\mathrm{d}t}x_t = -6x_t\frac{1}{\varepsilon}F(x_t, \lambda(t)). \tag{3.11}$$

The cases $|t| \geqslant (|a_0| \vee \varepsilon)^{1/2}$ and $a_0 \leqslant -\varepsilon/\gamma_0$ can be treated by expanding F around the equilibrium branch $x^\star_+(t)$ that x_t is tracking, and using the estimates (3.4), (3.5), (3.9) for $x_t - x^\star_+(t)$. The remaining case can be treated by considering (3.7) directly.

In addition, these estimates show that

$$\mathcal{A}(\varepsilon, 0) = 2\pi A \int_{-1/2}^{1/2} x_t \sin(2\pi t)\,\mathrm{d}t \asymp \varepsilon A. \tag{3.12}$$

Similar properties hold for solutions tracking $x^\star_-(t)$ and $x^\star_0(t)$, and the above arguments on the Poincaré map can be repeated to show the existence of two stable and one unstable periodic orbit.

3.2 The case $a_0 \geqslant \gamma_1\varepsilon$

Let t_c be the solution of $A\cos(2\pi t_c) = \lambda_c$ in $[0, 1/4]$. The equilibrium branches $x^\star_+$ and $x^\star_0$ bifurcate at the point $(-t_c, x_c)$, where $x_c = 1/\sqrt{3}$. The translation

$$t = -t_c + s, \qquad x = x_c + y \tag{3.13}$$

yields the equation

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}s} = \mu(s) - \sqrt{3}y^2 - y^3, \tag{3.14}$$

where

$$\begin{aligned}\mu(s) &= \lambda_{\mathrm{c}} - A\cos(2\pi(-t_{\mathrm{c}} + s)) \\ &= \lambda_{\mathrm{c}}(1 - \cos(2\pi s)) - \sqrt{A^2 - \lambda_{\mathrm{c}}^2}\,\sin(2\pi s) \\ &= -2\pi\sqrt{2a_0\lambda_{\mathrm{c}} + a_0^2}\,s + 2\pi^2\lambda_{\mathrm{c}}s^2 + \mathcal{O}(s^3). \end{aligned} \tag{3.15}$$

As before, one shows that y_s tracks the equilibrium branch $y_+^\star(s) = x_+^\star(-t_{\mathrm{c}} + s) - x_{\mathrm{c}}$ at a distance scaling like $\varepsilon/|s|$ for $s \leqslant -\varepsilon^{2/3}a_0^{-1/6}$. For larger times, we use the scaling

$$s = c\varepsilon^{2/3}a_0^{-1/6}u, \qquad y = \frac{1}{\sqrt{3}c}\varepsilon^{1/3}a_0^{1/6}z \tag{3.16}$$

which yields, for an appropriate choice of c, a perturbation of order $\varepsilon^{1/3}a_0^{1/6}$ of the Riccati equation

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \hat{\mu}(u) - z^2, \qquad \text{where } \hat{\mu}(u) = -u + \mathcal{O}(\varepsilon^{2/3}a_0^{-2/3}u^2). \tag{3.17}$$

For sufficiently large γ_1 (recall that $a_0 \geqslant \gamma_1\varepsilon$), one can show that z_u reaches a value of order -1 in a time of order 1, and thus y_s reaches order $-\varepsilon^{1/3}a_0^{1/6}$ for some s of order $\varepsilon^{2/3}a_0^{-1/6}$. Finally, the fact that the right-hand side of (3.14) is smaller than $-y^2$ for sufficiently small s and y can be used to show that y_s reaches values of order -1 after another time of order $\varepsilon^{2/3}a_0^{-1/6}$. For larger times, y_s is quickly attracted by the lower stable equilibrium branch. Later we will use the fact that the linearization around y_s satisfies

$$a(-t_{\mathrm{c}} + s) \coloneqq \frac{\partial}{\partial y}\bigl[\mu(s) - \sqrt{3}y^2 - y^3\bigr]\Big|_{y=y_s} \asymp -(|s| \vee a_0^{1/4}\sqrt{|s|} \vee \varepsilon^{1/3}a_0^{1/6}) \tag{3.18}$$

for $s \leqslant c_0\varepsilon^{2/3}a_0^{-1/6}$. Furthermore, one can check that $|a'(t)|$ is bounded above by a constant times $(a_0/\varepsilon)^{1/3}$ for $t \leqslant -t_{\mathrm{c}} + \mathcal{O}(\varepsilon^{2/3}a_0^{-1/6})$. It is easy to see that the solution we constructed encloses an area satisfying $\mathcal{A} - \mathcal{A}_0 \asymp \varepsilon^{2/3}a_0^{1/3}$, where the main contribution comes from the delayed jump from a neighbourhood of $x_+^\star(t)$ to a neighbourhood of $x_-^\star(t)$. Since $x_+^\star(t)$ is the only equilibrium branch near $t_0 = -1/2$, the Poincaré map contracts any interval of order 1 containing $x_+^\star(t)$ to an exponentially small neighbourhood of $x_{1/2}$, which implies the existence of a unique periodic orbit.

4 The random motion near stable equilibrium branches

We consider now the SDE

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\bigl[x_t - x_t^3 - A\cos(2\pi t)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \tag{4.1}$$

with a given (deterministic) initial condition $x_{t_0} = x_0$. Let x_t^{det} denote the solution of the deterministic equation (3.1) with the same initial condition. We will start by investigating the difference $x_t - x_t^{\mathrm{det}}$, and then derive some properties of the area delimited by this difference.

4.1 Noise-induced deviations from the deterministic solution

The difference $y_t = x_t - x_t^{\det}$ obeys the SDE

$$\mathrm{d}y_t = \frac{1}{\varepsilon}\big[a(t)y_t + b(y_t,t)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t, \qquad y_{t_0} = 0, \tag{4.2}$$

where

$$\begin{aligned} a(t) &= 1 - 3(x_t^{\det})^2 \\ b(y,t) &= -y^2\big[3x_t^{\det} + y\big]. \end{aligned} \tag{4.3}$$

In this section, we are interested in situations where $x_s^{\det}$ is attracting up to time t, that is, we assume that $a(s) < 0$ for $t_0 \leqslant s \leqslant t$. Results from the previous section (see (3.10) and (3.18)) show that this is true when the following condition is satisfied.

Assumption 4.1 (Stable case)**.** Assume

- either $a_0 \leqslant \gamma_0\varepsilon$ and t arbitrary
- or $a_0 > \gamma_1\varepsilon$, $t_0 \geqslant -1/2$ and $t \leqslant t^\star := -t_{\mathrm{c}} + c_0\varepsilon^{2/3}a_0^{-1/6}$.

For the sake of brevity, we will refer to these assumptions as *stable case.*

If we were to omit the nonlinear term $b(y,t)$ in (4.2), the solution y_t of the equation would be normally distributed with mean zero and variance

$$v(t) = \frac{\sigma^2}{\varepsilon}\int_{t_0}^t \mathrm{e}^{2\alpha(t,s)/\varepsilon}\,\mathrm{d}s, \qquad \text{where } \alpha(t,s) = \int_s^t a(u)\,\mathrm{d}u. \tag{4.4}$$

It is straightforward to show that the function

$$\zeta(t) := \frac{1}{2|a(t_0)|}\,\mathrm{e}^{2\alpha(t,t_0)/\varepsilon} + \frac{1}{\varepsilon}\int_{t_0}^t \mathrm{e}^{2\alpha(t,s)/\varepsilon}\,\mathrm{d}s \tag{4.5}$$

satisfies, in both cases summarized in Assumption 4.1,

$$\zeta(t) \asymp \frac{1}{2|a(t)|}. \tag{4.6}$$

Thus in the linear case, the standard deviation of y_t is smaller than $\sigma\sqrt{\zeta(t)}$. The following result applies to the whole path $\{y_s\}_{t_0\leqslant s\leqslant t}$ of the nonlinear equation (4.2).

Proposition 4.2. *Let*

$$\hat\zeta(t) = \sup_{t_0\leqslant s\leqslant t}\zeta(s). \tag{4.7}$$

In the stable case, there exists a constant h_0 ($h_0 \asymp 1$) such that for all $h \leqslant h_0\hat\zeta(t)^{-3/2}$,

$$\mathbb{P}^{t_0,0}\bigg\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}} > h\bigg\} \leqslant \bigg(\frac{|\alpha(t,t_0)|}{\varepsilon^2}+2\bigg)\exp\bigg\{-\frac{1}{2}\frac{h^2}{\sigma^2}\Big(1-\mathcal{O}(\varepsilon)-\mathcal{O}\big(h\hat\zeta(t)^{3/2}\big)\Big)\bigg\}. \tag{4.8}$$

Proof: The case $a_0 \leqslant 0$ is the one considered in [BG2, Theorem 2.6], and the other cases can be proved in exactly the same way. □

If we do not care for the precise value of the exponent in (4.8), an obvious modification in the proof yields the bound

$$\mathbb{P}^{t_0,0}\biggl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}}>h\biggr\}\leqslant C\biggl(\frac{t-t_0}{\varepsilon}+1\biggr)\mathrm{e}^{-\kappa h^2/\sigma^2}, \tag{4.9}$$

for all $h\leqslant h_0\hat{\zeta}(t)^{-3/2}$, where C and κ are positive constants. This estimate shows that in the time interval $[t_0,t]$, the typical spreading of paths is of order $\sigma\sqrt{\zeta(s)}$ for $\sigma\ll\hat{\zeta}(t)^{-3/2}$. It allows to bound the probability of deviations up to order $\hat{\zeta}(t)^{-1}$. On the other hand, the special (cubic) form of the drift term in Equation (4.2) allows for a bound on deviations of order larger than 1.

Proposition 4.3. *There exist constants $L_0,C,\kappa>0$ such that for all $L\geqslant L_0$ and all $y_0\leqslant L_0/2$,*

$$\mathbb{P}^{t_0,y_0}\biggl\{\sup_{t_0\leqslant s\leqslant t}y_s>L\biggr\}\leqslant C\biggl(\frac{t-t_0}{\varepsilon}+1\biggr)\mathrm{e}^{-\kappa L^4/\sigma^2}\,. \tag{4.10}$$

Proof: First note that we are working in slow time. The estimate is classical for $t-t_0\leqslant\varepsilon$, and starting from there, (4.10) can be obtained by considering a partition of the interval $[t_0,t]$ with spacing proportional to $\varepsilon/(t-t_0)$. □

Remark 4.4. Note that the preceding proposition holds for all a_0, but L_0 may depend on the amplitude A.

The following proposition gives bounds on the moments of y_t. These bounds hold whenever the estimates (4.9) and (4.10) are satisfied, and we do not need to assume that $a(s)<0$ holds for all s.

Proposition 4.5. *Fix $t>t_0$ such that $t-t_0$ is at most of order 1 and assume that there exists an $h_0(\varepsilon,t)>0$ such that (4.9) holds for all $h\leqslant h_0(\varepsilon,t)$. Then there exist constants $K,M>0$ such that*

$$\mathbb{E}^{t_0,0}\bigl\{|y_t|^{2k}\bigr\}\leqslant k!\,M^k\sigma^{2k}\zeta(t)^k\biggl(1+\frac{|\log\varepsilon|^k}{k!}\biggr),\qquad k\in\mathbb{N}\,, \tag{4.11}$$

whenever σ satisfies $K\sigma\sqrt{|\log(\varepsilon\sigma^2)|}\leqslant h_0(\varepsilon,t)$.

Proof: We will only prove the case $k=1$, as the general case follows along the same lines. Let $\gamma=K\sigma\sqrt{|\log\varepsilon|}$ for some constant $K>0$ to be chosen later and set $h=h_0(\varepsilon,t)$. Note that, under our condition on σ, we may assume $\gamma<h$. Let $\zeta_0=\inf_{s\in[t_0,t]}\zeta(s)$. We write the expectation of y_t^2 as E_1+E, where

$$E_1=\mathbb{E}^{t_0,0}\Bigl\{y_t^2\,1_{\bigl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}}\leqslant\gamma\bigr\}}\Bigr\}\qquad\text{and}\qquad E=\mathbb{E}^{t_0,0}\Bigl\{y_t^2\,1_{\bigl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}}>\gamma\bigr\}}\Bigr\}. \tag{4.12}$$

The first term E_1 can be estimated trivially, namely by $E_1\leqslant\gamma^2\zeta(t)$. To estimate the second term E, we employ integration by parts, thereby obtaining

$$E\leqslant\zeta(t)\int_0^\infty 2z\,\mathbb{P}^{t_0,0}\biggl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}}\geqslant\gamma\vee z\biggr\}\,\mathrm{d}z. \tag{4.13}$$

We now split the integral at γ, h and $L_0/\sqrt{\zeta_0}$ and estimate the resulting terms separately. By (4.9),

$$\begin{aligned} E_2 &= \zeta(t) \int_0^{L_0/\sqrt{\zeta_0}} 2z \, \mathbb{P}^{t_0,0}\Big\{ \sup_{t_0 \leqslant s \leqslant t} \frac{|y_s|}{\sqrt{\zeta(s)}} \geqslant \gamma \vee z \Big\} \, \mathrm{d}z \\ &\leqslant \zeta(t) C \Big(\frac{t-t_0}{\varepsilon} + 1 \Big) \Big[\Big(\gamma^2 + \frac{\sigma^2}{\kappa} \Big) \, \mathrm{e}^{-\kappa\gamma^2/\sigma^2} + \frac{L_0^2}{\zeta_0} \, \mathrm{e}^{-\kappa h^2/\sigma^2} \Big]. \end{aligned} \tag{4.14}$$

Estimating the remaining part of the integral with the help of (4.10),

$$E_3 = \zeta(t) \int_{L_0/\sqrt{\zeta_0}}^{\infty} 2z \, \mathbb{P}^{t_0,0}\Big\{ \sup_{t_0 \leqslant s \leqslant t} \frac{|y_s|}{\sqrt{\zeta(s)}} \geqslant \gamma \vee z \Big\} \, \mathrm{d}z \leqslant \zeta(t) C \Big(\frac{t-t_0}{\varepsilon} + 1 \Big) \frac{\sigma^2}{2\kappa L_0^2 \zeta_0} \, \mathrm{e}^{-\kappa L_0^4/\sigma^2} \tag{4.15}$$

follows. Since the expectation of y_t^2 is bounded above by $E_1 + E_2 + E_3$, (4.11) follows from the fact that we can choose K large enough to bound all three terms by some constant times $\sigma^2 \zeta(t) |\log \varepsilon|$. □

In the stable case, which is our major concern in this section, the previous bound can be improved as follows.

Corollary 4.6. *Fix t such that $t - t_0$ is at most of order 1. In the stable case, there exist constants $c_1 > 0$ and $M > 0$ such that, if $\sigma |\log \varepsilon| \hat{\zeta}(t)^{3/2} \leqslant c_1$, then*

$$\mathbb{E}^{t_0,0}\big\{ |y_t|^{2k} \big\} \leqslant k! \, M^k \sigma^{2k} \zeta(t)^k, \qquad k \in \mathbb{N}. \tag{4.16}$$

Proof: Let us again focus on $k = 1$. Estimate (4.16) is obtained in the same way as (4.11), the only difference lying in a more elaborate bound on E_1. We use the integral representation

$$y_t = \frac{1}{\varepsilon} \int_{t_0}^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, b(y_s, s) \, \mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}} \int_{t_0}^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s \tag{4.17}$$

of the SDE (4.2) (for $y_{t_0} = 0$), thereby obtaining

$$E_1 \leqslant 2\mathbb{E}^{t_0,0}\Big\{ \Big(\frac{1}{\varepsilon} \int_{t_0}^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, b(y_s, s) \, \mathrm{d}s \Big)^2 1_{\{\sup_{t_0 \leqslant s \leqslant t} \frac{|y_s|}{\sqrt{\zeta(s)}} \leqslant \gamma\}} \Big\} + 2\frac{\sigma^2}{\varepsilon} \int_{t_0}^t \mathrm{e}^{2\alpha(t,s)/\varepsilon} \, \mathrm{d}s. \tag{4.18}$$

The second term on the right-hand side is bounded above by $2\sigma^2 \zeta(t)$. The first one can be estimated by bounding $b(y_s, s)$ uniformly in s, with the help of the estimate $|b(y,s)| \leqslant M_0(y^2 + |y|^3)$, valid for all s, c.f. (4.3). The remaining integral behaves like $\varepsilon \zeta(t)$. Thus we obtain

$$E_1 \leqslant \mathit{const} \, \zeta(t)^2 \big(\gamma^2 \hat{\zeta}(t) + \gamma^3 \hat{\zeta}(t)^{3/2} \big)^2 + 2\sigma^2 \zeta(t), \tag{4.19}$$

while E can be estimated as before. Again choosing $\gamma = K\sigma\sqrt{|\log \varepsilon|}$ for K large, yields Estimate (4.16). □

4.2 Noise-induced deviations from the deterministic area

Let us now examine the behaviour of the surface delimited by the process y_t. We want to control the process

$$Y_t = -\int_{t_0}^t y_s \lambda'(s) \, \mathrm{d}s, \tag{4.20}$$

which measures the deviation of the area $\mathcal{A}(\varepsilon,\sigma)$ enclosed by x_t from the one enclosed by x_t^{det}. Using the representation (4.17) of y_t, we obtain, by a version of Fubini's theorem, that

$$Y_t = \frac{1}{\varepsilon}\int_{t_0}^t g(t,s) b(y_s,s)\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t g(t,s)\,\mathrm{d}W_s, \tag{4.21}$$

where

$$g(t,s) = -\int_s^t \mathrm{e}^{\alpha(u,s)/\varepsilon}\,\lambda'(u)\,\mathrm{d}u. \tag{4.22}$$

In particular, the term

$$Y_t^0 = \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t g(t,s)\,\mathrm{d}W_s \tag{4.23}$$

is a Gaussian random variable with mean zero and variance

$$\mathrm{Var}(Y_t^0) = \frac{\sigma^2}{\varepsilon}\int_{t_0}^t g(t,s)^2\,\mathrm{d}s. \tag{4.24}$$

In the sequel, we will use the following abbreviations:

$$\Gamma_i(t,t_0) = \frac{1}{\varepsilon}\int_{t_0}^t |g(t,s)|\zeta(s)^i\,\mathrm{d}s, \qquad i\in\{1,\tfrac{3}{2}\}, \tag{4.25}$$

$$\Gamma(t,t_0) = \frac{1}{\varepsilon^2}\int_{t_0}^t |g(t,s)|^2\,\mathrm{d}s, \tag{4.26}$$

$$\Lambda(t,t_0) = \int_{t_0}^t |\lambda'(s)|\,\mathrm{d}s. \tag{4.27}$$

In addition, we denote by $(2k-1)!!$ the product $\prod_{i=1}^k(2i-1)$.

Proposition 4.7. *Under the assumptions of Proposition 4.5, there is a constant $M_1>0$ such that for all $k\geqslant 1$,*

$$\mathbb{E}^{t_0,0}\bigl\{(Y_t^0)^{2k}\bigr\} = (2k-1)!!\,\bigl(\sigma^2\varepsilon\Gamma(t,t_0)\bigr)^k \tag{4.28}$$

$$\mathbb{E}^{t_0,0}\bigl\{|Y_t^0|^{2k-1}\bigr\} = k!\,2^{k-1}\sqrt{\frac{2}{\pi}}\bigl(\sigma^2\varepsilon\Gamma(t,t_0)\bigr)^{(2k-1)/2} \tag{4.29}$$

$$\mathbb{E}^{t_0,0}\bigl\{\bigl|Y_t-Y_t^0\bigr|^k\bigr\} \leqslant k!\,M_1^k\bigl(\sigma^2\Gamma_1(t,t_0)\bigr)^k\biggl[1+\frac{|\log\varepsilon|^k}{k!}\biggr] \\ \biggl(1+k!\,\sigma^{2k}\biggl(\frac{\Gamma_{3/2}(t,t_0)}{\Gamma_1(t,t_0)}\biggr)^{2k}\biggl[1+\frac{|\log\varepsilon|^k}{k!}\biggr]\biggr)^{1/2}. \tag{4.30}$$

Proof: Since (4.28) and (4.29) are an immediate consequence of the fact that Y_t^0 is Gaussian with variance (4.24), we only need to prove (4.30). We restrict our attention to the case $k=2$ as the case k even follows by an obvious adaptation and the case k odd is obtained from the case k even by an application of Schwarz' inequality. First note that

$$\mathbb{E}^{t_0,0}\bigl\{(Y_t-Y_t^0)^2\bigr\} \leqslant \frac{M_0^2}{\varepsilon^2}\int_{t_0}^t\int_{t_0}^t |g(t,u)||g(t,v)|\mathbb{E}^{t_0,0}\bigl\{(y_u^2+|y_u|^3)(y_v^2+|y_v|^3)\bigr\}\,\mathrm{d}v\,\mathrm{d}u. \tag{4.31}$$

Estimating the expectation of the product by Hölder's inequality and Proposition 4.5, (4.30) follows. □

Remark 4.8. In the stable case, under the assumptions of Corollary 4.6, the bound (4.30) simplifies to

$$\mathbb{E}^{t_0,0}\bigl\{\bigl|Y_t - Y_t^0\bigr|^k\bigr\} \leqslant k!\, M_1^k \bigl(\sigma^2\Gamma_1(t,t_0)\bigr)^k \bigl(1 + k!\,\sigma^{2k}\hat\zeta(t)^k\bigr)^{1/2}. \tag{4.32}$$

The following proposition gives bounds on the probability that the deviation of the area from the corresponding area in the deterministic case is large.

Proposition 4.9.

- *Assume that there exists an $h_0(\varepsilon,t) > 0$ such that (4.9) holds for all $h \leqslant h_0(\varepsilon,t)$. Then there exist constants $h_1, \kappa, C > 0$ such that for any $p \in (0,1)$,*

$$\mathbb{P}^{t_0,0}\bigl\{|Y_t| > H\bigr\} \leqslant \exp\Bigl\{-\frac{(1-p)^2}{2\Gamma(t,t_0)}\frac{H^2}{\sigma^2\varepsilon}\Bigr\} + C\Bigl(\frac{t-t_0}{\varepsilon}+1\Bigr)\exp\Bigl\{-\frac{\kappa}{\sigma^2}\frac{pH}{\Gamma_1(t,t_0)}\Bigr\}, \tag{4.33}$$

  whenever $pH \leqslant h_1\Gamma_1(t,t_0)(h_0(\varepsilon,t)^2 \wedge \hat\zeta(t)^{-1})$.
- *In addition,*

$$\mathbb{P}^{t_0,0}\bigl\{|Y_t| > H\bigr\} \leqslant C\Bigl(\frac{t-t_0}{\varepsilon}+1\Bigr)\exp\Bigl\{-\frac{\kappa}{\sigma^2}\Bigl(\frac{H}{\Lambda(t,t_0)}\Bigr)^4\Bigr\}, \tag{4.34}$$

  whenever $H \geqslant L_0\Lambda(t,t_0)$.

Proof: Consider first the case pH small. By (4.21), we have for any $p \in (0,1)$

$$\mathbb{P}^{t_0,0}\bigl\{|Y_t| > H\bigr\} \leqslant \mathbb{P}^{t_0,0}\bigl\{|Y_t^0| > (1-p)H\bigr\} + \mathbb{P}^{t_0,0}\Bigl\{\frac{1}{\varepsilon}\int_{t_0}^t |g(t,s)||b(y_s,s)|\,\mathrm{d}s > pH\Bigr\}. \tag{4.35}$$

The first term on the right-hand side immediately yields the first term in (4.33) due to the Gaussian nature of Y_t^0. We denote pH/M_0 by Q. For $q \in (0,1)$, the second term can be bounded by

$$\mathbb{P}^{t_0,0}\Bigl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}} > \Bigl(\frac{qQ}{\Gamma_1(t,t_0)}\Bigr)^{1/2}\Bigr\} + \mathbb{P}^{t_0,0}\Bigl\{\sup_{t_0\leqslant s\leqslant t}\frac{|y_s|}{\sqrt{\zeta(s)}} > \Bigl(\frac{(1-q)Q}{\Gamma_{3/2}(t,t_0)}\Bigr)^{1/3}\Bigr\}. \tag{4.36}$$

We choose q by

$$\frac{q}{1-q} = \frac{1}{h_0(\varepsilon,t)\wedge\hat\zeta(t)^{-1/2}}\frac{\Gamma_1(t,t_0)}{\Gamma_{3/2}(t,t_0)} \tag{4.37}$$

and estimate both summands in (4.36) by (4.9). Note that the first summand dominates the second one by our choice of q, since we assumed $pH \leqslant h_1\Gamma_1(t,t_0)(h_0(\varepsilon,t)^2 \wedge \hat\zeta(t)^{-1})$. Thus we obtain the bound (4.33).

For H large, we employ the trivial bound $|Y_t| \leqslant (\sup_{t_0\leqslant s\leqslant t}|y_s|)\Lambda(t,t_0)$ together with Estimate (4.10), thereby obtaining (4.34). □

Choosing $p = p(H,\varepsilon)$ in (4.33) by

$$\frac{p}{(1-p)^2} = \frac{H}{2\kappa\varepsilon}\frac{\Gamma_1(t,t_0)}{\Gamma(t,t_0)} \tag{4.38}$$

yields the following corollary.

Corollary 4.10. *There exist constants $h_1, \kappa, C > 0$ such that*

$$\mathbb{P}^{t_0,0}\{|Y_t| > H\} \leqslant C\left(\frac{t-t_0}{\varepsilon}+1\right)\exp\left\{-\frac{(1-p)^2}{2\Gamma(t,t_0)}\frac{H^2}{\sigma^2\varepsilon}\right\} \tag{4.39}$$

for all H satisfying $(1-p)^2H^2 \leqslant 2\kappa h_1\varepsilon\Gamma(t,t_0)(h_0(\varepsilon,t)^2 \wedge \hat{\zeta}(t)^{-1})$. Here p is defined by (4.38). Furthermore, whenever $H \leqslant \mathrm{const}\,\varepsilon\Gamma(t,t_0)/\Gamma_1(t,t_0)$, then $1-p$ is bounded away from zero.

In order to complete the proof of Theorem 2.3, we have to control the function $g(t,s)$, defined in (4.22). This task is simplified by using the following lemma.

Lemma 4.11. *Assume that $|a'(u)| \leqslant a_1(\varepsilon)$ and $a(u) \leqslant -c\sqrt{\varepsilon a_1(\varepsilon)}$ for all u in an interval $[s,t]$, where $c > 0$ is independent of a_0 and ε. Then there exists a constant $d > 0$, independent of s,t,c, such that*

$$g(t,s) \asymp -\frac{\varepsilon}{|a(s)|}\left[\lambda'(s) + \mathcal{O}\left(\frac{\varepsilon}{|a(s)|}\right)\right] \tag{4.40}$$

whenever $t - s \geqslant d\varepsilon/|a(s)|$.

Proof of Theorem 2.3. First note that Proposition 4.2 establishes the bound (2.13) on the probability of a sample path crossing the potential barrier. From (3.10), (4.6) and the preceding lemma, one easily obtains

$$\Gamma_1(\tfrac12,-\tfrac12) = \mathcal{O}\big(|\log(|a_0| \vee \varepsilon)|\big) \quad \text{and} \quad \Gamma(\tfrac12,-\tfrac12) \asymp 1 \tag{4.41}$$

in the stable case, while $\Lambda(\frac12,-\frac12) \asymp 1$ is trivial. Now the preceding results imply the stated estimates. □

5 The large noise regime

In this section, we consider those parameter regimes in which the noise intensity σ is large enough to allow for transitions from one potential well to the other one, with a probability close to 1. Depending on the amplitude, there are three cases to consider:

- $a_0 \leqslant -\varepsilon$ and $\sigma \geqslant |a_0|^{3/4}$;
- $|a_0| \leqslant \varepsilon$ and $\sigma \geqslant \varepsilon^{3/4}$;
- $a_0 \geqslant \varepsilon$ and $\sigma \geqslant (\varepsilon\sqrt{a_0})^{1/2}$.

Actually, we will need to assume that $\sigma \geqslant K|a_0|^{3/4}$, $\sigma \geqslant K\varepsilon^{3/4}$ or $\sigma \geqslant K(\varepsilon\sqrt{a_0})^{1/2}$, respectively, for some large constant K, but in order not to overburden notations, we will assume that $K = 1$ is a possible choice.

Recall that in the deterministic case, transitions are impossible if $a_0 \leqslant \gamma_0\varepsilon$, and occur only after time $-t_c + c_0\varepsilon^{2/3}a_0^{-1/6}$ if $a_0 \geqslant \gamma_1\varepsilon$. It turns out that under the above conditions on σ, transitions are likely to occur some time *before* the potential barrier reaches its minimal height or even vanishes. For brevity, we shall only discuss the case $|a_0| \leqslant \varepsilon$ in detail, but the other cases can be investigated similarly (since transitions occur early, they are not influenced by the details of the bifurcation or avoided bifurcation).

5.1 The transition time

We assume $|a_0| \leqslant \varepsilon$ unless stated otherwise. By symmetry, we may restrict our attention to a half-period, say $t \in [-1/4, 1/4]$. Let x_t be the solution of the SDE (2.11) starting at time $t_0 = -1/4$ in the upper well, i. e., near $x^\star_+(t_0)$. We define the transition time as the stopping time

$$\tau^0 = \inf\bigl\{s > t_0 \colon x_s \leqslant 0\bigr\} \in (t_0, \infty], \tag{5.1}$$

when x_s crosses the t-axis for the first time. The choice of $x_s = 0$ is purely for convenience, and the qualitative behaviour of τ^0 remains the same if 0 is replaced by any level between $-x_c - \delta$ and $x_c + \delta$ as long as $\delta > 0$ is chosen in such a way that $f(x,t) \leqslant 0$ holds for all $|x| \leqslant x_c + \delta$ and all t in question. The following result characterizes the distribution of τ^0.

Proposition 5.1. *There exist constants C, c_1, c_2, $\kappa > 0$ such that*

- *for $t_0 < t \leqslant -c_1\sigma^{2/3}$,*

$$\mathbb{P}^{t_0,x_0}\bigl\{\tau^0 < t\bigr\} \leqslant C\biggl(\frac{t-t_0}{\varepsilon} + 1\biggr)\exp\Bigl\{-\frac{\kappa}{\sigma^2\hat\zeta(t)^3}\Bigr\}; \tag{5.2}$$

- *for $-c_1\sigma^{2/3} + c_2\varepsilon \leqslant t \leqslant c_1\sigma^{2/3}$,*

$$\mathbb{P}^{t_0,x_0}\bigl\{\tau^0 > t\bigr\} \leqslant \frac{3}{2}\exp\Bigl\{-\frac{\kappa}{|\log\sigma|}\frac{1}{\varepsilon}\int_{-c_1\sigma^{2/3}}^{t}|a(s)|\,\mathrm{d}s\Bigr\} + \mathrm{e}^{-\kappa/\sigma^2}. \tag{5.3}$$

Recall that $a(s)$ is the linearization of the drift term along $x^{\mathrm{det},+}_s$ as defined in (3.10)*, and $\hat\zeta(t)$ is defined in* (4.7)*.*

Proof: The first part is a direct consequence of (4.9) with $h = h_1\hat\zeta(t)^{-3/2}$, where h_1 is chosen sufficiently small that the relation $|x_s - x^{\mathrm{det}}_s| \leqslant h_1/\zeta(s)$ for all $s \in [t_0, t]$ implies that $x_s > 0$ for these s.

The second part is an application of Theorem 2.7 in [BG2] (with $h = \mathit{const}\ \sigma|\log\varepsilon|^{1/2}$). Note that the theorem naturally extends to the case $0 < a_0 \leqslant \varepsilon$. In fact, the integrand in (5.3) should be the curvature of the potential at the deterministic solution tracking the saddle $x^\star_0(t)$, but the curvature behaves like $|a(t)|$, compare [BG2, Proposition 4.3]. □

The condition $\sigma \geqslant \varepsilon^{3/4}$ implies that $\hat\zeta(t)^{-1} \asymp |a(t)| \asymp |t|$ for $t \leqslant -c_1\sigma^{2/3}$, and thus the exponent in (5.2) scales like $|t|^3/\sigma^2$. The integral in (5.3) behaves like $\sigma^{2/3}(t + c_1\sigma^{2/3})$.

Proposition 5.1 shows that the transition is likely to occur close to time $t_1 = -c_1\sigma^{2/3}$, which satisfies $\lambda(t_1) + \lambda_c \asymp \sigma^{4/3}$. Therefore, we should compare the area $\mathcal{A}(\varepsilon,\sigma)$ to a reference area $\hat{\mathcal{A}}$ given by

$$\frac{1}{2}\hat{\mathcal{A}} = \int_{t_0}^{t_1} x^{\mathrm{det},+}_s(-\lambda'(s))\,\mathrm{d}s + \int_{t_1}^{t_2} x^{\mathrm{det},-}_s(-\lambda'(s))\,\mathrm{d}s, \tag{5.4}$$

where $t_0 = -1/4$, $t_1 = -c_1\sigma^{2/3}$, $t_2 = 1/4$, and we denote by $x^{\mathrm{det},+}_s$ the deterministic solution starting in x_0, which tracks $x^\star_+(s)$, and by $x^{\mathrm{det},-}_s$ a deterministic solution tracking $x^\star_-(s)$. It is easy to check that

$$\hat{\mathcal{A}} - \mathcal{A}_0 \asymp -\sigma^{4/3}. \tag{5.5}$$

(This relation does not depend on the initial conditions of $x^{\mathrm{det},\pm}_s$, as long as they are sufficiently close to $x^\star_+(t_0)$ or $x^\star_-(t_1)$, respectively).

Remark 5.2. Proposition 5.1 also holds for $a_0 < -\varepsilon$, with the same exponents.

In the case $a_0 > \varepsilon$, $\hat\zeta(t)^{-1}$ behaves like $|a(t)|$, given by (3.18). The bound (5.2) holds for $t_0 < t \leqslant t_1 = -t_{\rm c} - c_1(\sigma^{2/3} \wedge \sigma^{4/3} a_0^{-1/2})$, with the exponent replaced by $-\kappa a_0^{3/4}|t+t_{\rm c}|^{3/2}/\sigma^2$ if $\sigma \leqslant a_0^{3/4}$ and by $-\kappa|t+t_{\rm c}|^3/\sigma^2$ if $\sigma \geqslant a_0^{3/4}$. Note that in both cases, $\lambda(t_1) + \lambda_{\rm c} \asymp \sigma^{4/3}$.

The bound (5.3) holds for $t_1 + c_2\varepsilon \leqslant t \leqslant -t_{\rm c} + c_0\varepsilon^{2/3}a_0^{-1/6}$, with an exponent of the same order as in the other cases, namely $\sigma^{2/3}(t-t_1)/(\varepsilon|\log\sigma|)$. The behaviour for larger t will be discussed in Proposition 6.1 below.

5.2 Deviations from the reference area

Our aim is to characterize the deviations of the random variable $\mathcal{A}(\varepsilon,\sigma)$ from its deterministic reference value $\hat{\mathcal{A}}$ over one half-period. We focus again on the case $|a_0| \leqslant \varepsilon$. With a slight abuse of notation, we can write this deviation as $\frac12(\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}) = Y^+_{t_1} + Y^-_{t_2}$, where

$$Y^+_{t_1} := \int_{t_0}^{t_1} (x_s - x_s^{\det,+})(-\lambda'(s))\,\mathrm{d}s, \qquad Y^-_{t_2} := \int_{t_1}^{t_2} (x_s - x_s^{\det,-})(-\lambda'(s))\,\mathrm{d}s. \tag{5.6}$$

We will estimate separately the probability that each of these terms is larger than H or smaller than $-H$. To do so, we need a preparatory result allowing to extend the estimate (4.9) to larger values of h.

Proposition 5.3. *Define the stopping time*

$$\tau = \inf\bigl\{t\in[t_0,t_1] \colon x_t \leqslant x_t^{\det,+} - h_0\hat\zeta(t)^{-1}\bigr\} \in [t_0,t_1]\cup\{\infty\}, \tag{5.7}$$

where the constant h_0 is taken from Proposition 4.2. Then

$$\mathbb{P}^{t_0,x_0}\biggl\{\sup_{t_0\leqslant s\leqslant\tau} \frac{|x_s - x_s^{\det,+}|}{\sqrt{\zeta(s)}} > h\biggr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa h^2/\sigma^2} \tag{5.8}$$

for some C, $\kappa > 0$, all $t\in[t_0,t_1]$ and all $h > 0$.

Proof: The fact that the drift term F has a negative second derivative with respect to x for all $x > 0$ implies that x_s is unlikely to exit the strip of width $h\sqrt{\zeta(s)}$ through its upper boundary, as was proved for negative a_0 in [BG2, Proposition 4.5]. We also know by (4.9) that x_s is unlikely to exit the strip through its lower boundary if $\sigma \ll h \leqslant h_0\hat\zeta(t)^{-3/2}$. The stopping time τ has been defined in such a way that x_s cannot leave a strip of larger width before time τ. □

Note that by decreasing h_0 if necessary, we can arrange for $\tau < \tau_0$. We are now able to estimate deviations of $Y^+_{t_1}$.

Proposition 5.4. *There exist constants C, κ, $h_1 > 0$ such that*

$$\mathbb{P}^{t_0,x_0}\bigl\{Y^+_{t_1} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} \tag{5.9}$$

$$\mathbb{P}^{t_0,x_0}\bigl\{Y^+_{t_1} > +H\bigr\} \leqslant \mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} + \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2|\log\sigma|)} \tag{5.10}$$

for $0 \leqslant H \leqslant h_1\sigma^{2/3}|\log\sigma|$.

Proof: We decompose

$$\mathbb{P}^{t_0,x_0}\bigl\{Y^+_{t_1} < -H\bigr\} \leqslant \mathbb{P}^{t_0,x_0}\bigl\{Y^+_{\tau\wedge t_1} < -\tfrac12 H\bigr\} \\ + \mathbb{E}^{t_0,x_0}\biggl\{1_{\{\tau<t_1\}}\mathbb{P}^{\tau,x_\tau}\biggl\{\int_\tau^{t_1}(x_s - x_s^{\det,+})(-\lambda'(s))\,\mathrm{d}s < -\tfrac12 H\biggr\}\biggr\}, \tag{5.11}$$

where τ is defined in (5.7). The first term on the right-hand side can be estimated as in Proposition 4.9, as there is no need to distinguish positive and negative deviations for this term. However, Proposition 5.3 allows us to obtain bounds valid on a larger domain of H. Note that Proposition 4.9 remains valid when Y_t is replaced by $Y_{\tau\wedge t}$. This is a consequence of the $\Gamma_i(t,t_0)$ being monotone functions of $t\in[t_0,t_1]$ and a slightly more elaborate estimate showing that $\sup_{t_0\leqslant s\leqslant t}|Y^0_s|$ obeys the same bound as was used for $|Y^0_t|$ in (4.35). Thus we obtain the estimate

$$\mathbb{P}^{t_0,x_0}\bigl\{|Y^+_{\tau\wedge t_1}| > H\bigr\} \leqslant \exp\biggl\{-\frac{(1-p)^2}{2\Gamma(t_1,t_0)}\frac{H^2}{\sigma^2\varepsilon}\biggr\} + \frac{C}{\varepsilon}\exp\biggl\{-\frac{\kappa}{\Gamma_1(t_1,t_0)}\frac{pH}{\sigma^2}\biggr\}, \tag{5.12}$$

valid for $pH \leqslant \mathit{const}\,\Gamma_1(t_1,t_0)/\hat\zeta(t_1)$. An application of Lemma 4.11 shows that $\Gamma(t_1,t_0)\asymp 1$, $\Gamma_1(t_1,t_0)\asymp|\log\sigma|$, and we already know that $\hat\zeta(t_1)\asymp\sigma^{-2/3}$. Choosing $p\asymp 1$ provides an estimate of the form (5.10). The second term on the right-hand side of (5.11) can be estimated, using the monotonicity of λ in $[t_0,t_1]$, by the relation

$$\mathbb{P}^{\tau,x_\tau}\biggl\{\int_\tau^{t_1}(x_s - x_s^{\det,+})(-\lambda'(s))\,\mathrm{d}s < -H\biggr\} \\ \leqslant \mathbb{P}^{\tau,x_\tau}\Bigl\{\sup_{\tau\leqslant s\leqslant t_1}|x_s - x_s^{\det,+}| > L\Bigr\} + \mathbb{P}^{\tau,x_\tau}\bigl\{\lambda(\tau)-\lambda(t_1) > H/L\bigr\}. \tag{5.13}$$

The first term on the right-hand side can be estimated by Proposition 4.3, provided L is larger than some constant of order 1. It decreases like $\mathrm{e}^{-\mathit{const}/\sigma^2}/\varepsilon$. Using the fact that $\lambda(\tau)-\lambda(t_1)\asymp\tau^2-t_1^2$ for τ not too close to t_0 (note that the contribution of τ close to t_0 is even smaller) and (5.2) of Proposition 5.1, we obtain that

$$\mathbb{E}^{t_0,x_0}\Bigl\{1_{\{\tau<t_1\}}\mathbb{P}^{\tau,x_\tau}\bigl\{\lambda(\tau)-\lambda(t_1) > H/L\bigr\}\Bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\mathit{const}\,H^{3/2}/\sigma^2}\,. \tag{5.14}$$

This last term is easily seen to dominate all others, so that (5.9) is proved.

To estimate deviations in the positive direction, we split terms as in (5.11). The first term can also be bounded by (5.12). Using the fact (compare [BG2, Proposition 4.5]) that

$$\mathbb{P}^{\tau,x_\tau}\biggl\{\sup_{\tau\leqslant s\leqslant t_1}\frac{x_s - x_s^{\det,+}}{\sqrt{\zeta(s)}} > \sqrt{H/2}\biggr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/\sigma^2}, \tag{5.15}$$

it only remains to estimate

$$\mathbb{P}^{\tau,x_\tau}\biggl\{\int_\tau^{t_1}(x_s - x_s^{\det,+})(-\lambda'(s))\,\mathrm{d}s > H/2,\ \sup_{\tau\leqslant s\leqslant t_1}\frac{x_s - x_s^{\det,+}}{\sqrt{\zeta(s)}} \leqslant \sqrt{H/2}\biggr\} \\ \leqslant \mathbb{P}^{\tau,x_\tau}\biggl\{\int_\tau^{t_1}\sqrt{\zeta(s)}(-\lambda'(s))\,\mathrm{d}s > \sqrt{H/2}\biggr\} \\ \leqslant \mathbb{P}^{\tau,x_\tau}\bigl\{|\tau|^{3/2} > \sigma + \mathit{const}\,\sqrt{H}\bigr\}. \tag{5.16}$$

The expectation of this term also decreases like $\mathrm{e}^{-\kappa H/\sigma^2}/\varepsilon$ by Proposition 5.1. Taking (5.12) and (5.15) into account, we have proved (5.10). □

The term $Y^-_{t_2}$ can be controlled in a similar way:

Proposition 5.5. *There exist constants C, κ, h_2, $h_3 > 0$ such that for all $x_{t_1} \in [-L, L]$ ($L \asymp 1$), and all $H \leqslant h_3$,*

$$\mathbb{P}^{t_1,x_{t_1}}\bigl\{Y^-_{t_2} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/\sigma^2} + \frac{3}{2}\,\mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)} + \mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} \tag{5.17}$$

$$\mathbb{P}^{t_1,x_{t_1}}\bigl\{Y^-_{t_2} > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/\sigma^2} + \frac{3}{2}1_{[0,h_2\sigma^{4/3})}(H)\,\mathrm{e}^{-\kappa H/(\varepsilon|\log\sigma|)}\,. \tag{5.18}$$

Proof: The proof being similar to the one of the previous proposition, we only outline the main steps. Introduce a stopping time $\tau_{\mathrm{c}} = \inf\{s \in [t_1, t_2]\colon x_s \leqslant -x_{\mathrm{c}} - \delta\}$ for some small $\delta > 0$, cf. the comment on the definition of τ^0 in the beginning of the subsection. We first need to control the behaviour of

$$Y^-_{\tau_{\mathrm{c}}\wedge t_2} = \int_{t_1}^{\tau_{\mathrm{c}}\wedge 0} (x_s - x^{\det,-}_s)(-\lambda'(s))\,\mathrm{d}s + \int_{\tau_{\mathrm{c}}\wedge 0}^{\tau_{\mathrm{c}}\wedge t_2} (x_s - x^{\det,-}_s)(-\lambda'(s))\,\mathrm{d}s. \tag{5.19}$$

Observe that since $x_s > x^{\det,-}_s$ for $s \leqslant \tau_{\mathrm{c}}$, the first term on the right-hand side is positive, while the second one is negative or zero. First note that if x_s is bounded above by $L \asymp 1$, then $Y^-_{\tau_{\mathrm{c}}\wedge t_2}$ cannot exceed a value of order $\sigma^{4/3}$. Deviations of $Y^-_{\tau_{\mathrm{c}}\wedge t_2}$ in the positive direction can be bounded using a decomposition similar to (5.13) and applying (5.3) for τ_{c} instead of τ. We find

$$\mathbb{P}^{t_1,x_{t_1}}\bigl\{Y^-_{\tau_{\mathrm{c}}\wedge t_2} > H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa/\sigma^2} + \frac{3}{2}1_{[0,h_2\sigma^{4/3})}(H)\,\mathrm{e}^{-\kappa H/(\varepsilon|\log\sigma|)}, \tag{5.20}$$

valid for $H \geqslant \mathcal{O}(\varepsilon\sigma^{2/3})$.

Deviations of $Y^-_{\tau_{\mathrm{c}}\wedge t_2}$ in the negative direction can only be caused by the second term on the right-hand side of (5.19). However, there is no small lower bound for that term. The reason is that transitions to $x^{\det,-}_s$ are only probable in the window $t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$. If this opportunity is missed, which happens with a probability of order $\mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)}$, then x_s keeps tracking $x^{\det,+}_s$ and $Y^-_{\tau_{\mathrm{c}}\wedge t_2}$ may reach negative values of order 1.

To complete the proof, we need to show that on $\{\tau_{\mathrm{c}} < t_2\}$

$$\mathbb{P}^{\tau_{\mathrm{c}},x_{\tau_{\mathrm{c}}}}\biggl\{\biggl|\int_{\tau_{\mathrm{c}}}^{t_2} (x_s - x^{\det,-}_s)(-\lambda'(s))\,\mathrm{d}s\biggr| > H\biggr\} \leqslant \mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} + \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/\sigma^2}\,. \tag{5.21}$$

Let $x^{\det,\tau_{\mathrm{c}}}_s$ be the deterministic solution starting in $x_{\tau_{\mathrm{c}}}$ at time τ_{c}. This solution is attractive, and thus (5.21) holds with $x^{\det,\tau_{\mathrm{c}}}_s$ instead of $x^{\det,-}_s$ as a consequence of Proposition 4.9. But the distance between $x^{\det,\tau_{\mathrm{c}}}_s$ and $x^{\det,-}_s$ decreases exponentially in $(s - \tau_{\mathrm{c}})/\varepsilon$, which implies that the area between them is at most of order $\varepsilon\sigma^{2/3}$. Thus (5.21) holds for $H > \mathcal{O}(\varepsilon\sigma^{2/3})$. But for smaller H, it is trivially satisfied. □

We can summarize the properties obtained so far in the following way.

Proposition 5.6. *There exist constants C, κ, h_1, h_2, $h_3 > 0$ such that*

$$\mathbb{P}^{t_0,x_0}\bigl\{\mathcal{A} - \hat{\mathcal{A}} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} + \frac{3}{2}\,\mathrm{e}^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)} \tag{5.22}$$

$$\mathbb{P}^{t_0,x_0}\bigl\{\mathcal{A} - \hat{\mathcal{A}} > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2|\log\sigma|)} + \frac{3}{2}1_{[0,h_2\sigma^{4/3})}(H)\,\mathrm{e}^{-\kappa H/(\varepsilon|\log\sigma|)} \tag{5.23}$$

for $0 \leqslant H \leqslant h_1\sigma^{2/3}|\log\sigma|$. In addition, for all $H \geqslant h_3$ we have

$$\mathbb{P}^{t_0,x_0}\bigl\{|\mathcal{A}| > H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^4/\sigma^2}\,. \tag{5.24}$$

As an immediate consequence, we obtain the following estimates on the moments of the deviation of the area.

Corollary 5.7. *There exist positive constants C, c_1, c_2 such that*

$$\mathbb{E}^{t_0,x_0}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \geqslant -C\sigma^{4/3}|\log\varepsilon|^{2/3} \tag{5.25}$$

$$\mathbb{E}^{t_0,x_0}\bigl\{\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}}\bigr\} \leqslant C\bigl[(\varepsilon \vee \sigma^2|\log\varepsilon|)|\log\sigma|\bigr] \tag{5.26}$$

$$\mathbb{E}^{t_0,x_0}\bigl\{(\mathcal{A}(\varepsilon,\sigma) - \hat{\mathcal{A}})^2\bigr\} \leqslant C\bigl(\sigma^{4/3}|\log\varepsilon|^{2/3}\bigr)^2 \tag{5.27}$$

provided $c_1\varepsilon < \sigma^{4/3}/|\log\sigma|^2$ and $\sigma^{2/3}|\log\sigma| \leqslant c_2/|\log\varepsilon|$.

Proof: By partial integration,

$$\mathbb{E}^{t_0,x_0}\bigl\{\mathcal{A} - \hat{\mathcal{A}}\bigr\} = \int_0^\infty \mathbb{P}^{t_0,x_0}\bigl\{\mathcal{A} - \hat{\mathcal{A}} > H\bigr\}\,\mathrm{d}H - \int_0^\infty \mathbb{P}^{t_0,x_0}\bigl\{\mathcal{A} - \hat{\mathcal{A}} < -H\bigr\}\,\mathrm{d}H. \tag{5.28}$$

The first integral be evaluated by splitting it at $H = C(\varepsilon \vee \sigma^2|\log\varepsilon|)|\log\sigma|$, $h_2\sigma^{4/3}$ and h_3, and the second one at $C\sigma^{4/3}|\log\varepsilon|^{2/3}$ and h_3. Each time, the integral over the first interval dominates. The estimate (5.27) is obtained similarly. □

6 The large amplitude case

We consider finally the large amplitude case $a_0 \geqslant \gamma_1\varepsilon$, but with noise intensity σ satisfying $\sigma^2 < \sqrt{a_0}\varepsilon$. By symmetry, we may again concentrate on a half-period $[-1/2, 0]$. Let x_t be the solution of the SDE (2.11) starting at time $t_0 = -1/2$ in the upper well, i. e., near $x^\star_+(t)$. Recall that the solution $x^{\det}_t$ of the deterministic equation (3.1) with the same initial condition tracks $x^\star_+(t)$ until time $-t_{\mathrm{c}}$, and jumps to the other potential well at $x^\star_-(t)$ after a delay of order $\varepsilon^{2/3}a_0^{-1/6}$.

We introduce a time $t^\star = -t_{\mathrm{c}} + c_0\varepsilon^{2/3}a_0^{-1/6}$ just before the jump. Then we know that for $t_0 \leqslant t = -t_{\mathrm{c}} + s \leqslant t^\star$,

$$x^{\det}_t - x_{\mathrm{c}} \asymp \frac{1}{\zeta(t)} \asymp -a(t) \asymp |s| \vee a_0^{1/4}\sqrt{s} \vee (\varepsilon\sqrt{a_0})^{1/3}, \tag{6.1}$$

compare (3.18) and (4.6). The fact that $x^{\det}_t$ behaves in this way follows from the fact that $x^\star_+(t) - x_{\mathrm{c}} \asymp \sqrt{\mu(s)}$ dominates $x^\star(t)_+ - x^{\det}_t$ for $t \leqslant -t_{\mathrm{c}} - \varepsilon^{2/3}a_0^{-1/6}$, c.f. (3.14). For larger t, we know from (3.16) and (3.17) that $x^{\det}_t - x_{\mathrm{c}} \asymp (\varepsilon\sqrt{a_0})^{1/3}$.

6.1 The transition time

Let us again start by investigating the distribution of the stopping time

$$\tau^0 = \inf\bigl\{s > t_0\colon x_s = 0\bigr\}. \tag{6.2}$$

The following result shows that τ^0 is likely to be close to $t^\star$.

Proposition 6.1. *There exist constants C, c_1, $\kappa > 0$ such that*

- *for $t_0 \leqslant t \leqslant t^\star$,*

$$\mathbb{P}^{t_0,x_0}\big\{\tau^0 < t\big\} \leqslant C\Big(\frac{t-t_0}{\varepsilon}+1\Big)\exp\Big\{-\frac{\kappa}{\sigma^2\hat\zeta(t)^3}\Big\}. \tag{6.3}$$

- *for $t^\star \leqslant t \leqslant t^\star + c_1\sqrt{a_0}$, $0 \leqslant x_{t^\star} \leqslant \rho$ and any $\rho > x_{\rm c}$,*

$$\mathbb{P}^{t^\star,x_{t^\star}}\big\{\tau^0 > t\big\} \leqslant 3\exp\bigg\{-\frac{\varepsilon\sqrt{a_0}}{\sigma^2}\bigg[\frac{\kappa}{|\log(\varepsilon^2 a_0)|\vee\log\rho}\frac{a_0^{1/6}(t-t^\star)}{\varepsilon^{2/3}}-1\bigg]\bigg\}. \tag{6.4}$$

Proof: First note that (6.3) is a direct consequence of Proposition 4.2 as we are in the stable case.

In order to prove (6.4), we consider again the stochastic process $y_t = x_t - x_{\rm c}$, satisfying the SDE

$$\mathrm{d}y_t = \frac{1}{\varepsilon}\big[\mu(t+t_{\rm c}) - \sqrt{3}y_t^2 - y_t^3\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t, \qquad y_{t^\star} = x_{t^\star} - x_{\rm c}, \tag{6.5}$$

where (3.14) implies $\mu(t+t_{\rm c}) \leqslant -\varepsilon^{2/3}a_0^{1/3}$ for $t^\star \leqslant t \leqslant t^\star + c_1\sqrt{a_0}$. Note furthermore that $-\sqrt{3}y_t^2 - y_t^3 \leqslant -y_t^2$ for $t \leqslant \tau^0$. By Gronwall's inequality, it follows that $y_t \leqslant z_t$ for $t \leqslant \tau^0$, where z_t is defined as the solution of the time-homogeneous SDE

$$\mathrm{d}z_t = \frac{1}{\varepsilon}\big[-\varepsilon^{2/3}a_0^{1/3} - z_t^2\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t, \qquad z_{t^\star} = x_{t^\star} - x_{\rm c}. \tag{6.6}$$

For any $\delta_0 > 0$, we can write

$$\mathbb{P}^{t^\star,x_{t^\star}}\big\{\tau^0 > t\big\} \leqslant \mathbb{P}^{t^\star,x_{t^\star}}\Big\{\sup_{t^\star\leqslant s\leqslant t} z_s > \delta_0\Big\} + \mathbb{P}^{t^\star,x_{t^\star}}\big\{-x_{\rm c} \leqslant z_s \leqslant \delta_0\ \forall s\in[t^\star,t]\big\}. \tag{6.7}$$

Since (6.6) is an autonomous SDE, it is easy to see that the first term on the right-hand side can be bounded by

$$\mathbb{P}^{t^\star,x_{t^\star}}\Big\{\sup_{t^\star\leqslant s\leqslant t} z_s > \delta_0\Big\} \leqslant C\Big(\frac{t-t^\star}{\varepsilon}+1\Big)\,\mathrm{e}^{-\kappa\delta_0^3/\sigma^2}, \tag{6.8}$$

which can be made as small as we like by taking δ_0 sufficiently large. In order to estimate the second term, we introduce $\Delta = c\varepsilon^{2/3}a_0^{-1/6}$, where $c > 1$ will be chosen later, and define

$$Q = \sup_{-x_{\rm c}\leqslant z_0\leqslant\delta_0}\mathbb{P}^{0,z_0}\big\{-x_{\rm c} \leqslant z_s \leqslant \delta_0\ \forall s\in[0,\Delta]\big\}. \tag{6.9}$$

Using time homogeneity and the Markov property, we can write

$$\mathbb{P}^{t^\star,x_{t^\star}}\big\{-x_{\rm c} \leqslant z_s \leqslant \delta_0\ \forall s\in[t^\star,t]\big\} \leqslant Q^{(t-t^\star)/\Delta-1}. \tag{6.10}$$

The result is thus proved if we manage to bound Q by a term exponentially small in $\varepsilon\sqrt{a_0}/\sigma^2$.

In order to estimate Q, it is convenient to introduce the process $\tilde z_t = -z_t/(\varepsilon^{1/3}a_0^{1/6})$, which obeys the SDE

$$\mathrm{d}\tilde z_t = \frac{a_0^{1/6}}{\varepsilon^{2/3}}\big[1+\tilde z_t^2\big]\,\mathrm{d}t - \frac{\sigma}{\varepsilon^{5/6}a_0^{1/6}}\,\mathrm{d}W_t, \qquad \tilde z_{t^\star} = -\frac{x_{t^\star}-x_{\rm c}}{\varepsilon^{1/3}a_0^{1/6}}. \tag{6.11}$$

Let $\rho = x_c/(\varepsilon^{1/3}a_0^{1/6})$ and $\delta = \delta_0/(\varepsilon^{1/3}a_0^{1/6})$. Using again Markov property and time-homogeneity shows that $Q \leqslant Q_1 + Q_2 + Q_3$, where

$$\begin{aligned} Q_1 &= \sup_{-\delta\leqslant\tilde z_0\leqslant -1} \mathbb{P}^{0,\tilde z_0}\bigl\{\tilde z_s < -1 \;\forall s\in[0,\Delta/3]\bigr\} \\ Q_2 &= \sup_{-1\leqslant\tilde z_0\leqslant 1} \mathbb{P}^{0,\tilde z_0}\bigl\{\tilde z_s < 1 \;\forall s\in[0,\Delta/3]\bigr\} \\ Q_3 &= \sup_{1\leqslant\tilde z_0\leqslant \rho} \mathbb{P}^{0,\tilde z_0}\bigl\{\tilde z_s < \rho \;\forall s\in[0,\Delta/3]\bigr\}. \end{aligned} \tag{6.12}$$

Since $1+\tilde z^2 \geqslant 1\vee|\tilde z|$, each term can be easily estimated by comparison with an appropriate linear or $\tilde z$-independent equation. Consider for instance Q_1. We know that $\tilde z_t$ lies above the solution $\tilde z_t^0$ of the linear SDE

$$\mathrm{d}\tilde z_t^0 = -\frac{a_0^{1/6}}{\varepsilon^{2/3}}\tilde z_t^0\,\mathrm{d}t - \frac{\sigma}{\varepsilon^{5/6}a_0^{1/6}}\,\mathrm{d}W_t, \qquad\qquad \tilde z_0^0 = -\delta, \tag{6.13}$$

the solution of which at time $\Delta/3$ is a Gaussian random variable with mean $-\delta\,\mathrm{e}^{-c/3}$ and variance $(1-\mathrm{e}^{-2c/3})\sigma^2/(2\varepsilon\sqrt{a_0})$. We can thus estimate

$$Q_1 \leqslant \mathbb{P}^{0,-\delta}\bigl\{\tilde z_{\Delta/3}^0 < -1\bigr\} \leqslant \exp\Bigl\{-\frac{\varepsilon\sqrt{a_0}}{\sigma^2}\frac{(1-\delta\,\mathrm{e}^{-c/3})^2}{1-\mathrm{e}^{-2c/3}}\Bigr\}, \tag{6.14}$$

provided $\delta\,\mathrm{e}^{-c/3} < 1$, i. e., $c > 3\log\delta$. Now, Q_2 and Q_3 allow for similar bounds, and the result thus follows from (6.8) and (6.10), taking δ_0 and c sufficiently large. □

6.2 The case IIa

We now examine the process Y_t defined in (4.20), which describes deviations from the deterministic area. Using Lemma 4.11 and (6.1), it is easy to check that

$$\Gamma_1(t^\star, t_0) \asymp |\log(\varepsilon^{2/3}a_0^{-1/6})| \quad\text{and}\quad \Gamma(t^\star,t_0)\asymp 1, \tag{6.15}$$

where $t_0 = -1/2$. Applying Remark 4.8 and Corollary 4.10, it is straightforward to check that the distribution of Y_t is close to a Gaussian with variance proportional to $\sigma^2\varepsilon$.

The situation changes, however, for $t > t^\star$, because the deterministic solution crosses a zone of instability between x_c and $-x_c$, compare (4.3). This instability causes a spreading of paths which we will now analyse in more detail. Let us introduce times $t_1^\star$ and $t_2^\star$ such that

$$x_{t_1^\star}^{\det} = x_c - c_1(\varepsilon\sqrt{a_0})^{1/3}, \qquad\qquad x_{t_2^\star}^{\det} = -x_c - \delta, \tag{6.16}$$

where $c_1 > 0$ and $\delta < x_c$. Then $t_1^\star - t^\star$ and $t_2^\star - t^\star$ are both of order $\varepsilon^{2/3}a_0^{-1/6}$. We now proceed to determining the behaviour of $\zeta(t)$, defined in (4.5), which measures the spreading of paths around the deterministic solution.

Proposition 6.2. *Let $z_t = x_c - x_t^{\det}$. Then there exist constants C, $K > 0$ such that*

$$\zeta(t) \asymp (\varepsilon\sqrt{a_0})^{-1/3} \qquad \text{for } t^\star \leqslant t \leqslant t_1^\star \tag{6.17}$$

$$\zeta(t) \asymp (\varepsilon\sqrt{a_0})^{-5/3} z_t^4 \qquad \text{for } t_1^\star \leqslant t \leqslant t_2^\star \tag{6.18}$$

$$\zeta(t) \leqslant C\bigl[(\varepsilon\sqrt{a_0})^{-5/3}\,\mathrm{e}^{-K(t-t_2^\star)/\varepsilon} + 1\bigr] \qquad \text{for } t_2^\star \leqslant t \leqslant 0. \tag{6.19}$$

PROOF: (6.17) follows by an elementary calculation. Next, consider the time interval $t_1^\star \leqslant t \leqslant t_2^\star$. The variable z_t satisfies the differential equation

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = \frac{1}{\varepsilon}\bigl[-\mu(t) + \sqrt{3}z_t^2 - z_t^3\bigr], \qquad\qquad \mu(t) = \lambda_{\mathrm{c}} - A\cos(2\pi t). \tag{6.20}$$

Note that z_t is monotonously decreasing and $\mu(t) \leqslant 0$ for the times under consideration. The linearization of the drift term F at z_t is $a(t) = 2\sqrt{3}z_t - 3z_t^2$. It follows that for $t_1^\star \leqslant s \leqslant t \leqslant t_2^\star$,

$$\alpha(t,s) = \int_s^t a(u)\,\mathrm{d}u \leqslant \varepsilon\int_{z_s}^{z_t} \frac{2\sqrt{3}z - 3z^2}{\sqrt{3}z^2 - z^3}\,\mathrm{d}z = \varepsilon\log\biggl(\frac{z_t^2(\sqrt{3}-z_t)}{z_s^2(\sqrt{3}-z_s)}\biggr), \tag{6.21}$$

and thus

$$\mathrm{e}^{\alpha(t,s)/\varepsilon} \leqslant \frac{z_t^2(\sqrt{3}-z_t)}{z_s^2(\sqrt{3}-z_s)} \asymp \frac{z_t^2}{z_s^2}. \tag{6.22}$$

More careful estimates, based on the inequalities

$$\frac{1}{\varepsilon}\bigl[\sqrt{3}z_t^2 - z_t^3\bigr] \leqslant \frac{\mathrm{d}z}{\mathrm{d}t} \leqslant \frac{1}{\varepsilon}\bigl[ct + \sqrt{3}z_t^2\bigr], \tag{6.23}$$

show that $\mathrm{e}^{\alpha(t,s)/\varepsilon}$ is also bounded below by a constant times $(z_t/z_s)^2$. Now $\zeta(t)$ can be computed in the same way, by performing the change of variables $s \mapsto z_s$, yielding (6.18). Finally, (6.19) follows easily from the fact that we are again in the stable case for $t \geqslant t_2^\star$. □

We now return to the SDE (4.2) for $a(t)$ given by (3.18) and $b(y,t) = F(y,t) - a(t)y$. Following the proof of [BG2, Proposition 3.10], it is easy to establish (4.9) for all $h \leqslant h_0(\varepsilon\sqrt{a_0})^{5/6}$ and all t. The condition on h stems from the fact that the linear term $a(t)y_t$ should dominate the nonlinear term $b(y_t,t)$ for all realizations ω satisfying $|y_t(\omega)| \leqslant h\sqrt{\zeta(t)}$ $\forall t$.

The condition on h implies that we need to require $\sigma \leqslant (\varepsilon\sqrt{a_0})^{5/6}$ for (4.9) to be of interest. Then the maximal spreading of paths will typically be of order $\sigma\sqrt{\zeta(t_2^\star)} \asymp \sigma(\varepsilon\sqrt{a_0})^{-5/6}$.

Since $|a'(t)|$ is no longer bounded for $t_1^\star \leqslant t \leqslant t_2^\star$, we cannot apply Lemma 4.11 to compute the integrals (4.25) and (4.26). However, using the same change of variables as in the proof of Proposition 6.2, it is not difficult to establish that

$$\Gamma_1(0,t_0) \asymp (\varepsilon\sqrt{a_0})^{-2/3} \quad\text{and}\quad \Gamma(0,t_0) \asymp \varepsilon^{-2/3}a_0^{1/6}, \tag{6.24}$$

where again $t_0 = -1/2$.

The following proposition now follows immediately from Corollary 4.10.

Proposition 6.3. *Assume that $\sigma \leqslant (\varepsilon\sqrt{a_0})^{5/6}$. There exists a constant h_1 such that for $H \leqslant h_1\varepsilon\sqrt{a_0}$,*

$$\mathbb{P}^{t_0,x_0}\bigl\{|Y_0| > H\bigr\} \leqslant \frac{C}{\varepsilon}\exp\Bigl\{-\kappa\frac{H^2}{\sigma^2(\varepsilon\sqrt{a_0})^{1/3}}\Bigr\}. \tag{6.25}$$

Finally, Proposition 4.7 can also be applied to show that for $\sigma|\log\varepsilon| \leqslant$ *const* $(\varepsilon\sqrt{a_0})^{5/6}$,

$$\bigl|\mathbb{E}^{t_0,x_0}\{Y_0\}\bigr| = \mathcal{O}\Bigl(\frac{\sigma^2|\log\varepsilon|}{(\varepsilon\sqrt{a_0})^{2/3}}\Bigr) \tag{6.26}$$

$$\operatorname{Var}(Y_0) \asymp \sigma^2(\varepsilon\sqrt{a_0})^{1/3}. \tag{6.27}$$

6.3 The case IIb

For σ larger than $(\varepsilon\sqrt{a_0})^{5/6}$, the strong dispersion of trajectories near time $t_2^\star$ prevents us from applying methods of Section 4. However, methods similar to those of Section 5 can be applied to obtain some information.

The following result allows to estimate deviations of $Y_{t^\star}$ in a larger domain than Proposition 4.9.

Proposition 6.4. *There exist constants C, κ, $h_1 > 0$ such that*

$$\mathbb{P}^{t_0,x_0}\bigl\{Y_{t^\star} < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa(H^{3/2}\vee\varepsilon\sqrt{a_0})/\sigma^2} \tag{6.28}$$

$$\mathbb{P}^{t_0,x_0}\bigl\{Y_{t^\star} > +H\bigr\} \leqslant \mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} + \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/(\sigma^2\Gamma_1(t^\star,t_0))} \tag{6.29}$$

for $0 \leqslant H \leqslant h_1(\varepsilon\sqrt{a_0})^{1/3}\Gamma_1(t^\star,t_0)$, where $\Gamma_1(t^\star,t_0) \asymp |\log(\varepsilon^{2/3}a_0^{-1/6})|$.

Proof: The proof is almost the same as the proof of Proposition 5.4, the only difference lying in the different behaviour of $\zeta(t)$, given in (6.1), which requires to distinguish between $\tau + t_{\mathrm{c}} \leqslant -\sqrt{a_0}$, $-\sqrt{a_0} \leqslant \tau + t_{\mathrm{c}} \leqslant -\varepsilon^{2/3}a_0^{-1/6}$, and the remaining τ up to $t^\star$. □

Proceeding as in the proof of Proposition 5.5, but using Proposition 6.1 for the transition time, we obtain

Proposition 6.5. *There exist constants C, κ, h_2, $h_3 > 0$ such that for all $x_{t_1} \in [-L, L]$ ($L \asymp 1$), and $h_2(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})| \leqslant H \leqslant h_3$,*

$$\mathbb{P}^{t^\star,x_{t^\star}}\bigl\{Y_0 < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H/\sigma^2} + \mathrm{e}^{-\kappa H^2/(\sigma^2\varepsilon)} \tag{6.30}$$

$$\mathbb{P}^{t^\star,x_{t^\star}}\bigl\{Y_0 > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa(\varepsilon\sqrt{a_0})^{1/3}H/(\sigma^2|\log(\varepsilon\sqrt{a_0})|)}\,. \tag{6.31}$$

Corollary 6.6. *For $h_2(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})| \leqslant h_1(\varepsilon\sqrt{a_0})^{1/3}\Gamma_1(t^\star,t_0)$,*

$$\mathbb{P}^{t_0,x_0}\bigl\{Y_0 < -H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa H^{3/2}/\sigma^2} \tag{6.32}$$

$$\mathbb{P}^{t_0,x_0}\bigl\{Y_0 > +H\bigr\} \leqslant \frac{C}{\varepsilon}\,\mathrm{e}^{-\kappa(\varepsilon\sqrt{a_0})^{1/3}H/(\sigma^2|\log(\varepsilon\sqrt{a_0})|)}\,. \tag{6.33}$$

The required lower bound on H only allows us to conclude that expectation and standard deviation of Y_0 are smaller than a constant times $(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})|$, although the above estimates are already very small for $H = h_2(\varepsilon\sqrt{a_0})^{2/3}|\log(\varepsilon\sqrt{a_0})|$.

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