Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus

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Abstract

We consider slowly time-dependent singular stochastic partial differential equations on the two-dimensional torus, driven by weak space-time white noise, and renormalised in the Wick sense. Our main results are concentration results on sample paths near stable equilibrium branches of the equation without noise, measured in appropriate Besov and Hölder norms. We also discuss a case involving a pitchfork bifurcation. These results extend to the two-dimensional torus those obtained in [3] for finite-dimensional SDEs, and in [7] for SPDEs on the one-dimensional torus.

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1 Introduction

In this work, we are interested in slowly time-dependent singular stochastic partial differential equations (SPDEs) on the two-dimensional torus, of the form

$$\mathrm{d}\phi(t,x) = \left[\Delta\phi(t,x) + :F(\varepsilon t, \phi(t,x)):\right] \mathrm{d}t + \sigma\, \mathrm{d}W(t,x) \;, \tag{1.1}$$

where $:F:$ denotes Wick renormalisation (see below), and $\mathrm{d}W(t,x)$ denotes space-time white noise. While analogous SPDEs on the one-dimensional torus are well-posed, without the need for any renormalisation procedure, it is well known that renormalisation is required in dimension two and higher, because space-time white noise is a distribution-valued process that is too singular.

The well-posedness problem on the two-dimensional torus was first solved by Giuseppe Da Prato and Arnaud Debussche in the landmark work [12]. The main idea of their approach is to write an equation for the difference between the solution and the stochastic convolution, which solves a linear equation. It turns out that unlike the stochastic convolution, which is distribution-valued, the difference is an actual function. Solutions to the equation can then be constructed by a fixed-point argument in an appropriate Besov space, provided the equation is renormalised in the sense of Wick. While the method has been spelled out for time-independent systems, extending it to time-dependent equations of the form (1.1) is straightforward.

The work [12] has later given rise to far-reaching generalisations, that allow to solve large classes of singular SPDEs. These generalisations include the theory of regularity structures, introduced by Martin Hairer in the work [15] and further developed with Ajay Chandra, Yvain Bruned, Ilya Chevyrev and Lorenzo Zambotti in [10, 11, 9], and the theory of paracontrolled distributions, introduced in [14] by Massimiliano Gubinelli, Peter Imkeller and Nicholas Perkowski. Most of these more general singular SPDEs require more refined renormalisation methods than Wick renormalisation.

For time-independent versions of the equation (1.1) on the two-dimensional torus, many results going beyond well-posedness and existence/uniqueness of solutions have been obtained. For instance, the fact that their solutions satisfy the Markov property and are reversible with respect to the Gibbs measure was proved in [24] using Dirichlet forms, while uniqueness of the Gibbs measure and convergence to it were obtained in [23]. The fact that solutions satisfy the strong Feller property and are exponentially mixing was shown in [26] using a dissipative bound, while the strong Feller property was also proved (for more general equations) in [17], using the theory of regularity structures. The work [18] provided a large-deviation principle, valid for a class of two- and three-dimensional singular SPDEs. In the particular case of the Allen–Cahn equation, sharper asymptotics on transition times between metastable states than those provided by large-deviation estimates have been obtained in [2] and [27].

In the present work, we are interested in obtaining more detailed non-equilibrium properties for time-dependent SPDEs of the form (1.1) on the two-dimensional torus. The case of the one-dimensional torus has been previously considered in the work [7]. The first main result of that work concerned the motion near so-called stable equilibrium branches of the equation. These are curves of the form $t \mapsto \phi^*(t,x)$ on which the right-hand side of the equation vanishes in the absence of noise. The deterministic equation admits particular solutions that stay at distance of order ε, in the H^1 Sobolev norm, from ϕ^*, and it was proved that solutions of the stochastic equation remain with high probability in a neighborhood of size of order σ, measured in the H^s Sobolev norm for $s < \frac{1}{2}$. This result provides an extension to the infinite-dimensional setting of similar results previously obtained in [5, 6] for finite-dimensional stochastic differential equations.

The other results in [7] concerned certain situations involving bifurcations, or avoided bifurcations. These occur when the equilibrium branch $t \mapsto \phi^*(t,x)$ (almost) loses stability at some time, usually because of the presence of a nearby unstable equilibrium branch. This can result in interesting phenomena such as stochastic resonance, where solutions of the equation make fast jumps in a close-to-periodic way. Those results were an infinite-dimensional generalisation of one-dimensional results obtained in [4].

The aim of the present work is to obtain similar results in the case of the two-dimensional torus, where Wick renormalisation is needed. The main result, Theorem 2.4, shows that Wick powers of the stochastic convolution remain concentrated near zero with high probability. Their size is measured here in the Besov norm $\|\cdot\|_{\mathcal{B}^{\alpha}_{2,\infty}}$ for any parameter $\alpha < 0$. Theorem 2.11 shows that this estimate implies concentration properties of solutions in a neighborhood of a stable equilibrium branch $\{\phi^*(t,x)\}_{0 \leqslant t \leqslant T}$. In particular, the difference between a solution of (1.1) and the stochastic convolution is likely to remain small, in a stronger Hölder norm of positive index.

Despite this concentration result, one may be concerned that it is of little practical use, because it does only concern the difference between a solution and the more singular stochastic convolution. Theorems 2.12 and 2.14 show that this is not the case, by discussing the particular situation of a dynamic pitchfork bifurcation, which was previously considered in [3] for one-dimensional stochastic differential equations.

The remainder of this paper is organized as follows. Section 2 contains a precise description of the set-up, a short introduction to Besov spaces, and the concentration results in the stable case, and in a case involving a pitchfork bifurcation. Section 3 contains the proof of Theorem 2.4 on Wick powers of the stochastic convolution. Section 4 contains the proofs of the other concentration results. Three appendices provide further information on Besov spaces, Wick calculus, and some technical proofs.

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2 Set-up and main results

2.1 A family of Wick-renormalised singular SPDEs

We are interested in renormalised versions of the SPDE

$$\mathrm{d}\phi(t,x) = \bigl[\Delta\phi(t,x) + F(\varepsilon t, \phi(t,x))\bigr]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)\;, \tag{2.1}$$

where time t belongs to an interval $I = [0,T] \subset \mathbb{R}_+$, the spatial variable x belongs to the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, and the solution $\phi(t,x)$ is real-valued. In addition, we assume that

- $\varepsilon > 0$ and $\sigma \geqslant 0$ are small positive parameters;
- F is polynomial, of the form

$$F(t,\phi) = \sum_{j=0}^{n} A_j(t)\phi^j \tag{2.2}$$

  for some odd $n \geqslant 3$, where the coefficients $A_j : I \to \mathbb{R}$ are of class $\mathcal{C}^1$, and the leading coefficient $A_n(t)$ is strictly negative for all $t \in I$, to avoid blow-up of solutions;
- $\mathrm{d}W(t,x)$ denotes space-time white noise on $I \times \mathbb{T}^2$.

It is well-known (see for instance [12]) that the SPDE (2.1) is not well-posed, and that a renormalisation procedure is required to define a notion of solution. There exist several slightly different ways of doing this. For our purposes, it will be convenient to work with spectral Galerkin approximations. Let

$$\bigl\{e_k(x) = \mathrm{e}^{2\pi\,\mathrm{i}\,k\cdot x}\bigr\}_{k\in\mathbb{Z}^2}$$

denote a complex Fourier basis of $L^2(\mathbb{T}^2)$, and write any $\phi \in L^2(\mathbb{T}^2)$ as

$$\phi(x) = \sum_{k\in\mathbb{Z}^2} \phi_k e_k(x)\;. \tag{2.3}$$

Note that since $\phi(x)$ is assumed to be real, the coefficients ϕ_k satisfy the reality condition

$$\phi_{-k} = \overline{\phi_k} \qquad \forall k \in \mathbb{Z}\;.$$

For any cut-off $N \in \mathbb{N}$, we define the spectral Galerkin approximation at order N of ϕ by

$$\phi_N(x) = (P_N\phi)(x) := \sum_{k\in\mathbb{Z}^2\colon\, |k|\leqslant N} \phi_k e_k(x)\;,$$

where $|k| = |k_1| + |k_2|$. We denote the eigenvalues of the Laplacian on $\mathbb{T}^2$ by $-\mu_k$, with

$$\mu_k := (2\pi)^2\,\|k\|^2\;, \qquad k \in \mathbb{Z}^2\;, \tag{2.4}$$

where $\|k\|$ denotes the Euclidean norm of k, and define the renormalisation constant

$$C_N = \frac{\sigma^2}{2}\,\mathrm{Tr}\Bigl([-P_N\Delta + 1]^{-1}\Bigr) = \sigma^2 \sum_{k\in\mathbb{Z}^2\colon\, |k|\leqslant N} \frac{1}{2(\mu_k+1)}\;. \tag{2.5}$$

One easily checks that C_N diverges like $\sigma^2 \log N/(2\pi)$ as $N \to \infty$. Note that the shift $+1$ in the definition (2.5) of C_N is only there to avoid problems with the $k = 0$ mode, and can be replaced by any other strictly positive constant.

Recall that the Hermite polynomials with variance C_N are defined recursively by

$$H_0(x; C_N) := 1 , \qquad H_{m+1}(x; C_N) := xH_m(x; C_N) - C_N \frac{\mathrm{d}}{\mathrm{d}x} H_m(x; C_N) \quad \forall m \in \mathbb{N}_0 .$$

The mth Wick power of ϕ_N is defined by

$$:\phi_N^m: \; = \; :\phi_N^m:_{C_N} := H_m(\phi_N; C_N) .$$

For instance, we have

$$\begin{aligned}
:\phi_N(x)^1: &= \phi_N(x) , \\
:\phi_N(x)^2: &= \phi_N(x)^2 - C_N , \\
:\phi_N(x)^3: &= \phi_N(x)^3 - 3C_N\phi_N(x), \\
:\phi_N(x)^4: &= \phi_N(x)^4 - 6C_N\phi_N(x)^2 + 3C_N^2 .
\end{aligned}$$

The renormalised version of (2.1) we want to study is given by the limit, as $N \to \infty$, of

$$\mathrm{d}\phi_N(t,x) = \big[\Delta\phi_N(t,x) + :F(\varepsilon t, \phi_N(t,x)):_{C_N}\big]\,\mathrm{d}t + \sigma\,\mathrm{d}W_N(t,x) , \tag{2.6}$$

where $\mathrm{d}W_N = P_N\,\mathrm{d}W$, and

$$:F(t,\phi):_{C_N} := \sum_{j=0}^{n} A_j(t) :\phi^j:_{C_N} .$$

As proved in [12], solutions of the renormalised equation (2.6) do admit a well-defined limit as $N \to \infty$, in appropriate Besov spaces that we define below. The limiting equation is denoted by

$$\mathrm{d}\phi(t,x) = \big[\Delta\phi(t,x) + :F(\varepsilon t, \phi(t,x)):\big]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x) .$$

In what follows, it will be convenient to rescale time by a factor ε, which results in the SPDE

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + :F(t, \phi(t,x)):\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x) . \tag{2.7}$$

2.2 Besov spaces and the "Da Prato–Debussche trick"

As mentioned above, the importance of Besov spaces in solving singular SPDEs of the form (2.7) was realised in the seminal work [12]. We recall one of their definitions here.

Definition 2.1 (Besov spaces)**.** *Let ϕ admit the Fourier series* (2.3)*. We define a collection of annuli by setting $\mathcal{A}_0 = \{(0,0)\}$ and $\mathcal{A}_q = \{k \in \mathbb{Z}^2 \colon 2^{q-1} \leqslant |k| < 2^q\}$ for any $q \in \mathbb{N}$. The projection of ϕ on $\mathcal{A}_q$ is defined by*

$$\delta_q\phi(x) := \sum_{k \in \mathcal{A}_q} \phi_k e_k(x) .$$

For $\alpha \in \mathbb{R}$ and $p, r \in [1, \infty]$, define the norm

$$\|\phi\|_{\mathcal{B}^{\alpha}_{p,r}} := \left\| \left\{ 2^{rq\alpha} \|\delta_q \phi\|_{L^p} \right\}_{q \geqslant 0} \right\|_{\ell^r}$$
$$:= \begin{cases} \left(\displaystyle\sum_{q \geqslant 0} 2^{rq\alpha} \|\delta_q \phi\|_{L^p} \right)^{1/r} & \text{if } 1 \leqslant r < \infty , \\ \displaystyle\sup_{q \geqslant 0} 2^{q\alpha} \|\delta_q \phi\|_{L^p} & \text{if } r = \infty . \end{cases}$$

Then the Besov space $\mathcal{B}^{\alpha}_{p,r} = \mathcal{B}^{\alpha}_{p,r}(\mathbb{T}^2)$ is defined as the set of all ϕ such that $\|\phi\|_{\mathcal{B}^{\alpha}_{p,r}} < \infty$.

The Besov space $\mathcal{B}^{\alpha}_{p,r}$ is a Banach space for all $\alpha \in \mathbb{R}$ and $p, r \in [1, \infty]$. In particular,

$$\mathcal{C}^{\alpha} := \mathcal{B}^{\alpha}_{\infty,\infty} \qquad \text{and} \qquad H^{\alpha} := \mathcal{B}^{\alpha}_{2,2}$$

coïncide with the usual Hölder and (fractional) Sobolev spaces respectively.

We will use the following results, which can be found, for instance, in [12, Proposition 2.1], in [12, Lemma 3.3] and [20].

Proposition 2.2 (Embeddings and products)**.**

1. *If $\alpha \in \mathbb{R}$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, then $\mathcal{B}^{\alpha}_{p_1,q_1}$ is continuously embedded in $\mathcal{B}^{\beta}_{p_2,q_2}$, where $\beta = \alpha - 2(\frac{1}{p_1} - \frac{1}{p_2})$.*
2. *If $\alpha_1 < \alpha_2 \in \mathbb{R}$ and $p, q \in [1, \infty]$, then $\mathcal{B}^{\alpha_1}_{p,q}$ is compactly embedded in $\mathcal{B}^{\alpha_2}_{p,q}$.*
3. *Let $p, r \geqslant 1$ and let $\alpha, \beta < \frac{2}{p}$ satisfy $\alpha + \beta > 0$. Then, if $\phi \in \mathcal{B}^{\alpha}_{p,r}$ and $\psi \in \mathcal{B}^{\beta}_{p,r}$, one has*
$$\phi\psi \in \mathcal{B}^{\gamma}_{p,r} \qquad \text{and} \qquad \|\phi\psi\|_{\mathcal{B}^{\gamma}_{p,r}} \lesssim \|\phi\|_{\mathcal{B}^{\alpha}_{p,r}} \|\psi\|_{\mathcal{B}^{\beta}_{p,r}} , \tag{2.8}$$
where $\gamma = \alpha + \beta - \frac{2}{p}$.
4. *Let $n \in \mathbb{N}$, $p, r \geqslant 1$ and $-\frac{2}{p(2n+1)} < \alpha < 0$. Set $s = \frac{2}{p} + 2\alpha$. Then, if $\phi \in \mathcal{B}^{s}_{p,r}$ and $\psi \in \mathcal{B}^{\alpha}_{p,r}$, one has*
$$\phi^{\ell}\psi \in \mathcal{B}^{(2\ell+1)\alpha}_{p,r} \qquad \text{and} \qquad \|\phi^{\ell}\psi\|_{\mathcal{B}^{(2\ell+1)\alpha}_{p,r}} \lesssim \|\phi\|^{\ell}_{\mathcal{B}^{s}_{p,r}} \|\psi\|_{\mathcal{B}^{\alpha}_{p,r}} \tag{2.9}$$
for $\ell \in \{0, \dots, n-1\}$, with a constant depending on α, s, p, r and n.

Let ψ denote the stochastic convolution, that is, the solution of the linear equation

$$\mathrm{d}\psi(t,x) = \Delta\psi(t,x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

with initial condition $\psi(0,x) = 0$. It is known (see, for instance, [12, Lemma 3.2]) that the stochastic convolution belongs to all Besov spaces with *negative* regularity α, but not with positive α. This means that ψ is a distribution, but not a function. The central idea in [12] is that the difference $\phi_1 = \phi - \psi$ enjoys much better regularity properties:

Theorem 2.3 ([12, Theorem 4.2])**.** *For any $p > n$ and $r \geqslant 1$, let α and s satisfy*

$$0 > \alpha > \max\left\{ -\frac{2}{p(n+1)}, -\frac{1}{n-1}\left(1 - \frac{n}{p}\right) \right\} , \qquad s = \frac{2}{p} + 2\alpha .$$

Then, for almost any initial condition (with respect to a natural probability measure), the renormalised SPDE admits for any $T \geqslant 0$ a unique solution ϕ such that

$$\phi - \psi \in \mathcal{C}([0,T], \mathcal{B}^{\alpha}_{p,r}) \cap L^p([0,T], \mathcal{B}^{s}_{p,r}) .$$

Note in particular that $s > 0$, implying that the difference $\phi - \psi$ takes values in the space of functions $\mathcal{B}^{s}_{p,r}$, which have some Hölder regularity in space.

2.3 Main results: Wick powers of the stochastic convolution

Our first main results concern the stochastic convolution and its Wick powers. Let $a : I \to \mathbb{R}$ be a continuously differentiable function satisfying

$$-a_+ < a(t) < -a_- \qquad \forall t \in I \tag{2.10}$$

for some constants $a_+ > a_- > 0$. The (time-inhomogeneous) stochastic convolution is defined as the solution of the linear equation

$$\mathrm{d}\psi(t,x) = \frac{1}{\varepsilon}\big[\Delta\psi(t,x) + a(t)\psi(t,x)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x) \tag{2.11}$$

with initial condition $\psi(0,x) = 0\ \forall x \in \mathbb{T}^2$. The following estimate is the main result of this section.

Theorem 2.4 (Tail estimates on Wick powers of the stochastic convolution)**.** *For any $\alpha < 0$ and for any $m \in \mathbb{N}$, there exist constants $C_m(T,\varepsilon,\alpha)$ and $\kappa_m(\alpha)$, independent of the cut-off N, such that*

$$\mathbb{P}\biggl\{\sup_{0\leqslant t\leqslant T} \|:\psi(t,\cdot)^m:\|_{\mathcal{B}^{\alpha}_{2,\infty}} > h^m\biggr\} \leqslant C_m(T,\varepsilon,\alpha)\,\mathrm{e}^{-\kappa_m(\alpha)h^2/\sigma^2}$$

holds for all $h > 0$. Furthermore, there are constants c_0, c_1, uniform in m, α, T and ε, such that

$$\kappa_m(\alpha) \geqslant c_0\frac{\alpha^2}{m^7}\;, \qquad C_m(T,\varepsilon,\alpha) \leqslant c_1\frac{T}{\varepsilon}\frac{m^{3/2}\,\mathrm{e}^m\,m^m}{|\alpha|}\;.$$

Remark 2.5. Comparable results cannot be expected to hold in any Besov space $\mathcal{B}^{\alpha}_{p,\infty}$. For instance, we have

$$\mathbb{E}\big[\|\psi(t,\cdot)\|_{\mathcal{B}^{\alpha}_{\infty,\infty}}\big] = \sup_{q\geqslant 0} 2^{q\alpha} \sum_{k\in\mathcal{A}_q} \mathbb{E}\big[|\psi_k(t)|\big]\;.$$

Since the random variables $\psi_k(t)$ follow centred normal distributions of variance of order $\|k\|^{-2}$, the sum over $k \in \mathcal{A}_q$ of the expectations of $|\psi_k(t)|$ has order 2^q. Therefore, the expectation of $\|\psi(t,\cdot)\|_{\mathcal{B}^{\alpha}_{\infty,\infty}}$ diverges with the cut-off N as $N^{\alpha+1}$ if $\alpha > -1$. Since the limiting random variable does not admit a first moment, its tail probabilities have to decay more slowly than $1/h$. $\diamondsuit$

Remark 2.6. The following observation may provide some intuition on what it means for a distribution to be concentrated in a ball in the Besov space $\mathcal{B}^{\alpha}_{2,\infty}$. Let $\eta : \mathbb{T}^2 \to \mathbb{R}$ be a compactly supported test function of class $\mathcal{C}^1$, of unit $\mathcal{C}^1$-norm, and set

$$\eta_\rho(x) = \frac{1}{\rho}\eta\biggl(\frac{x}{\rho}\biggr)$$

for any $\rho \in (0,1]$. Note that the scaled test functions η_ρ have constant L^2-norm, instead of constant L^1-norm, as one would require when working with $\mathcal{B}^{\alpha}_{\infty,\infty}$. Then we have

$$\big|\langle :\psi^m: , \eta_{2^{-q_0}}\rangle\big| \lesssim 2^{|\alpha|q_0}\,\|:\psi^m:\|_{\mathcal{B}^{\alpha}_{2,\infty}}\;,$$

for all $q_0 \in \mathbb{N}_0$ (cf. Lemma A.1), so that Theorem 2.4 implies

$$\mathbb{P}\biggl\{\sup_{0\leqslant t\leqslant T}\langle :\psi(t)^m: , \eta_{2^{-q_0}}\rangle > h^m\biggr\} \leqslant C_m(T,\varepsilon)\exp\biggl\{-\kappa_m(\alpha)2^{-2|\alpha|q_0/m}\frac{h^2}{\sigma^2}\biggr\}$$

for any $m \in \mathbb{N}$ and any $q_0 \in \mathbb{N}_0$. This shows that sample paths of $\langle :\psi(t)^m: , \eta_{2^{-q_0}}\rangle$ are concentrated in a strip of width $\sigma^m 2^{|\alpha|q_0}|\alpha|^{-m}$. The same holds of course for $\eta_\rho(x - x_0)$, for any $x_0 \in \mathbb{T}^2$. $\diamondsuit$

2.4 Main results: concentration around stable equilibrium branches

The main part of our results concern the effect of weak space-time white noise on the dynamics near a stable equilibrium branch of the unperturbed equation.

Assumption 2.7 (Stable case)**.** *There exists a map $\phi^* : I \to \mathbb{R}$ such that*

$$F(t, \phi^*(t)) = 0 \; \forall t \in I \; .$$

Furthermore, the linearisation $a(t) = \partial_\phi F(t, \phi^*(t))$ satisfies (2.10).

In the deterministic case $\sigma = 0$, the SPDE (2.7) reduces to

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\phi(t,x) + F(t,\phi(t,x))\bigr]\,\mathrm{d}t \; , \tag{2.12}$$

since the renormalisation counterterm C_N vanishes for $\sigma = 0$. We then have the following generalisation of Tihonov's theorem (cf. [25]).

Proposition 2.8 (Deterministic case)**.** *There exist constants $\varepsilon_0, C > 0$ such that, when $0 < \varepsilon < \varepsilon_0$,* (2.12) *admits a particular solution $\bar\phi(t)$ satisfying*

$$\bigl\|\bar\phi(t,\cdot) - \phi^*(t)e_0\bigr\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I \; .$$

The difference $\phi_0 = \phi - \bar\phi$ satisfies the SPDE

$$\mathrm{d}\phi_0(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\phi_0(t,x) + {:}F_0(t,x,\phi_0(t,x)){:}\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x) \; , \tag{2.13}$$

where

$${:}F_0(t,x,\phi_0(t,x)){:} = {:}F(t,\bar\phi(t,x) + \phi_0(t,x)){:} - F(t,\bar\phi(t,x))$$

has similar properties as F, and satisfies in addition $F_0(t,x,0) = 0$ for all $t \in I$ and all $x \in \mathbb{T}^2$. More precisely, we have the following result.

Lemma 2.9. *The renormalised forcing term is given by*

$${:}F_0(t,x,\phi_0(t,x)){:} = a(t)\phi_0(t,x) + \sum_{j=1}^n \hat A_j(t,x)\,{:}\phi_0(t,x)^j{:} \; , \tag{2.14}$$

where the $\hat A_j(t,\cdot)$ belong to H^1 (which is embedded in $\mathcal{B}^1_{2,\infty}$) for all $t \in I$, and are given by

$$\hat A_j(t,x) = \begin{cases} \displaystyle\sum_{i=2}^n iA_i(t)\bigl[\bar\phi(t,x)^{i-1} - \phi^*(t)^{i-1}e_0(x)\bigr] \; , & j = 1 \; , \\ \displaystyle\sum_{i=j}^n \binom{i}{j} A_i(t)\bar\phi(t,x)^{i-j} \; , & j = 2,\dots,n \; . \end{cases}$$

Remark 2.10. The proof of [7, Theorem 2.4] contains a small mistake, which is, however, easily corrected. In [7, Equation (3.5)], $\bar a(t)$ should be defined as $\bar a(t) = \partial_\phi f(t,\phi^*(t)e_0)$ instead of $\bar a(t) = \partial_\phi f(t,\bar\phi(t,x))$, in order to obtain a value independent of x. The only change to me made in the proof is that the nonlinear term b has order $h^2 + \varepsilon h$ instead of h^2. ♢

We rewrite (2.13) as

$$\mathrm{d}\phi_0(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi_0(t,x) + a(t)\phi_0(t,x) + {:}\,b(t,x,\phi_0(t,x)){:}\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)\;,$$

where ${:}\,b{:}$ denotes the sum over j in (2.14). Note that ${:}\,b{:}$ contains a term linear in ϕ_0. However, it has a coefficient of order ε, since $\bar\phi$ and $\phi^\star$ are at a distance of order ε.

We now apply the Da Prato–Debussche trick, and consider the difference $\phi_1 = \phi_0 - \psi$. It satisfies the equation

$$\mathrm{d}\phi_1(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi_1(t,x) + a(t)\phi_1(t,x) + {:}\,b(t,x,\psi(t,x)+\phi_1(t,x)){:}\big]\,\mathrm{d}t\;, \tag{2.15}$$

where

$${:}\,b(t,x,\psi(t,x)+\phi_1(t,x)){:} = \sum_{j=1}^{n}\hat A_j(t,x)\sum_{\ell=0}^{j}\binom{j}{\ell}\phi_1(t,x)^{j-\ell}{:}\,\psi(t,x)^\ell{:}\;.$$

It follows from Proposition 2.2 that if $\phi_1 \in \mathcal{B}^{\beta}_{2,\infty}$ and ${:}\,\psi^\ell{:} \in \mathcal{B}^{\alpha}_{2,\infty}$ for $\alpha < 0$ and $\ell = 0,\dots,n-1$, then

$${:}\,b(t,x,\psi+\phi_1){:} \in \mathcal{B}^{\bar\alpha}_{2,\infty} \qquad \forall\bar\alpha < (2n-1)\alpha\;,$$

provided $\beta \geqslant 1+2\alpha$. By the Schauder estimate recalled in Proposition A.2, the solution of (2.15) belongs to $\mathcal{B}^{\gamma}_{2,\infty}$ for $\gamma < 2-(2n+1)|\alpha|$, which allows to close the fixed-point argument, in accordance with Theorem 2.3.

By the embedding $\mathcal{B}^{\gamma}_{2,\infty} \hookrightarrow \mathcal{B}^{\gamma-1}_{\infty,\infty} = \mathcal{C}^{\gamma-1}$, we see that the solution of (2.15) is Hölder continuous, with exponent almost 1. In other words, the solution is almost Lipschitz continuous. Our main result is then the following.

Theorem 2.11 (Concentration estimate for ϕ_1)**.** *For any choice of $\gamma < 2$ and $\nu < 1-\frac{\gamma}{2}$, there exist constants $C(T,\varepsilon), M, \kappa, h_0, \varepsilon_0 > 0$ such that, whenever $\varepsilon < \varepsilon_0$ and $h < h_0\varepsilon^\nu$, one has*

$$\mathbb{P}\bigg\{\sup_{t\in[0,T]}\|\phi_1(t)\|_{\mathcal{B}^{\gamma}_{2,\infty}} > M\varepsilon^{-\nu}h(h+\varepsilon)\bigg\} \leqslant C(T,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2}\;,$$

$$\mathbb{P}\bigg\{\sup_{t\in[0,T]}\|\phi_1(t)\|_{\mathcal{C}^{\gamma-1}} > M\varepsilon^{-\nu}h(h+\varepsilon)\bigg\} \leqslant C(T,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2}\;.$$

This result shows in particular that sample paths of ϕ_1 are concentrated in a ball in $\mathcal{C}^{\gamma-1}$-norm of size

$$\varepsilon^{-\nu}\sigma(\sigma+\varepsilon) \asymp \begin{cases} \varepsilon^{-\nu}\sigma^2 & \text{if } \sigma > \varepsilon\;,\\ \varepsilon^{1-\nu}\sigma & \text{if } \sigma \leqslant \varepsilon\;.\end{cases}$$

2.5 The case of bifurcations

In this section, we comment on how the results of the last section can be extended to situations where the nonlinearity F fails to satisfy Assumption 2.7, that is, in the case of a bifurcation. In the work [7], which concerned SPDEs on the one-dimensional torus, we considered the case of an avoided transcritical bifurcation, where F is given locally by

$$F(t,\phi) = \delta + t^2 - \phi^2 + \mathcal{O}\big((|t|+|\phi|)^3\big)$$

with $0 < \delta \ll 1$. In that case, there is a stable equilibrium branch $\phi^*_+(t) \simeq \sqrt{\delta + t^2}$ approaching an unstable branch $\phi^*_-(t) \simeq -\sqrt{\delta + t^2}$ at distance $2\sqrt{\delta}$. While the linearization $a(t) = \partial_\phi F(t, \phi^*(t))$ remains positive, its value becomes small in terms of δ near $t = 0$. As a result, while the system still behaves as in the stable case when $\sigma \ll (\delta \vee \varepsilon)^{3/4}$, a new behaviour emerges for $\sigma \gg (\delta \vee \varepsilon)^{3/4}$: it becomes likely for sample paths to cross the unstable equilibrium branch, and travel in a short time to a distant region of space.

Here we will illustrate how these results can be transposed to singular SPDEs on the two-dimensional torus. However, for a change, we are going to take as an example the equation

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\phi(t,x) + a(t)\phi(t,x) - {:}\phi(t,x)^3{:}\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x) , \qquad (2.16)$$

which describes a pitchfork bifurcation when $a(t)$ changes from being negative to being positive at a time t^*. In the deterministic case $\sigma = 0$, there is a phenomenon known as bifurcation delay: solutions attracted by the stable equilibrium branch $\phi^*(t) = 0$ for $t < t^*$ remain close to 0 for a time of order 1 beyond the bifurcation time t^*, even though the equilibrium branch has become unstable. This is due to the solution becoming exponentially close to 0 during the stable phase, and a time of order 1 being required for the solution to reach again values of order 1.

In the one-dimensional SDE case, the effect of noise on such a system has been studied in [3]. The main result of that work is that sample paths remain with high probability at a distance of order $\sigma\varepsilon^{-1/4}$ from zero up to a time $t^* + \mathcal{O}(\varepsilon^{1/2})$, but are unlikely to remain close to 0 after times of order $t^* + \mathcal{O}((\varepsilon\log(\sigma^{-1}))^{1/2})$. The effect of noise is thus to reduce the bifurcation delay from order 1 to order $(\varepsilon\log(\sigma^{-1}))^{1/2}$.

In order to analyse the SPDE (2.16), we start by carrying out the change of variables

$$\phi(t,x) = \psi_\perp(t,x) + \phi_1(t,x) ,$$

where the stochastic convolution $\psi_\perp$ solves the SPDE

$$\mathrm{d}\psi_\perp(t,x) = \frac{1}{\varepsilon}\bigl[\Delta_\perp\psi_\perp(t,x) + a(t)\psi_\perp(t,x)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_\perp(t,x)$$

with zero initial condition. Here the noise $\mathrm{d}W_\perp$ acts only on non-zero Fourier modes, implying that the spatial average of $\psi_\perp(t,x)$ always remains equal to zero. We use the notation $\Delta_\perp$ to emphasize that the Laplacian only acts on non-zero Fourier modes, although it has the same effect as the usual Laplacian. The resulting equation for ϕ_1 reads

$$\mathrm{d}\phi_1(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\phi_1(t,x) + a(t)\phi_1(t,x) + {:}F(\psi_\perp(t,x),\phi_1(t,x)){:}\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t,x) ,$$

where

$${:}F(\psi_\perp,\phi_1){:} = -{:}\psi_\perp^3{:} - 3\phi_1{:}\psi_\perp^2{:} - 3\phi_1^2\psi_\perp - \phi_1^3 .$$

The next step is to split ϕ_1 into its mean and oscillating spatial part, writing

$$\phi_1(t,x) = \phi_1^0(t)e_0(x) + \phi_1^\perp(t,x) , \qquad \phi_1^0(t) = \langle e_0, \phi_1(t,\cdot)\rangle .$$

This results in the coupled SDE–SPDE system

$$\mathrm{d}\phi_1^0(t) = \frac{1}{\varepsilon}\bigl[a(t)\phi_1^0(t) - \phi_1^0(t)^3 + F_0(\psi_\perp,\phi_1^0,\phi_1^\perp)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t) , \qquad (2.17)$$

$$\mathrm{d}\phi_1^\perp(t,x) = \frac{1}{\varepsilon}\bigl[\Delta_\perp\phi_1^\perp(t,x) + a(t)\phi_1^\perp(t,x) + {:}F_\perp(\psi_\perp,\phi_1^0,\phi_1^\perp){:}\bigr]\,\mathrm{d}t , \qquad (2.18)$$

where F_0 and $:F_\perp:$ are nonlocal nonlinearities given by

$$\begin{aligned} F_0(\psi_\perp, \phi_1^0, \phi_1^\perp) &= \langle e_0, :F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp):\rangle , \\ &= \delta_0(:F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp):) , \\ F_\perp(\psi_\perp, \phi_1^0, \phi_1^\perp) &= :F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp): - F_0(\psi_\perp, \phi_1^0, \phi_1^\perp) . \end{aligned}$$

We start by describing concentration properties of $\phi_1^\perp$. For that, given a parameter $H_0 > 0$, we introduce the stopping time

$$\tau_0(H_0) = \inf\{t \in [0,T] \colon |\phi_1^0(t)| > H_0\} .$$

Theorem 2.12 (Concentration estimate for $\phi_1^\perp$)**.** *Assume there exists a constant $a_0 > 0$ such that $a(t) \leqslant (2\pi)^2 - a_0$ for all $t \in [0,T]$. Then for any choice of $\gamma < 2$ and $\nu < 1 - \frac{\gamma}{2}$, there exist constants $C(T,\varepsilon), M, \kappa, h_0 > 0$ such that, whenever $h + H_0 \leqslant h_0\varepsilon^{\nu/2}$, one has*

$$\mathbb{P}\biggl\{\sup_{t\in[0,T\wedge\tau_0(H_0)]} \|\phi_1^\perp(t)\|_{\mathcal{C}^{\gamma-1}} > M\varepsilon^{-\nu}(h+H_0)^3\biggr\} \leqslant C(T,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2} .$$

Note in particular the weaker condition on $a(t)$: instead of having to stay negative, $a(t)$ may become positive, as long as it stays smaller than $(2\pi)^2$. This is because the eigenvalues of the Laplacian $\Delta_\perp$ are bounded above by $-(2\pi)^2$.

Remark 2.13. One can easily extend the result to cases where $a(t)$ exceeds the value $(2\pi)^2$ by incorporating more Fourier modes in the variables ϕ_1^0. ◊

It is now relatively straightforward to extend the one-dimensional results from [3] to the SDE (2.17) governing the zeroth Fourier mode. The idea is that its solution is likely to remain close, on some time interval, to the solution of the linearised equation

$$\mathrm{d}\phi^\circ(t) = \frac{1}{\varepsilon}\, a(t)\phi^\circ(t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_0(t) , \tag{2.19}$$

which is a Gaussian process, with variance

$$v^\circ(t) = v^\circ(0) + \frac{\sigma^2}{\varepsilon}\int_0^t \mathrm{e}^{2\alpha(t,t_1)/\varepsilon}\,\mathrm{d}t_1 , \qquad \alpha(t,t_1) = \int_{t_1}^t a(t,t_2)\,\mathrm{d}t_2 .$$

One can show (see [3, Lemma 4.2]) that for an initial variance $v^\circ(0)$ of order σ^2, bounded away from zero, one has

$$v^\circ(t) \asymp \begin{cases} \dfrac{\sigma^2}{|t-t^*|} & \text{for } 0 \leqslant t \leqslant t^* - \sqrt{\varepsilon} , \\ \dfrac{\sigma^2}{\sqrt{\varepsilon}} & \text{for } -\sqrt{\varepsilon} \leqslant t - t^* \leqslant \sqrt{\varepsilon} , \\ \dfrac{\sigma^2}{\sqrt{\varepsilon}}\,\mathrm{e}^{2\alpha(t,t^*)/\varepsilon} & \text{for } t \geqslant t^* + \sqrt{\varepsilon} . \end{cases}$$

Note that the variance increases slowly up to time $t^* + \sqrt{\varepsilon}$, and then increases exponentially fast. This suggests defining sets

$$\begin{aligned} \mathcal{B}_-(h_-) &= \biggl\{(t,\phi_1^0) \in [0, t^* + \sqrt{\varepsilon}] \times \mathbb{R} \colon |\phi_1^0| \leqslant \frac{h_-}{\sigma} v^\circ(t)\biggr\} , \\ \mathcal{B}_+(h_+) &= \biggl\{(t,\phi_1^0) \in [t^* + \sqrt{\varepsilon}, T] \times \mathbb{R} \colon |\phi_1^0| \leqslant \frac{h_+}{\sqrt{a(t)}}\biggr\} . \end{aligned}$$

The first set is a union of confidence intervals associated with the variance $v^\circ(t)$. The second set is motivated by the fact that the variance of processes obtained by linearization grows like

$$\frac{\sigma^2}{2a(t)}\,\mathrm{e}^{2\alpha(t,t^*)/\varepsilon}\;.$$

One then has the following generalisation of [3, Theorem 2.10] and [3, Proposition 4.7].

Theorem 2.14 (Behaviour of $\phi_1^0(t)$ near a pitchfork bifurcation)**.** *There exist positive constants M, ε_0, h_0 such that, for any $\varepsilon < \varepsilon_0$ and $h_- \leqslant h_0\varepsilon^{1/2}$, and any $t \leqslant t^* + \varepsilon^{1/2}$, one has*

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}_-(h_-)} \leqslant t\bigr\} \leqslant C(t,\varepsilon)\exp\biggl\{-\frac{h_-^2}{2\sigma^2}\biggl[1-\mathcal{O}(\sqrt{\varepsilon}\,)-\mathcal{O}\biggl(\frac{h_-^2}{\varepsilon}\biggr)\biggr]\biggr\}\;, \tag{2.20}$$

where $C(t,\varepsilon) = \mathcal{O}(\alpha(t)/\varepsilon^2)$. Furthermore, for $h_+ = \sigma\log(\sigma^{-1})^{1/2}$ and any $t \geqslant t^* + \varepsilon^{1/2}$, one has

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}_+(h_+)} \geqslant t\bigr\} \leqslant \frac{h_+}{\sigma}\exp\biggl\{-\kappa\frac{\alpha(t,t^*)}{\varepsilon}\biggr\} + C(t,\varepsilon)\,\mathrm{e}^{-\kappa\log(\sigma^{-1})/\sqrt{\varepsilon}} \tag{2.21}$$

for a constant $\kappa > 0$.

The bound (2.20) shows that when $\sigma \ll h_- \ll \sqrt{\varepsilon}$, sample paths are likely to stay in $\mathcal{B}_-(h_-)$ up to time $t^* + \sqrt{\varepsilon}$. At time $t^* + \sqrt{\varepsilon}$, typical fluctuations have a size of order $\sigma\varepsilon^{-1/4}$. Since $\alpha(t,t^*)$ grows like $(t-t^*)^2$, the bound (2.21) shows that sample paths are likely to leave a neighborhood of size σ of 0 at times of order $\sqrt{\varepsilon\log(\sigma^{-1})}$.

3 Proof of Theorem 2.4

3.1 Relation between Hermite polynomials and binomial formula

We will first give a proof of Theorem 2.4 in the particular case where $a(t) = -1$ for all t. That is, we consider the linear equation

$$\mathrm{d}\psi(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\psi(t,x) - \psi(t,x)\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)\;.$$

Its projection on the kth basis vector e_k is given by

$$\mathrm{d}\psi_k(t) = -\frac{1}{\varepsilon}(\mu_k+1)\psi_k(t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_k(t)\;, \tag{3.1}$$

where the μ_k are the eigenvalues (2.4) of the Laplacian, and the $\{W_k(t)\}_{t\geqslant 0}$ are independent Wiener processes (actually, since we use complex Fourier series, we have $\mathbb{E}[\mathrm{d}W_{k_1}\,\mathrm{d}W_{k_2}] = \delta_{k_1,-k_2}\,\mathrm{d}t$, but this yields equivalent results). We write $\alpha_k(t,t_1) = -(\mu_k+1)(t-t_1)$ and $\alpha_k(t,0) = \alpha_k(t)$ for brevity. The solution of (3.1) is an Ornstein–Uhlenbeck process, which can be represented using Duhamel's principle by the Ito integral

$$\psi_k(t) = \mathrm{e}^{\alpha_k(t)/\varepsilon}\,\psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{\alpha_k(t,t_1)/\varepsilon}\,\mathrm{d}W_k(t_1)\;. \tag{3.2}$$

At any time $t \geqslant 0$, $\psi_k(t)$ is a zero-mean Gaussian random variable of variance

$$\begin{aligned}
v_k(t) &= \mathrm{Var}\bigl(\psi_k(0)\bigr)\,\mathrm{e}^{2\alpha_k(t)/\varepsilon} + \frac{\sigma^2}{\varepsilon}\int_0^t \mathrm{e}^{2\alpha_k(t,t_1)/\varepsilon}\,\mathrm{d}t_1 \\
&= \mathrm{Var}\bigl(\psi_k(0)\bigr)\,\mathrm{e}^{2\alpha_k(t)/\varepsilon} + \frac{\sigma^2}{2(\mu_k+1)}\Bigl[1-\mathrm{e}^{2\alpha_k(t)/\varepsilon}\Bigr]\;.
\end{aligned}$$

In order to obtain a stationary process, we will assume that the initial conditions $\psi_k(0)$ follow centred normal distributions with variance $v_k = \sigma^2/[2(\mu_k+1)]$, which are mutually independent, and independent of the Wiener processes. In this way, we have $v_k(t) = v_k$ for all t.

Fix $\alpha < 0$. By Definition 2.1 of Besov norms,

$$
\begin{aligned}
P_m(h) &:= \mathbb{P}\Big\{\sup_{0\leqslant t\leqslant T} \|{:}\psi(t,\cdot)^m{:}\|_{\mathcal{B}^\alpha_{2,\infty}} > h^m\Big\} \\
&= \mathbb{P}\Big\{\sup_{0\leqslant t\leqslant T}\sup_{q_0\geqslant 0} 2^{-|\alpha|q_0}\,\|\delta_{q_0}({:}\psi(t,\cdot)^m{:})\|_{L^2} > h^m\Big\} \\
&= \mathbb{P}\Big\{\exists q_0\geqslant 0;\ \sup_{0\leqslant t\leqslant T}\|\delta_{q_0}({:}\psi(t,\cdot)^m{:})\|_{L^2} > h^m 2^{|\alpha|q_0}\Big\} \\
&\leqslant \sum_{q_0\geqslant 0}\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant T}\|\delta_{q_0}({:}\psi(t,\cdot)^m{:})\|_{L^2} > h^m 2^{|\alpha|q_0}\Big\}\ . \qquad (3.3)
\end{aligned}
$$

Remark 3.1. At any fixed time t, the law of $\psi(t,\cdot)$ is that of the truncated Gaussian free field, with variance

$$
\sum_{k\in\mathbb{Z}^2\colon |k|\leqslant N} v_k = C_N\ ,
$$

which diverges like $\sigma^2\log(N)$, as mentioned in (2.5). $\diamondsuit$

In what follows, it will be convenient to use multiindex notations. For any $\mathbf{n}\in\mathbb{N}^{\mathbb{N}}$ with finitely many nonzero components, we write

$$
|\mathbf{n}| = \sum_{q\geqslant 0}\mathbf{n}_q \qquad\text{and}\qquad \mathbf{n}! := \prod_{q\geqslant 0}\mathbf{n}_q!\ .
$$

Since $\mathbf{n}$ has finitely many nonzero components, these quantities are indeed well-defined. Let

$$
[\mathbf{n}] = \#\{q : \mathbf{n}_q > 0\}
$$

be the number of these nonzero components. We can order them as $q_1 < q_2 < \dots < q_{[\mathbf{n}]}$, where

$$
q_{[\mathbf{n}]} = \max\{q : \mathbf{n}_q > 0\}
$$

is the index of the largest nonzero entry of $\mathbf{n}$. In what follows, we will always assume that $|\mathbf{n}| = m$. We notice that this implies $[\mathbf{n}]\leqslant m$.

The projection of $\psi(t,\cdot)$ on the annulus $\mathcal{A}_q$ has constant variance

$$
c_q := \mathbb{E}\Big[\|\delta_q\psi(t,\cdot)\|^2_{L^2}\Big] = \sum_{k\in\mathcal{A}_q} v_k\ .
$$

In fact, $\delta_q\psi(t,x)$ has variance c_q for all $x\in\mathbb{T}^2$. An important feature of the projections is that

$$
c_q \lesssim \sum_{k\in\mathcal{A}_q}\frac{\sigma^2}{1+\|k\|^2} \lesssim 2\sigma^2\int_{2^{q-1}}^{2^q}\frac{r\,\mathrm{d}r}{1+r^2} = \sigma^2\log\bigg(\frac{1+2^{2q}}{1+2^{2(q-1)}}\bigg) \leqslant 2\sigma^2\log 2
$$

for all q. Finally note that the cut-off condition $|k|\leqslant N$ implies $q\leqslant \ell_N = \lfloor\log_2 N\rfloor$, and that

$$
C_N = \sum_{q=0}^{\ell_N} c_q\ .
$$

With these notations in place, we can introduce the binomial formula for Hermite polynomials, see Lemma B.4 in Appendix B.

Lemma 3.2. *For any $m \in \mathbb{N}$, we have*

$$H_m(\psi(t,\cdot);C_N) = \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\psi(t,\cdot),c_q) \;.$$

It thus follows from (3.3) that we have

$$\begin{aligned}
P_m(h) &\leqslant \sum_{q_0\geqslant 0} \mathbb{P}\Bigl\{\sup_{0\leqslant t\leqslant T} \bigl\|\delta_{q_0}\bigl(\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q\geqslant 0} :\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:\bigr)\bigr\|_{L^2} > h^m 2^{|\alpha|q_0}\Bigr\} \\
&= \sum_{q_0\geqslant 0} \mathbb{P}\Bigl\{\sup_{0\leqslant t\leqslant T} \bigl\|\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \delta_{q_0}\bigl(\prod_{q\geqslant 0} :\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:\bigr)\bigr\|_{L^2} > h^m 2^{|\alpha|q_0}\Bigr\} \\
&\leqslant \sum_{q_0\geqslant 0} \mathbb{P}\Bigl\{\sup_{0\leqslant t\leqslant T} \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \bigl\|\delta_{q_0}\bigl(\prod_{q\geqslant 0} :\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:\bigr)\bigr\|_{L^2} > h^m 2^{|\alpha|q_0}\Bigr\} \;.
\end{aligned}$$

Remark 3.3. Note that for any $q_1,\ q_2 \geqslant 0$, one has $2^{q_1} + 2^{q_2} \leqslant 2^{\max\{q_1,q_2\}+1}$. Therefore,

$$\delta_{q_0}\bigl(\prod_{q\geqslant 0} :\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:\bigr) \neq 0 \qquad \Rightarrow \qquad q_0 \leqslant \max_{i\leqslant q_{[\mathbf{n}]}}\{q_i + \mathbf{n}_{q_i}\} \leqslant q_{[\mathbf{n}]} + \mathbf{n}_{q_{[\mathbf{n}]}}$$

for any $\mathbf{n}$, which will be useful in restricting the domains of the sums. ♢

For any decomposition $h^m = \sum_{|\mathbf{n}|=m} h^m_{\mathbf{n}}$, one has

$$P_m(h) \leqslant \sum_{q_0\geqslant 0} \sum_{|\mathbf{n}|=m} \mathbb{P}\Bigl\{\sup_{0\leqslant t\leqslant T} \bigl\|\delta_{q_0}\bigl(\prod_{q\geqslant 0} :\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:\bigr)\bigr\|_{L^2} > \frac{\mathbf{n}!}{m!} h^m_{\mathbf{n}} 2^{|\alpha|q_0}\Bigr\} \;. \tag{3.4}$$

3.2 Martingale and partition

In this section, we fix $q_0 \geqslant 0$ and $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ with $|\mathbf{n}| = m$. Our aim is to estimate one term in the double sum (3.4). We notice that the stochastic integral $\psi_k(t)$ is not a martingale. However,

$$\mathrm{e}^{-\alpha_k(t)/\varepsilon}\,\psi_k(t) = \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{-\alpha_k(t_1)/\varepsilon}\,\mathrm{d}W_k(t_1) \tag{3.5}$$

is a martingale of variance $\mathrm{e}^{-2\alpha_k(t)/\varepsilon}\,v_k$. The variances of $\psi_k(t)$ and $\hat\psi_k(t)$ are too different on the whole time interval $[0,T]$ to allow a useful comparison of the two processes. This is why we introduce a partition $0 = u_0 \leqslant u_1 < \cdots < u_L = T$ of this interval. Given $\gamma_0 > 0$ and any $k_0 \in \mathbb{Z}^2$ such that $|k_0| = 2^{q_{[\mathbf{n}]}}$, we define the partition by

$$\alpha_{k_0}(u_{l+1},u_l) = -\gamma_0\varepsilon \qquad \text{for } 1 \leqslant l \leqslant L = \biggl\lfloor \frac{((2\pi)^2\,\|k_0\|^2 + 1)T}{\gamma_0\varepsilon} \biggr\rfloor \;, \tag{3.6}$$

and write $I_l = [u_l, u_{l+1}]$. Multiplying (3.5) by $\mathrm{e}^{\alpha_k(u_{l+1})/\varepsilon}$, we obtain the martingale

$$\hat\psi_k(t) := \mathrm{e}^{\alpha_k(u_{l+1},t)/\varepsilon}\,\psi_k(t) = \mathrm{e}^{\alpha_k(u_{l+1})/\varepsilon}\,\psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\alpha_k(u_{l+1},t_1)/\varepsilon}\,\mathrm{d}W_k(t_1) \;, \tag{3.7}$$

where we do not indicate the l-dependence of $\hat\psi_k(t)$ in order not to overload the notations. The variance of $\hat\psi_k(t)$ is

$$\hat v_k(t) = v_k\,\mathrm{e}^{2\alpha_k(u_{l+1},t)/\varepsilon} \;. \tag{3.8}$$

They key observation is the following property of Hermite polynomials, which is proved in Appendix B.

Lemma 3.4. *For any* $m \geqslant 1$, $\{H_m(\hat{\psi}_k(t);\hat{v}_k(t))\}_{t\geqslant 0}$ *is a martingale with respect to the canonical filtration* $\{\mathcal{F}_t\}_t$ *of* $(W_k(t))_t$.

This observation will allow us to deal with the supremum over times in (3.4), by using Doob's submartingale inequality. We will thus be interested in the martingales

$$\delta_q\hat{\psi}(t,x) = \sum_{k\in\mathcal{A}_q}\hat{\psi}_k(t)e_k(x) \;,$$

as well as of the related quantities

$$X_{\mathbf{n}}^2(t) = \big\|\delta_{q_0}\big(\prod_{q\geqslant 0} :\delta_q\hat{\psi}(t,\cdot)^{\mathbf{n}_q}:\big)\big\|_{L^2}^2 \;.$$

Later on, we will extend the obtained bounds to functions of $\delta_q\psi(t,x)$.

Proposition 3.5. *Fix a constant* $\gamma_{q_0,q_{[\mathbf{n}]}} \in \mathbb{R}$ *and* $l \in \{0,\dots,L\}$. *Then the bound*

$$\mathbb{P}\biggl\{\sup_{t\in I_l} X_{\mathbf{n}}^2(t) > \Bigl(\frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\Bigr)^2\biggr\} \leqslant C_{\mathbf{n}}(l,\varepsilon)\exp\biggl\{-\frac{H_{\mathbf{n}}(q_0,l)}{\sigma^2}\biggr\}$$

holds, where

$$\begin{aligned} C_{\mathbf{n}}(l,\varepsilon) &= \mathrm{e}^{m-1} + \mathbb{E}\biggl[\exp\biggl\{\frac{\gamma_{q_0,q_{[\mathbf{n}]}}}{\sigma^2}\bigl[X_{\mathbf{n}}^2(u_{l+1})\bigr]^{1/m}\biggr\}\biggr] \;, \\ H_{\mathbf{n}}(q_0,l) &= \gamma_{q_0,q_{[\mathbf{n}]}}\biggl(\frac{\mathbf{n}!}{m!}2^{|\alpha|q_0}h_{\mathbf{n}}^m\biggr)^{2/m} \;. \end{aligned} \tag{3.9}$$

PROOF: The process $(X_{\mathbf{n}}^2(t))_{t\in I_l}$ is a submartingale, because it is the projection of a sum of squares of independent martingales. We note that the function $f_\gamma:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$f_\gamma(x) = \max\bigl\{\mathrm{e}^{m-1},\mathrm{e}^{\gamma x^{1/m}}\bigr\} = \begin{cases} \mathrm{e}^{m-1} & \text{if } x \leqslant \Bigl(\dfrac{m-1}{\gamma}\Bigr)^m \;, \\ \mathrm{e}^{\gamma x^{1/m}} & \text{if } x > \Bigl(\dfrac{m-1}{\gamma}\Bigr)^m \end{cases}$$

is non-decreasing and convex. By Doob's submartingale inequality, we get

$$\begin{aligned} \mathbb{P}\biggl\{\sup_{t\in I_l} X_{\mathbf{n}}^2(t) > \Bigl(\frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\Bigr)^2\biggr\} &= \mathbb{P}\biggl\{\sup_{t\in I_l} f_\gamma\bigl(X_{\mathbf{n}}^2(t)\bigr) > f_\gamma\Bigl(\Bigl(\frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\Bigr)^2\Bigr)\biggr\} \\ &\leqslant \frac{1}{f_\gamma\Bigl(\bigl(\frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\bigr)^2\Bigr)}\mathbb{E}\Bigl[f_\gamma\bigl(X_{\mathbf{n}}^2(u_{l+1})\bigr)\Bigr] \;. \end{aligned}$$

In the denominator, we bound $f_\gamma(x)$ below by $\mathrm{e}^{\gamma x^{1/m}}$. In the expectation, we bound the maximum defining f_γ above by the sum. Setting $\gamma = \gamma_{q_0,q_{[\mathbf{n}]}}/\sigma^2$ yields the result. □

Remark 3.6. Corollary 6.13 in [19] implies that if X is a polynomial of degree m in the field, then $\mathbb{E}\bigl[\mathrm{e}^{t|X|}\bigr]$ is finite for $m = 2$, and is in general infinite if $m \geqslant 3$. This explains the mth root in Proposition 3.5. ♢

In order to bound the exponential moment in (3.9), we provide the following technical lemma, whose proof is postponed to Appendix C.

Lemma 3.7. *There exists a numerical constant $C_0 \geqslant 0$ such that for any l, one has*

$$\mathbb{E}\big[X_{\mathbf{n}}^2(u_{l+1})\big] \leqslant C_m \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}} \tag{3.10}$$

where $C_m = C_0^m m!$.

The bound (3.10) says that although high frequency modes, of order $2^{q_{[\mathbf{n}]}}$, have some influence on lower modes of order 2^{q_0}, this influence decreases exponentially in their ratio.

Proposition 3.8. *For any $\gamma_{q_0,q_{[\mathbf{n}]}} < \mathrm{e}^{-1}(C_m)^{-1/m} 2^{2(q_{[\mathbf{n}]}-q_0)/m}$, one has*

$$\mathbb{E}\biggl[\exp\biggl\{\frac{\gamma_{q_0,q_{[\mathbf{n}]}}}{\sigma^2}\big[X_{\mathbf{n}}^2(u_{l+1})\big]^{1/m}\biggr\}\biggr] < \frac{1}{1-\gamma_{q_0,q_{[\mathbf{n}]}}\,\mathrm{e}\, C_m^{1/m} 2^{2(q_0-q_{[\mathbf{n}]})/m}}\;.$$

Proof: Expanding the exponential, we get

$$\mathbb{E}\biggl[\exp\biggl\{\frac{\gamma_{q_0,q_{[\mathbf{n}]}}}{\sigma^2}\big[X_{\mathbf{n}}^2(u_{l+1})\big]^{1/m}\biggr\}\biggr] = \sum_{p\geqslant 0}\frac{\gamma_{q_0,q_{[\mathbf{n}]}}^p}{\sigma^{2p}p!}\mathbb{E}\Bigl[\bigl(X_{\mathbf{n}}^2(u_{l+1})\bigr)^{p/m}\Bigr]\;.$$

By Jensen's inequality (or Hölder's inequality with conjugates m and $\frac{m}{m-1}$), we have

$$\mathbb{E}\Bigl[\bigl(X_{\mathbf{n}}^2(u_{l+1})\bigr)^{p/m}\Bigr] \leqslant \mathbb{E}\Bigl[\bigl(X_{\mathbf{n}}^2(u_{l+1})\bigr)^{p}\Bigr]^{1/m}\;.$$

Since $X_{\mathbf{n}}^2(u_{l+1})$ belongs to the $2m$th Wiener chaos, we can use for even p equivalence of norms (see Lemma B.5) to obtain the bound

$$\mathbb{E}\Bigl[\bigl(X_{\mathbf{n}}^2(u_{l+1})\bigr)^{p}\Bigr] \leqslant (p-1)^{mp}\mathbb{E}\big[X_{\mathbf{n}}^2(u_{l+1})\big]^p \leqslant \Bigl[(p-1)^m C_m \sigma^{2m} 2^{2(q_0-q_{[\mathbf{n}]})}\Bigr]^p\;,$$

where we have used Lemma 3.7 in the last inequality. A similar bound follows for odd p by the Cauchy–Schwarz inequality. Combining these bounds, we get

$$\mathbb{E}\biggl[\exp\biggl\{\frac{\gamma_{q_0,q_{[\mathbf{n}]}}}{\sigma^2}\big[X_{\mathbf{n}}^2(u_{l+1})\big]^{1/m}\biggr\}\biggr] \leqslant \sum_{p\geqslant 0}\frac{(p-1)^p}{p!}\Bigl[\gamma_{q_0,q_{[\mathbf{n}]}} C_m^{1/m} 2^{2(q_0-q_{[\mathbf{n}]})/m}\Bigr]^p\;.$$

Stirling's formula yields $p^p/p! \leqslant \mathrm{e}^p$. The result follows by summing a geometric series. □

Choosing $\gamma_{q_0,q_{[\mathbf{n}]}} = (2\,\mathrm{e}\, C_m^{1/m})^{-1} 2^{2(q_{[\mathbf{n}]}-q_0)/m}$, we obtain

$$\begin{aligned} C_{\mathbf{n}}(l,\varepsilon) &\leqslant 2 + \mathrm{e}^{m-1}\;,\\ H_{\mathbf{n}}(q_0,l) &= \frac{1}{2\,\mathrm{e}\, C_m^{1/m}}\biggl(\frac{\mathbf{n}!}{m!}2^{|\alpha|q_0}2^{q_{[\mathbf{n}]}-q_0}h_{\mathbf{n}}^m\biggr)^{2/m}\;. \end{aligned}$$

This motivates the choice

$$h_{\mathbf{n}}^m = \frac{1}{K_m(q_0)}h^m\frac{m!}{\mathbf{n}!}\frac{1}{2^{(q_{[\mathbf{n}]}-q_0)/2}}1_{\{q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}}\geqslant q_0\}}\;,$$

where the indicator is due to Remark 3.3, which yields

$$\mathbb{P}\biggl\{\sup_{t\in I_l}X_{\mathbf{n}}^2(t) > \Bigl(\frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\Bigr)^2\biggr\} \leqslant (2+\mathrm{e}^{m-1})\exp\biggl\{-\frac{h^2}{2\,\mathrm{e}\,\sigma^2}\biggl(\frac{2^{(q_{[\mathbf{n}]}-q_0)/2}2^{|\alpha|q_0}}{K_m(q_0)C_m^{1/2}}\biggr)^{2/m}\biggr\}\;. \tag{3.11}$$

The condition $h^m = \sum_{|\mathbf{n}|=m} h^m_{\mathbf{n}}$ imposes

$$K_m(q_0) = \sum_{\substack{|\mathbf{n}|=m \\ q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}}\geqslant q_0}} \frac{m!}{\mathbf{n}!} \frac{1}{2^{(q_{[\mathbf{n}]}-q_0)/2}} \;. \tag{3.12}$$

The proof of the following bound is postponed to Appendix C.

Lemma 3.9. *There exist numerical constants $c_0, c_1, c_2 > 0$ such that*

$$K_m(q_0) \leqslant c_0 m!(m+c_2)^m (q_0+c_1)^m \;.$$

Substituting in (3.11) yields the bound

$$\mathbb{P}\biggl\{\sup_{t\in I_l} X^2_{\mathbf{n}}(t) > \Bigl(\frac{h^m 2^{|\alpha|q_0}}{K_m(q_0)2^{(q_{[\mathbf{n}]}-q_0)/2}}\Bigr)^2\biggr\} \leqslant (2+\mathrm{e}^{m-1})\exp\biggl\{-\kappa_m \frac{2^{(q_{[\mathbf{n}]}-q_0)/m}2^{2|\alpha|q_0/m}}{(q_0+c_1)^2}\frac{h^2}{\sigma^2}\biggr\} \;, \tag{3.13}$$

where

$$\kappa_m = \frac{1}{2\,\mathrm{e}\, c_0^{2/m}(C_m(m!)^2)^{1/m}(m+c_2)^2} = \mathcal{O}\biggl(\frac{1}{m^5}\biggr) \;.$$

We now have to convert the estimate (3.13) into an estimate involving Wick powers of $\delta_q\psi(t,\cdot)$ instead of $\delta_q\hat\psi(t,\cdot)$. For that, we are going to use the following, rather rough bound. For any $\mathbf{l}\in\mathbb{N}^{\mathbb{N}}$ with finitely many nonzero components, we write

$$|\mathbf{l}| := \sum_{q\geqslant0}\mathbf{l}_q \;, \qquad \mathbf{l}! := \prod_{q\geqslant0}\mathbf{l}_q! \;, \qquad \text{and} \qquad \mathbf{l}\leqslant\lfloor\tfrac{\mathbf{n}}{2}\rfloor \quad\Leftrightarrow\quad \mathbf{l}_q\leqslant\lfloor\tfrac{\mathbf{n}_q}{2}\rfloor\;\forall q\geqslant0 \;.$$

We introduce the shorthands

$$\begin{aligned}
\varphi(t,\cdot) &= \prod_{q\geqslant0} H_{\mathbf{n}_q}(\delta_q\psi(t,\cdot);c_q) \;, \\
\hat\varphi(t,\cdot) &= \prod_{q\geqslant0} H_{\mathbf{n}_q}(\delta_q\hat\psi(t,\cdot);\hat c_q(t)) \;.
\end{aligned}$$

The proof of the following result is postponed to Appendix C.

Proposition 3.10. *There is a numerical constant c_1 such that for all $t\in I_l$, one has*

$$\begin{aligned}
&\bigl\|\delta_{q_0}\bigl(\varphi(t,\cdot)-\hat\varphi(t,\cdot)\bigr)\bigr\|_{L^2} \\
&\qquad\leqslant 2^{q_0}\bigl(c_1\gamma_0\bigr)^{[\mathbf{n}]}\Biggl(\prod_{\substack{q\geqslant0\\ \mathbf{n}_q>0}}\mathbf{n}_q\Biggr)\sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor} A_{\mathbf{n}\mathbf{l}}\hat c(t)^{|\mathbf{l}|}\prod_{q\geqslant0}\bigl\|\delta_q\hat\psi(t,\cdot)\bigr\|_{L^2}^{\mathbf{n}_q-2\mathbf{l}_q} \;,
\end{aligned} \tag{3.14}$$

where

$$\hat c(t) = \sup_{q\geqslant0}\hat c_q(t) \qquad \textit{and} \qquad A_{\mathbf{n}\mathbf{l}} = \frac{\mathbf{n}!}{2^{|\mathbf{l}|}\mathbf{l}!(\mathbf{n}-2\mathbf{l})!}2^{2q_{[\mathbf{n}]}(|\mathbf{n}|-2|\mathbf{l}|)} \;.$$

Note that the first product over q in (3.14) *can be bounded above by $m^{[\mathbf{n}]}$.*

We can now derive the main estimate of this section.

Proposition 3.11. *There is a constant $Q_m = \mathcal{O}(m^{-1/2})$ such that, if one chooses γ_0 of order $q_0^m Q_m 2^{-(m+1)q_{[\mathbf{n}]}}$, there exists a constant $\bar\kappa_m$, comparable to κ_m, such that*

$$\mathbb{P}\biggl\{\sup_{t\in I_\ell}\|\delta_{q_0}\varphi(t,\cdot)\|_{L^2} > \frac{2^{|\alpha|q_0}h^m}{K_m(q_0)2^{(q_{[\mathbf{n}]}-q_0)/2}}\biggr\} \leqslant \bar C_m(\mathbf{n})\exp\biggl\{-\bar\kappa_m\frac{2^{(q_{[\mathbf{n}]}-q_0)/m}2^{2|\alpha|q_0/m}}{(q_0+c_1)^2}\frac{h^2}{\sigma^2}\biggr\}$$

holds for all $h\geqslant\sigma$, where $\bar C_m(\mathbf{n}) = 2+\mathrm{e}^{m-1}+c_0(q_{[\mathbf{n}]}+m)$ for a numerical constant c_0.

Proof: The argument is essentially deterministic. We introduce the two events

$$\Omega_1(\tilde h) = \biggl\{\forall q\leqslant q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}},\ \sup_{t\in I_\ell}\|\delta_q\hat\psi(t,\cdot)\|_{L^2}\leqslant\tilde h\biggr\}\;,$$
$$\Omega_2(h,q_0) = \biggl\{\sup_{t\in I_\ell}\|\delta_{q_0}\hat\varphi(t,\cdot)\|_{L^2}\leqslant\frac12\frac{2^{|\alpha|q_0}h^m}{K_m(q_0)2^{(q_{[\mathbf{n}]}-q_0)/2}}\biggr\}\;.$$

The estimate (3.13) provides an upper bound on $\mathbb{P}(\Omega_2(h,q_0)^c)$. As for $\Omega_1(\tilde h)$, the bound

$$\mathbb{P}\bigl(\Omega_1(\tilde h)^c\bigr)\leqslant c_0\bigl(q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}}\bigr)\,\mathrm{e}^{-\kappa_0\tilde h^2/\sigma^2}$$

follows as a particular case of (3.13), applied separately to all $\mathbf{n}$ of size $|\mathbf{n}|=1$. We now choose $\tilde h$ is such a way that

$$\kappa_0\tilde h^2 = \kappa_m\frac{2^{(q_{[\mathbf{n}]}-q_0)/m}2^{2|\alpha|q_0/m}}{(q_0+c_1)^2}h^2\;,$$

so that $\mathbb{P}(\Omega_1(\tilde h)^c)$ and $\mathbb{P}(\Omega_2(h,q_0)^c)$ are of comparable size. This allows us to bound the quantity

$$\sup_{t\in I_\ell}\|\delta_{q_0}\varphi(t,\cdot)\|_{L^2}\leqslant\sup_{t\in I_\ell}\|\delta_{q_0}\hat\varphi(t,\cdot)\|_{L^2}+\sup_{t\in I_\ell}\|\delta_{q_0}\varphi(t,\cdot)-\delta_{q_0}\hat\varphi(t,\cdot)\|_{L^2}$$

on $\Omega_1(\tilde h)\cap\Omega_2(h,q_0)$. By Proposition 3.10, we have

$$\sup_{t\in I_\ell}\|\delta_{q_0}\varphi(t,\cdot)-\delta_{q_0}\hat\varphi(t,\cdot)\|_{L^2}\leqslant 2^{q_0}(c_1\gamma_0 m)^{[\mathbf{n}]}\sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor}\frac{\mathbf{n}!}{\mathbf{l}!(\mathbf{n}-2\mathbf{l})!}\biggl(\frac{\hat c(t)}{2}\biggr)^{|\mathbf{l}|}\bigl(2^{2q_{[\mathbf{n}]}}\tilde h\bigr)^{|\mathbf{n}|-2|\mathbf{l}|}\;.$$

Note the relation

$$\begin{aligned}\sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor}\frac{\mathbf{n}!}{\mathbf{l}!(\mathbf{n}-2\mathbf{l})!}a^{\mathbf{l}}b^{\mathbf{n}-2\mathbf{l}} &= \prod_{q\geqslant0}\sum_{\mathbf{l}_q\leqslant\lfloor\mathbf{n}_q/2\rfloor}\frac{\mathbf{n}_q!}{\mathbf{l}_q!(\mathbf{n}_q-2\mathbf{l}_q)!}a^{\mathbf{l}_q}b^{\mathbf{n}_q-2\mathbf{l}_q}\\ &\leqslant\prod_{q\geqslant0}\frac{\mathbf{n}_q!}{(\mathbf{n}_q/2)!}\bigl(\sqrt a+b\bigr)^{\mathbf{n}_q}\leqslant 2^{|\mathbf{n}|}\bigl(\sqrt a+b\bigr)^{|\mathbf{n}|}\;.\end{aligned}$$

Since $|\mathbf{n}|=m$, it follows that

$$\sup_{t\in I_\ell}\|\delta_{q_0}\varphi(t,\cdot)\|_{L^2}\leqslant\frac12\frac{2^{|\alpha|q_0}h^m}{K_m(q_0)2^{(q_{[\mathbf{n}]}-q_0)/2}}+2^{q_0}(c_1\gamma_0 m)^m 2^m\bigl(\hat c(t)+2^{2q_{[\mathbf{n}]}}\tilde h\bigr)^m$$

holds on $\Omega_1(\tilde h)\cap\Omega_2(h,q_0)$. Choosing γ_0 such that both summands are equal yields the result. □

Corollary 3.12. *We have*

$$\mathbb{P}\Big\{\sup_{t\in I}\big\|\delta_{q_0}(\prod_{q\geqslant 0}:\delta_q\psi(t,\cdot)^{\mathbf{n}_q}:)\big\|_{L^2} > \frac{\mathbf{n}!}{m!}h_{\mathbf{n}}^m 2^{|\alpha|q_0}\Big\}$$
$$\leqslant \frac{T}{\varepsilon}\tilde{C}_m(\mathbf{n})q_0^{-m}2^{(m+3)q_{[\mathbf{n}]}}\exp\Big\{-\bar{\kappa}_m\frac{2^{(q_{[\mathbf{n}]}-q_0)/m}2^{2|\alpha|q_0/m}}{(q_0+c_1)^2}\frac{h^2}{\sigma^2}\Big\}\;,$$

where $\tilde{C}_m(\mathbf{n}) = Q_m^{-1}\bar{C}_m(\mathbf{n})$.

PROOF: It suffices to sum the previous estimate over all $\ell\in\{1,\dots,L\}$, where L has been introduced in (3.6). □

3.3 Summing over q_0 and $\mathbf{n}$

Replacing the bound obtained in Corollary 3.12 in (3.4), we get

$$P_m(h)\leqslant\frac{T}{\varepsilon}\sum_{q_0\geqslant 0}q_0^{-m}\sum_{|\mathbf{n}|=m}\tilde{C}_m(\mathbf{n})2^{(m+3)q_{[\mathbf{n}]}}\exp\Big\{-\beta(m,q_0)2^{(q_{[\mathbf{n}]}-q_0)/m}\Big\}\;,$$

where

$$\beta(m,q_0) = \bar{\kappa}_m\frac{2^{2|\alpha|q_0/m}}{(q_0+c_1)^2}\frac{h^2}{\sigma^2}\;.$$

We will first perform the sum over $\mathbf{n}$. To this end, we write

$$\bar{K}_{m,b}(q_0) = \sum_{|\mathbf{n}|=m}q_{[\mathbf{n}]}^b 2^{(m+3)q_{[\mathbf{n}]}}\exp\Big\{-\beta(m,q_0)2^{(q_{[\mathbf{n}]}-q_0)/m}\Big\}\;,$$

The following lemma is obtained in a similar way as Lemma 3.9. We give its proof in Appendix C.

Lemma 3.13. *There are numerical constants c_1,β_0 such that for all $\beta(m,q_0)\geqslant\beta_0$, one has the bound*

$$\bar{K}_{m,b}(q_0)\leqslant c_1 q_0^{m+b}m^m 2^{(m+3)q_0}\,\mathrm{e}^{-\beta(m,q_0)}\;.$$

Using the expression for $\bar{C}_m(\mathbf{n})$ given in Proposition 3.11, we thus obtain

$$P_m(h)\leqslant\frac{T}{\varepsilon}Q_m^{-1}\sum_{q_0\geqslant 0}q_0^{-m}\Big[(2+\mathrm{e}^{m-1}+c_0m)\bar{K}_{m,0}(q_0)+\bar{K}_{m,1}(q_0)\Big]$$
$$\leqslant\frac{T}{\varepsilon}c_1m^mQ_m^{-1}\sum_{q_0\geqslant 0}\Big[2+\mathrm{e}^{m-1}+c_0m+c_0q_0\Big]2^{(m+3)q_0}\,\mathrm{e}^{-\beta(m,q_0)}\;. \tag{3.15}$$

It remains to perform the sums over q_0. These are of the form

$$\sum_{q_0\geqslant 0}f(q_0)\;,\qquad f(x)=x^b2^{ax}\exp\Big\{-\gamma\frac{2^{|\alpha|x/m}}{(x+c_1)^2}\Big\}\;,\qquad a=m+3\;,\qquad \gamma=\bar{\kappa}_m\frac{h^2}{\sigma^2}\;.$$

One checks that f is decreasing, so that one has the upper bound

$$\sum_{q_0\geqslant 0}f(q_0)\leqslant f(0)+f(1)+\int_1^\infty f(x)\,\mathrm{d}x\;.$$

The terms $f(0)$ and $f(1)$ are both exponentially small in h^2/σ^2. To evaluate the integral, we can absorb the constant c_1 in γ, and the term x^b into 2^{ax}, by changing slightly the definitions of γ and a. We first consider the case where the term x^2 in the denominator is absent, where the changes of variables $y = 2^{|\alpha|x/m}$ and $z = \gamma y$ yield

$$\begin{aligned}\int_1^\infty 2^{ax}\,\mathrm{e}^{-\gamma 2^{|\alpha|x/m}}\,\mathrm{d}x &= \frac{m}{|\alpha|\log 2}\int_{2^{|\alpha|/m}}^\infty y^{am/|\alpha|-1}\,\mathrm{e}^{-\gamma y}\,\mathrm{d}y \\ &\leqslant \frac{m}{|\alpha|\log 2}\frac{1}{\gamma^{\lambda+1}}\int_\gamma^\infty z^\lambda\,\mathrm{e}^{-z}\,\mathrm{d}z\;,\end{aligned} \tag{3.16}$$

with $\lambda = am/(2|\alpha|)$. The asymptotics of the incomplete Gamma function shows that

$$\int_1^\infty 2^{ax}\,\mathrm{e}^{-\gamma 2^{|\alpha|x/m}}\,\mathrm{d}x \lesssim \frac{m}{|\alpha|}\frac{\mathrm{e}^{-\gamma}}{\gamma} \lesssim \frac{m}{|\alpha|}\,\mathrm{e}^{-\gamma/2}\;.$$

In order to incorporate the effect of the denominator x^2, we use the upper bound

$$x^2 2^{-|\alpha|x/m} \leqslant \frac{4\,\mathrm{e}^{-2}}{(\log 2)^2}\frac{m^2}{|\alpha|^2}\;.$$

We can thus bound the integral of f by the integral (3.16), with γ multiplied by a constant times $|\alpha|^2/m^2$, and α divided by 2. In other words, we get

$$\int_1^\infty f(x)\,\mathrm{d}x \leqslant c\frac{m}{|\alpha|}\,\mathrm{e}^{-\kappa|\alpha|^2\gamma/m^2}\;.$$

Replacing this in (3.15) yields a bound on $P_m(h)$, completing the proof of Theorem 2.4 in the case of a constant linearisation $a(t)$.

3.4 The case of a general linearisation $a(t)$

Recall that we actually want to consider the more general linear equation (2.11)

$$\mathrm{d}\tilde\psi(t,x) = \frac{1}{\varepsilon}\big[\Delta\tilde\psi(t,x) + a(t)\tilde\psi(t,x)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)\;,$$

where here $a(t)$ satisfies (2.10), and we write $\psi = \tilde\psi$ to avoid confusion in the notations. Projecting (3.17) on the kth basis vector e_k, we obtain

$$\mathrm{d}\tilde\psi_k(t) = \frac{1}{\varepsilon}a_k(t)\tilde\psi_k(t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_k(t)\;, \tag{3.17}$$

where $a_k(t) = -\mu_k + a(t)$ and the $\{W_k(t)\}_{t\geqslant 0}$ are the same independent Wiener processes as before. The solution of (3.17) with the same initial condition $\psi_k(0)$ as in (3.2), is given by

$$\tilde\psi_k(t) = \mathrm{e}^{\tilde\alpha_k(t)/\varepsilon}\,\psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{\tilde\alpha_k(t,t_1)/\varepsilon}\,\mathrm{d}W_k(t_1)\;,$$

where

$$\tilde\alpha_k(t,t_1) = \alpha_k(t,t_1) + \int_{t_1}^t (1 + a(t_2))\,\mathrm{d}t_2 = \alpha_k(t,t_1) + \mathcal{O}(|t-t_1|)\;.$$

For given q_0, we use the same partition of $[0,T]$ into intervals I_l as before. On each interval, we can write

$$\hat{\psi}_k(t) := \mathrm{e}^{\tilde{\alpha}_k(u_{l+1},t)/\varepsilon}\,\tilde{\psi}_k(t) = \mathrm{e}^{\tilde{\alpha}_k(u_{l+1})/\varepsilon}\,\psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{\tilde{\alpha}_k(u_{l+1},t_1)/\varepsilon}\,\mathrm{d}W_k(t_1)\;.$$

$\hat{\psi}_k(t)$ is again a martingale, so that its supremum over the interval I_l can be estimated as before. Note however that the variance of the associated sums over $k \in \mathcal{A}_q$ is not exactly equal to $\hat{c}_q(t)$: it is rather of the form

$$\hat{V}_q(t) = \hat{c}_q(t)\bigl[1 + \mathcal{O}(\gamma_0 2^{-2(q_{[\mathbf{n}]}-q)})\bigr]\;.$$

Therefore, the Wick powers of this martingale with respect to $\hat{c}_q(t)$ are not martingales. One can however estimate the supremum of the Wick powers with the correct variance as before, and then compare the two types of Wick powers. In fact, we want to bound the supremum over $t \in I_l$ of

$$\bigl\|\delta_{q_0}\bigl(\prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\tilde{\psi}(t,\cdot),c_q)\bigr)\bigr\|_{L^2}\;.$$

By the binomial formula introduced in Lemma B.2, we obtain

$$\begin{aligned}
&\prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\tilde{\psi}(t,x),c_q)\\
&\quad= \prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\hat{\psi}(t,x) + \delta_q\tilde{\psi}(t,x) - \delta_q\hat{\psi}(t,x), \hat{V}_q(t) + c_q - \hat{V}_q(t))\\
&\quad= \sum_{0\leqslant|\mathbf{l}|\leqslant|\mathbf{n}|}\binom{\mathbf{n}}{\mathbf{l}}\prod_{q\geqslant 0} H_{\mathbf{l}_q}(\delta_q\tilde{\psi}(t,x) - \delta_q\hat{\psi}(t,x), c_q - \hat{V}_q(t)) H_{\mathbf{n}_q-\mathbf{l}_q}(\delta_q\hat{\psi}(t,x),\hat{V}_q(t))\\
&\quad= \prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\hat{\psi}(t,x),\hat{V}_q(t))\\
&\qquad+ \sum_{1\leqslant|\mathbf{l}|\leqslant|\mathbf{n}|}\binom{\mathbf{n}}{\mathbf{l}}\prod_{q\geqslant 0} H_{\mathbf{l}_q}(\delta_q\tilde{\psi}(t,x) - \delta_q\hat{\psi}(t,x), c_q - \hat{V}_q(t)) H_{\mathbf{n}_q-\mathbf{l}_q}(\delta_q\hat{\psi}(t,x),\hat{V}_q(t))\;.
\end{aligned}$$

Therefore, by the triangle inequality we get

$$\begin{aligned}
&\bigl\|\delta_{q_0}\bigl(\prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\tilde{\psi}(t,\cdot),c_q)\bigr)\bigr\|_{L^2}\\
&\quad\leqslant \bigl\|\delta_{q_0}\bigl(\prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\hat{\psi}(t,\cdot),\hat{V}_q(t))\bigr)\bigr\|_{L^2}\\
&\qquad+ \sum_{1\leqslant|\mathbf{l}|\leqslant|\mathbf{n}|}\binom{\mathbf{n}}{\mathbf{l}} 2^{q_0}\bigl\|\prod_{q\geqslant 0} H_{\mathbf{l}_q}(\delta_q\tilde{\psi}(t,\cdot) - \delta_q\hat{\psi}(t,\cdot), c_q - \hat{V}_q(t))\bigr\|_{L^2}\\
&\qquad\quad\times \bigl\|\prod_{q\geqslant 0} H_{\mathbf{n}_q-\mathbf{l}_q}(\delta_q\hat{\psi}(t,\cdot),\hat{V}_q(t))\bigr\|_{L^2}\;,
\end{aligned}$$

where the last inequality is a rough bound, obtained by Cauchy–Schwarz's inequality. The first norm can be bounded as in (3.13), and one can see the last norm as a particular case of it. In the same spirit as in the proof of Proposition 3.10, for $|\mathbf{l}| \geqslant 1$ one can bound

$$\bigl\|\prod_{q\geqslant 0} H_{\mathbf{l}_q}(\delta_q\tilde{\psi}(t) - \delta_q\hat{\psi}(t), c_q - \hat{V}_q(t))\bigr\|_{L^2} \leqslant \bigl(c_1\gamma_0\bigr)^{|\mathbf{l}|/2}\sum_{\mathbf{p}:\mathbf{p}\leqslant\lfloor\mathbf{l}/2\rfloor} A_{\mathbf{l}\mathbf{p}}\hat{c}(t)^{|\mathbf{p}|}\prod_{q\geqslant 0}\bigl\|\delta_q\hat{\psi}(t,\cdot)\bigr\|_{L^2}^{\mathbf{l}_q-2\mathbf{p}_q}\;,$$

with

$$\hat{c}(t) = \sup_{q\geqslant 0} \hat{c}_q(t) \qquad \text{and} \qquad A_{\mathbf{lp}} = \frac{\mathbf{l}!}{2^{|\mathbf{p}|}\mathbf{p}!(\mathbf{l}-2\mathbf{p})!} 2^{2q_{[\mathbf{n}]}(|\mathbf{l}|-2|\mathbf{p}|)} .$$

Notice that

$$\left|c_q - \hat{V}_q(t)\right| = \left|\hat{c}_q(t)\right|\left|\mathrm{e}^{-2\alpha_{k(q)}(u_{l+1},t)/\varepsilon} - 1 - \mathcal{O}(\gamma_0 2^{-2(q_{[\mathbf{n}]}-q)})\right| \leqslant c_0\gamma_0.$$

Choosing γ_0 small enough in the sense taken in Proposition 3.11 yields the result.

Finally, similar results holds for processes $\tilde{\psi}(t,x)$ with another initial condition. In particular, when it is equal to 0, one can proceed exactly in the same way and bound the variance of the associated martingale by $\hat{V}_q(t)$.

4 Proof of the concentration results

4.1 Proof of Proposition 2.8

The proof of Proposition 2.8 is almost the same as the proof of Proposition 2.3 in [7], the only difference being that x belongs to the two-dimensional torus. We give here only a few hints of the proof. Recall that the proposition concerns the deterministic equation

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\left[\Delta\phi(t,x) + F(t,\phi(t,x))\right] \mathrm{d}t ,$$

where F satisfies (2.2) and Assumption 2.7. We consider the difference $\psi(t) = \phi(t) - \phi^*(t)e_0$. Using Taylor's formula to expand $F(t,\psi(t)+\phi^*(t)e_0)$, we obtain that ψ satisfies the equation

$$\varepsilon\partial_t\psi(t,x) = \Delta\psi(t,x) + a(t)\psi(t,x) + b(t,\psi(t,x)) - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\phi^*(t)e_0(x) ,$$

where

$$\begin{aligned} a(t) &= \partial_\phi F(t,\phi^*(t)e_0) , \\ b(t,\psi) &= \frac{1}{2}\partial_\phi^2 F\big(t,\phi^*(t)+\theta\psi\big)\psi^2 \qquad \text{for some } \theta\in[0,1] . \end{aligned}$$

We define the Lyapounov function

$$V(\psi) = \frac{1}{2}\left\|\psi\right\|_{H^1}^2 = \frac{1}{2}\left\|\psi\right\|_{L^2}^2 + \frac{L^2}{2\pi^2}\left\|\nabla\psi\right\|_{L^2}^2 .$$

Its time derivative satisfies

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}V(\psi) \leqslant 2a(t)V(\psi) + \langle\psi,b(t,\psi)\rangle - \frac{L^2}{\pi^2}\langle\Delta\psi,b(t,\psi)\rangle - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\phi^*(t)\langle\psi,e_0\rangle . \tag{4.1}$$

We introduce for a fixed $C_0>0$, $\bar{\tau}$, the first exist-time from the set $\big\{V(\psi(t,\cdot))\leqslant C_0\big\}$. Then, we bound the different terms in (4.1) similarly to the proof in [7]. We arrive at the relation

$$\varepsilon\dot{V} \leqslant -\frac{1}{2}C_1V + \varepsilon C_2V^{1/2}$$

for all $t\leqslant\bar{\tau}$, and some constants $C_1,C_2>0$. Using Gronwall's inequality, we find that there exists a particular solution satisfying $V(\psi(t,\cdot)) = \mathcal{O}(\varepsilon^2)$ for all $t\leqslant\bar{\tau}$. The result extends for all $t\in I$.

4.2 Proof of Lemma 2.9

The binomial formula for Hermite polynomials yields

$$:\phi(t,x)^i: \;=\sum_{j=0}^{i}\binom{i}{j}\bar\phi(t,x)^{i-j}:\phi_0(t,x)^j:\;.$$

Using the definition (2.2) of F and swapping the sums, we obtain

$$:F_0(t,x,\phi_0(t,x)):\;=\sum_{j=1}^{n}\biggl[\sum_{i=j}^{n}\binom{i}{j}A_i(t)\bar\phi(t,x)^{i-j}\biggr]:\phi_0(t,x)^j:$$

(note that the terms $j=0$ cancel). This proves the claim for the terms with $j\geqslant 2$. For $j=1$, we note that

$$a(t)=\partial_\phi F(t,\phi^*(t))=\sum_{i=1}^{n}iA_i(t)\phi^*(t)^{i-1}\;.$$

Rearranging terms yields the claimed result. Proposition 2.8 shows that $\bar\phi(t,\cdot)\in H^1$, and that $\|\hat A_1(t,\cdot)\|_{H^1}=\mathcal{O}(\varepsilon)$. By [8, Théorème 7], powers of $\bar\phi$ belong to H^1 as well. □

4.3 Proof of Theorem 2.11

Recall that $\phi_1(t,x)$ solves the equation

$$\mathrm{d}\phi_1(t,x)=\frac{1}{\varepsilon}\bigl[\Delta\phi_1(t,x)+a(t)\phi_1(t,x)+\;:b(t,x,\psi(t,x)+\phi_1(t,x)):\bigr]\,\mathrm{d}t\;,$$

where $\psi(t,x)$ is the stochastic convolution, and

$$:b(t,x,\psi(t,x)+\phi_1(t,x)):\;=\sum_{j=1}^{n}\hat A_j(t,x)\sum_{\ell=0}^{j}\binom{j}{\ell}\phi_1(t,x)^{j-\ell}:\psi(t,x)^\ell:\;. \tag{4.2}$$

Assume that $\psi(t,\cdot)\in\mathcal{B}^\alpha_{2,\infty}$ and $\phi_1(t,\cdot)\in\mathcal{B}^\beta_{2,\infty}$ for all $t\in[0,T]$. The bound (2.9) on products in Besov spaces shows that

$$\|\phi_1(t,\cdot)^{j-\ell}:\psi(t,\cdot)^\ell:\|_{\mathcal{B}^{(2(j-\ell)+1)\alpha}_{2,\infty}}\leqslant M_1\,\|\phi_1(t,\cdot)\|^{j-\ell}_{\mathcal{B}^\beta_{2,\infty}}\,\|:\psi(t,\cdot)^\ell:\|_{\mathcal{B}^\alpha_{2,\infty}}$$

for a constant $M_1=M_1(\beta,\alpha,n)$, provided $\beta\geqslant 1+2\alpha$.

We treat separately the term $j=1$ in the sum (4.2) and the remaining terms. For $j=1$, we use the fact that $\hat A_1(t,\cdot)\in\mathcal{B}^1_{2,\infty}$ has a norm of order ε and (2.8) to obtain that

$$\|\hat A_1(t,\cdot)(\phi_1(t,\cdot)+\psi(t,\cdot))\|_{\mathcal{B}^{\bar\alpha}_{2,\infty}}\leqslant M_2\varepsilon\Bigl(\|\phi_1(t,\cdot)\|_{\mathcal{B}^\beta_{2,\infty}}+\|\psi(t,\cdot)\|_{\mathcal{B}^\alpha_{2,\infty}}\Bigr)$$

for any $\bar\alpha<\alpha$. For $j\geqslant 2$, we have a similar bound, but without the factor ε, since the $\hat A_j$ are of order 1 in H^1. Now let $h,H\in(0,1]$ be constants such that

$$\max_{1\leqslant\ell\leqslant n}\|:\psi(t,\cdot)^\ell:\|_{\mathcal{B}^\alpha_{2,\infty}}\leqslant h\;,\qquad \|\phi_1(t,\cdot)\|_{\mathcal{B}^\beta_{2,\infty}}\leqslant H\;.$$

Summing over j, since $h, H \leqslant 1$ we get the existence of constants M_3, M_4 such that

$$\begin{aligned}\|{:}\,b(t,x,\psi+\phi_1)\,{:}\|_{\mathcal{B}^{\bar\alpha}_{2,\infty}} &\leqslant M_2\varepsilon(H+h) + M_3\biggl[H^2 + \sum_{j=2}^{n}\sum_{\ell=1}^{j} H^{j-\ell}h^\ell\biggr] \\ &\leqslant M_4(H+h)(H+h+\varepsilon)\end{aligned}$$

for any $\bar\alpha < (2n-1)\alpha$. We now fix a $\gamma < \bar\alpha + 2$ and introduce the stopping time

$$\tau = \inf\Bigl\{t\in[0,T]\colon\ \|\phi_1(t,\cdot)\|_{\mathcal{B}^{\gamma}_{2,\infty}} > H\Bigr\}\;.$$

Then we have

$$\begin{aligned}\mathbb{P}\{\tau<T\} \leqslant \mathbb{P}\biggl\{&\exists\ell\in\{1,\dots,n\}\colon\ \sup_{t\leqslant T}\|{:}\,\psi(t,\cdot)^\ell\,{:}\|_{\mathcal{B}^{\alpha}_{2,\infty}} > h^\ell\biggr\} \\ &+ \mathbb{P}\biggl\{\tau<T,\ \sup_{t\leqslant T}\|{:}\,\psi(t,\cdot)^\ell\,{:}\|_{\mathcal{B}^{\alpha}_{2,\infty}} \leqslant h^\ell\ \forall\ell\in\{1,\dots,n\}\biggr\}\;. \qquad (4.3)\end{aligned}$$

The first term on the right-hand side can be bounded using Theorem 2.4. As for the second term, we use the fact that under the condition on the Wick powers of the stochastic convolution being small, the Schauder estimate given in Corollary A.3 yields

$$\|\phi_1(T\wedge\tau,\cdot)\|_{\mathcal{B}^{\gamma}_{2,\infty}} \leqslant M\varepsilon^{-\nu}(H+h)(H+h+\varepsilon)\;, \qquad \nu = 1-\frac{\gamma-\bar\alpha}{2}$$

for a constant M. Choosing first $H = 2M\varepsilon^{-\nu}h(h+\varepsilon)$, and then ε small enough and $h < h_0\varepsilon^\nu$ for a sufficiently small h_0, one can ensure that $(H+h)(H+h+\varepsilon) < 2h(h+\varepsilon)$, so that the second probability is actually equal to zero.

To conclude the proof, we first pick a $\gamma < 2$, and then $\bar\alpha \in (\gamma-2, 0)$, and finally $\alpha \in (\frac{\bar\alpha}{2n-1}, 0)$. We also require that $\beta \leqslant \gamma$, which is possible by choosing $\beta = 1+2\alpha$ and asking that $\alpha \leqslant -\frac12(1-\gamma)$. This yields the claimed result, thanks to the embedding $\mathcal{B}^{\gamma}_{2,\infty} \hookrightarrow \mathcal{B}^{\gamma-1}_{\infty,\infty} = \mathcal{C}^{\gamma-1}$. □

4.4 Proof of Theorem 2.12

The proof is very similar to the proof of Theorem 2.11, so we only comment on the differences. Given $\alpha < 0$ and $H_\perp > 0$, we introduce stopping times

$$\begin{aligned}\tau_\psi(h) &= \inf\Bigl\{t\in[0,T]\colon\ \max_{1\leqslant\ell\leqslant3}\|{:}\,\psi_\perp(t,\cdot)^\ell\,{:}\|_{\mathcal{B}^{\alpha}_{2,\infty}} > h\Bigr\}\;,\\ \tau_\perp(H_\perp) &= \inf\Bigl\{t\in[0,T]\colon\ \|\phi_1^\perp(t,\cdot)\|_{\mathcal{B}^{\gamma}_{2,\infty}} > H_\perp\Bigr\}\;.\end{aligned}$$

For any $\bar\alpha < 5\alpha$, writing $\tau = \tau_\psi(h)\wedge\tau_\perp(H_\perp)\wedge\tau_0(H_0)$, one obtains the existence of a constant M such that, for any $t\leqslant\tau$, one has

$$\|{:}\,F_\perp(\psi_\perp(t,\cdot),\phi_1^0(t,\cdot),\phi_1^\perp(t,\cdot))\,{:}\|_{\mathcal{B}^{\bar\alpha}_{2,\infty}} \leqslant M(h+H_0+H_\perp)^3\;.$$

Using Duhamel's formula to write the solution of (2.18) in integral form, and the Schauder estimate in Corollary A.3 (adapted to the eigenvalues of the new linear part), one obtains

$$\|\phi_1^\perp(t\wedge\tau)\|_{\mathcal{B}^{\gamma}_{2,\infty}} \leqslant M_1\varepsilon^{-\nu}(h+H_0+H_\perp)^3$$

for $\nu < 1-\frac12(\gamma-\bar\alpha)$ and a constant $M_1(\nu) > 0$, provided $1+2\alpha \geqslant \gamma$. Then it suffices to decompose the probability as in (4.3). Choosing $H_\perp = 2M_1\varepsilon^{-\nu}(h+H_0)^3$ and $h+H_0$ of order $\varepsilon^{\nu/2}$ yields the result. □

4.5 Proof of Theorem 2.14

The proof is essentially the same as the proof of [3, Theorem 2.10] and [3, Proposition 4.7], except that one has to account for the effect of the extra term $F_0(\psi_\perp, \phi_1^0, \phi_1^\perp)$ in the equation. The solution of (2.17) admits the integral representation

$$\phi_1^0(t) = \phi^\circ(t) + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,t_1)/\varepsilon}\bigl[-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))\bigr]\,\mathrm{d}t_1 \;, \tag{4.4}$$

where ϕ° is the solution of the linear equation (2.19). Proposition 4.3 in [3] provides a similar estimate as (2.20) for ϕ°. Furthermore, up to time $\tau_{\mathcal{B}_-(h_-)} \wedge \tau_\psi(h) \wedge \tau_\perp(H_\perp)$, we have the bound

$$\bigl|-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))\bigr| \leqslant M\biggl(\frac{h_-}{\varepsilon^{1/4}} + h + H_\perp\biggr)^3$$

for a constant M. The integral over t_1 in (4.4) yields an extra factor $1/\sqrt{\varepsilon}$. This allows to bound the supremum of $\phi_1^0(t)$ in terms of the supremum of $\phi^\circ(t)$ on the event

$$\Omega_{h,H_\perp} = \bigl\{\tau_\psi(h) \wedge \tau_\perp(H_\perp) > \tau_{\mathcal{B}_-(h_-)}\bigr\} \;.$$

The probability of the complement $\Omega_{h,H_\perp}^c$ can be estimated by Theorems 2.4 and 2.12. Choosing $H_0 = h_-\varepsilon^{-1/4}$, $h = h_-$ and $H_\perp = \varepsilon^{-\nu}(h + H_0)^3$, one finds that $\mathbb{P}(\Omega_{h,H_\perp}^c)$ is negligible with respect to the probability of ϕ° leaving $\mathcal{B}_-(h_-[1 - \mathcal{O}(h_-^2/\varepsilon)])$, which proves (2.20).

To prove (2.21), we use the fact that for $t \leqslant \tau_{\mathcal{B}_+(h_+)} \wedge \tau_\psi(h) \wedge \tau_\perp(H_\perp)$, one has

$$\begin{aligned}\bigl|-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))\bigr| &\leqslant M_1\biggl(\frac{h_+}{a(t_1)^{1/2}} + h + H_\perp\biggr)^3 \\ &\leqslant M_2\biggl(\frac{h_+}{\varepsilon^{1/4}} + h + H_\perp\biggr)^3\end{aligned}$$

for constants M_1, M_2. This motivates the choice $h = H_\perp = \varepsilon^{-1/4}h_+$. Proceeding as in the proof of [3, Proposition 4.7], one obtains

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}_+(h_+)} > t, \tau_\psi(h) \vee \tau_\perp(H_\perp) \geqslant t\bigr\} \leqslant \frac{h_+}{\sigma}\exp\biggl\{-\kappa\frac{\alpha(t,t^*)}{\varepsilon}\biggr\} \;.$$

The second term on the right-hand side of (2.21) bounds $\mathbb{P}\{\tau_\psi(h) \vee \tau_\perp(H_\perp) < t\}$. □

A Besov spaces

Lemma A.1. *Let $\eta : \mathbb{T}^2 \to \mathbb{R}$ be a compactly supported function of class $\mathcal{C}^1$, with $\|\eta\|_{\mathcal{C}^1} = 1$. For any $p \in [2,\infty]$ and any $\rho \in (0,1]$, let*

$$\eta_\rho^{(p)}(x) = \frac{1}{\rho^{2(1-1/p)}}\eta\biggl(\frac{x}{\rho}\biggr) \;.$$

Then $\|\eta_\rho^{(p)}\|_{L^r} = \|\eta\|_{L^r}$ for all $\rho \in (0,1]$, where $r = (1-1/p)^{-1}$ is the Hölder conjugate of p. Moreover, for any $\psi \in \mathcal{B}_{p,\infty}^\alpha$ with $\alpha \in (-1,0)$, and any $q \in \mathbb{N}_0$, one has

$$\bigl|\langle\psi, \eta_{2^{-q_0}}^{(p)}\rangle\bigr| \lesssim 2^{|\alpha|q_0}\,\|\psi\|_{\mathcal{B}_{p,\infty}^\alpha} \;.$$

PROOF: We have

$$\|\eta_\rho^{(p)}\|_{L^r}^r = \frac{1}{\rho^2}\int_{\mathbb{T}^2} \eta\biggl(\frac{x}{\rho}\biggr)^r \mathrm{d}x = \int_{\mathbb{T}^2} \eta(y)^r \,\mathrm{d}y = \|\eta\|_{L^r}^r \;\;,$$

where we have used the change of variables $x = \rho y$, and the fact that the integration domain does not change because η is compactly supported. For the same reason, we have

$$\bigl|\langle e_k, \eta_\rho^{(p)}\rangle\bigr| = \rho^{2/p}\biggl|\int_{\mathbb{T}^2} \mathrm{e}^{-\mathrm{i}\,\rho k\cdot y}\,\eta(y)\,\mathrm{d}y\biggr| \leqslant \frac{\rho^{2/p}}{1\vee\rho^2|k_1k_2|}\;,$$

where we have used one integration by parts if $\rho^2|k_1k_2| \leqslant 1$. In particular, for $k \in \mathcal{A}_q$ and $\rho = 2^{-q_0}$, this yields

$$\bigl|\langle e_k, \eta_{2^{-q_0}}^{(p)}\rangle\bigr| \leqslant \frac{2^{-2q_0/p}}{1\vee 2^{2(q-q_0)}}\;.$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
\bigl|\langle \delta_q\psi, \eta_{2^{-q_0}}^{(p)}\rangle\bigr| &= \bigl|\langle \delta_q\psi, \delta_q\eta_{2^{-q_0}}^{(p)}\rangle\bigr| \leqslant \|\delta_q\psi\|_{L^p}\,\|\delta_q\eta_{2^{-q_0}}^{(p)}\|_{L^r} \\
&\leqslant \|\delta_q\psi\|_{L^p}\Biggl(\sum_{k\in\mathcal{A}_q}\bigl|\langle e_k, \eta_{2^{-q_0}}^{(p)}\rangle\bigr|^p\Biggr)^{1/p} \\
&\leqslant 2^{|\alpha|q}\,\|\psi\|_{\mathcal{B}_{p,\infty}^\alpha}\,\frac{2^{2(q-q_0)/p}}{1\vee 2^{2(q-q_0)}}\;.
\end{aligned}$$

The result then follows by summing over all $q \in \mathbb{N}_0$, noticing that this sum is dominated by the term $q = q_0$. □

Proposition A.2 (Schauder estimate on the heat kernel)**.** *Let $g \in \mathcal{B}_{2,\infty}^\alpha$ for some $\alpha \in \mathbb{R}$, and let $\mathrm{e}^{t\Delta}$ denote the heat kernel. Then there exists a constant M_0 depending on $\beta - \alpha$ such that*

$$\bigl\|\mathrm{e}^{t\Delta} g\bigr\|_{\mathcal{B}_{2,\infty}^\beta} \leqslant M_0 t^{-\frac{\beta-\alpha}{2}}\,\|g\|_{\mathcal{B}_{2,\infty}^\alpha}$$

holds for all $t \geqslant 0$ and all $\beta \leqslant \alpha + 2$.

PROOF: Denoting by μ_k the eigenvalues of the Laplacian (cf. (2.4)) and by $(g_k)_{k\in\mathbb{Z}^2}$ the Fourier coefficients of g, there is a constant $c > 0$ such that

$$\bigl\|\delta_q(\mathrm{e}^{t\Delta} g)\bigr\|_{L^2}^2 = \sum_{k\in\mathcal{A}_q} \mathrm{e}^{-2\mu_k t}\,|g_k|^2 \leqslant \mathrm{e}^{-c2^{2q}t}\,\|\delta_q g\|_{L^2}^2 \leqslant \mathrm{e}^{-c2^{2q}t}\,2^{-2q\alpha}\,\|g\|_{\mathcal{B}_{2,\infty}^\alpha}^2$$

for all q. Therefore,

$$\bigl\|\mathrm{e}^{t\Delta} g\bigr\|_{\mathcal{B}_{2,\infty}^\beta} \leqslant \sup_{q\geqslant 0} 2^{q(\beta-\alpha)}\,\mathrm{e}^{-\frac12 c2^{2q}t}\,\|g\|_{\mathcal{B}_{2,\infty}^\alpha}\;.$$

Now we observe that for any $\gamma \geqslant 0$,

$$2^{q(\beta-\alpha)}\,\mathrm{e}^{-\frac12 c2^{2q}t} = 2^{q(\beta-\alpha-2\gamma)}t^{-\gamma}(2^{2q}t)^\gamma\,\mathrm{e}^{-\frac12 c2^{2q}t} \leqslant M_0(2\gamma)2^{q(\beta-\alpha-2\gamma)}t^{-\gamma}$$

by boundedness of the map $x \mapsto x^\gamma\,\mathrm{e}^{-x}$. Choosing $\gamma = \frac{\beta-\alpha}{2}$ yields the result. □

Corollary A.3 (Schauder estimate on convolutions with the heat kernel)**.** *Let $g(t) \in \mathcal{B}^{\alpha}_{2,\infty}$ for all $t \in [0,T]$, where $\alpha \in \mathbb{R}$. Let ϕ be the solution of*

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + a(t)\phi(t,x) + g(t,x)\big]\,\mathrm{d}t\;, \tag{A.1}$$

starting from 0, where $a \in \mathcal{C}^1([0,T],\mathbb{R}_-)$ is bounded away from 0 (cf. (2.10)). Then $\phi(t) \in \mathcal{B}^{\beta}_{2,\infty}$ for all $\beta < \alpha + 2$ and all $t \in [0,T]$, and there is a constant $M = M(\beta - \alpha)$ such that

$$\|\phi(t,\cdot)\|_{\mathcal{B}^{\beta}_{2,\infty}} \leqslant M\varepsilon^{\frac{\beta-\alpha}{2}-1} \sup_{t_1\in[0,T]} \|g(t_1,\cdot)\|_{\mathcal{B}^{\alpha}_{2,\infty}}$$

holds for all $\beta < \alpha + 2$ and all $t \in [0,T]$.

Proof: The solution of (A.1) can be written as

$$\phi(t,x) = \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,t_1)/\varepsilon}\big(\mathrm{e}^{\frac{t-t_1}{\varepsilon}\Delta}\,g\big)(t_1,x)\,\mathrm{d}t_1\;,$$

where $\alpha(t,t_1) = \int_{t_1}^t a(t_2)\,\mathrm{d}t_2$ is negative whenever $t \geqslant t_1$. Therefore

$$\begin{aligned}\|\phi(t,\cdot)\|_{\mathcal{B}^{\beta}_{2,\infty}} &\leqslant \frac{1}{\varepsilon}\int_0^t \|\big(\mathrm{e}^{\frac{t-t_1}{\varepsilon}\Delta}\,g\big)(t_1,\cdot)\|_{\mathcal{B}^{\beta}_{2,\infty}}\,\mathrm{d}t_1 \\ &\leqslant \frac{1}{\varepsilon}M_0(\beta-\alpha)\int_0^t\bigg(\frac{t-t_1}{\varepsilon}\bigg)^{-\frac{\beta-\alpha}{2}}\|g(t_1,\cdot)\|_{\mathcal{B}^{\alpha}_{2,\infty}}\,\mathrm{d}t_1 \\ &\leqslant \varepsilon^{\frac{\beta-\alpha}{2}-1}M_0(\beta-\alpha)\sup_{t_1\in[0,T]}\|g(t_1,\cdot)\|_{\mathcal{B}^{\alpha}_{2,\infty}}\int_0^t (t-t_1)^{-\frac{\beta-\alpha}{2}}\,\mathrm{d}t_1\;.\end{aligned}$$

The integral is bounded whenever $\beta < \alpha + 2$. □

B Wick calculus

This appendix summarises some properties of Hermite polynomials and Wick calculus needed in this work. Proofs of these properties can be found, for instance, in the monographs [21, 22], the lecture notes [16], and Section 4.2.2 and Appendix D of [1].

Hermite polynomials with variance C have been introduced in Section 2.1. Some of the above references consider the special case $C = 1$, but results for that case can easily be converted into results for the general case by using the scaling property

$$H_n(x;C) = C^{n/2}H_n(C^{-1/2}x;1)\;.$$

The first n Hermite polynomials and the monomials $1,\dots,x^n$ both form a basis of the vector space of polynomials of degree n, where the change of basis is given by the formulas

$$\begin{aligned}H_n(x;C) &= \sum_{\ell=0}^{\lfloor n/2\rfloor} a_{n\ell}C^{\ell}x^{n-2\ell}\;, \qquad & a_{n\ell} &= \frac{(-1)^{\ell}n!}{2^{\ell}\ell!(n-2\ell)!}\;, \\ x^n &= \sum_{\ell=0}^{\lfloor n/2\rfloor} b_{n\ell}C^{\ell}H_{n-2\ell}(x;C)\;, \qquad & b_{n\ell} &= \frac{n!}{2^{\ell}\ell!(n-2\ell)!} = |a_{n\ell}|\;.\end{aligned} \tag{B.1}$$

The Hermite polynomials admit the generating function

$$G(t,x;C) := \mathrm{e}^{tx - Ct^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x;C) , \tag{B.2}$$

which can be used to establish the following identity.

Lemma B.1 (Expectation of products of Wick powers)**.** *Let X and Y be jointly Gaussian centred random variables, of respective variance C_1 and C_2. Then for any $n, m \geqslant 0$, one has*

$$\mathbb{E}\big[H_n(X;C_1)H_m(Y;C_2)\big] = \begin{cases} n!\mathbb{E}\big[XY\big]^n & \textit{if } n = m , \\ 0 & \textit{otherwise} . \end{cases}$$

Another consequence of the expression (B.2) of the generating function is the following binomial formula.

Lemma B.2 (Binomial formula for Hermite polynomials)**.** *For any $x, y \in \mathbb{R}$, $C_1, C_2 \geqslant 0$ and $n \in \mathbb{N}_0$,*

$$H_n(x+y;C_1+C_2) = \sum_{m=0}^{n} \binom{n}{m} H_m(x;C_1)H_{n-m}(y;C_2) . \tag{B.3}$$

A direct consequence is Lemma 3.4, which we recall here. We recall that $\hat{\psi}_k(t)$ is the martingale introduced in (3.7), and that $\hat{v}_k(t)$ is its variance defined in (3.8).

Lemma B.3 (Martingale property)**.** *For any $m \geqslant 1$, $H_m(\hat{\psi}_k(t);v_k(t))$ is a martingale with respect to the canonical filtration $\mathcal{F}_t$ of the Wiener process $(W_k(t))_{t\geqslant 0}$.*

Proof: We write $H_m(\hat{\psi}_k(t);v_k(t)) = H_m(\hat{\psi}_k(t))$ in order not to overload the notation. For any $0 \leqslant s < t$, we have

$$\mathbb{E}\big[H_m(\hat{\psi}_k(t)) \bigm| \mathcal{F}_s\big] = \mathbb{E}\Big[H_m\big(\hat{\psi}_k(s) + (\hat{\psi}_k(t) - \hat{\psi}_k(s))\big) \bigm| \mathcal{F}_s\Big]. \tag{B.4}$$

By the binomial formula (B.3) and additivity of the v_k,

$$H_m\big(\hat{\psi}_k(s) + (\hat{\psi}_k(t) - \hat{\psi}_k(s))\big) = \sum_{n=0}^{m} \binom{m}{n} H_n(\hat{\psi}_k(s))H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s)) .$$

We replace this expression in (B.4). Since $H_n(\hat{\psi}_k(s))$ is $\mathcal{F}_s$-measurable and $H_{m-n}(\hat{\psi}_k(t)-\hat{\psi}_k(s))$ is independent of $\mathcal{F}_s$ we obtain

$$\begin{aligned}
\mathbb{E}\big[H_m(\hat{\psi}_k(t)) \bigm| \mathcal{F}_s\big] &= \sum_{n=0}^{m} \binom{m}{n} \mathbb{E}\big[H_n(\hat{\psi}_k(s))H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s)) \bigm| \mathcal{F}_s\big] \\
&= \sum_{n=0}^{m} \binom{m}{n} H_n(\hat{\psi}_k(s))\mathbb{E}\big[H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s))\big] \\
&= H_m(\hat{\psi}_k(s)) .
\end{aligned}$$

The last equality is due to the fact that mth Hermite polynomials are centred variables for $m \geqslant 1$ and for $m = n$, $H_0(\hat{\psi}_k(t) - \hat{\psi}_k(s)) = 1$. □

The following generalisation of the binomial formula (B.3) is obtained by induction.

Lemma B.4 (Multinomial formula for Hermite polynomials)**.** *Let $(a_q)_{q\geqslant 0}$ be a sequence of real numbers in ℓ^2. Then for any convergent sequence $(x_q)_{q\geqslant 0}$, one has*

$$H_m\Big(\sum_{q\geqslant 0} x_q; \sum_{q\geqslant 0} a_q^2\Big) = \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q\geqslant 0} H_{\mathbf{n}_q}(x_q; a_q^2) ,$$

where the sum runs over all $\mathbf{n} \in \mathbb{N}_0^{\mathbb{N}_0}$ such that $|\mathbf{n}| := \sum_{q\geqslant 0} \mathbf{n}_q = m$, and $\mathbf{n}! := \prod_{q\geqslant 0} \mathbf{n}_q!$.

Given a set $\{\psi_q\}_q$ of independent centred Gaussian random variables, one defines the mth homogeneous Wiener chaos $\mathcal{H}_m$ as the vector space generated by all Wick powers of the ψ_q of total degree m, that is, all

$$\Phi_{\mathbf{n}} = \prod_{q\geqslant 0} H_{\mathbf{n}_q}\big(\psi_q; \mathrm{Var}(\psi_q)\big)$$

with $|\mathbf{n}| = m$. Then one has the following result on equivalence of norms, which is a consequence of hypercontractivity of the Ornstein–Uhlenbeck semigroup. See for instance [13, Theorem 4.1] or [21, Theorem 1.4.1].

Lemma B.5 (Equivalence of moments)**.** *Let X be a random variable, belonging to the m-th homogeneous Wiener chaos. Then for any $p \geqslant 1$ one has*

$$\mathbb{E}\big[X^{2p}\big] \leqslant (2p-1)^{mp}\mathbb{E}\big[X^2\big]^p .$$

C Some technical proofs for Section 3

C.1 Proof of Lemma 3.7

We divide the proof of the lemma into the following two parts.

Lemma C.1. *For any $q_0 \geqslant 0$, $t \in I_l$ and $\mathbf{n}$, one has*

$$\mathbb{E}\big[X_{\mathbf{n}}^2\big] = \mathbf{n}! \sum_{\substack{k_1^{(q)},k_2^{(q)},\dots,k_{\mathbf{n}_q}^{(q)}\in\mathcal{A}_q\ \forall q \\ \sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q} k_i^{(q)}\in\mathcal{A}_{q_0}}} \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t) . \tag{C.1}$$

Proof: Let $\varphi(t,\cdot) = \prod_{q\geqslant 0} :\delta_q\hat{\psi}(t,\cdot)^{\mathbf{n}_q}:$. The L^2-norm of its projection on $\mathcal{A}_{q_0}$ is given by

$$\big\|\delta_{q_0}\varphi(t,\cdot)\big\|_{L^2}^2 = \sum_{k\in\mathcal{A}_{q_0}} |(P_k\varphi)(t,\cdot)|^2 ,$$

where $(P_k\varphi)(t,x)$ is the projection of φ on the kth Fourier basis vector $e_k(x)$, given by

$$(P_k\varphi)(t,x) = \int_{\mathbb{T}^2} e_{-k}(x_1)\varphi(t,x_1)\,\mathrm{d}x_1 e_k(x) .$$

Therefore,

$$\mathbb{E}\Big[\big\|\delta_{q_0}\varphi(t,\cdot)\big\|_{L^2}^2\Big] = \sum_{k\in\mathcal{A}_{q_0}} \mathbb{E}\Big[|(P_k\varphi)(t,\cdot)|^2\Big].$$

For a fixed $k \in \mathcal{A}_{q_0}$, we have

$$\begin{aligned}\mathbb{E}\Big[|(P_k\varphi)(t,x)|^2\Big] &= \mathbb{E}\Big[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} e_{-k}(x_1)\varphi(t,x_1)e_k(x_2)\bar{\varphi}(t,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 e_k(x)e_{-k}(x)\Big] \\ &= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} e_{-k}(x_1-x_2)\mathbb{E}\big[\varphi(t,x_1)\bar{\varphi}(t,x_2)\big]\,\mathrm{d}x_1\,\mathrm{d}x_2\ ,\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}\big[\varphi(t,x_1)\bar{\varphi}(t,x_2)\big] &= \mathbb{E}\Big[\prod_{q\geqslant 0} :\delta_q\hat{\psi}(t,x_1)^{\mathbf{n}_q}:\overline{:\delta_q\hat{\psi}(t,x_2)^{\mathbf{n}_q}:}\Big] \\ &= \prod_{q\geqslant 0}\mathbb{E}\big[:\delta_q\hat{\psi}(t,x_1)^{\mathbf{n}_q}:\overline{:\delta_q\hat{\psi}(t,x_2)^{\mathbf{n}_q}:}\big]\ ,\end{aligned}$$

since the projections δ_q and $\delta_{q'}$ are independent for $q \neq q'$. By Lemma B.1, we get

$$\mathbb{E}\big[:\delta_q\hat{\psi}(t,x_1)^{\mathbf{n}_q}:\overline{:\delta_q\hat{\psi}(t,x_2)^{\mathbf{n}_q}:}\big] = \mathbf{n}_q!\mathbb{E}\big[\delta_q\hat{\psi}(t,x_1)\overline{\delta_q\hat{\psi}(t,x_2)}\big]^{\mathbf{n}_q}\ ,$$

where

$$\begin{aligned}\mathbb{E}\big[\delta_q\hat{\psi}(t,x_1)\overline{\delta_q\hat{\psi}(t,x_2)}\big] &= \sum_{k_1,k_2\in\mathcal{A}_q}\mathbb{E}\big[\hat{\psi}_{k_1}(t)\overline{\hat{\psi}_{k_2}(t)}\big]e_{k_1}(x_1)e_{-k_2}(x_2) \\ &= \sum_{k_1,k_2\in\mathcal{A}_q}\hat{v}_{k_1}(t)\delta_{k_1,k_2}e_{k_1}(x_1)e_{-k_2}(x_2) \\ &= \sum_{k_1\in\mathcal{A}_q}\hat{v}_{k_1}(t)e_{k_1}(x_1-x_2)\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\big[\varphi(t,x_1)\bar{\varphi}(t,x_2)\big] &= \prod_{q\geqslant 0}\mathbf{n}_q!\Big(\sum_{k_1\in\mathcal{A}_q}\hat{v}_{k_1}(t)e_{k_1}(x_1-x_2)\Big)^{\mathbf{n}_q} \\ &= \Big(\prod_{q\geqslant 0}\mathbf{n}_q!\Big)\prod_{q\geqslant 0}\Big(\sum_{k_1,\dots,k_{\mathbf{n}_q}\in\mathcal{A}_q}\hat{v}_{k_1}(t)\cdots\hat{v}_{k_{\mathbf{n}_q}}(t)e_{k_1+\dots k_{\mathbf{n}_q}}(x_1-x_2)\Big)\ .\end{aligned}$$

Integrating over x_1 and x_2, we get

$$\begin{aligned}\mathbb{E}\Big[|(P_k\varphi)(t,x)|^2\Big] &= \mathbf{n}!\sum_{k_1^{(q)},\dots,k_{\mathbf{n}_q}^{(q)}\in\mathcal{A}_q,\ \forall q}\ \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q}\hat{v}_{k_i^{(q)}}(t) \\ &\quad\times\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} e_{-k}(x_1-x_2)e_{\sum_{q\geqslant 0}k_1^{(q)}+\dots+k_{\mathbf{n}_q}^{(q)}}(x_1-x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \mathbf{n}!\sum_{k_1^{(q)},\dots,k_{\mathbf{n}_q}^{(q)}\in\mathcal{A}_q,\ \forall q}\ \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q}\hat{v}_{k_i^{(q)}}(t)\mathbb{1}_{\big\{\sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q}k_i^{(q)}=k\big\}} \\ &= \mathbf{n}!\sum_{\substack{k_1^{(q)},\dots,k_{\mathbf{n}_q}^{(q)}\in\mathcal{A}_q,\ \forall q \\ \sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q}k_i^{(q)}=k}}\ \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q}\hat{v}_{k_i^{(q)}}(t)\ .\end{aligned}$$

Summing over $k_0 \in \mathcal{A}_{q_0}$ yields the claimed result. □

Lemma C.2. *There exists a numerical constant C_0 such that*

$$\mathbb{E}\big[X_{\mathbf{n}}^2\big] \leqslant C_0^m \mathbf{n}! \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}} \leqslant C_0^m m! \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}} .$$

PROOF: We have to evaluate the sum given by (C.1). Recall that $q_1 < q_2 < \cdots < q_{[\mathbf{n}]}$ denote the indices of the $[\mathbf{n}]$ nonzero entries of $\mathbf{n}$, and that there is a numerical constant c_0 such that

$$\hat{v}_k(t) \leqslant \frac{c_0\sigma^2}{1 + \|k\|^2}$$

for all $t \in I_l$. For a fixed $k_0 \in \mathcal{A}_{q_0}$, we get the bound

$$S_{\mathbf{n},k_0} := \sum_{\substack{k_1^{(q)},\dots,k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q,\ \forall q \\ \sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k_0}} \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t) \leqslant \sum_{\substack{k_1^{(q)},\dots,k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q,\ \forall q \\ \sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k_0}} \prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q} \frac{c_0\sigma^2}{1 + \big\|k_i^{(q)}\big\|^2} .$$

Note that

$$\prod_{q\geqslant 0}\prod_{i=1}^{\mathbf{n}_q}(c_0\sigma^2) = \prod_{q\geqslant 0}(c_0\sigma^2)^{\mathbf{n}_q} = (c_0\sigma^2)^{\sum_{q\geqslant 0}\mathbf{n}_q} = (c_0\sigma^2)^m ,$$

and that we can write

$$k_1^{(q_1)} = k_0 - \sum_{j=2}^{[\mathbf{n}]} k_1^{(q_j)} - \sum_{j=1}^{[\mathbf{n}]}\sum_{i=2}^{\mathbf{n}_q} k_i^{(q_j)} . \tag{C.2}$$

Since $\big\|k_i^{(q_1)}\big\| < \big\|k_i^{(q_2)}\big\| < \dots < \big\|k_i^{(q_{[\mathbf{n}]})}\big\|$ and $\big\|k_0\big\| \leqslant \big\|k_i^{(q_{[\mathbf{n}]})}\big\|$, by the second triangle inequality, we get

$$\begin{aligned}\bigg\|k_0 - \sum_{j=2}^{[\mathbf{n}]} k_1^{(q_j)} - \sum_{j=1}^{[\mathbf{n}]}\sum_{i=2}^{\mathbf{n}_q} k_i^{(q_j)}\bigg\| &\geqslant \bigg|\|k_0\| - \sum_{j=2}^{[\mathbf{n}]}\|k_1^{(q_j)}\| - \sum_{j=1}^{[\mathbf{n}]}\sum_{i=2}^{\mathbf{n}_q}\|k_i^{(q_j)}\|\bigg| \\ &\geqslant \sum_{j=2}^{[\mathbf{n}]}\|k_1^{(q_j)}\| + \sum_{j=1}^{[\mathbf{n}]}\sum_{i=2}^{\mathbf{n}_q}\|k_i^{(q_j)}\| - \|k_0\| \\ &\geqslant c\,\Big\|k_1^{(q_{[\mathbf{n}]})}\Big\|\end{aligned} \tag{C.3}$$

for a numerical constant $c > 0$. Replacing $k_1^{(q_1)}$ by (C.2) and bounding its norm by (C.3), we obtain

$$S_{\mathbf{n},k_0} \lesssim (c_0\sigma^2)^m \prod_{j=1}^{[\mathbf{n}]}\Bigg(\sum_{k_1^{(q_j)},\dots,k_{\mathbf{n}_{q_j}}^{(q_j)} \in \mathcal{A}_{q_j}} \frac{1}{\|k_i^{(q_j)}\|^2}\Bigg)^{\mathbf{n}_{q_j}} \sum_{k_1^{(q_{[\mathbf{n}]})} \in \mathcal{A}_{q_{[\mathbf{n}]}}} \frac{1}{\|k_1^{(q_{[\mathbf{n}]})}\|^4} .$$

For a fixed q_j, we view these sums as Riemann sums, and integrating using polar coordinates yields

$$\sum_{k_1^{(q_j)},\dots,k_{\mathbf{n}_{q_j}}^{(q_j)} \in \mathcal{A}_{q_j}} \frac{1}{\|k_i^{(q_j)}\|^2} \lesssim \int_{2^{q_j-1}}^{2^{q_j}} \frac{r}{r^2}\,\mathrm{d}r \leqslant \log(2^{q_j}) - \log(2^{q_j-1}) = \log(2) ,$$

and

$$\sum_{k_1^{(q_{[\mathbf{n}]})}\in\mathcal{A}_{q_{[\mathbf{n}]}}} \frac{1}{\|k_1^{(q_{[\mathbf{n}]})}\|^4} \lesssim \int_{2^{q_{[\mathbf{n}]}-1}}^{2^{q_{[\mathbf{n}]}}} \frac{r}{r^4}\,\mathrm{d}r \lesssim \frac{1}{2^{2q_{[\mathbf{n}]}}} .$$

We conclude that

$$S_{\mathbf{n},k_0} \lesssim (c_0\sigma^2)^m \prod_{j=1}^{[\mathbf{n}]}(c_1\log(2))^{\mathbf{n}_{q_j}} \frac{1}{2^{2q_{[\mathbf{n}]}}} = C_0^m\sigma^{2m}\frac{1}{2^{2q_{[\mathbf{n}]}}}$$

for some numerical constant c_1, C_0. The result follows again by summing over $k \in \mathcal{A}_{q_0}$. $\square$

C.2 Proof of Lemma 3.9

We decompose the sum (3.12) as

$$K_m(q_0) = \sum_{[\mathbf{n}]=1}^{m} S_m([\mathbf{n}],0) , \tag{C.4}$$

where for $a \in \{1,\dots,m\}$ and $b \in \mathbb{N}_0$, we define

$$S_m(a,b) = \sum_{\substack{\mathbf{n}:|\mathbf{n}|=m\\ [\mathbf{n}]=a,\ q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}}\geqslant q_0}} \frac{m!}{\mathbf{n}!}\frac{q_{[\mathbf{n}]}^b}{2^{(q_{[\mathbf{n}]}-q_0)/2}} .$$

We will estimate this sum by induction on a, for arbitrary $b \in \mathbb{N}_0$. For $a = 1$, the only possible $\mathbf{n}$ are those with one component, say q, equal to m, and all other components equal to 0. Therefore,

$$S_m(1,b) \leqslant \sum_{q\geqslant(q_0-m)\vee 0} \frac{q^b}{2^{(q-q_0)/2}}$$

(since one must have $q \geqslant 0$). The sum can be computed via the inequality

$$\sum_{q=0}^{\infty}(q+1)^b z^q \leqslant \frac{b!}{(1-z)^{b+1}} ,$$

valid for any $z \in [0,1)$ and $b \in \mathbb{N}$, which follows directly from the definitions of the polylogarithm function and Eulerian numbers. Setting $1/\sqrt{2} = z$, we have

$$\begin{aligned} S_m(1,b) &\leqslant \sum_{q\geqslant(q_0-m)\vee 0} q^b z^{q-q_0} = z^{-m}\sum_{p\geqslant 0\vee(m-q_0)}(q_0-m+p)^b z^p \\ &\leqslant z^{-m}\sum_{\ell=0}^{b}\binom{b}{\ell}q_0^{b-\ell}\sum_{p\geqslant 0}p^\ell z^p \\ &\leqslant z^{-m}\frac{b!}{1-z}\sum_{\ell=0}^{b}\binom{b}{\ell}\frac{q_0^{b-\ell}}{(1-z)^\ell} \\ &= 2^{m/2}b!\,c_1(q_0+c_1)^b , \end{aligned}$$

where $c_1 = (1-z)^{-1} = 2+\sqrt{2}$.

Assume now that $a \geqslant 1$ and $[\mathbf{n}] = a+1$. We decompose $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, with $[\mathbf{n}_1] = a$ and $[\mathbf{n}_2] = 1$, and $|\mathbf{n}_1| = m_1$, $|\mathbf{n}_2| = m_2$ with $m_1 + m_2 = m$. We may assume that the largest nonzero component of $\mathbf{n}$ appears in $\mathbf{n}_1$, so that $q_{[\mathbf{n}_1]} = q_{[\mathbf{n}]}$ and $q_{[\mathbf{n}_2]} = q < q_{[\mathbf{n}]}$. It follows that

$$\begin{aligned} S_m(a+1,b) &= \sum_{m_1+m_2=m} \sum_{q<q_{[\mathbf{n}]}} \sum_{\substack{|\mathbf{n}_1|=m_1 \\ [\mathbf{n}_1]=a \\ q_{[\mathbf{n}_1]}+\mathbf{n}_{q_{[\mathbf{n}_1]}} \geqslant q_0}} \frac{m!}{\mathbf{n}_1! m_2!} \frac{q_{[\mathbf{n}_1]}^b}{2^{(q_{[\mathbf{n}_1]}-q_0)/2}} \\ &\leqslant \sum_{m_1=1}^{m-1} \binom{m}{m_1} S_{m_1}(a,b+1) \;, \end{aligned}$$

where we have bounded the sum over q by $q_{[\mathbf{n}]} = q_{[\mathbf{n}_1]}$. It is then straightforward to show by induction that

$$S_m(a,b) \leqslant c_1(\sqrt{2}+a-1)^m (q_0+c_1)^{a+b-1}(a+b-1)!$$

for all a, b. In particular,

$$S_m(a,0) \leqslant c_1(\sqrt{2}+a-1)^m (q_0+c_1)^{a-1}(a-1)! \;.$$

Replacing this in (C.4) yields the result, with $c_2 = \sqrt{2}-1$. □

C.3 Proof of Proposition 3.10

Using the relation (B.1) between Wick polynomials and monomials, we get

$$\begin{aligned} \prod_{q\geqslant 0} H_{\mathbf{n}_q}(\delta_q\psi(t,\cdot);c_q) &= \prod_{q\geqslant 0}\biggl(\sum_{\mathbf{l}_q=0}^{\lfloor \mathbf{n}_q/2\rfloor} a_{\mathbf{n}_q\mathbf{l}_q} c_q^{\mathbf{l}_q} \bigl(\delta_q\psi(t,\cdot)\bigr)^{\mathbf{n}_q-2\mathbf{l}_q}\biggr) \\ &= \sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor \mathbf{n}/2\rfloor} a_{\mathbf{n}\mathbf{l}} \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} c_q^{\mathbf{l}_q} \bigl(\delta_q\psi(t,\cdot)\bigr)^{\mathbf{n}_q-2\mathbf{l}_q} \;, \end{aligned}$$

where

$$a_{\mathbf{n}\mathbf{l}} = \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} a_{\mathbf{n}_q\mathbf{l}_q} = \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} \frac{(-1)^{\mathbf{l}_q}\mathbf{n}_q!}{2^{\mathbf{l}_q}\mathbf{l}_q!(\mathbf{n}_q-2\mathbf{l}_q)!} = \frac{(-1)^{|\mathbf{l}|}\mathbf{n}!}{2^{|\mathbf{l}|}\mathbf{l}!(\mathbf{n}-2\mathbf{l})!} \;.$$

Recall that

$$\delta_q\psi(t,x) = \sum_{k\in\mathcal{A}_q} \psi_k(t) e_k(x) \;,$$

which implies

$$\begin{aligned} \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} \bigl(\delta_q\psi(t,x)\bigr)^{\mathbf{n}_q-2\mathbf{l}_q} &= \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} \biggl(\sum_{k_1,\dots,k_{\mathbf{n}_q-2\mathbf{l}_q}\in\mathcal{A}_q} \psi_{k_1}(t)\dots\psi_{k_{\mathbf{n}_q-2\mathbf{l}_q}}(t) e_{k_1+\dots+k_{\mathbf{n}_q-2\mathbf{l}_q}}(x)\biggr) \\ &= \sum_{k_1^{(q)},\dots,k^{(q)}_{\mathbf{n}_q-2\mathbf{l}_q}\in\mathcal{A}_q\ \forall q} \prod_{\substack{q\geqslant 0 \\ \mathbf{n}_q>0}} \prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q} \psi_{k_i^{(q)}}(t) e_{\sum_{q\geqslant 0:\ \mathbf{n}_q>0}\sum_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q} k_i^{(q)}}(x) \;, \end{aligned}$$

whose projection on the k_0th Fourier basis vector is given by

$$P_{k_0}\biggl(\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\bigl(\delta_q\psi(t,x)\bigr)^{\mathbf{n}_q-2\mathbf{l}_q}\biggr) = \sum_{\mathcal{B}(k_0)}\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\psi_{k_i^{(q)}}(t)e_{k_0}(x)\;,$$

where the sum runs over all tuples $(k_1^{(q)},\dots,k_{\mathbf{n}_q-2\mathbf{l}_q}^{(q)})_{q>0}$ in the set

$$\mathcal{B}(k_0) = \biggl\{k_1^{(q)},\dots,k_{\mathbf{n}_q-2\mathbf{l}_q}^{(q)}\in\mathcal{A}_q\;\forall q\colon\sum_{q\geqslant 0}\sum_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}k_i^{(q)} = k_0\biggr\}\;.$$

Similar relations hold with $\hat\psi(t,x)$. We now note that

$$\bigl\|\delta_{q_0}\bigl(\varphi(t,\cdot)-\hat\varphi(t,\cdot)\bigr)\bigr\|_{L^2}^2 = \sum_{k_0\in\mathcal{A}_{q_0}}\bigl|\langle e_{k_0},P_{k_0}\varphi(t,\cdot)-P_{k_0}\hat\varphi(t,\cdot)\rangle\bigr|^2\;,$$

where

$$\begin{aligned}
&\langle e_{k_0},P_{k_0}\varphi(t,\cdot)-P_{k_0}\hat\varphi(t,\cdot)\rangle\\
&\quad= \sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor}a_{\mathbf{nl}}\sum_{\mathcal{B}(k_0)}\biggl[\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}c_q^{\mathbf{l}_q}\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\psi_{k_i^{(q)}}(t) - \prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\hat c_q(t)^{\mathbf{l}_q}\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\hat\psi_{k_i^{(q)}}(t)\biggr]\;.
\end{aligned}\tag{C.5}$$

Observe that

$$\begin{aligned}
\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\psi_{k_i^{(q)}}(t) &= \exp\biggl\{-\frac{1}{\varepsilon}\sum_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\alpha_{k_i^{(q)}}(u_{l+1},t)\biggr\}\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\hat\psi_{k_i^{(q)}}(t)\;,\\
c_q^{\mathbf{l}_q} &\leqslant \exp\biggl\{-\frac{2}{\varepsilon}\mathbf{l}_q\alpha_{k(q)}(u_{l+1},t)\biggr\}\hat c_q(t)^{\mathbf{l}_q}
\end{aligned}$$

for some $k(q)\in\mathcal{A}_q$. The definition of the partition implies that $|u_{l+1}-t|$ has order $2^{-2q_{[\mathbf{n}]}}\gamma_0\varepsilon$, and therefore there is a numerical constant c_0 such that

$$-\frac{1}{\varepsilon}\alpha_k(u_{l+1},t)\leqslant c_0 2^{-2(q_{[\mathbf{n}]}-q)}\gamma_0$$

holds for all $k\in\mathcal{A}_q$. Therefore,

$$-\frac{1}{\varepsilon}\sum_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\alpha_{k_i^{(q)}}(u_{l+1},t)-\frac{2}{\varepsilon}\mathbf{l}_q\alpha_{k(q)}(u_{l+1},t)\leqslant c_0\gamma_0|\mathbf{n}_q|2^{-2(q_{[\mathbf{n}]}-q)}\;.$$

Replacing this in (C.5) yields

$$\begin{aligned}
&\bigl|\langle e_{k_0},P_{k_0}\varphi(t,\cdot)-P_{k_0}\hat\varphi(t,\cdot)\rangle\bigr|\\
&\quad\leqslant \sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor}|a_{\mathbf{nl}}|\sum_{\mathcal{B}(k_0)}\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\Bigl(\mathrm{e}^{c_0\gamma_0|\mathbf{n}_q|2^{-2(q_{[\mathbf{n}]}-q)}}-1\Bigr)\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\biggl(\hat c_q(t)^{\mathbf{l}_q}\prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q}\bigl|\hat\psi_{k_i^{(q)}}(t)\bigr|\biggr)\;.
\end{aligned}\tag{C.6}$$

Since the exponent $c_0\gamma_0|\mathbf{n}_q|2^{-2(q_{[\mathbf{n}]}-q)}$ is bounded, we can write, for a numerical constant c_1,

$$\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\Bigl(\mathrm{e}^{c_0\gamma_0|\mathbf{n}_q|2^{-2(q_{[\mathbf{n}]}-q)}}-1\Bigr)\leqslant\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\Bigl(c_1\gamma_0|\mathbf{n}_q|2^{-2(q_{[\mathbf{n}]}-q)}\Bigr)$$
$$\leqslant\bigl(c_1\gamma_0\bigr)^{[\mathbf{n}]}\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\mathbf{n}_q\ ,$$

since the product of powers of 2 is bounded by 1 (in fact, it can even be bounded by $2^{-2([\mathbf{n}]-1)}$, but this just decreases the constant c_1). Since

$$\bigl\|\delta_q\hat{\psi}(t,\cdot)\bigr\|_{L^2}^2=\sum_{k\in\mathcal{A}_q}|\hat{\psi}_k(t)|^2,$$

we have the rough bound

$$|\hat{\psi}_k(t)|^2\leqslant\bigl\|\delta_q\hat{\psi}(t,\cdot)\bigr\|_{L^2}^2\quad\forall k\in\mathcal{A}_q\ .$$

Plugging the last bounds into (C.6), we get

$$\begin{aligned}&\bigl|\langle e_{k_0},P_{k_0}\varphi(t,\cdot)-P_{k_0}\hat{\varphi}(t,\cdot)\rangle\bigr|\\&\qquad\leqslant\bigl(c_1\gamma_0\bigr)^{[\mathbf{n}]}\Biggl(\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\mathbf{n}_q\Biggr)\sum_{\mathbf{l}:\mathbf{l}\leqslant\lfloor\mathbf{n}/2\rfloor}|a_{\mathbf{n}\mathbf{l}}|\hat{c}(t)^{|\mathbf{l}|}\prod_{\substack{q\geqslant 0\\ \mathbf{n}_q>0}}\Bigl(\bigl\|\delta_q\hat{\psi}(t,\cdot)\bigr\|_{L^2}^{\mathbf{n}_q-2\mathbf{l}_q}\Bigr)\#\mathcal{B}(k_0)\ .\end{aligned}$$

Finally, by counting the number of choices of the $k_i^{(q)}$, we obtain

$$\#\mathcal{B}(k_0)\leqslant 2^{2q_{[\mathbf{n}]}(|\mathbf{n}|-2|\mathbf{l}|)}\ .$$

This yields the claimed result, noticing that this bound is independent of k_0, so summing over all $k_0\in\mathcal{A}_{q_0}$ only yields an extra factor 2^{2q_0} in the L^2-norm squared. □

C.4 Proof of Lemma 3.13

We decompose the sum as

$$\bar{K}_{m,b}(q_0)=\sum_{a=1}^{m}S_{m,m}(a,b)\ ,$$

where for $a\in\{1,\dots,m\}$ and $b\in\mathbb{N}_0$, we define

$$S_{m,m_0}(a,b)=\sum_{\substack{\mathbf{n}:|\mathbf{n}|=m\\ [\mathbf{n}]=a,\ q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}}\geqslant q_0}}q_{[\mathbf{n}]}^b2^{(m_0+3)q_{[\mathbf{n}]}}\exp\Bigl\{-\beta(m_0,q_0)2^{(q_{[\mathbf{n}]}-q_0)/m_0}\Bigr\}\ .$$

We will proceed similarly to the proof of Lemma 3.13, and estimate this sum by induction on a, for arbitrary $b\in\mathbb{N}_0$. For $a=1$, the only possible $\mathbf{n}$ are those with one component, say q, equal to

m, and all other components equal to 0. Then $q_{[\mathbf{n}]} = q$, and writing $x_+ = x \vee 0$ we get

$$\begin{aligned}
S_{m,m_0}(1,b) &\leqslant \sum_{q \geqslant (q_0 - m)_+} q^b 2^{(m_0+3)q} \exp\Bigl\{-\beta(m_0,q_0) 2^{(q-q_0)/m_0}\Bigr\} \\
&= \sum_{p \geqslant (m-q_0)_+} (q_0 - m + p)^b 2^{(m_0+3)(p+q_0-m)} \exp\Bigl\{-\beta(m_0,q_0) 2^{(p-m)/m_0}\Bigr\} \\
&= 2^{(m_0+3)(q_0-m)} \sum_{p \geqslant (m-q_0)_+} (q_0 - m + p)^b 2^{(m_0+3)p} \exp\Bigl\{-\beta(m_0,q_0) 2^{(p-m)/m_0}\Bigr\} .
\end{aligned}$$

One checks that for $\beta(m_0, q_0)$ larger than a numerical constant of order 1, the general term of this sum is decreasing in p. Estimating the sum by an integral, we get

$$S_{m,m_0}(1,b) \leqslant c_1 (q_0 - m)^b 2^{(m_0+3)(q_0-m)_+} \exp\Bigl\{-2^{-m/m_0} \beta(m_0,q_0)\Bigr\}$$

for a numerical constant c_1. Assume now that $a \geqslant 1$ and $[\mathbf{n}] = a+1$. We decompose $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, with $[\mathbf{n}_1] = a$ and $[\mathbf{n}_2] = 1$, and $|\mathbf{n}_1| = m_1$, $|\mathbf{n}_2| = m_2$ with $m_1 + m_2 = m$. We may assume that the largest nonzero component of $\mathbf{n}$ appears in $\mathbf{n}_1$, so that $q_{[\mathbf{n}_1]} = q_{[\mathbf{n}]}$ and $q_{[\mathbf{n}_2]} = q < q_{[\mathbf{n}]}$. It follows that

$$\begin{aligned}
S_{m,m_0}(a+1,b) &= \sum_{m_1+m_2=m} \sum_{q<q_{[\mathbf{n}]}} \sum_{\substack{|\mathbf{n}_1|=m_1 \\ [\mathbf{n}_1]=a \\ q_{[\mathbf{n}_1]}+\mathbf{n}_{q_{[\mathbf{n}_1]}} \geqslant q_0}} q_{[\mathbf{n}]}^b 2^{(m_0+3)q_{[\mathbf{n}]}} \exp\Bigl\{-\beta(m_0,q_0) 2^{(q_{[\mathbf{n}]}-q_0)/m_0}\Bigr\} \\
&\leqslant \sum_{m_1=1}^{m-1} S_{m_1,m_0}(a, b+1) .
\end{aligned}$$

where we have bounded the sum over q by $q_{[\mathbf{n}]} = q_{[\mathbf{n}_1]}$. It is then straightforward to show by induction that

$$S_{m,m_0}(a,b) \leqslant c_1 m^{a-1} q_0^{a+b-1} 2^{(m_0+3)q_0} \exp\Bigl\{-\beta(m_0,q_0)\Bigr\}$$

for all a, b. Summing over a and setting $m = m_0$ yields the result. □

References

[1] Nils Berglund. *An Introduction to Singular Stochastic PDEs.* EMS Press, 2022.

[2] Nils Berglund, Giacomo Di Gesù, and Hendrik Weber. An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two. *Electron. J. Probab.*, 22:1–27, 2017.

[3] Nils Berglund and Barbara Gentz. Pathwise description of dynamic pitchfork bifurcations with additive noise. *Probab. Theory Related Fields*, 122(3):341–388, 2002.

[4] Nils Berglund and Barbara Gentz. A sample-paths approach to noise-induced synchronization: stochastic resonance in a double-well potential. *Ann. Appl. Probab.*, 12(4):1419–1470, 2002.

[5] Nils Berglund and Barbara Gentz. Geometric singular perturbation theory for stochastic differential equations. *J. Differential Equations*, 191:1–54, 2003.

[6] Nils Berglund and Barbara Gentz. *Noise-induced phenomena in slow-fast dynamical systems. A sample-paths approach*. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2006.

[7] Nils Berglund and Rita Nader. Stochastic resonance in stochastic pdes. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2022.

[8] Gérard Bourdaud. Le calcul symbolique dans certaines algèbres de type Sobolev. In *Recent developments in fractals and related fields*, Appl. Numer. Harmon. Anal., pages 131–144. Birkhäuser Boston, Boston, MA, 2010.

[9] Y. Bruned, A. Chandra, I. Chevyrev, and M. Hairer. Renormalising SPDEs in regularity structures. *arXiv:1711.10239*, pages 1–85, 2017. To appear in J. Eur. Math. Soc.

[10] Y. Bruned, M. Hairer, and L. Zambotti. Algebraic renormalisation of regularity structures. *Invent. Math.*, 215(3):1039–1156, Mar 2019.

[11] Ajay Chandra and Martin Hairer. An analytic BPHZ theorem for regularity structures. Preprint `arXiv:1612.08138`, 2016.

[12] Giuseppe Da Prato and Arnaud Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.

[13] Giuseppe Da Prato and Luciano Tubaro. Wick powers in stochastic PDEs: an introduction. Technical Report UTM 711, University of Trento, 2007.

[14] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[15] Martin Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[16] Martin Hairer. Introduction to Malliavin calculus. Lecture notes, available at `http://hairer.org/notes/Malliavin.pdf`, 2021.

[17] Martin Hairer and Jonathan Mattingly. The strong Feller property for singular stochastic PDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1314–1340, 2018.

[18] Martin Hairer and Hendrik Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(1):55–92, 2015.

[19] Svante Janson. *Gaussian Hilbert spaces*, volume 129 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, reprint of the 1997 hardback ed. edition, 2008.

[20] Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[21] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[22] Giovanni Peccati and Murad S. Taqqu. *Wiener chaos: moments, cumulants and diagrams*, volume 1 of *Bocconi & Springer Series*. Springer, Milan; Bocconi University Press, Milan, 2011. A survey with computer implementation, Supplementary material available online.

[23] Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Ergodicity for the stochastic quantization problems on the 2D-torus. *Comm. Math. Phys.*, 352(3):1061–1090, 2017.

[24] Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications. *J. Funct. Anal.*, 272(10):4263–4303, 2017.

[25] A. N. Tihonov. Systems of differential equations containing small parameters in the derivatives. *Mat. Sbornik N. S.*, 31:575–586, 1952.

[26] Pavlos Tsatsoulis and Hendrik Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Ann. Inst. H. Poincaré Probab. Statist.*, 54(3):1204–1249, 08 2018. `arXiv:1609.08447`.

[27] Pavlos Tsatsoulis and Hendrik Weber. Exponential loss of memory for the 2-dimensional Allen–Cahn equation with small noise. *Probability Theory and Related Fields volume*, 177:257–322, 2020. `arXiv:1808.04171`.

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