The dynamic saddle–node bifurcation with noise on the slow variable

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Abstract

In this work, we analyse the effect of adding Gaussian white noise to the slow variable of a slow–fast system passing through a saddle–node (or fold) bifurcation. This problem is mainly motivated by applications to non-equilibrium energy sinks. While the effect of adding noise to the fast variable, which is important for noise-induced tipping, has been previously analysed in detail, the case where the slow variable is perturbed by noise has not been considered before. Our main result is that the noise increases the slow variable on average. We compute the effect of the noise, to lowest order, on the expectation and variance of the slow variable after the bifurcation. The contribution of the noise can be explicitly expressed in terms of Airy functions. We also provide numerical simulations, which show that the expansion to lowest order matches the observations for fairly large values of the noise intensity.

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1 Introduction

In this work, we are interested in two-dimensional slow–fast systems near a saddle–node, or fold bifurcation, in the presence of noise. In the deterministic case, the study of these systems goes back to the works [21] and [10]. If ε denotes the time scale separation, the fast variable is known to track the stable part of the critical manifold at a distance that grows up to order $\varepsilon^{1/3}$ when approaching the bifurcation point. After a delay of order $\varepsilon^{2/3}$, the fast variable quickly exits the neighbourhood of the bifurcation point.

The case when Gaussian white noise is added to the fast variable has been studied in [3] (see also [4, Section 3.3]). The main effect in that case is the existence of a threshold for the noise intensity, scaling like $\sqrt{\varepsilon}$. When the noise intensity is below this threshold, the system behaves with high probability like the deterministic system. However, when the noise intensity exceeds the threshold, the fast variable is likely to cross the unstable critical manifold some time before reaching the bifurcation point. This has important consequences on noise-induced tipping [17] and early warning signs [24], and has been further investigated in many works, see for instance [15, 18, 16, 12, 11, 13, 23].

By contrast, in this work we are interested in the case where noise is added to the slow variable of the system, which leads to a very different behaviour. This study is motivated by the investigation of nonlinear energy sinks (NESs) [8, 25], particularly when they are employed to mitigate self-sustained oscillations. In this context, the dynamical system under consideration generally consists of a mechanical self-sustained oscillator (SSO) – whose oscillations we seek to attenuate – coupled with the NES, i.e., a small mass connected to the SSO via a linear damper and an essentially nonlinear spring (typically purely cubic). After applying an averaging procedure, the coupled

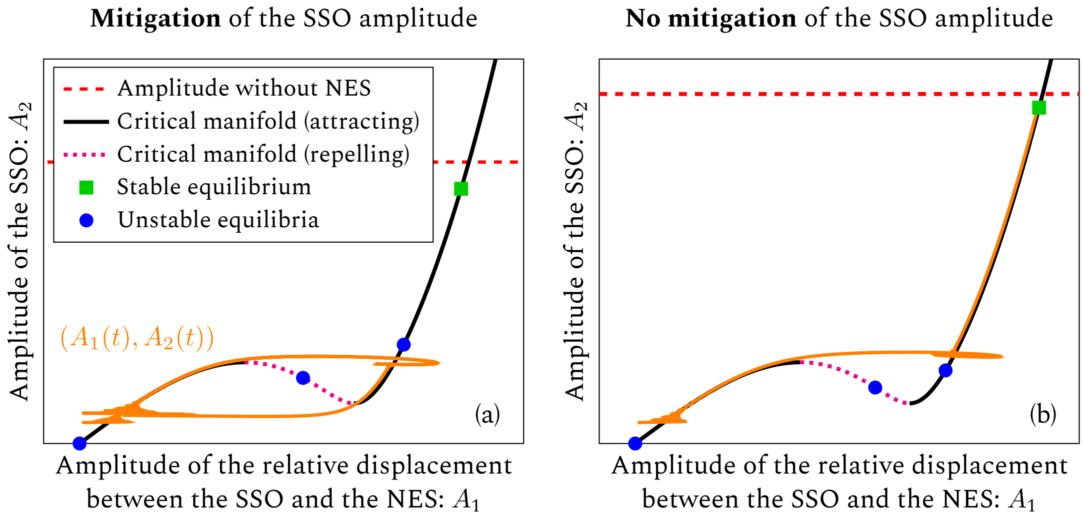

Figure 1: Behavior of the deterministic SSO–NES system in the vicinity of the tipping toward unmitigated responses. (a) Parameter below the tipping value (mitigated response): the system exhibits relaxation oscillations whose amplitudes remain smaller than those of the self-sustained oscillations that would occur without the NES. (b) Parameter above the tipping value (unmitigated response): the trajectory converges to a stable equilibrium, corresponding to an SSO amplitude close to that observed in the absence of the NES.

system reduces to a (2,1)-fast–slow system. The two fast variables relate to the amplitude of the relative displacement between the SSO and the NES, and the phase difference between this relative displacement and the displacement of the SSO. The slow variable is related to the amplitude of the SSO motion. As presented in Figure 1, the deterministic dynamics of this fast–slow system are primilary governed (i) by its S-shaped one-dimensional critical manifold and (ii) by its equilibria, whose stability and position on the critical manifold depend on a bifurcation parameter. In particular, certain values of this parameter result in a configuration supporting relaxation oscillations with amplitudes smaller than those of the self-sustained oscillations that would occur without the NES (see Figure 1(a)). For larger values of the parameter, these relaxation oscillations are no longer possible, and a stable equilibrium is reached, corresponding to an SSO amplitude close to that observed in the absence of the NES (see Figure 1(b)). It therefore appears that the position of the trajectory's endpoint on the right-hand attracting branch of the critical manifold, together with the location of the rightmost unstable equilibrium on this branch, determines which situation occurs: if the endpoint lies below the equilibrium, mitigation occurs; if it lies above, there is no mitigation. This means that, for a certain value of the bifurcation parameter, a tipping occurs between the two situations. In the seminal works of Gendelman *et al.* [9, 7], a multiple time scales analysis [19] is used to estimate the tipping value of the parameter by (i) obtaining analytically the equilibria and (ii) approximating the endpoint's ordinate with that of the critical manifold's left fold point, where the manifold loses stability via a saddle–node bifurcation. This approximation, sometimes referred to as the zero-order approximation, does not account for the particular dynamics that occur near the fold points, where the normal hyperbolicity of the critical manifold is lost. By means of the center manifold theorem, Bergeot [1] addressed this limitation by reducing the dynamics near the left fold point to the normal form of a dynamic saddle-node bifurcation, which can be solved analytically. This approach allowed the author to provide an improved theoretical estimate of the parameter's tipping value.

The influence of noise on this class of systems was investigated in [2], where a small-amplitude white-noise forcing was applied to the SSO. Using stochastic averaging – following Roberts and Spanos [22] – the study first demonstrated that the SSO–NES stochastic system can be accurately reduced to a (2,1)-fast–slow system with noise acting only the slow variable, the drift part being approximated by the corresponding deterministic system. A Monte Carlo simulation approach then showed that the stochastic forcing can significantly alter the dynamic behavior of the corresponding deterministic system, in particular by rendering certain configurations dangerous (i.e., unmitigated) that were safe in the deterministic case. The results also reveal complex stochastic dynamics, showing that the reasoning used to predict the system's deterministic behavior can fail in the presence of noise. The present work paves the way for identifying the underlying mechanisms that govern the stochastic behavior. To do this, we exploit the fact that, near the left fold point of the deterministic critical manifold, the SSO–NES stochastic system reduces to the normal form of a dynamic saddle–node bifurcation (as done for the deterministic system in [1]), with noise acting on the slow variable.

Our main result, Theorem 2.1, provides the first relevant terms of an expansion in the noise intensity of the first two moments of the slow variable after passing through the bifurcation. The most notable feature is that the expectation of the slow variable is increased by the noise. In terms of the SSO–NES system, this suggests that noise may slightly promote the unmitigated responses. While the effect is relatively small for weak noise, since the expectation changes by an amount proportional to the variance of the noise, numerical simulations indicate that our expansion is accurate for fairly large values of that variance.

The remainder of this article is structured as follows. In Section 2, we give a precise formulation of the mathematical set-up, state and discuss the main results, and present numerical simulations. The remaining sections are devoted to the proofs of the main results. Section 3 gives a brief overview of the structure of the proofs. Section 4 provides a detailed analysis in a small interval of the fast variable. In Section 5, the small intervals are stitched together, to provide the proof of the main theorem on moments of the slow variable. Section 6 contains the proof of an auxiliary result, giving explicit expressions for the moments in terms of Airy functions. Appendix A gives a short summary of some useful properties of Airy functions.

2 Results

2.1 Set-up

We are interested in the slow–fast system

$$
\begin{aligned}
\mathrm{d}\bar{x}_{\bar{t}} &= \frac{1}{\varepsilon}(\bar{y}_{\bar{t}} + \bar{x}_{\bar{t}}^2)\,\mathrm{d}\bar{t} \\
\mathrm{d}\bar{y}_{\bar{t}} &= \mathrm{d}\bar{t} + \bar{\sigma}\,\mathrm{d}\overline{W}_{\bar{t}} \;,
\end{aligned} \tag{2.1}
$$

describing the normal form at a saddle-node bifurcation point. Here $\varepsilon > 0$ is a small parameter measuring time scale separation, and $(\overline{W}_{\bar{t}})_{\bar{t}\geqslant 0}$ is a standard Wiener process describing white noise, while $\bar{\sigma} > 0$ is a small parameter measuring the noise intensity.

The scaling

$$
\bar{t} = \varepsilon^{2/3} t \;, \qquad \bar{x}_{\bar{t}} = \varepsilon^{1/3} x_t \;, \qquad \bar{y}_{\bar{t}} = \varepsilon^{2/3} y_t
$$

results in the system

$$
\begin{aligned}
\mathrm{d}x_t &= (y_t + x_t^2)\,\mathrm{d}t \\
\mathrm{d}y_t &= \mathrm{d}t + \sigma\,\mathrm{d}W_t \;,
\end{aligned} \tag{2.2}
$$

where $(W_t)_{t\geqslant 0}$ is again a standard Wiener process, owing to the scaling property of Brownian motion, and

$$
\sigma = \varepsilon^{1/3}\bar{\sigma}
$$

is a rescaled noise intensity. We will keep in mind the fact that results obtained for the rescaled system (2.2) on a rectangle $[-a, a] \times [-b, b]$ will translate into results for the original system (2.1) on the scaled rectangle $[-\varepsilon^{1/3}a, \varepsilon^{1/3}a] \times [-\varepsilon^{2/3}b, \varepsilon^{2/3}b]$. It is therefore of interest to obtain results for possibly large values of a and b.

The deterministic case $\sigma = 0$ is well-known [21, 10]. The slow–fast system (2.1) has a critical manifold $\{(\bar{x}, \bar{y}) \colon \bar{y} = -\bar{x}^2\}$, separating orbits with increasing and decreasing $\bar{x}$. After scaling, this manifold takes the equivalent form $\{(x, y) \colon y = -x^2\}$.

In the deterministic case $\sigma = 0$, using the theory of Riccati equations, the general solution of (2.2) with initial condition $(x_{\text{in}}, y_{\text{in}})$ can be written

$$\begin{aligned} x^{\text{det}}(t) &= \frac{\text{Ai}'(-y_{\text{in}} - t) + K\,\text{Bi}'(-y_{\text{in}} - t)}{\text{Ai}(-y_{\text{in}} - t) + K\,\text{Bi}(-y_{\text{in}} - t)} \;, \\ y^{\text{det}}(t) &= y_{\text{in}} + t \;, \end{aligned} \tag{2.3}$$

where Ai and Bi are Airy function (cf. Appendix A), and K is determined by the condition $x^{\text{det}}(0) = x_{\text{in}}$. It will sometimes be convenient to write, with a slight abuse of notation,

$$x^{\text{det}}(y) = \frac{\text{Ai}'(-y) + K\,\text{Bi}'(-y)}{\text{Ai}(-y) + K\,\text{Bi}(-y)} \tag{2.4}$$

for the solution parametrised in terms of the y coordinate. We will be particularly interested in the case $K = 0$, where we have the particular solution (called *slow solution* in the case of the unscaled system (2.1))

$$\begin{aligned} x_0^{\text{det}}(t) &= \frac{\text{Ai}'(-y_{\text{in}} - t)}{\text{Ai}(-y_{\text{in}} - t)} , \\ y_0^{\text{det}}(t) &= y_{\text{in}} + t \;. \end{aligned} \tag{2.5}$$

This is because the asymptotics (A.2) of Airy functions imply that $(x_0^{\text{det}}(t), y_0^{\text{det}}(t))$ converges to the critical manifold as $t \to -\infty$. On the other hand, we have

$$\lim_{t \to t^\star} x_0^{\text{det}}(t) = +\infty \;, \qquad \lim_{t \to t^\star} y_0^{\text{det}}(t) = y^\star \;,$$

where $t^\star = y^\star - y_{\text{in}}$, and

$$y^\star = 2.338107410459767\ldots \tag{2.6}$$

is the negative of the largest zero of the Airy function Ai (cf. the Online Encyclopedia of Integer Sequences `OEIS A096714`).

Figure 2 shows an example of the general solution (2.3) of (2.2) with $\sigma = 0$, i.e., the parametric curve $\big(x^{\text{det}}(t), y^{\text{det}}(t)\big)$ for the initial condition $(x_{\text{in}}, y_{\text{in}}) = (-3, -2)$. The figure also displays the reference trajectory $\big(x_0^{\text{det}}(t), y_0^{\text{det}}(t)\big)$ (see (2.5)), the critical manifold $y = -x^2$, and the limit value $y^\star$.

2.2 Main result

To formulate our main result, we fix an initial value $(x_{\text{in}}, y_{\text{in}})$ with $x_{\text{in}} < 0$ and $x_{\text{in}}^2 + y_{\text{in}} > 0$, as well as a final value $x_{\text{fin}} > 0$. Given the solution (x_t, y_t) of (2.2) with this initial condition, we denote by

$$\tau = \inf\big\{t > 0 \colon x_t = x_{\text{fin}}\big\} \tag{2.7}$$

the first-hitting time of x_{fin}. We are interested in properties of the random variable y_τ. We will also need the deterministic analogue of (2.7), given by

$$T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \inf\big\{t > 0 \colon x^{\text{det}}(t) = x_{\text{fin}}\big\} \;, \tag{2.8}$$

where $(x^{\text{det}}(t), y^{\text{det}}(t))$ denotes the deterministic solution (2.3) of the system (2.2) with the same initial condition $(x_{\text{in}}, y_{\text{in}})$. We write

$$y_{\text{fin}} = y^{\text{det}}(T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})) = y_{\text{in}} + T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$$

for the final value of y^{det}.

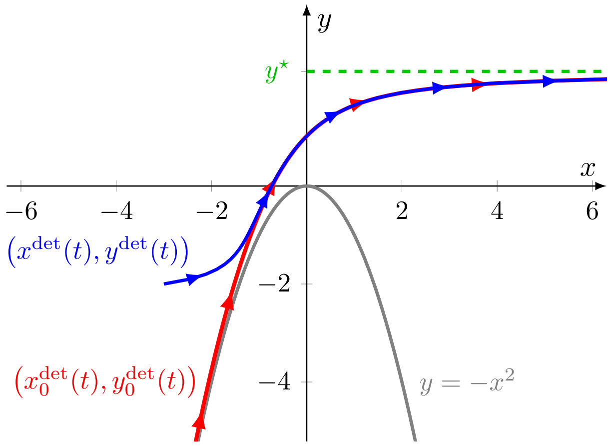

Figure 2: Dynamic saddle-node bifurcation. An example of the general solution $\big(x^{\rm det}(t), y^{\rm det}(t)\big)$ of (2.2) (see (2.3)) for the initial condition $(x_{\rm in}, y_{\rm in}) = (-3, -2)$, the reference trajectory $\big(x_0^{\rm det}(t), y_0^{\rm det}(t)\big)$ (see (2.5)), the critical manifold $y = -x^2$, and the limit value $y^\star$ (see (2.6)).

Theorem 2.1. *Fix a constant $h_0 > 0$. Let Ω_0 be the event*

$$\Omega_0 = \left\{\omega\colon \sup_{0\leqslant t\leqslant\tau} \big|y_t(\omega) - y^{\rm det}(t)\big| \leqslant h_0\right\} . \tag{2.9}$$

There exists a constant $\kappa > 0$, depending only on h_0, such that

$$\mathbb{P}(\Omega_0^c) \leqslant \frac{y_{\rm fin} - y_{\rm in}}{\sigma}\, {\rm e}^{-\kappa/\sigma} . \tag{2.10}$$

Furthermore, the expectation and variance of $y_\tau 1_{\Omega_0}$ satisfy

$$\mathbb{E}^{(x_{\rm in}, y_{\rm in})}\big[y_\tau \mathbf{1}_{\Omega_0}\big] = y_{\rm fin} + \frac{1}{2}\sigma^2 D(x_{\rm in}, y_{\rm in}; x_{\rm fin}) + \mathcal{O}(\sigma^3) , \tag{2.11}$$

$$\operatorname{Var}^{(x_{\rm in}, y_{\rm in})}\big[y_\tau \mathbf{1}_{\Omega_0}\big] = \sigma^2 V(x_{\rm in}, y_{\rm in}; x_{\rm fin}) + \mathcal{O}(\sigma^3) . \tag{2.12}$$

The functions D and V are given by

$$\begin{aligned} D(x_{\rm in}, y_{\rm in}; x_{\rm fin}) &= \int_{y_{\rm in}}^{y_{\rm fin}} \partial_{yy} T(x^{\rm det}(y), y; x_{\rm fin})\, {\rm d}y , \\ V(x_{\rm in}, y_{\rm in}; x_{\rm fin}) &= \int_{y_{\rm in}}^{y_{\rm fin}} \big(1 + \partial_y T(x^{\rm det}(y), y; x_{\rm fin})\big)^2\, {\rm d}y , \end{aligned} \tag{2.13}$$

where we write $\partial_y T$ and $\partial_{yy} T$ for derivatives of the deterministic time (2.8) with respect to its second argument $y_{\rm in}$.

The functions D and V can be made more explicit if we assume that the initial condition lies on the slow solution (2.5), that is, if

$$x_{\rm in} = x_0^{\rm det}(0) = \frac{{\rm Ai}'(-y_{\rm in})}{{\rm Ai}(-y_{\rm in})} . \tag{2.14}$$

Note that the slow solution is attracting for $x < 0$, so that small changes in $x_{\rm in}$ will have almost no effect on the result.

Proposition 2.2. *If x_{in} is given by (2.14), then*

$$\lim_{x_{\text{fin}}\to\infty} D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{3}{4} + \frac{1}{\text{Ai}'(-y^\star)^2}\left[2\pi\left(\frac{\text{Bi}'(-y^\star)}{\text{Ai}'(-y^\star)}\mathcal{F}(-y_{\text{in}}) - \mathcal{G}(-y_{\text{in}})\right) - f(-y_{\text{in}})\right] ,$$
$$\lim_{x_{\text{fin}}\to\infty} V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{1}{2}y^\star + \frac{\mathcal{F}(-y_{\text{in}})}{\text{Ai}'(-y^\star)^4} , \tag{2.15}$$

where

$$f(z) = \text{Ai}'(z)^2 - z\,\text{Ai}(z)^2 , \tag{2.16}$$
$$\mathcal{F}(z) = \left(\frac{z^3}{2} + \frac{1}{8}\right)\text{Ai}(z)^4 - \frac{z}{2}\,\text{Ai}(z)^3\,\text{Ai}'(z) - z^2\,\text{Ai}(z)^2\,\text{Ai}'(z)^2 + \frac{1}{2}\,\text{Ai}(z)\,\text{Ai}'(z)^3 + \frac{z}{2}\,\text{Ai}'(z)^4 , \tag{2.17}$$
$$\mathcal{G}(z) = \left[\left(\frac{z^3}{2} + \frac{1}{8}\right)\text{Ai}(z)^3 - \frac{3z}{8}\,\text{Ai}(z)^2\,\text{Ai}'(z) - \frac{z^2}{2}\,\text{Ai}(z)\,\text{Ai}'(z)^2 + \frac{1}{8}\,\text{Ai}'(z)^3\right]\text{Bi}(z) + \left[-\frac{z}{8}\,\text{Ai}(z)^3 - \frac{z^2}{2}\,\text{Ai}(z)^2\,\text{Ai}'(z) + \frac{3}{8}\,\text{Ai}(z)\,\text{Ai}'(z)^2 + \frac{z}{2}\,\text{Ai}'(z)^3\right]\text{Bi}'(z) . \tag{2.18}$$

In particular, the asymptotics of Airy functions (cf. (A.2)) imply

$$\lim_{x_{\text{in}}\to-\infty}\lim_{x_{\text{fin}}\to\infty} D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{3}{4} ,$$
$$\lim_{x_{\text{in}}\to-\infty}\lim_{x_{\text{fin}}\to\infty} V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{1}{2}y^\star = 1.169053705229883\dots . \tag{2.19}$$

Furthermore,

$$\lim_{x_{\text{fin}}\to\infty} D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) > 0$$

for any $x_{\text{in}} \in \mathbb{R}$ (or equivalently, for any $y_{\text{in}} < y^\star$).

Figure 3 shows graphs of the explicit expressions for the limits of the functions D and V as $x_{\text{fin}} \to \infty$, as given in equation (2.15). These limits tend to $\frac{3}{4}$ and $\frac{1}{2}\,y^\star$, respectively, when $x_{\text{in}} \to -\infty$, in agreement with equation (2.19).

Remark 2.3. We also have explicit expressions for the functions D and V when x_{fin} is finite, but they are more complicated, see Proposition 6.1. ♢

2.3 Discussion

The most interesting feature of our results is that the expectation of y_τ increases with the noise intensity σ, since D is positive. Stricly speaking, we have proved this only in the limit $y_{\text{fin}} \to y^\star$, but numerical simulations indicate that $\partial_{yy}T$ is always positive. One caveat of our result is that the expansion in powers of σ is only guaranteed to be accurate for small σ. In that case, the expected difference, of order σ^2, is much smaller than the standard deviation of y_τ, which has order σ. Therefore, the probability of having $y_\tau > y_\tau^{\text{det}}$ should be roughly $\frac{1}{2} + \mathcal{O}(\sigma)$, which is a small effect. This is one of the reasons the analysis is quite technical.

However, in the next section we will show numerical simulations that indicate that the expansions (2.11) and (2.12) are actually surprisingly accurate, even for $\sigma = 1$. In that case, the expected difference and standard deviation are of comparable size, to that the effect of the noise is no longer small.

One may wonder whether the presence of the indicator function $\mathbf{1}_{\Omega_0}$ is necessary, or only a technical limitation. The answer is that the indicator is indeed important. The main reason is that if a sample path exits the h_0-neighbourhood of the deterministic solution, it has a larger probability

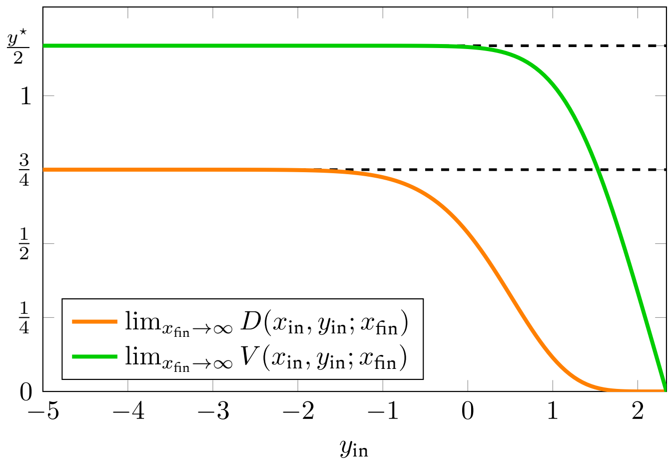

Figure 3: Plots of the explicit limit expressions of the functions D and V as $x_{\text{fin}} \to \infty$, derived in equation (2.15), as a function of y_{in}.

to reach the critical manifold $y = -x^2$. In that case, the sample path may move to the left for a while, before crossing the critical manifold again and moving to the right. This is a rare event, but it can potentially have a large effect on expectations, which we do not take into account.

One question of interest is about the effect of undoing the scaling transforming the system (2.1) into the system (2.2). The change in expectation now becomes of order $\varepsilon^{4/3}\bar{\sigma}^2$, while the variance has order $\varepsilon^2\bar{\sigma}^2$. One may also wonder whether our results are only valid for x_{in} and x_{fin} of order 1, or whether they can be extended to x_{in} and x_{fin} of order $\varepsilon^{-1/3}$, as would be required to describe the system for $(\bar{x}, \bar{y})$ of order 1. While we do not provide here a rigorous proof of that fact, we expect this extension to be possible, since the quantities D and V vary little when $|x_{\text{in}}|$ and x_{fin} are large, and the error terms in Proposition 4.12 actually decrease when x^2 increases. The main effect is that the prefactor in the probability (2.10) would scale like $(\bar{y_{\text{fin}}} - \bar{y_{\text{in}}})/(\varepsilon\bar{\sigma})$, which is negligible compared to the exponential term $\mathrm{e}^{-\kappa/\sigma} = \mathrm{e}^{-\kappa/(\varepsilon^{1/3}\bar{\sigma})}$.

For the SSO–NES system motivating this work, our results show that after the trajectories pass the left fold point of the critical manifold (see Figure 1), the slow variable – associated with the amplitude of the SSO – has an expectation larger than the corresponding deterministic solution, with the difference becoming larger as the noise intensity increases. This suggests that noise may promote unmitigated responses, increasing the probability that trajectories end on the right-hand attracting branch above the rightmost unstable equilibrium (Figure 1b). This hypothesis will require future verification through numerical and theoretical analyses. The numerical results in the following section show that we can be confident in the validity of our results even for large values of σ. However, the central manifold theorem, which allows the reduction to the normal form of a dynamic saddle–node bifurcation, is valid only near the left fold point. For the deterministic system, this reduction remains relevant beyond the immediate vicinity of the fold, particularly concerning the trajectory's arrival on the right-hand attracting branch of the critical manifold (see Figure 5 of [1]), which is crucial for determining if mitigation occurs. Whether this extends to the stochastic case will be explored in future work, taking into account the stochastic slow dynamics after the system reaches the right-hand attracting branch, which also appears to influence the system's tipping, as suggested by results in [2].

2.4 Numerical simulations

In this section the results stated in Theorem 2.1 and Proposition 2.2 are compared with numerical simulations. For this purpose, the statistics $\mathbb{E}^{(x_{\text{in}},y_{\text{in}})}[y_\tau]$ and $\operatorname{Var}^{(x_{\text{in}},y_{\text{in}})}[y_\tau]$ are numerically estimated via a Monte Carlo approach for 20 noise levels σ, ranging from 0.05 to 1, with $3 \cdot 10^5$ realizations of system (2.2) generated for each value. Note that the indicator function $\mathbf{1}_{\Omega_0}$ is not

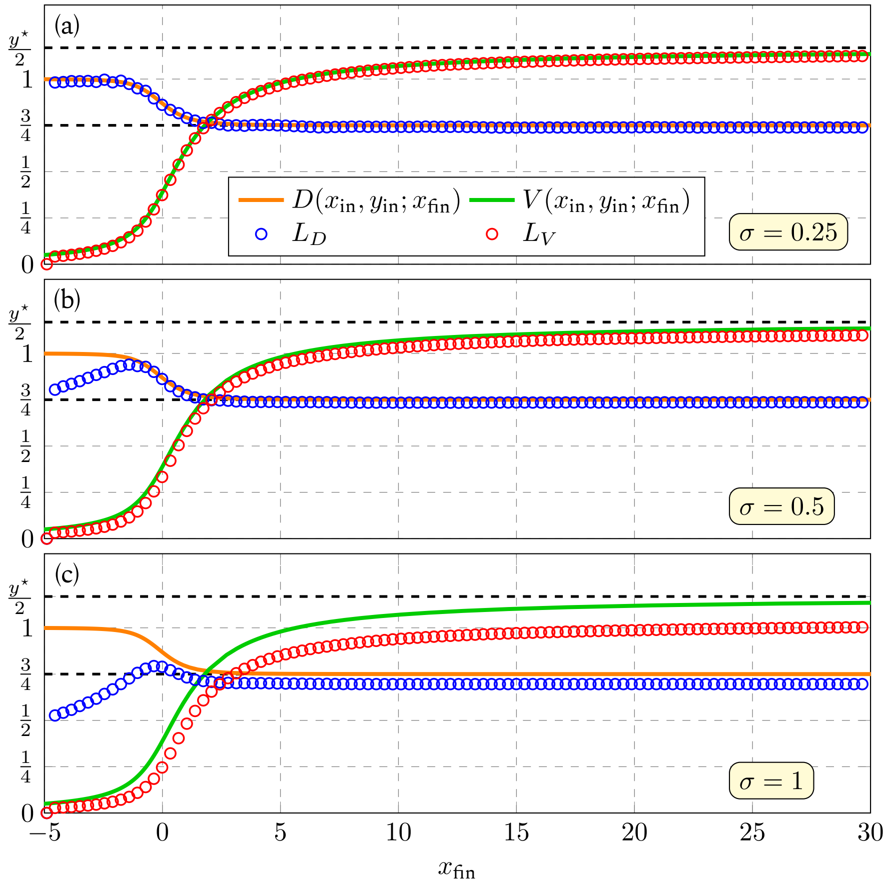

Figure 4: Comparison between the results of Theorem 2.1 and numerical simulations. Numerically estimated statistics L_D and L_V (see equation (2.20)) and their analytical counterparts, $D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ and $V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ (see equation (2.13)), are plotted as functions of x_{fin} for (a) $\sigma = 0.25$, (b) $\sigma = 0.5$ and (c) $\sigma = 1$.

used here, since the expectation and variance are computed numerically over all generated samples. This should not make any difference for small σ, since $\mathbb{P}(\Omega_0^c)$ is exponentially small, but may have an effect for larger noise levels. The following initial conditions are used: $x_{\text{in}} = -5$ and $y_{\text{in}} = -x_{\text{in}}^2 + 0.1$ (i.e. near the critical manifold $y = -x^2$). The simulation of the process (2.2) is performed using an Euler-Maruyama scheme (see e.g. [14]) with an integration time step equal to 10^{-5}. The required standard normal random variables are generated using the Box–Muller algorithm.

In Figure 4, the integral forms (2.13) of the functions $D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ and $V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ are plotted with respect to x_{fin} and compared with their numerically estimated counterparts

$$L_D = \frac{2}{\sigma^2}\left[\mathbb{E}^{(x_{\text{in}},y_{\text{in}})}[y_\tau] - y_{\text{fin}}\right] \quad \text{and} \quad L_V = \frac{1}{\sigma^2}\,\text{Var}^{(x_{\text{in}},y_{\text{in}})}[y_\tau]\,, \tag{2.20}$$

respectively. Only the results obtained for $\sigma = 0.25$ (Figure 4(a)), $\sigma = 0.5$ (Figure 4(b)) and $\sigma = 1$ (Figure 4(c)) are shown here.

Figure 5 displays, as functions of σ for a fixed value $x_{\text{fin}} = 30$, the quantities

$$M_D = \frac{3}{8}\left[\mathbb{E}^{(x_{\text{in}},y_{\text{in}})}[y_\tau] - y_{\text{fin}}\right] \quad \text{and} \quad M_V = \frac{2}{y^\star}\,\text{Var}^{(x_{\text{in}},y_{\text{in}})}[y_\tau]\,, \tag{2.21}$$

which, according to (2.19), are theoretically equal to $\sigma^2 + \mathcal{O}(\sigma^3)$. The goal is to investigate the range of validity of the explicit asymptotic expressions for $D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ and $V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ given in

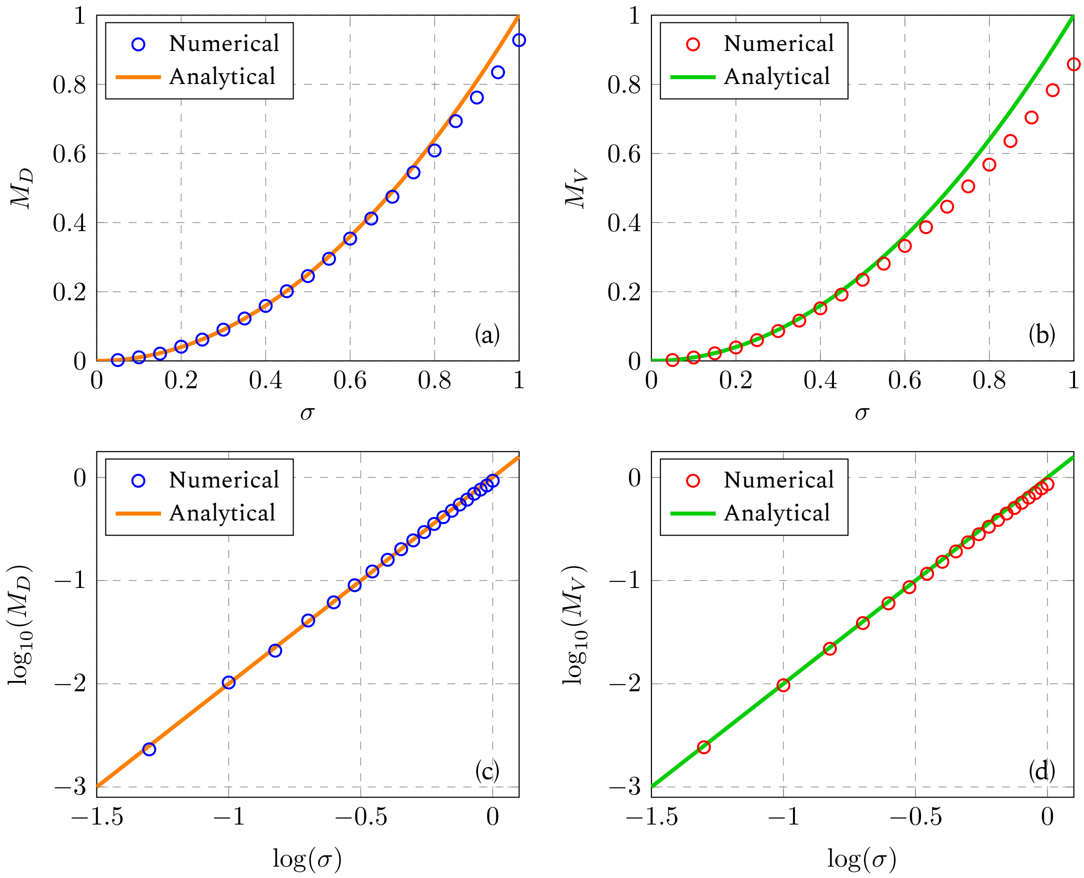

Figure 5: Comparison between Proposition 2.2 and numerical simulations. (a,b) Numerical estimates of M_D and M_V (see equation (2.21)) compared to their analytical counterpart σ^2 as functions of σ for $x_{\text{fin}} = 30$. (c,d) $\log_{10}(M_D)$ and $\log_{10}(M_V)$ plotted against $\log_{10}(\sigma)$, which theoretically yields a straight line with slope 2.

equation (2.19). The statistics (2.21) are first directly plotted as functions of σ in Figures 5(a,b) and compared to the theoretical parabola σ^2. Moreover, the root mean square error (RMSE) between theory and data is computed for both M_D and M_V, we obtain 0.0270897 and 0.0576924, respectively. Finally, $\log_{10}(M_D)$ and $\log_{10}(M_V)$ are plotted in Figures 5(c,d) with respect to $\log_{10}(\sigma)$ which theoretically correspond to a line with slope equal to 2. Again, RSME are computed between theory and data leading to 0.0185379 and 0.0380149 for $\log_{10}(M_D)$ and $\log_{10}(M_V)$, respectively.

These comparisons show excellent agreement between the analytical and numerical estimates, even for values of σ that appear to lie beyond the validity range of the small-σ expansions used in this study. It should, however, be noted (see Figure 4) that the discrepancy between the numerical and analytical results, as σ increases, is greater when $x_{\text{fin}} < 0$. The reason for this discrepancy is not clear at this point – a possible explanation is that it has to do with the indicator function $\mathbf{1}_{\Omega_0}$. As it only occurs for rather large σ, however, this does not contradict our asymptotic results. Also, it does not correspond to the most relevant scenarios, particularly in the context of our SSO–NES application.

3 Structure of the proofs

To prove Theorem 2.1, we start by fixing an initial condition $(x_{\text{in}}, y_{\text{in}})$ satisfying $x_{\text{in}}^2 + y_{\text{in}} > 0$, and a final value $x_{\text{fin}} > x_{\text{in}}$. The main idea is to introduce a partition

$$x_{\text{in}} = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = x_{\text{fin}}$$

of the interval $[x_{\text{in}}, x_{\text{fin}}]$, and to study the equation (2.2) separately on each interval. The partition could be chosen equally spaced, but the proof simplifies slightly if it is chosen "equally spaced in y", meaning that the deterministic solution starting in $(x_{\text{in}}, y_{\text{in}})$ reaches $x = x_n$ when $y(t) = y_n^{\text{det}} = y_{\text{in}} + n\gamma$, where $\gamma = (y_{\text{fin}} - y_{\text{in}})/N$. This means that the deterministic reference solution always takes the same time γ to go from one section to the next one. Let

$$\tau_n = \inf\{t > 0 \colon x_t = x_n\}$$

be the first-hitting time of the line $\{x = x_n\}$. By the strong Markov property, to obtain the distribution of $y_\tau = y_{\tau_N}$, it is sufficient to know, for each $n \in \{0, 1, \dots, N-1\}$, the distribution of $y_{\tau_{n+1}}$, conditionned on the value of y_{τ_n}.

The analysis of a single "slice" $[x_n, x_{n+1}]$ is performed in detail in Section 4. The main result is Proposition 4.12, which provides moment estimates related to $y_{\tau_{n+1}}$, given the value of y_{τ_n}. To obtain these estimates, we first analyse the distribution of the time required to reach the section $\{x = x_{n+1}\}$, and then use this information to describe the exit location.

Section 5 provides the proof of Theorem 2.1 by combining the information obtained on each separate slice. This yields a recursive relation between moments, that is solved by going backwards in time, starting with the interval $[x_{N-1}, x_N]$ and going all the way to $\{x = x_0\}$. The moments can be written as sums over n, that are then approximated by integrals.

Finally, Section 6 contains an analysis of the functions D and V that appear in Theorem 2.1, and provides the proof of Proposition 2.2. This relies on properties of Airy's equation and Airy functions, that are used to derive explicit expressions for $\partial_y T$ and $\partial_{yy} T$ and analyse their integrals.

4 Analysis of one slice

In this section, we analyse the SDE (2.2) on one "slice" of width $\delta > 0$. That is, we fix an initial condition (x_0, y_0) satisfying

$$x_0^2 + y_0 > 0 ,$$

and define the stopping time

$$\tau = \inf\bigl\{t > 0 \colon x_t = x_0 + \delta\bigr\} . \tag{4.1}$$

We will always assume that $x_0^2 + y_0$ is bounded below by a positive constant independent of δ. We consider δ as a small parameter, but such that $\sigma^2 \ll \delta$.

The aim is to characterise the distribution of y_τ, in particular its expectation and its variance. We divide this task into several steps:

- In Section 4.1, we perform a time change and a linearisation around the deterministic solution with the same initial condition, in order to reduce the problem to a standardised first-hitting problem, namely of the first time integrated Brownian motion hits a curve that crosses the x-axis with slope -1.
- In Section 4.2, we consider the linearised equation. We determine a sharp approximation of the density of the first-hitting time for this linearised process, based on results by Durbin on first-hitting densities of Gaussian processes to general, curved boundaries [6].
- In Section 4.3, we use the results of the previous section to determine the expectation and variance of the first-hitting time of the linearised process.
- In Section 4.4, we quantify the error made by linearising the equation.
- Finally, in Section 4.5, we derive properties of the distribution of y_τ.

4.1 Time change

Let

$$u_t = x_t - x^{\text{det}}(t)$$

denote the difference between the x-components of the solutions of the SDE (2.2) and its deterministic counterpart, for the same initial condition (x_0, y_0). It satisfies the equation

$$\mathrm{d}u_t = (\sigma W_t + 2x^{\text{det}}(t)u_t + u_t^2)\,\mathrm{d}t\;,$$

with initial condition $u_0 = 0$. The stopping time (4.1) can be rewritten

$$\tau = \inf\bigl\{t > 0 \colon u_t = d(t)\bigr\}\;,$$

where

$$d(t) = \delta + x_0 - x^{\text{det}}(t)\;.$$

In the deterministic case $\sigma = 0$, τ is equal to the deterministic value

$$\begin{aligned} T(x_0, y_0) = T(x_0, y_0; x_0 + \delta) &= \inf\bigl\{t > 0 \colon x^{\text{det}}(t) = x_0 + \delta\bigr\} \\ &= \inf\bigl\{t > 0 \colon d(t) = 0\bigr\}\;. \end{aligned}$$

Since

$$x^{\text{det}}(t) = x_0 + (x_0^2 + y_0)t + \mathcal{O}(t^2)\;,$$

one has

$$d(t) = \delta - (x_0^2 + y_0)t + \mathcal{O}(t^2)\;, \tag{4.2}$$

and therefore

$$T(x_0, y_0) = \frac{\delta}{x_0^2 + y_0} + \mathcal{O}(\delta^2)\;. \tag{4.3}$$

We focus first on the linearised equation

$$\mathrm{d}u_t^0 = (\sigma W_t + 2x^{\text{det}}(t)u_t^0)\,\mathrm{d}t\;,$$

again with initial condition $u_0^0 = 0$. Its solution can be written

$$u_t^0 = \sigma\,\mathrm{e}^{2\alpha(t)} \int_0^t \mathrm{e}^{-2\alpha(s)}\,W_s\,\mathrm{d}s\;,$$

where

$$\alpha(t) = \int_0^t x^{\text{det}}(s)\,\mathrm{d}s = x_0 t + \mathcal{O}(t^2)\;. \tag{4.4}$$

Let

$$\begin{aligned} \tau^0 &= \inf\bigl\{t > 0 \colon u_t^0 = d(t)\bigr\} \\ &= \inf\biggl\{t > 0 \colon \sigma \int_0^t \mathrm{e}^{-2\alpha(s)}\,W_s\,\mathrm{d}s = \mathrm{e}^{-2\alpha(t)}\,d(t)\biggr\}\;. \end{aligned}$$

The following result shows that up to a time change, we can always assume that $d(t)$ vanishes in $t = \delta$, where its slope is close to -1, and that u_t^0 is proportional to integrated Brownian motion.

Proposition 4.1. *For sufficiently small δ, there exists an invertible time change*

$$t = cg(\tilde{t}) = c\bigl[\tilde{t} + \mathcal{O}(\tilde{t}^2)\bigr]\;,$$

with $c > 0$ and g a strictly increasing function, close to identity, such that

$$\tau^0 = cg(\tilde{\tau}^0)\;, \tag{4.5}$$

where

$$\tilde{\tau}^0 = \inf\{\tilde{t} > 0 \colon \tilde{\sigma} X_{\tilde{t}} = \tilde{d}(\tilde{t})\} . \tag{4.6}$$

Here

$$\tilde{\sigma} = \sigma c^{3/2}(1 + \mathcal{O}(\delta)) , \tag{4.7}$$

$X_{\tilde{t}}$ is the integrated Brownian motion

$$X_{\tilde{t}} = \int_0^{\tilde{t}} W_s \, \mathrm{d}s ,$$

and $\tilde{d}$ is a strictly decreasing function satisfying

$$\tilde{d}(\delta) = 0 , \qquad \tilde{d}'(\delta) = -1 . \tag{4.8}$$

Furthermore, $\tilde{d}'(\tilde{t}) = -1 + \mathcal{O}(\delta)$ whenever $\tilde{t} = \mathcal{O}(\delta)$.

PROOF: Assuming that g is indeed strictly increasing, we can write $\tau^0 = cg(\tilde{\tau}^0)$ where

$$\tilde{\tau}^0 = \inf\{\tilde{t} > 0 \colon \sigma \tilde{X}_{\tilde{t}} = \tilde{d}(\tilde{t})\} , \tag{4.9}$$

with

$$\widetilde{X}_{\tilde{t}} = \int_0^{cg(\tilde{t})} \mathrm{e}^{-2\alpha(s)} \, W_s \, \mathrm{d}s ,$$
$$\tilde{d}(\tilde{t}) = \mathrm{e}^{-2\alpha(cg(\tilde{t}))} \, d(cg(\tilde{t})) .$$

We now perform the change of variables $s = cg(\tilde{s})$ in the integral defining $\widetilde{X}_{\tilde{t}}$. We have $\mathrm{d}s = cg'(\tilde{s}) \, \mathrm{d}\tilde{s}$, and the scaling property of Brownian motion implies that

$$W_{cg(\tilde{s})} = \sqrt{\frac{cg(\tilde{s})}{\tilde{s}}} W_{\tilde{s}} ,$$

where equality is in distribution. Therefore,

$$\widetilde{X}_{\tilde{t}} = c^{3/2} \int_0^{\tilde{t}} g'(\tilde{s}) \sqrt{\frac{g(\tilde{s})}{\tilde{s}}} \, \mathrm{e}^{-2\alpha(cg\tilde{s}))} \, W_{\tilde{s}} \, \mathrm{d}\tilde{s} .$$

We now require that g satisfies the ODE

$$g'(\tilde{s}) = \sqrt{\frac{\tilde{s}}{g(\tilde{s})}} \, \mathrm{e}^{2\alpha(cg\tilde{s}))} \tag{4.10}$$

with initial condition $g(0) = 0$. Then we will have

$$\sigma \tilde{X}_{\tilde{t}} = c^{3/2} \sigma X_{\tilde{t}} ,$$

so that (4.9) is equivalent to (4.6) with $\tilde{\sigma} = c^{3/2}\sigma$. Separating variables in (4.10), we get

$$\int_0^{g(\tilde{s})} \sqrt{g} \, \mathrm{e}^{-2\alpha(cg)} \, \mathrm{d}g = \int_0^{\tilde{s}} \sqrt{\tilde{s}} \, \mathrm{d}\tilde{s} = \frac{2}{3} \tilde{s}^{3/2} .$$

Since the left-hand side is strictly increasing in $g(\tilde{s})$, this equation has a unique solution for any $\tilde{s} \geqslant 0$, which is strictly increasing in $\tilde{s}$, as required. Using (4.4) and a Taylor expansion, one finds

$$g(\tilde{s}) = \tilde{s} + \frac{4}{5} x_0 c \tilde{s}^2 + \mathcal{O}(\tilde{s}^3) . \tag{4.11}$$

It remains to choose the value of c in such a way that $\tilde{d}$ satisfies (4.8). Setting

$$c = \frac{T(x_0, y_0)}{g(\delta)} = \frac{1}{x_0^2 + y_0} + \mathcal{O}(\delta) \tag{4.12}$$

entails $d(cg(\delta)) = 0$, and therefore $\tilde{d}(\delta) = 0$. Furthermore, we obtain

$$\begin{aligned}\tilde{d}'(\delta) &= c\,\mathrm{e}^{-2\alpha(cg(\delta))}\,d'(cg(\delta))g'(\delta) \\ &= cd'(T(x_0,y_0))(1+\mathcal{O}(\delta)) \\ &= -1+\mathcal{O}(\delta)\;,\end{aligned}$$

where we have used (4.2) and (4.4). A similar computation shows that $\tilde{d}'(\tilde{t})$ has the same order whenever $\tilde{t} = \mathcal{O}(\delta)$. We can achieve $\tilde{d}'(\delta) = -1$ by scaling $\tilde{d}(\tilde{t})$ by a factor $1+\mathcal{O}(\delta)$, and modifying the definition of $\tilde{\sigma}$ by an equivalent amount. □

4.2 Linearised equation: density of hitting times

Thanks to Proposition 4.1, we have reduced the problem to the simpler one of characterising the first time integrated Brownian motion crosses a decreasing curve, intersecting the time axis with slope -1. Dropping the tildes for now, we can rewrite this as characterising

$$\tau^0 = \inf\bigl\{t > 0 \colon \sigma X_t = d(t)\bigr\}\;,$$

where X_t is the integrated Brownian motion

$$X_t = \int_0^t W_s\,\mathrm{d}s\;, \tag{4.13}$$

and d is strictly decreasing, with $d(\delta) = 0$ and $d'(\delta) = -1$. As a consequence, we have

$$d(t) = -(t-\delta) + \mathcal{O}((t-\delta)^2)\;. \tag{4.14}$$

We will rely for that on results in [6] on first-passage densities of Gaussian processes through a time-dependent boundary. These results depend on the covariance function of the process, which we compute in the following lemma.

Lemma 4.2. *For any choice of* $0 \leqslant s \leqslant u \leqslant t$*, one has*

$$\begin{aligned}\rho(s,u,t) &:= \mathbb{E}\bigl[(X_u - X_s)(X_t - X_s)\bigr] \\ &= \frac{2}{3}s^3 - \frac{1}{6}u^3 + \frac{1}{2}\bigl[u^2t - s^2u - s^2t\bigr]\;.\end{aligned}$$

PROOF: This follows by writing

$$\rho(s,u,t) = \int_s^u \int_s^t \mathbb{E}\bigl[W_{v_1}W_{v_2}\bigr]\,\mathrm{d}v_1\,\mathrm{d}v_2\;,$$

using the property $\mathbb{E}\bigl[W_{v_1}W_{v_2}\bigr] = v_1 \wedge v_2$ of Brownian motion, and splitting the integral along the line $\{v_1 = v_2\}$. □

In particular, the covariance and variance of X_t are given by

$$\begin{aligned}\mathbb{E}\bigl[X_sX_t\bigr] &= \rho(s,t) := \rho(0,s,t) = -\frac{1}{6}s^3 + \frac{1}{2}s^2t\;, \\ \mathbb{E}\bigl[X_t^2\bigr] &= \rho(t,t) = \frac{1}{3}t^3\;.\end{aligned}$$

The distribution of X_t is centred Gaussian with this covariance, and therefore the density of u_t^0 at $d(t)$ is given by

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi\rho(t,t)}}\exp\biggl\{-\frac{d(t)^2}{2\sigma^2\rho(t,t)}\biggr\}\;. \tag{4.15}$$

By [6, Eq. (2)], the density of the first-passage time τ^0 can be written as

$$\psi(t) = b(t)\phi(t) \;, \tag{4.16}$$

where the function $b(t)$ solves a certain fixed-point equation, and is well-approximated by

$$b_0(t) = \frac{d(t)}{\rho(t,t)}\partial_s\rho(t,t) - d'(t) = \frac{3}{2t}d(t) - d'(t) \;.$$

More precisely, we have from [6, Eq. (7)] that

$$\psi(t) = b_0(t)\phi(t) + \int_0^t \tilde{b}(s,t)\phi(t|s)\psi(s)\,\mathrm{d}s \;, \tag{4.17}$$

where

$$\phi(t|s) = \frac{1}{\sigma\sqrt{2\pi\rho(s,t,t)}} \exp\biggl\{-\frac{[d(t)-d(s)]^2}{2\sigma^2\rho(s,t,t)}\biggr\} \tag{4.18}$$

is analogous to $\phi(t)$, but for a starting point $(s, d(s))$ with $s \in [0,t]$, and

$$\tilde{b}(s,t) = d'(t) - \beta_1(s,t)d(s) - \beta_2(s,t)d(t) \;, \tag{4.19}$$

where β_1 and β_2 solve the linear system

$$\begin{pmatrix} \rho(s,s) & \rho(s,t) \\ \rho(s,t) & \rho(t,t) \end{pmatrix} \begin{pmatrix} \beta_1(s,t) \\ \beta_2(s,t) \end{pmatrix} = \begin{pmatrix} \partial_t\rho(s,t) \\ \partial_s\rho(t,t) \end{pmatrix} \;. \tag{4.20}$$

Substituting (4.18) in (4.17) and using (4.16), we obtain that $b(t)$ satisfies the fixed-point equation

$$b(t) = b_0(t) + \frac{1}{\sigma\sqrt{2\pi}} \int_0^t \sqrt{\frac{\rho(t,t)}{\rho(s,t,t)\rho(s,s)}}\tilde{b}(s,t)b(s)\,\mathrm{e}^{-r(s,t)/(2\sigma^2)}\,\mathrm{d}s \;, \tag{4.21}$$

where

$$r(s,t) = \frac{d(s)^2}{\rho(s,s)} - \frac{d(t)^2}{\rho(t,t)} + \frac{[d(t)-d(s)]^2}{\rho(s,t,t)} \;.$$

We will use Banach's fixed point theorem to show that the solution b of (4.21) is indeed close to b_0. For this, the following two lemmas will be useful.

Lemma 4.3. *For sufficiently small δ, there exists a constant $r_{\min} > 0$, independent of δ, s and t, such that*

$$r(s,t) \geqslant \frac{r_{\min}t^2}{s^3} \geqslant \frac{r_{\min}}{s}$$

holds whenever $0 \leqslant s \leqslant t \leqslant T_{\max} = 2\delta$.

PROOF: We first note that Lemma 4.2 implies that

$$\rho(s,t,t) = \frac{2}{3}s^3 + \frac{1}{3}t^3 - s^2t = \frac{1}{3}(t-s)^2(t+2s) \;. \tag{4.22}$$

It follows that

$$r(s,t) = 3\biggl[\frac{d(s)^2}{s^3} - \frac{d(t)^2}{t^3} + \frac{[d(t)-d(s)]^2}{(t-s)^2(t+2s)}\biggr] \;.$$

It follows from the mean value theorem and the fact that $d'(t) = -1 + \mathcal{O}(\delta)$ that

$$\frac{[d(t)-d(s)]^2}{(t-s)^2(t+2s)} = \frac{1+\mathcal{O}(\delta)}{t+2s} \;.$$

This shows in particular that $r(t,t)$ is well-defined (it is equal to $d'(t)^2/t$). It is convenient to make the change of variables

$$t = \delta x\;, \qquad s = \delta p x\;, \tag{4.23}$$

since this maps the domain $0 \leqslant s \leqslant t \leqslant 2\delta$ to $0 \leqslant x \leqslant 2$, $0 \leqslant p \leqslant 1$. Using the fact that $t + 2s = \delta x(1+2p) \leqslant 3\delta x$ and $d(t)^2 = (\delta - t)^2(1 + \mathcal{O}(\delta))$, we obtain

$$r(\delta x, \delta p x) \geqslant \frac{3}{\delta p^3 x^3} g(x,p)(1 + \mathcal{O}(\delta))\;, \tag{4.24}$$

where

$$\begin{aligned} g(x,p) &= (1-px)^2 - p^3(1-x)^2 + \frac{p^3x^2}{3} \\ &= ax^2 - 2bx + c\;, \end{aligned}$$

with $a = p^2(1 - \frac{2}{3}p)$, $b = p(1-p^2)$, and $c = 1 - p^3$. One can easily compute the minimum of $g(x,p)$ over $x \in [0,2]$. However, it turns out to be preferable to minimise $g(x,p)/x^2$. One finds that

$$\frac{\partial}{\partial x}\biggl(\frac{g(x,p)}{x^2}\biggr) = \frac{2(bx-c)}{x^3}$$

vanishes in

$$x^* = \frac{c}{b} = \frac{p^2+p+1}{p(1+p)} = 1 + \frac{1}{p^2+p}\;.$$

There are two cases to consider:

1. If $p \leqslant p^* = \frac{\sqrt{5}-1}{2}$, then $x^* \geqslant 2$, so that $g(x,p)/x^2$ is decreasing for $x \in [0,2]$, and therefore

$$\frac{g(x,p)}{x^2} \geqslant \frac{g(2,p)}{4} = \frac{1}{4}\biggl[\frac{1}{3}p^3 + (1-2p)^2\biggr]\;.$$

   One easily checks that this is bounded below by a positive constant for $p \in [0, p^*]$.

2. If $p^* < p \leqslant 1$, then $x^* \in (0,2)$ and

$$\frac{g(x,p)}{x^2} \geqslant \frac{g(x^*,p)}{(x^*)^2} = a - \frac{b^2}{c} = p^2\biggl[1 - \frac{2}{3}p - \frac{(1-p^2)(1+p)}{1+p+p^2}\biggr]\;,$$

   which is also bounded below by a positive constant for $p \in [p^*, 1]$.

The result follows by replacing this lower bound in (4.24), together with the relations

$$x = \frac{t}{\delta}\;, \qquad p = \frac{s}{\delta x} = \frac{s}{t}$$

which are the inverse of the transformation (4.23). □

The next lemma gives a bound on the prefactor of the exponential term in (4.21).

Lemma 4.4. *There exists a constant $M_0 > 0$ such that for any $0 < s \leqslant t \leqslant T_{\max}$, one has*

$$\biggl|\sqrt{\frac{\rho(t,t)}{\rho(s,t,t)\rho(s,s)}}\,\tilde{b}(s,t)\biggr| \leqslant \frac{M_0\delta}{s^{5/2}}\;.$$

PROOF: The matrix $M(s,t)$ appearing in (4.20) has determinant

$$\det M(s,t) = \frac{s^3}{36}\bigl[4t^3 - 9st^2 + 6s^2t - s^3\bigr] = \frac{1}{36}s^3(t-s)^2(4t-s)\;.$$

This allows to solve (4.20) for β_1 and β_2. Using the fact that

$$\partial_t \rho(s,t) = \frac{1}{2}s^2 , \qquad \partial_s \rho(t,t) = \frac{1}{2}s^2 ,$$

we obtain

$$\beta_1(s,t) = -\frac{3t^2}{s(t-s)(4t-s)} , \qquad \beta_2(s,t) = \frac{3(2t-s)}{(t-s)(4t-s)} .$$

By Taylor's formula, there exists $\theta \in (s,t)$ such that

$$d(s) = d(t) - (t-s)d'(t) + \frac{1}{2}(t-s)^2 d''(\theta) .$$

Plugging the last two relations into (4.19) and rearranging terms yields

$$\tilde{b}(s,t) = \frac{t-s}{s(4t-s)}\left[3d(t) - (3t-s)d'(t) + \frac{3}{2}t^2 d''(\theta)\right] .$$

Using $4t-s \geqslant 3t$ and $3t-s \leqslant 3t$, we obtain

$$|\tilde{b}(s,t)| \leqslant \frac{t-s}{st}\left[|d(t)| + t|d'(t)| + \frac{1}{2}t^2|d''(\theta)|\right] .$$

The result then follows by using the lower bound $\rho(s,t,t) \geqslant \frac{1}{3}(t-s)^2 t$, which follows from (4.22), together with the facts that $d(t) = \mathcal{O}(\delta)$, while d' and d'' are bounded on $[0, T_{\max}]$. □

We can now show the main result of this subsection, using a classical fixed-point argument.

Proposition 4.5. *For sufficiently small σ, there exists a constant $\kappa > 0$ such that (4.21) has a unique solution satisfying*

$$\big|b(t) - b_0(t)\big| \leqslant \frac{\delta^3}{t}\,\mathrm{e}^{-\kappa/\sigma^2}$$

for all $t \in [0, T_{\max}]$.

PROOF: We will apply Banach's fixed-point theorem to the space $\mathcal{B} = \mathcal{C}([0,T_{\max}],\mathbb{R})$ of continuous functions $b : [0,T_{\max}] \to \mathbb{R}$, equipped with the weighted supremum norm

$$\|b\| = \sup_{0\leqslant t\leqslant T_{\max}} t|b(t)| .$$

This definition guarantees that $\|b_0\| \leqslant M_1\delta$ for a constant M_1 independent of t and δ. We write (4.21) as the fixed-point equation $b = \Gamma b$, where $\Gamma : \mathcal{B} \to \mathcal{B}$ is the integral operator

$$(\Gamma b)(t) = b_0(t) + \int_0^t k(s,t) b(s)\,\mathrm{d}s ,$$

with

$$k(s,t) = \frac{1}{\sigma\sqrt{2\pi}}\sqrt{\frac{\rho(t,t)}{\rho(s,t,t)\rho(s,s)}}\,\tilde{b}(s,t)\,\mathrm{e}^{-r(s,t)/(2\sigma^2)} .$$

Lemmas 4.3 and 4.4 imply that

$$\big|k(s,t)\big| \leqslant \frac{M_0\delta}{\sigma\sqrt{2\pi}s^{5/2}}\,\mathrm{e}^{-r_{\min}/(2\sigma^2 s)} .$$

It follows that

$$\big|(\Gamma b)(t)\big| \leqslant \frac{M_1\delta}{t} + \frac{M_0\delta}{\sqrt{2\pi}}\int_0^t \frac{\mathrm{e}^{-r_{\min}/(2\sigma^2 s)}}{\sigma s^{7/2}}\,\mathrm{d}s\,\|b\| .$$

Using the change of variables $x = r_{\min}/(2\sigma^2 s)$, one checks that the integral is bounded by $\mathrm{e}^{-\kappa/\sigma^2}$ for a constant κ comparable to $r_{\min}$. This implies

$$\|\Gamma b\| \leqslant M_2\delta\bigl[1 + \delta\,\mathrm{e}^{-\kappa/\sigma^2}\,\|b\|\bigr]$$

for a constant $M_2 > 0$. Choosing for instance $R = 2M_2\delta$, we see that for $\|b\| \leqslant R$ and σ small enough, one has $\|\Gamma b\| \leqslant R$, so that Γ maps the ball of radius R into itself.

A similar argument shows that if b_1 and b_2 belong to that ball, then

$$\|\Gamma b_2 - \Gamma b_1\| \leqslant M_2\delta^2\,\mathrm{e}^{-\kappa/\sigma^2}\,\|b_2 - b_1\| \;.$$

Therefore, for sufficiently small σ, Γ is a contraction on the ball $\{\|b\| \leqslant R\}$, with contraction constant $\lambda = M_2\delta^2\,\mathrm{e}^{-\kappa/\sigma^2} < 1$, so that Banach's fixed point theorem implies that the equation $\Gamma b = b$ admits a unique solution. Furthermore, the sequence of $b_n = \Gamma^n b_0$ converges to the fixed point b, so that

$$\begin{aligned}\|b_0 - b\| &\leqslant \|b_0 - b_1\| + \|b_1 - b_2\| + \dots \\ &\leqslant (1 + \lambda + \lambda^2 + \dots)\,\|b_0 - b_1\| \\ &= \frac{1}{1-\lambda}\,\|b_0 - b_1\| \;.\end{aligned}$$

Since $b_0 = \Gamma(0)$, we have $\|b_0 - b_1\| \leqslant \lambda\,\|b_0\| \leqslant M_1\delta\lambda$. By slightly decreasing the value of κ, we can absorb any multiplicative constant in the term $\mathrm{e}^{-\kappa/\sigma^2}$, so that $\|b_0 - b\| \leqslant \delta^3\,\mathrm{e}^{-\kappa/\sigma^2}$, which implies the result. □

Owing to (4.17), a direct consequence of this result is that the density of τ^0 is given on $[0, T_{\max}]$ by

$$\psi(t) = \biggl[\frac{3}{2t}d(t) - d'(t) + \mathcal{O}\biggl(\frac{\delta^3}{t}\,\mathrm{e}^{-\kappa/\sigma^2}\biggr)\biggr]\phi(t) \;. \tag{4.25}$$

This will allow us to compute moments of τ^0.

4.3 Linearised equation: moments of hitting times

We now compute the expectation and variance of the first hitting time τ^0, using the density obtained in the previous section. To this end, we introduce the event

$$\Omega_1 = \bigl\{\omega\colon \tau^0(\omega) < T_{\max}\bigr\} \;, \tag{4.26}$$

where we recall that $T_{\max} = 2\delta$. We will show in Corollary 4.9 below that Ω_1 has a probability exponentially close to 1.

Proposition 4.6. *For sufficiently small σ and δ, the first two moments of $\tau^0 \mathbf{1}_{\Omega_1}$ admit the expansions*

$$\mathbb{E}\bigl[\tau^0 \mathbf{1}_{\Omega_1}\bigr] = \delta + \frac{1}{2}\delta^2\sigma^2 + \mathcal{O}(\delta^3\sigma^2) \;, \tag{4.27}$$

and

$$\operatorname{Var}\bigl[\tau^0 \mathbf{1}_{\Omega_1}\bigr] = \frac{1}{3}\delta^3\sigma^2 + \mathcal{O}(\delta^4\sigma^2) \;. \tag{4.28}$$

Furthermore, the third moment satisfies

$$\mathbb{E}\bigl[(\tau^0)^3 \mathbf{1}_{\Omega_1}\bigr] = \delta^3 + \mathcal{O}(\delta^4\sigma^2) \;. \tag{4.29}$$

PROOF: The expected value of $\tau^0 \mathbf{1}_{\Omega_1}$ is given by the integral

$$\mathbb{E}[\tau^0 \mathbf{1}_{\Omega_1}] = \int_0^{T_{\max}} t\,\psi(t)\,\mathrm{d}t \;, \tag{4.30}$$

where (4.25), (4.14) and (4.15) imply that the density ψ is given by

$$\psi(t) = \frac{\sqrt{3}}{\sigma\sqrt{2\pi t^3}}\biggl[\biggl(\frac{3\delta}{2t} - \frac{1}{2}\biggr) + \mathcal{O}(\delta - t) + \mathcal{O}\biggl(\frac{\delta^3}{t}\,\mathrm{e}^{-\kappa/\sigma^2}\biggr)\biggr] \exp\biggl\{-\frac{3d(t)^2}{2\sigma^2 t^3}\biggr\} .$$

Substituting this in the integral (4.30) gives

$$\mathbb{E}[\tau^0 \mathbf{1}_{\Omega_1}] = \frac{1}{\sigma}\sqrt{\frac{3}{2\pi}} \int_0^{T_{\max}} \Bigl[q_\delta(t) + r_1(t) + \bar{r}_1(t)\Bigr] \exp\biggl\{-\frac{p_\delta(t)}{\sigma^2}(1 + r_2(t))\biggr\}\,\mathrm{d}t ,$$

where

$$q_\delta(t) = \frac{1}{\sqrt{t}}\biggl(\frac{3\delta}{2t} - \frac{1}{2}\biggr) , \qquad p_\delta(t) = \frac{3}{2}\frac{(\delta - t)^2}{t^3} ,$$

and the remainders satisfy $r_1(t), r_2(t) = \mathcal{O}(\delta - t)$ and $\bar{r}_1(t) = \mathcal{O}(\delta^3 t^{-1}\,\mathrm{e}^{-\kappa/\sigma^2})$. We note that the remainder $\bar{r}_1(t)$ yields only an exponentially small error term, despite the fact that it diverges like t^{-1}, thanks to the behaviour in t^{-3} of $p_\delta(t)$. Since this is negligible with respect to any algebraic order term in σ, we can ignore it from now on. We now perform the change of variables

$$z = \frac{\sqrt{3}}{\delta^{3/2}\sigma}(\delta - t) ,$$

which yields

$$\begin{aligned}\mathbb{E}[\tau^0 \mathbf{1}_{\Omega_1}] = \frac{\delta^{3/2}}{\sqrt{2\pi}} \int_{-\sqrt{3}/(\sqrt{\delta}\sigma)}^{\sqrt{3}/(\sqrt{\delta}\sigma)} &\biggl[q_\delta\biggl(\delta - \frac{\delta^{3/2}\sigma}{\sqrt{3}}z\biggr) + \tilde{r}_1(z)\biggr] \\ &\times \exp\biggl\{-\frac{1}{\sigma^2}p_\delta\biggl(\delta - \frac{\delta^{3/2}\sigma}{\sqrt{3}}z\biggr)(1 + \tilde{r}_2(z))\biggr\}\,\mathrm{d}z ,\end{aligned}$$

with remainders $\tilde{r}_1(z), \tilde{r}_2(z) = \mathcal{O}(\delta^{3/2}\sigma z)$. To simplify this expression, we introduce

$$\tilde{\sigma} = \frac{\delta^{1/2}\sigma}{\sqrt{3}} , \tag{4.31}$$

and note that

$$q_\delta\biggl(\delta - \frac{\delta^{3/2}\sigma}{\sqrt{3}}z\biggr) = q_\delta\bigl(\delta(1 - \tilde{\sigma}z)\bigr) = \frac{1}{\sqrt{\delta}}q_1\bigl(1 - \tilde{\sigma}z\bigr) ,$$

and

$$p_\delta\biggl(\delta - \frac{\delta^{3/2}\sigma}{\sqrt{3}}z\biggr) = \frac{z^2}{2}\frac{\delta^3}{(\delta - \delta\tilde{\sigma}z)^3}\sigma^2 = \frac{z^2}{2}\frac{1}{(1 - \tilde{\sigma}z)^3}\sigma^2 .$$

The expectation then becomes

$$\mathbb{E}[\tau^0 \mathbf{1}_{\Omega_1}] = \frac{\delta}{\sqrt{2\pi}} \int_{-1/\tilde{\sigma}}^{1/\tilde{\sigma}} \bigl[q_1(1 - \tilde{\sigma}z) + \tilde{r}_1(z)\bigr] \exp\biggl\{-\frac{z^2}{2}\frac{1 + \tilde{r}_2(z)}{(1 - \tilde{\sigma}z)^3}\biggr\}\,\mathrm{d}z . \tag{4.32}$$

For sufficiently small $\tilde{\sigma}$, we can use the Taylor expansions

$$q_1(1 - \tilde{\sigma}z) = q_1(1) - q_1'(1)\tilde{\sigma}z + \frac{1}{2}q_1''(1)\tilde{\sigma}^2 z^2 + \mathcal{O}(\tilde{\sigma}^3 z^3) ,$$

and

$$\begin{aligned}\exp\biggl\{-\frac{z^2}{2}\frac{1 + \tilde{r}_2(z)}{(1 - \tilde{\sigma}z)^3}\biggr\} &= \exp\biggl\{-\frac{z^2}{2}(1 + 3\tilde{\sigma}z + 6\tilde{\sigma}^2 z^2 + \mathcal{O}(\tilde{\sigma}^3 z^3))(1 + \tilde{r}_2(z))\biggr\} \\ &= \exp\biggl\{-\frac{3}{2}\tilde{\sigma}z^3 - 3\tilde{\sigma}^2 z^4 + \mathcal{O}(\tilde{\sigma}^3 z^5) + \mathcal{O}\bigl(z^2\tilde{r}_2(z)\bigr)\biggr\}\,\mathrm{e}^{-z^2/2} \\ &= \biggl[1 - \frac{3}{2}z^3\tilde{\sigma} + \biggl(-3z^4 + \frac{9}{8}z^6\biggr)\tilde{\sigma}^2 + \mathcal{O}(\tilde{\sigma}^3 z^5) + \mathcal{O}\bigl(z^2\tilde{r}_2(z)\bigr)\biggr]\,\mathrm{e}^{-z^2/2} .\end{aligned}$$

Using the fact that $\tilde{r}_i(z) = a_i\delta\tilde{\sigma}z + \mathcal{O}(\delta^2\tilde{\sigma}^2z^2)$, $i = 1, 2$, for some constants a_1, a_2, as well as the expression

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^k\,\mathrm{e}^{-z^2/2}\,\mathrm{d}z = \begin{cases} \prod_{i=1}^{k/2}(2i-1) & \text{if } k \text{ is even ,} \\ 0 & \text{if } k \text{ is odd ,} \end{cases}$$

for the moments of the standard normal law, we find, upon substituting the expansions into (4.32),

$$\mathbb{E}[\tau^0\mathbf{1}_{\Omega_1}] = \delta\biggl[q_1(1) + \biggl(\frac{63}{8}q_1(1) + \frac{9}{2}q_1'(1) + \frac{1}{2}q_1''(1)\biggr)\tilde{\sigma}^2 + \mathcal{O}(\delta\tilde{\sigma}^2)\biggr] .$$

Here we used the fact that extending the bounds in the integral (4.32) to $\pm\infty$ only produces exponentially small error terms. The Taylor coefficients of q_1 at 1 are found to be $q_1(1) = 1$, $q_1'(1) = -2$, and $q_1''(1) = \frac{21}{4}$, which yields

$$\mathbb{E}[\tau^0\mathbf{1}_{\Omega_1}] = \delta + \frac{3}{2}\delta\tilde{\sigma}^2 + \mathcal{O}(\delta^2\tilde{\sigma}^2) .$$

Plugging in the value (4.31) of $\tilde{\sigma}$ gives (4.27). A similar computation, in which only the expression for q_δ is changed, can be made for the second moment of τ^0. One obtains

$$\mathbb{E}[(\tau^0)^2\mathbf{1}_{\Omega_1}] = \delta^2 + \frac{4}{3}\delta^3\sigma^2 + \mathcal{O}(\delta^4\sigma^2) ,$$

which yields (4.28). The proof of (4.29) is similar. □

Recall that the moments of τ^0 have been computed after performing the time change defined in Section 4.1 and dropping the tildes. We now have to undo this time change. We first note that the event Ω_1 becomes

$$\Omega_1 = \bigl\{\tau^0(\omega) < \widehat{T}_{\max}\bigr\} , \qquad \text{where } \widehat{T}_{\max} = cg(T_{\max}) .$$

The result on moments is then as follows, where we introduced the shorthand

$$r_0 = x_0^2 + y_0 .$$

Corollary 4.7. *For sufficiently small σ and δ, the moments of τ^0 before the time change satisfy*

$$\begin{aligned}
\mathbb{E}\bigl[\tau^0\mathbf{1}_{\Omega_1}\bigr] &= T(x_0,y_0) + \frac{1}{2}\frac{T(x_0,y_0)^2\sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^3\sigma^2}{r_0^4}\biggr) , \\
\mathrm{Var}\bigl[\tau^0\mathbf{1}_{\Omega_1}\bigr] &= \frac{1}{3}\frac{T(x_0,y_0)^3\sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^4\sigma^2}{r_0^5}\biggr) , \\
\mathbb{E}\bigl[(\tau^0)^3\mathbf{1}_{\Omega_1}\bigr] &= T(x_0,y_0)^3 + \mathcal{O}\biggl(\frac{\delta^4\sigma^2}{r_0^6}\biggr) .
\end{aligned}$$

PROOF: Recall the relation $\tau^0 = cg(\tilde{\tau}^0)$, cf. (4.5), where $g(t) = t + \mathcal{O}(t^2)$, cf. (4.11), and the constant c is given by (4.12). To compute the expectation of τ^0, we write

$$\begin{aligned}
\mathbb{E}\bigl[\tau^0\mathbf{1}_{\Omega_1}\bigr] - cg(\delta)\mathbb{P}(\Omega_1) &= \mathbb{E}\bigl[(\tau^0 - cg(\delta))\mathbf{1}_{\Omega_1}\bigr] \\
&= c\mathbb{E}\bigl[(g(\tilde{\tau}^0) - g(\delta))\mathbf{1}_{\Omega_1}\bigr] \\
&= c\mathbb{E}\biggl[\int_\delta^{\tilde{\tau}_0} g'(t)\,\mathrm{d}t\,\mathbf{1}_{\Omega_1}\biggr] \\
&= c\mathbb{E}\Bigl[\bigl(\tilde{\tau}^0 - \delta + \mathcal{O}((\tilde{\tau}^0)^2 - \delta^2)\bigr)\mathbf{1}_{\Omega_1}\Bigr] \\
&= c\biggl[\delta + \frac{1}{2}\delta^2\tilde{\sigma}^2 + \mathcal{O}(\delta^3\tilde{\sigma}^2) - \delta\mathbb{P}(\Omega_1)\biggr] + \mathcal{O}(c\delta^3\tilde{\sigma}^2) ,
\end{aligned}$$

where $\tilde{\sigma}$ has been defined in (4.7). As mentioned before, we will show in Corollary 4.9 that $\mathbb{P}(\Omega_1)$ is exponentially close to 1. The result for the expectation follows from the definition (4.12) of c and the relation (4.3) between δ and $T(x_0, y_0)$. The other moments are computed in an analogous way. □

4.4 Nonlinear equation

Recall that the moments of τ^0 that we computed correspond to the linearised equation

$$\mathrm{d}u_t^0 = (\sigma W_t + 2x^{\det}(t)u_t^0)\,\mathrm{d}t \;. \tag{4.33}$$

We now extend these results to the non-linear equation

$$\mathrm{d}u_t = (\sigma W_t + 2x^{\det}(t)u_t + u_t^2)\,\mathrm{d}t \;. \tag{4.34}$$

To do that, we show that $z_t = u_t - u_t^0$ remains small with high probability. We begin with a preparatory estimate.

Lemma 4.8. *There exist constants $M_0, \kappa_0 > 0$ such that for any $H > 0$ and $0 \leqslant t \leqslant T_{\max}$, one has*

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} |u_s^0| > H\biggr\} \leqslant M_0 H t \exp\biggl\{-\kappa_0\frac{H^2}{\sigma^2 t^3}\biggr\} \;.$$

PROOF: One can use the bound in [5, Theorem 2.2] on the supremum of a Gaussian process. This requires a bound of the form

$$\mathbb{E}\bigl[(X_t - X_s)^2\bigr] \leqslant G|t-s|^\gamma$$

to hold. This is indeed the case with $\gamma = 2$, thanks to the computation of the covariance made in (4.22). □

This result allows us to show that the probability of the event Ω_1 introduced in (4.26) is exponentially close to 1.

Corollary 4.9. *There exist constant $M_1, \kappa_1 > 0$ such that*

$$\mathbb{P}(\Omega_1^c) \leqslant M_1 \frac{\delta^2}{r_0}\,\mathrm{e}^{-\kappa_1 r_0^3/(\delta\sigma^2)} \;.$$

PROOF: It suffices to apply Lemma 4.8 with $t = \widehat{T}_{\max}$, which scales like δ/r_0, and an H of order δ, chosen in such a way that $|u_t^0| \leqslant H$ for all $t \leqslant \widehat{T}_{\max}$ implies $\tau^0 < \widehat{T}_{\max}$. □

We are now able to bound the probability that the difference $z_t = u_t - u_t^0$ becomes large.

Lemma 4.10. *For δ small enough, there exist constants $\kappa, M, h_0 > 0$ such that for any $h \in (0, h_0]$ and $0 \leqslant t \leqslant \widehat{T}_{\max}$, one has*

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} |z_s| > h\biggr\} \leqslant M \exp\biggl\{-\kappa\frac{h}{\sigma^2 t^4}\biggr\} \;.$$

PROOF: Taking the difference of (4.34) and (4.33), we find that z_t satisfies

$$\mathrm{d}z_t = \bigl[(u_t^0)^2 + 2(x_t^0 + u_t^0)z_t + z_t^2\bigr]\,\mathrm{d}t \;.$$

We introduce the stopping time

$$\tau = \inf\bigl\{s > 0 \colon |z_s| > h\bigr\} \;.$$

For any decomposition $h = h_1 + h_2 + h_3 + h_4$, we can write

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} |z_s| > h\biggr\} \leqslant \sum_{j=1}^4 \mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t\wedge\tau} |r_j(t)| > h_j\biggr\} =: \sum_{j=1}^4 P_j \;,$$

where

$$r_1(t) = \int_0^t (u_s^0)^2\,\mathrm{d}s \;, \quad r_2(t) = 2\int_0^t u_s^0 z_s\,\mathrm{d}s \;, \quad r_3(t) = \int_0^t z_s^2\,\mathrm{d}s \;, \quad r_4(t) = 2\int_0^t x_s^0 z_s\,\mathrm{d}s \;.$$

Using Lemma 4.8, we obtain the existence of constants $\kappa_1, \kappa_2, M_1, M_2 > 0$ such that

$$P_1 \leqslant M_1 \exp\biggl\{-\kappa_1 \frac{h_1}{\sigma^2 t^4}\biggr\} , \qquad P_2 \leqslant M_2 \exp\biggl\{-\kappa_2 \frac{h_2^2}{h^2\sigma^2 t^5}\biggr\} .$$

Choosing h_1 of order h and h_2 of order $h^{3/2}t^{1/2}$ yields comparable P_1 and P_2. Taking $h_3 = th^2$ and h_4 of order th yields $P_3 = P_4 = 0$. Since $t \leqslant \widehat{T}_{\max} = \mathcal{O}(\delta)$, such a choice of the h_i is possible. Combining the bounds yields the result. $\square$

We have now everything needed to extend the moment estimates in Corollary 4.7 to τ. It will be convenient to define the event

$$\Omega_2 = \biggl\{\omega\colon \sup_{0\leqslant t\leqslant \widehat{T}_{\max}} |u_t(\omega)| \leqslant a\widehat{T}_{\max}\biggr\} ,$$

where the constant a will be chosen in a suitable way.

Proposition 4.11. *There exist constants $M, \kappa, a > 0$ such that for sufficiently small σ and δ,*

$$\mathbb{P}(\Omega_2^c) \leqslant M\,\mathrm{e}^{-\kappa r_0/(\delta\sigma^2)} . \tag{4.35}$$

Furthermore, the moments of τ satisfy

$$\begin{aligned}
\mathbb{E}\bigl[\tau \mathbf{1}_{\Omega_2}\bigr] &= T(x_0,y_0) + \frac{1}{2}\frac{T(x_0,y_0)^2\sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^3\sigma^2}{r_0^4}\biggr) + \mathcal{O}\bigl(\mathrm{e}^{-\kappa r_0/\delta}\bigr) ,\\
\operatorname{Var}\bigl[\tau \mathbf{1}_{\Omega_2}\bigr] &= \frac{1}{3}\frac{T(x_0,y_0)^3\sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^4\sigma^2}{r_0^5}\biggr) + \mathcal{O}\bigl(\mathrm{e}^{-\kappa r_0/\delta}\bigr) ,\\
\mathbb{E}\bigl[\tau^3 \mathbf{1}_{\Omega_2}\bigr] &= T(x_0,y_0)^3 + \mathcal{O}\biggl(\frac{\delta^4\sigma^2}{r_0^6}\biggr) + \mathcal{O}\bigl(\mathrm{e}^{-\kappa r_0/\delta}\bigr) .
\end{aligned}$$

PROOF: Given $h \in [0, a\widehat{T}_{\max}]$, we introduce the events

$$\begin{aligned}
\Omega_0(h) &= \biggl\{\omega\colon \sup_{0\leqslant t\leqslant \widehat{T}_{\max}} |u_t^0(\omega)| \leqslant a\widehat{T}_{\max} - h\biggr\} ,\\
\Omega_3(h) &= \biggl\{\omega\colon \sup_{0\leqslant t\leqslant \widehat{T}_{\max}} |u_t(\omega) - u_t^0(\omega)| \leqslant h\biggr\} .
\end{aligned}$$

It follows from Lemmas 4.8 and 4.10 that

$$\begin{aligned}
\mathbb{P}(\Omega_0(h)^c) &\leqslant M_0 a\widehat{T}_{\max}^2 \exp\biggl\{-\kappa_0 \frac{(a\widehat{T}_{\max} - h)^2}{\sigma^2\widehat{T}_{\max}^3}\biggr\} ,\\
\mathbb{P}(\Omega_3(h)^c) &\leqslant M_1 \exp\biggl\{-\kappa_1 \frac{h}{\sigma^2\widehat{T}_{\max}^4}\biggr\}
\end{aligned}$$

for some $M_0, M_1, \kappa_0, \kappa_1 > 0$. Since $\Omega_2 \subset \Omega_0(h) \cap \Omega_3(h)$ for any admissible h, we have $\mathbb{P}(\Omega_2^c) \leqslant \mathbb{P}(\Omega_0(h)^c) + \mathbb{P}(\Omega_3(h)^c)$, and the bound (4.35) follows by choosing

$$h = \widehat{T}_{\max}^3 .$$

Regarding the expectations, we observe that if $\omega \in \Omega_2 \cap \Omega_3(h)$, then $|u_t^0(\omega)| \leqslant a\widehat{T}_{\max} + h$ for all $t \in [0, \widehat{T}_{\max}]$. If a is a sufficiently small constant (independent of δ and h), this implies $\tau^0 < \widehat{T}_{\max}$. Therefore,

$$\Omega_2 \cap \Omega_3(h) \subset \Omega_1 . \tag{4.36}$$

For any $h_1 \in \mathbb{R}$, we write

$$\tau^0(h_1) = \inf\{t > 0\colon u_t^0 = d(t) + h_1\} .$$

Then by construction, on $\Omega_3(h)$ one has

$$\tau^0(-h) \leqslant \tau \leqslant \tau^0(h) \ .$$

Using (4.36), this yields the upper bound

$$\begin{aligned}
\mathbb{E}[\tau \mathbf{1}_{\Omega_2}] &= \mathbb{E}[\tau \mathbf{1}_{\Omega_2} \mathbf{1}_{\Omega_3(h)}] + \mathbb{E}[\tau \mathbf{1}_{\Omega_2} \mathbf{1}_{\Omega_3(h)^c}] \\
&\leqslant \mathbb{E}[\tau^0(h) \mathbf{1}_{\Omega_2 \cap \Omega_3(h)}] + \widehat{T}_{\max} \mathbb{P}(\Omega_3(h)^c) \\
&\leqslant \mathbb{E}[\tau^0(h) \mathbf{1}_{\Omega_1}] + \widehat{T}_{\max} \mathbb{P}(\Omega_3(h)^c) \ ,
\end{aligned}$$

and the lower bound

$$\begin{aligned}
\mathbb{E}[\tau \mathbf{1}_{\Omega_2}] &\geqslant \mathbb{E}[\tau^0(-h) \mathbf{1}_{\Omega_2 \cap \Omega_3(h)}] \\
&= \mathbb{E}[\tau^0(-h) \mathbf{1}_{\Omega_1}] - \mathbb{E}[\tau^0(-h) \mathbf{1}_{\Omega_1 \cap (\Omega_2 \cap \Omega_3(h))^c}] \\
&\geqslant \mathbb{E}[\tau^0(-h) \mathbf{1}_{\Omega_1}] - \widehat{T}_{\max} \bigl[\mathbb{P}(\Omega_2^c) + \mathbb{P}(\Omega_3(h)^c))\bigr] \ .
\end{aligned}$$

Corollary 4.7 implies that for any h of order $\widehat{T}_{\max}$,

$$\mathbb{E}[\tau^0(h) \mathbf{1}_{\Omega_1}] = T(x_0, y_0) + \frac{h + \mathcal{O}(\delta h)}{r_0} + \frac{1}{2} \frac{T(x_0, y_0)^2 \sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^3 \sigma^2}{r_0^4}\biggr) \ .$$

The choice

$$h = \sigma^2 \widehat{T}_{\max}^3$$

ensures that the term of order h can be incorporated in the error term, since $\widehat{T}_{\max}$ scales like δ/r_0. Furthermore, with this choice of h, $\mathbb{P}(\Omega_3(h))^c$ has order $\mathrm{e}^{-\kappa/\widehat{T}_{\max}}$. This yields the expression for $\mathbb{E}[\tau \mathbf{1}_{\Omega_2}]$. The other moments are treated in a similar way. □

4.5 Moments of the exit location

Having obtained the moments of τ, we can now proceed to deriving expressions for the moments of y_τ, which is the main result of this entire section. Here it will be useful to work with the event

$$\Omega_4 = \Omega_2 \cap \biggl\{\omega \colon \sup_{0 \leqslant t \leqslant \widehat{T}_{\max}} \sigma |W_t(\omega)| \leqslant b \widehat{T}_{\max} \biggr\} \tag{4.37}$$

where $b > 0$ will be chosen later on.

Proposition 4.12. *There exist constants $M, \kappa > 0$, depending only on b, such that for sufficiently small σ and δ,*

$$\mathbb{P}(\Omega_4^c) \leqslant M \, \mathrm{e}^{-\kappa\delta/(r_0\sigma^2)} \ . \tag{4.38}$$

Furthermore, the first three moments of $(y_\tau - y_0 - T(x_0, y_0)) \mathbf{1}_{\Omega_4}$ satisfy

$$\begin{aligned}
\mathbb{E}\bigl[\bigl(y_\tau - y_0 - T(x_0, y_0)\bigr) \mathbf{1}_{\Omega_4}\bigr] &= \frac{1}{2} \frac{T(x_0, y_0)^2 \sigma^2}{r_0^2} + \mathcal{O}\biggl(\frac{\delta^3 \sigma^2}{r_0^4}\biggr) + \mathcal{O}\bigl(\mathrm{e}^{-\kappa r_0/\delta}\bigr) \ , \qquad (4.39) \\
\mathbb{E}\bigl[\bigl(y_\tau - y_0 - T(x_0, y_0)\bigr)^2 \mathbf{1}_{\Omega_4}\bigr] &= T(x_0, y_0) \sigma^2 - \frac{T(x_0, y_0)^2}{r_0} \sigma^2 + \mathcal{O}\biggl(\frac{\delta^3 \sigma^2}{r_0^4}\biggr) + \mathcal{O}\bigl(\mathrm{e}^{-\kappa r_0/\delta}\bigr) \ , \\
\mathbb{E}\bigl[\bigl(y_\tau - y_0 - T(x_0, y_0)\bigr)^3 \mathbf{1}_{\Omega_4}\bigr] &= \mathcal{O}(\sigma^3) + \mathcal{O}\biggl(\frac{\delta^3}{r_0^3}\biggr) \ .
\end{aligned}$$

PROOF: Using either the Bernstein-type estimate [4, Lemma B.1.3], or [5, Theorem 2.2], applied this time to Brownian motion, we obtain

$$\mathbb{P}\biggl\{\sup_{0 \leqslant t \leqslant \widehat{T}_{\max}} \sigma |W_t| > b \widehat{T}_{\max} \biggr\} \leqslant M_0 \, \mathrm{e}^{-\kappa_0 b^2 \widehat{T}_{\max}^2/(\sigma^2 \widehat{T}_{\max})} = M_0 \, \mathrm{e}^{-\kappa_0 b^2 \widehat{T}_{\max}/\sigma^2} \ .$$

Since $\widehat{T}_{\max}$ scales like δ/r_0, this yields (4.38), considering the bound (4.35) on $\mathbb{P}(\Omega_2^c)$.

Regarding the moments, since $y_t = y_0 + t + \sigma W_t$, we have immediately

$$\mathbb{E}\big[y_\tau \mathbf{1}_{\Omega_2}\big] = y_0 \mathbb{P}(\Omega_2) + \mathbb{E}\big[\tau \mathbf{1}_{\Omega_2}\big] + \sigma \mathbb{E}\big[W_\tau \mathbf{1}_{\Omega_2}\big] \;.$$

Since $\tau \wedge \widehat{T}_{\max}$ is a stopping time, and $(W_t)_{t\geqslant 0}$ is a martingale, we have

$$\begin{aligned} 0 &= \mathbb{E}\big[W_{\tau\wedge\widehat{T}_{\max}}\big] \\ &= \mathbb{E}\big[W_{\tau\wedge\widehat{T}_{\max}} \mathbf{1}_{\Omega_2}\big] + \mathbb{E}\big[W_{\tau\wedge\widehat{T}_{\max}} \mathbf{1}_{\Omega_2^c}\big] \\ &= \mathbb{E}\big[W_\tau \mathbf{1}_{\Omega_2}\big] + \mathbb{E}\big[W_{\tau\wedge\widehat{T}_{\max}} \mathbf{1}_{\Omega_2^c}\big] \;, \end{aligned}$$

since $\omega \in \Omega_2$ implies $\tau(\omega) \leqslant \widehat{T}_{\max}$. The Cauchy–Schwarz inequality implies

$$\big|\mathbb{E}\big[W_{\tau\wedge\widehat{T}_{\max}} \mathbf{1}_{\Omega_2^c}\big]\big| \leqslant \sqrt{\mathbb{E}\big[W^2_{\tau\wedge\widehat{T}_{\max}}\big]}\sqrt{\mathbb{P}(\Omega_2^c)} \;.$$

Since $(W_t^2)_{t\geqslant 0}$ is a submartingale,

$$\mathbb{E}\big[W^2_{\tau\wedge\widehat{T}_{\max}}\big] \leqslant \mathbb{E}\big[W^2_{\widehat{T}_{\max}}\big] = \widehat{T}_{\max} = \mathcal{O}(c\delta) \;,$$

and the bound (4.39) on the expectation, with $\mathbf{1}_{\Omega_2}$ instead of $\mathbf{1}_{\Omega_4}$, follows from (4.35). To extend the bound to $\mathbf{1}_{\Omega_4}$, we write

$$\mathbb{E}\big[y_\tau \mathbf{1}_{\Omega_4}\big] = \mathbb{E}\big[y_\tau \mathbf{1}_{\Omega_2}\big] + \mathbb{E}\big[y_\tau \mathbf{1}_{\Omega_4\cap\Omega_2^c}\big] \;.$$

Since y_τ is bounded on Ω_4, the second term on the right-hand side is bounded by a constant times $\mathbb{P}(\Omega_2^c)$, which yields a negligible exponentially small error term.

To estimate the second moment, we first compute

$$\mathbb{E}\big[(y_\tau - y_0)^2 \mathbf{1}_{\Omega_2}\big] = \mathbb{E}\big[\tau^2 \mathbf{1}_{\Omega_2}\big] + 2\sigma \mathbb{E}\big[\tau W_\tau \mathbf{1}_{\Omega_2}\big] + \sigma^2 \mathbb{E}\big[W_\tau^2 \mathbf{1}_{\Omega_2}\big] \;.$$

The term $\mathbb{E}[\tau^2 \mathbf{1}_{\Omega_2}]$ has been estimated in Proposition 4.11. We consider next the term $\mathbb{E}[W_\tau^2 \mathbf{1}_{\Omega_2}]$. Here we use the fact that the process $(M_t)_{t\geqslant 0}$ given by

$$M_t = W_t^2 - t$$

is a martingale. By a similar stopping argument as before, we get

$$\mathbb{E}\big[W_\tau^2 \mathbf{1}_{\Omega_2}\big] = \mathbb{E}\big[\tau \mathbf{1}_{\Omega_2}\big] + \mathcal{O}\Big(\sqrt{\mathbb{E}\big[M^2_{\tau\wedge\widehat{T}_{\max}}\big]}\sqrt{\mathbb{P}(\Omega_2^c)}\Big) \;.$$

To compute $\mathbb{E}[\tau W_\tau \mathbf{1}_{\Omega_2}]$, we observe that since $\mathrm{d}(tW_t) = t\,\mathrm{d}W_t + W_t\,\mathrm{d}t$, we have the integration-by-parts relation

$$tW_t = X_t + \int_0^t s\,\mathrm{d}W_s \;,$$

where X_t is defined in (4.13). The stochastic integral being a martingale, that we denote M'_t, we obtain

$$\mathbb{E}\big[(\tau W_\tau)\mathbf{1}_{\Omega_2}\big] = \mathbb{E}\big[X_\tau \mathbf{1}_{\Omega_2}\big] - \mathbb{E}\big[M'_{\tau\wedge\widehat{T}_{\max}} \mathbf{1}_{\Omega_2^c}\big] \;.$$

Here we can reuse the results from Section 4.3, since σX_t is equal to u_t^0 in the particular case $\alpha(t) = 0$, and $u_\tau^0 = d(\tau)$ for a suitable boundary d. This yields

$$\sigma \mathbb{E}\big[X_\tau \mathbf{1}_{\Omega_2}\big] = T(x_0, y_0) - \mathbb{E}\big[\tau \mathbf{1}_{\Omega_2}\big] = -\frac{1}{2}\frac{T(x_0,y_0)^2\sigma^2}{r_0^2} + \mathcal{O}\Big(\frac{\delta^3\sigma^2}{r_0^4}\Big) \;.$$

Combining the different estimates, and using a similar argument to replace $\mathbf{1}_{\Omega_2}$ by $\mathbf{1}_{\Omega_4}$ yields the expression for the second moment. The proof of the third moment is similar. □

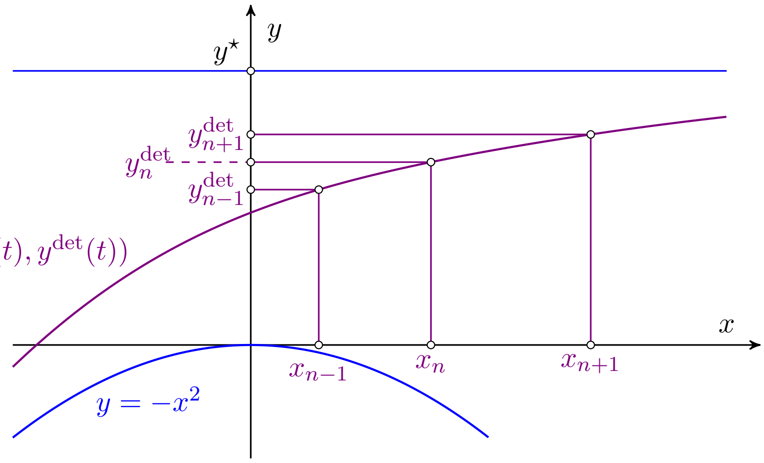

Figure 6: Definition of the partition of $[x_{\text{in}}, x_{\text{fin}}]$. The difference $y_{n+1}^{\text{det}} - y_n^{\text{det}}$ is constant, equal to γ. The difference $\delta_n = x_{n+1} - x_n$ is increasing with n.

5 Combining the slices

In this section, we provide the proof of Theorem 2.1. We fix an initial condition $(x_{\text{in}}, y_{\text{in}})$ satisfying $x_{\text{in}}^2 + y_{\text{in}} > 0$, and a final y-value $y_{\text{fin}} < y^\star$. Let $(x^{\text{det}}(t), y^{\text{det}}(t))$ be the deterministic solution (2.3) with this initial condition, and let $x_{\text{fin}} = x^{\text{det}}(y_{\text{fin}})$.

5.1 Partition

We first choose a partition $(x_n)_{0 \leqslant n \leqslant N}$ of $[x_{\text{in}}, x_{\text{fin}}]$ that is "equidistant in y" (see Figure 6). Given an integer $N \geqslant 1$, that will be taken large, the partition is defined by

$$x_n = x^{\text{det}}(n\gamma) , \qquad \gamma = \frac{y_{\text{fin}} - y_{\text{in}}}{N} .$$

To lighten notations, we write

$$y_n^{\text{det}} = y^{\text{det}}(n\gamma) = y_{\text{in}} + n\gamma .$$

We use the superscript $^{\text{det}}$ for y_n^{det} to distinguish it from general starting points y_n on the section $\{x = x_n\}$. Since such a distinction is not needed for x_n, we omit the superscript in that case. Note that since

$$x_{n+1} = x_n + \gamma(x_n^2 + y_n^{\text{det}}) + \mathcal{O}(\gamma^2) ,$$

we have

$$\delta_n := x_{n+1} - x_n = \gamma(x_n^2 + y_n^{\text{det}}) + \mathcal{O}(\gamma^2) .$$

Given $y_n > -x_n^2$ and $0 \leqslant n < m \leqslant N$, we use the notation

$$T_{n,m}(y_n) = T(x_n, y_n; x_m)$$

for the deterministic time needed for the solution starting in (x_n, y_n) to reach the line $\{x = x_m\}$ (the notation $T(x_n, y_n; x_m)$ has been introduced in (2.8)). In particular, we have

$$T_{n,m}(y_n^{\text{det}}) = y_m^{\text{det}} - y_n^{\text{det}} = (m - n)\gamma .$$

In other words, the deterministic reference solution $(x^{\text{det}}(t), y^{\text{det}}(t))$ takes always the same time γ to move from a line $\{x = x_n\}$ to the next line $\{x = x_{n+1}\}$.

In order to work with bounded values of y_n, we fix a strictly increasing function $h : [x_{\text{in}}, x_{\text{fin}}] \to (0, \infty)$, and introduce the increasing family of sets

$$B_n = \left\{(x, y) \in \mathbb{R}^2 \colon x_{\text{in}} \leqslant x \leqslant x_n, |y - y^{\text{det}}(x)| \leqslant 2h(x)\right\} ,$$

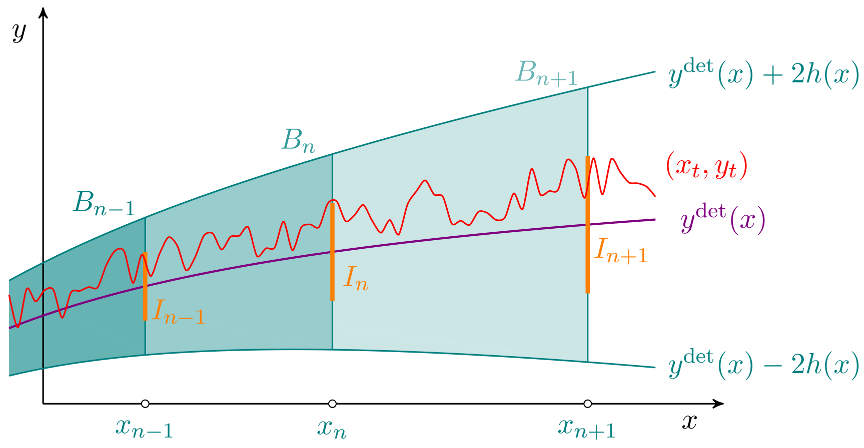

Figure 7: Definition of the sets B_n. Each set extends all the way to $x = x_{\text{in}}$, so the sets are nested ($B_{n-1} \subset B_n \subset B_{n+1}$). The sample path $(x_t, y_t)_t$ is an element of $\widehat{\Omega}_{n-1,n+1}$.

where $y^{\text{det}}(x)$ stands for the deterministic reference solution parametrized by x (the reciprocal of the function $x^{\text{det}}(y)$ defined in (2.4)). We also introduce the intervals

$$I_n = [y_n^{\text{det}} - h(x_n), y_n^{\text{det}} + h(x_n)]$$

(see Figure 7). Denoting by τ_n the first-exit time of the sample path $(x_t, y_t)_{t \geqslant 0}$ from B_n, we define the events

$$\widehat{\Omega}_n = \left\{ x_{\tau_n} = x_n, y_{\tau_n} \in I_n \right\} .$$

These events correspond to the sample path remaining between the curves $y = y^{\text{det}}(x) - 2h(x)$ and $y = y^{\text{det}}(x) + 2h(x)$, and exiting the set B_n through its right-hand boundary, at a point (x_n, y_{τ_n}) such that y_{τ_n} lies in I_n. The reason for the factor 2 in the definition of B_n is as follows. Since

$$T(x_n, y_n; x_{n+1}) = \gamma + \mathcal{O}(\gamma^2) ,$$

the deterministic solution starting in (x_n, y_n) with $y_n \in I_n$ will cross the next section $\{x = x_{n+1}\}$ at height $y(T)$ satisfying

$$y(T) = y_n + \gamma + \mathcal{O}(\gamma^2) ,$$

so that

$$\left| y(T) - y_{n+1}^{\text{det}} \right| \leqslant \left| y_n - y_n^{\text{det}} \right| + \mathcal{O}(\gamma^2) \leqslant h(x_n) + \mathcal{O}(\gamma^2) .$$

Since h is strictly increasing, we have $h(x_{n+1}) > h(x_n) + b\gamma$ for some $b > 0$, and therefore $y(T) \in I_{n+1}$ for γ small enough. This guarantees that the stochastic sample path starting in (x_n, y_n) will, with high probability, reach the line $\{x = x_{n+1}\}$ at a point in I_{n+1}. More precisely, we have the following bound.

Proposition 5.1. *There exist constants $\kappa, M > 0$ such that for γ small enough and any $y_n \in I_n$, one has*

$$\mathbb{P}^{x_n, y_n}(\widehat{\Omega}_{n+1}^c) \leqslant M \, \mathrm{e}^{-\kappa\gamma/\sigma^2} . \tag{5.1}$$

PROOF: Recall that $y_t = y_t^{\text{det}} + \sigma W_t$, and that we have constructed the sets B_n and I_n in such a way that the deterministic solution remains at distance $b\gamma$ from the boundaries of B_n and I_{n+1} for some $b > 0$. Therefore, the set $\Omega_4 = \Omega_{4,n}$ introduced in (4.37) is included in $\widehat{\Omega}_n$, for this choice of b. Hence, the result follows from (4.38) and the fact that δ/r_0 scales like γ. □

An immediate consequence of this result is that we can bound the probability of events

$$\widehat{\Omega}_{n,m} = \bigcap_{p=n+1}^{m} \widehat{\Omega}_p , \qquad 0 \leqslant n < m \leqslant N ,$$

which means that sample paths remain between the curves $y = y^{\det}(x) - 2h(x)$ and $y = y^{\det}(x) + 2h(x)$ at least until x_t reaches x_m, and cross each line $\{x = x_p\}$ at height $y_{\tau_p} \in I_p$. See Figure 7 for an example.

Corollary 5.2. *For any $y_n \in I_n$ and $0 \leqslant n < m \leqslant N$, one has*

$$\mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}^c_{n,m}\bigr) \leqslant (m-n)\,\mathrm{e}^{-\kappa\gamma/\sigma^2} \;.$$

PROOF: The decomposition $\widehat{\Omega}^c_{n,m} = \widehat{\Omega}^c_{n+1} \cup \widehat{\Omega}^c_{n+1,m}$ allows us to write

$$\begin{aligned}
\mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}^c_{n,m}\bigr) &= \mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}^c_{n+1} \cup \widehat{\Omega}^c_{n+1,m}\bigr) \\
&= \mathbb{P}^{x_n,y_n}\Bigl(\widehat{\Omega}^c_{n+1} \cup \bigl(\widehat{\Omega}_{n+1} \cap \widehat{\Omega}^c_{n+1,m}\bigr)\Bigr) \\
&= \mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}^c_{n+1}\bigr) + \mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}_{n+1} \cap \widehat{\Omega}^c_{n+1,m}\bigr) \;.
\end{aligned}$$

The first term can be bounded by (5.1), while the strong Markov property implies

$$\mathbb{P}^{x_n,y_n}\bigl(\widehat{\Omega}_{n+1} \cap \widehat{\Omega}^c_{n+1,m}\bigr) \leqslant \sup_{y_{n+1}\in I_{n+1}} \mathbb{P}^{x_{n+1},y_{n+1}}\bigl(\widehat{\Omega}^c_{n+1,m}\bigr) \;.$$

The result then follows by induction, starting with $n = m-1$ and moving backward in time. □

Since the event Ω_0 introduced in (2.9) contains $\widehat{\Omega}_{0,N}$ provided $h(x) \leqslant h_0$ for all x, we have $\mathbb{P}(\Omega_0^c) \leqslant \mathbb{P}(\widehat{\Omega}^c_{0,N})$, and the bound (2.10) will follow from our choice of γ.

5.2 Expectation of y_τ

Our aim is now to compute the expectations

$$E_{n,N}(y_n) = \mathbb{E}^{(x_n,y_n)}\bigl[y_{\tau_N}\mathbf{1}_{\widehat{\Omega}_{n,N}}\bigr]$$

for $y_n \in I_n$. Since the starting point y_n may be different from the reference solution $y_n^{\det}$, we introduce the notation

$$\Pi_n(y_n) = y_n + T_{n,n+1}(y_n) \tag{5.2}$$

for its image on $\{x = x_{n+1}\}$ by the deterministic flow. In other words, Π_n is the deterministic Poincaré map from the section $\{x = x_n\}$ to the next section $\{x = x_{n+1}\}$. In particular, we have $\Pi_n(y_n^{\det}) = y_n^{\det} + \gamma = y_{n+1}^{\det}$.

Then, we have the following inductive relation, which is based on a Taylor expansion of $E_{n,N}$.

Proposition 5.3. *Given $1 \leqslant n \leqslant N$ and an initial condition $y_{n-1} \in I_{n-1}$, let Z_n be the random variable*

$$Z_n = \frac{y_{\tau_n} - \Pi_{n-1}(y_{n-1})}{\sigma} \;.$$

Then one has

$$\begin{aligned}
E_{n-1,N}(y_{n-1}) = {}& E_{n,N}\bigl(\Pi_{n-1}(y_{n-1})\bigr)\mathbb{P}^{x_{n-1},y_{n-1}}\bigl(\widehat{\Omega}_n\bigr) \\
&+ \sigma E'_{n,N}\bigl(\Pi_{n-1}(y_{n-1})\bigr)\mathbb{E}^{(x_{n-1},y_{n-1})}\bigl[Z_n\mathbf{1}_{\widehat{\Omega}_n}\bigr] \\
&+ \frac{1}{2}\sigma^2 E''_{n,N}\bigl(\Pi_{n-1}(y_{n-1})\bigr)\mathbb{E}^{(x_{n-1},y_{n-1})}\bigl[Z_n^2\mathbf{1}_{\widehat{\Omega}_n}\bigr] \\
&+ \frac{1}{6}\sigma^3\mathbb{E}^{(x_{n-1},y_{n-1})}\bigl[Z_n^3\mathbf{1}_{\widehat{\Omega}_n}E'''_{n,N}\bigl(\Pi_{n-1}(y_{n-1}) + \theta\sigma Z_n\bigr)\bigr]
\end{aligned} \tag{5.3}$$

for some $\theta \in [0,1]$.

PROOF: Let χ_n be the density of y_{τ_n}, restricted to I_n (this can be viewed as the density of the process killed if it does not leave B_n through I_n). Since $\mathbf{1}_{\widehat\Omega_{n-1,N}} = \mathbf{1}_{\widehat\Omega_{n-1}}\mathbf{1}_{\widehat\Omega_{n,N}}$, the strong Markov property applied on $\{x = x_n\}$ implies

$$\begin{aligned}
E_{n-1,N}(y_{n-1}) &= \mathbb{E}^{(x_{n-1},y_{n-1})}\big[y_{\tau_N}\mathbf{1}_{\widehat\Omega_{n-1}}\mathbf{1}_{\widehat\Omega_{n,N}}\big] \\
&= \int_{I_n} \chi_n(z) E_{n,N}\big(\Pi(y_{n-1}) + \sigma z\big)\,\mathrm{d}z \;.
\end{aligned}$$

We use the Taylor expansion with remainder

$$E_{n,N}\big(\Pi(y_{n-1}) + \sigma z\big) = \sum_{k=0}^{2} \frac{1}{k!}\sigma^k z^k E^{(k)}_{n,N}\big(\Pi(y_{n-1})\big) + \frac{1}{3!}\sigma^3 z^3 E'''_{n,N}\big(\Pi(y_{n-1}) + \theta\sigma z\big) \;.$$

Since

$$\int_{I_n} z^k \chi_n(z)\,\mathrm{d}z = \mathbb{P}^{(x_{n-1},y_{n-1})}\big\{Z_n^k \mathbf{1}_{\widehat\Omega_n}\big\}$$

for any $k \geqslant 0$, the first three terms in the Taylor expansion yield the first three terms in (5.3). The remainder is dealed with in a similar way, except that the function $E'''_{n,N}$ has to remain inside the expectation. □

From now on, we make the choice

$$\gamma = \sigma \;,$$

since this will yield near-optimal error terms, and simplify their expressions. Since the event $\Omega_4 = \Omega_{4,n}$ introduced in (4.37) is included in $\widehat\Omega_n$, Proposition 4.12 implies

$$\begin{aligned}
\mathbb{E}^{(x_{n-1},y_{n-1})}\big[Z_n\mathbf{1}_{\widehat\Omega_n}\big] &= \frac{1}{2}\sigma\frac{T_{n-1,n}(y_{n-1})^2}{(x_n^2 + y_n^{\det})^2} + \mathcal{O}(\sigma^3) \;, \\
\mathbb{E}^{(x_{n-1},y_{n-1})}\big[Z_n^2\mathbf{1}_{\widehat\Omega_n}\big] &= T_{n-1,n}(y_{n-1}) + \mathcal{O}(\sigma^2) \;. \\
\mathbb{E}^{(x_{n-1},y_{n-1})}\big[Z_n^3\mathbf{1}_{\widehat\Omega_n}\big] &= \mathcal{O}(1) \;.
\end{aligned}$$

Here we have replaced the indicator functions $\mathbf{1}_{\Omega_{4,n}}$ by $\mathbf{1}_{\widehat\Omega_n}$, since Z_n is bounded on $\widehat\Omega_n$ and the $\Omega_{4,n}$ have an exponentially small probability. This allows us to obtain the following inductive relation on the $E_{n,N}(y_n)$.

Proposition 5.4. *For every $0 \leqslant n \leqslant N-1$, one has*

$$E_{n,N}(y_n) = y_n + T_{n,N}(y_n) + \sigma^2 D_{n,N}(y_n) + \sigma^3 R_{n,N}(y_n,\sigma) \;, \tag{5.4}$$

where the second-order corrections $D_{n,N}$ and remainders $R_{n,N}$ satisfy

$$D_{n-1,N}(y_{n-1}) = D_{n,N}\big(\Pi_{n-1}(y_{n-1})\big) + \frac{1}{2}\sigma T''_{n,N}\big(\Pi_{n-1}(y_{n-1})\big) + \mathcal{O}(\sigma^2) \tag{5.5}$$

$$R_{n-1,N}(y_{n-1},\sigma) = R_{n,N}\big(\Pi_{n-1}(y_{n-1}),\sigma\big) + \mathcal{O}(\sigma) \tag{5.6}$$

for $1 \leqslant n \leqslant N-1$.

PROOF: We first note that if $\sigma = 0$, then $\widehat\Omega_{n,N} = \widehat\Omega$ and

$$E_{n,N}(y_n) = \mathbb{E}^{(x_n,y_n)}\big[y_{\tau_N}\mathbf{1}_{\widehat\Omega_{n,N}}\big] = y_n + T_{n,N}(y_n) \;,$$

so that (5.4) holds. We now proceed by induction over n, going backwards in time. The relation (5.5) is true for $n = N$ with $T_{N,N}(y_N) = D_{N,N}(y_N) = R_{N,N}(y_N) = 0$. Assuming it holds for some n between 1 and N, we have

$$\begin{aligned}
E'_{n,N}(y) &= 1 + T'_{n,N}(y) + \mathcal{O}(\sigma^2) \;, \\
E''_{n,N}(y) &= T''_{n,N}(y) + \mathcal{O}(\sigma^2) \;.
\end{aligned} \tag{5.7}$$

We now plug the expansion (5.3) into (5.4) and expand into powers of σ. At order 1, we obtain

$$y_{n-1} + T_{n-1,N}(y_{n-1}) = \Pi_{n-1}(y_{n-1}) + T_{n,N}\big(\Pi_{n-1}(y_{n-1})\big) ,$$

which is true in view of (5.2), since $T_{n-1,N}(y_{n-1}) = T_{n,N}(y_{n-1}) + T_{n,N}(\Pi_{n-1}(y_{n-1}))$ (this follows from the semi-group property of the deterministic flow). At order σ^2, we get

$$\begin{aligned}D_{n-1,N}(y_{n-1}) = \lim_{\sigma\to0}\bigg[&D_{n,N}\big(\Pi_{n-1}(y_{n-1})\big) + \frac12 E'_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)\frac{T_{n-1,n}(y_{n-1})^2}{(x_n^2+y_n^{\text{det}})^2}\\ &+\frac12 E''_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)T_{n-1,n}(y_{n-1}) + \mathcal{O}(\sigma^2)\bigg] .\end{aligned}$$

The bound (5.5) follows from (5.7) and the fact that $T_{n-1,n}(y_{n-1}) = \mathcal{O}(\sigma)$. For the remainder, we obtain the relation

$$\begin{aligned}R_{n-1,N}(y_{n-1},\sigma) = R_{n,N}\big(\Pi_{n-1}(y_{n-1}),\sigma\big) &+ \frac{\mathbb{P}^{x_{n-1},y_{n-1}}(\widehat{\Omega}_n^c)}{\sigma^3}E_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)\\ &+\frac16\mathbb{E}^{(x_{n-1},y_{n-1})}\big[Z_n^3\mathbf{1}_{\widehat{\Omega}_{n,N}}E'''_{n,N}(\Pi_{n-1}(y_{n-1}+\theta\sigma Z_n))\big] + \mathcal{O}(\sigma^3) ,\end{aligned}$$

which implies the remainder bound (5.6), since $\mathbb{P}^{x_{n-1},y_{n-1}}(\widehat{\Omega}_n^c)$ is exponentially small. □

Since $\Pi_{n-1}(y_{n-1}^{\text{det}}) = y_n^{\text{det}}$, iterating the inductive relation (5.5) shows that

$$D_{0,N}(y_0^{\text{det}}) = \frac12\sigma\sum_{n=1}^N T''_{n-1,n}(y_n^{\text{det}}) + \mathcal{O}(N\sigma^2) , \tag{5.8}$$

and therefore, viewing the sum as a Riemann sum and since $\sigma = \gamma = \mathcal{O}(1/N)$,

$$\begin{aligned}\mathbb{E}^{(x_{\text{in}},y_{\text{in}})}\big[y_{\tau_N}\mathbf{1}_{\widehat{\Omega}_{0,N}}\big] &= y_{\text{fin}} + \frac12\sigma^3\sum_{n=1}^N T''_{n,N}(y_n^{\text{det}}) + \mathcal{O}(N\sigma^4) + \mathcal{O}(\sigma^3)\\ &= y_{\text{fin}} + \frac12\sigma^2\int_{y_{\text{in}}}^{y_{\text{fin}}}\partial_{yy}T(x^{\text{det}}(y),y;x_{\text{fin}})\,\mathrm{d}y + \mathcal{O}(\sigma^3) .\end{aligned}$$

Replacing $\mathbf{1}_{\widehat{\Omega}_{0,N}}$ by $\mathbf{1}_{\Omega_0}$ only results in an error term of order $\mathbb{P}(\widehat{\Omega}_{0,N}^c)$, which is exponentially small. We have thus proved the bound (2.11) on the expectation.

5.3 Variance of y_τ

The computation of the variance of y_τ is quite similar to the computation of the expectation, so we only comment on what changes. The quantity

$$M_{n,N}(y_{n-1}) = \mathbb{E}^{(x_n,y_n)}\big[y_{\tau_N^2}\mathbf{1}_{\widehat{\Omega}_{n,N}}\big]$$

satisfies the expansion (5.3) with $E_{n,N}$ replaced by $M_{n,N}$. Then the analogue of Proposition 5.4 reads

$$M_{n,N}(y_n) = \big[y_n + T_{n,N}(y_n)\big]^2 + \sigma^2F_{n,N}(y_n) + \mathcal{O}(\sigma^3) ,$$

where

$$\begin{aligned}F_{n-1,N}(y_{n-1}) = F_{n,N}&\big(\Pi_{n-1}(y_{n-1})\big)\\ &+\sigma\Big[\Big(1+T'_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)\Big)^2\\ &\qquad+\Big(\Pi_{n-1}(y_{n-1}) + T_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)\Big)T''_{n,N}\big(\Pi_{n-1}(y_{n-1})\big)\Big]\\ &+\mathcal{O}(\sigma^2) .\end{aligned}$$

Since $y_n^{\det} + T_{n,N}(y_n^{\det}) = y_N^{\det} = y_{\text{fin}}$, this entails

$$F_{0,N}(y_0^{\det}) = \sigma \sum_{n=1}^{N} \Big[\big(1 + T'_{n,N}(y_n^{\det})\big)^2 + y_N^{\det} T''_{n,N}(y_n^{\det}) \Big] + \mathcal{O}(\sigma^2) \; .$$

Therefore, using (5.8) one obtains

$$\begin{aligned}
\operatorname{Var}^{(x_{\text{in}},y_{\text{in}})}\{y_{\tau_N} \mathbf{1}_{\widehat{\Omega}_{0,N}}\} &= \sigma^2 \big[F_{0,N} - D_{0,N}^2\big] + \mathcal{O}(\sigma^3) \\
&= \sigma^3 \sum_{n=1}^{N} \big(1 + T'_{n,N}(y_n^{\det})\big)^2 + \mathcal{O}(\sigma^3) \\
&= \sigma^2 \int_{y_{\text{in}}}^{y_{\text{fin}}} \big(1 + \partial_y T(x^{\det}(y), y; x_{\text{fin}})\big)^2 \, \mathrm{d}y + \mathcal{O}(\sigma^3) \; .
\end{aligned}$$

As in the case of the expectation, replacing $\mathbf{1}_{\widehat{\Omega}_{0,N}}$ by $\mathbf{1}_{\Omega_0}$ only results in an exponentially small error term, so that we have proved the bound (2.12) on the variance. The proof of Theorem 2.1 is thus complete.

6 Analysis of the functions D and V

In this section, we analyse the functions $D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ and $V(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ that appear in the statement of Theorem 2.1, in the particular case where the deterministic reference solution

$$(x^{\det}(t), y^{\det}(t)) = (x_0^{\det}(t), y_0^{\det}(t))$$

is given by (2.5). This requires us to find expressions for the derivatives $\partial_y T(x_0^{\det}(y), y; x_{\text{fin}})$ and $\partial_{yy} T(x_0^{\det}(y), y; x_{\text{fin}})$ of the deterministic time to reach x_{fin}, which we obtain in Section 6.1. Section 6.2 is dedicated to the asymptotic behaviour as $x_{\text{fin}} \to \infty$, providing the proof of Proposition 2.2.

6.1 General expressions for the derivatives of $T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$

Since all trajectories in this section are deterministic, we suppress the superscript $^{\det}$ from now on. Thus, the deterministic reference solution (2.5) is denoted $(x_0(t), y_0(t))$.

Proposition 6.1. *Assume the initial condition $(x_{\text{in}}, y_{\text{in}})$ belongs to the deterministic reference solution (2.5). Fix $x_{\text{fin}} > x_{\text{in}}$, and denote by y_{fin} the corresponding y-coordinate. Then the deterministic travel time $T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ satisfies*

$$\begin{aligned}
\partial_y T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) &= -\frac{A}{r_{\text{fin}}} \; , \\
\partial_{yy} T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) &= \frac{2x_{\text{fin}} r_{\text{fin}} - 1}{r_{\text{fin}}^3} A^2 + \frac{2A}{r_{\text{fin}}^2} - \frac{B}{r_{\text{fin}}} \; ,
\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}
A &= r_{\text{fin}} - \frac{c_{\text{in}}}{\operatorname{Ai}(-y_{\text{fin}})^2} \; , \\
B &= 2r_{\text{fin}} x_{\text{fin}} + 1 + \frac{2}{\operatorname{Ai}(-y_{\text{fin}})^2} \bigg\{ - c_{\text{in}}(2x_{\text{fin}} - x_{\text{in}}) - \frac{1}{2} \operatorname{Ai}(-y_{\text{in}})^2 \\
&\qquad + \pi c_{\text{in}}^2 \bigg[\frac{\operatorname{Bi}(-y_{\text{in}})}{\operatorname{Ai}(-y_{\text{in}})} - \frac{\operatorname{Bi}(-y_{\text{fin}})}{\operatorname{Ai}(-y_{\text{fin}})} \bigg] \bigg\} \; ,
\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
r_{\text{fin}} &= x_{\text{fin}}^2 + y_{\text{fin}} \; , \\
c_{\text{in}} &= \operatorname{Ai}(-y_{\text{in}})^2 (x_{\text{in}}^2 + y_{\text{in}}) = y_{\text{in}} \operatorname{Ai}(-y_{\text{in}})^2 + \operatorname{Ai}'(-y_{\text{in}})^2 \; .
\end{aligned} \tag{6.3}$$

PROOF: In order to determine the effect of a change in the initial y-coordinate, we write

$$\begin{aligned} x(t) &= x_0(t) + \varepsilon x_1(t) + \frac{1}{2}\varepsilon^2 x_2(t) + \mathcal{O}(\varepsilon^3) \\ y(t) &= y_0(t) + \varepsilon , \end{aligned}$$

where $x_1(0) = x_2(0) = 0$. Plugging this into the system (2.2) with $\sigma = 0$ yields

$$\begin{aligned} \dot{x}_1 &= 2x_0(t)x_1 + 1 , \\ \dot{x}_2 &= 2x_0(t)x_2 + 2x_1(t)^2 . \end{aligned}$$

These are linear inhomogenous equations, which can be solved by the method of variation of the constant, yielding

$$\begin{aligned} x_1(t) &= \frac{I(t)}{\mathrm{Ai}(-y_{\text{in}} - t)^2} , \\ x_2(t) &= \frac{2}{\mathrm{Ai}(-y_{\text{in}} - t)^2} \int_0^t \frac{I(s)^2}{\mathrm{Ai}(-y_{\text{in}} - s)^2} \,\mathrm{d}s . \end{aligned} \tag{6.4}$$

Here

$$I(t) = \int_0^t \mathrm{Ai}(-y_{\text{in}} - s)^2 \,\mathrm{d}s = f(-y_{\text{in}} - t) - f(-y_{\text{in}}) ,$$

since f, which has been introduced in (2.16), satisfies $f'(z) = -\mathrm{Ai}(z)^2$. We now expand

$$T = T(x_{\text{in}}, y_{\text{in}} + \varepsilon; x_{\text{fin}}) = T_0 + \varepsilon T_1 + \frac{1}{2}\varepsilon^2 T_2 + \mathcal{O}(\varepsilon^3) ,$$

where $T_0 = T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$, while T_1 and T_2 coincide with the y-derivatives we want to determine. The condition $x(T) = x_{\text{fin}}$, expanded into powers of ε, yields

$$\begin{aligned} T_1\dot{x}_0(T_0) + x_1(T_0) &= 0 \\ T_2\dot{x}_0(T_0) + T_1^2\ddot{x}_0(T_0) + 2T_1\dot{x}_1(T_0) + x_2(T_0) &= 0 , \end{aligned}$$

whose solution reads

$$\begin{aligned} T_1 &= -\frac{x_1(T_0)}{\dot{x}_0(T_0)} \\ T_2 &= -\frac{1}{\dot{x}_0(T_0)} \left[\frac{x_1(T_0)^2\ddot{x}_0(T_0)}{\dot{x}_0(T_0)^2} - 2\frac{x_1(T_0)\dot{x}_1(T_0)}{\dot{x}_0(T_0)} + x_2(T_0) \right] . \end{aligned}$$

Writing $A = x_1(T_0)$, $B = x_2(T_0)$, and using the differential equations satisfied by $x_0(t)$ and $x_1(t)$ yields

$$\begin{aligned} \dot{x}_0(T_0) &= x_{\text{fin}}^2 + y_{\text{fin}} = r_{\text{fin}} \\ \ddot{x}_0(T_0) &= 2x_{\text{fin}}^3 + 2x_{\text{fin}}y_{\text{fin}} + 1 = 2x_{\text{fin}}r_{\text{fin}} + 1 \\ \dot{x}_1(T_0) &= 2x_{\text{fin}}A + 1 , \end{aligned}$$

which implies the expressions (6.1) for the derivatives $\partial_y T = T_1$ and $\partial_{yy} T = T_2$. It remains to check the expressions (6.2) of A and B. Regarding A, we note that (2.16) implies

$$f(-y_0(t)) = \mathrm{Ai}(-y_0(t)^2)\big[y_0(t) + x_0(t)^2\big] .$$

In particular, $f(-y_{\text{in}}) = c_{\text{in}}$, and therefore

$$I(t) = \mathrm{Ai}(-y_0(t))^2\big[y_0(t) + x_0(t)^2\big] - c_{\text{in}} . \tag{6.5}$$

This yields

$$I(T_0) = \mathrm{Ai}(-y_{\text{fin}})^2 r_{\text{fin}} - c_{\text{in}} ,$$

which by (6.4) implies the expression for A in (6.2). Regarding B, we replace one factor $I(s)$ in the expression (6.4) of $x_2(t)$ by (6.5), which yields

$$B = \frac{2}{\mathrm{Ai}(-y_{\text{fin}})^2} \int_0^{T_0} I(s) \biggl[\dot{x}_0(s) - \frac{c_{\text{in}}}{\mathrm{Ai}(-y_0(s))^2}\biggr] \,\mathrm{d}s , \tag{6.6}$$

and split the integral into two parts. For the first part, integration by parts yields

$$\begin{aligned} \int_0^{T_0} I(s)\dot{x}_0(s)\,\mathrm{d}s &= I(T_0)x_{\text{fin}} - \int_0^{T_0} \dot{I}(s)x_0(s)\,\mathrm{d}s \\ &= \bigl[\mathrm{Ai}(-y_{\text{fin}})^2 r_{\text{fin}} - c_{\text{in}}\bigr]x_{\text{fin}} - \int_0^{T_0} \mathrm{Ai}(-y_0(s))\,\mathrm{Ai}'(-y_0(s))\,\mathrm{d}s \\ &= \bigl[\mathrm{Ai}(-y_{\text{fin}})^2 r_{\text{fin}} - c_{\text{in}}\bigr]x_{\text{fin}} + \frac{1}{2}\bigl[\mathrm{Ai}(-y_{\text{fin}})^2 - \mathrm{Ai}(-y_{\text{in}})^2\bigr] . \end{aligned} \tag{6.7}$$

For the second part, we use the fact that (6.5) implies

$$\int_0^{T_0} \frac{I(s)}{\mathrm{Ai}(-y_0(s))^2}\,\mathrm{d}s = x_{\text{fin}} - x_{\text{in}} - c_{\text{in}} \int_0^{T_0} \frac{\mathrm{d}s}{\mathrm{Ai}(-y_0(s))^2} , \tag{6.8}$$

where the integral can be computed thanks to the relation

$$\pi \frac{\mathrm{d}}{\mathrm{d}z}\biggl(\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}\biggr) = \frac{1}{\mathrm{Ai}(z)^2} ,$$

which follows from the expression (A.1) of the Wronskian of Airy functions. Replacing (6.7) and (6.8) in (6.6) and rearranging yields the expression for B in (6.2). □

6.2 Asymptotics for large x_{fin}

The expressions for the derivatives $\partial_y T$ and $\partial_{yy} T$ simplify somewhat in the limit $x_{\text{fin}} \to \infty$, as shows the following result.

Proposition 6.2. *Assume the initial condition* $(x_{\text{in}}, y_{\text{in}})$ *belongs to the deterministic solution* (2.5)*. Then the deterministic travel time* $T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}})$ *satisfies*

$$\lim_{x_{\text{fin}}\to\infty} \partial_y T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{c_{\text{in}}}{\mathrm{Ai}'(-y^\star)^2} - 1 , \tag{6.9}$$

$$\begin{aligned} \lim_{x_{\text{fin}}\to\infty} \partial_{yy} T(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) = \frac{2}{\mathrm{Ai}'(-y^\star)^2} \biggl\{ &\pi c_{\text{in}} \bigl[-y_{\text{in}}\,\mathrm{Ai}(-y_{\text{in}})\,\mathrm{Bi}(-y_{\text{in}}) - \mathrm{Ai}'(-y_{\text{in}})\,\mathrm{Bi}'(-y_{\text{in}})\bigr] \\ &+ \pi c_{\text{in}}^2 \frac{\mathrm{Bi}'(-y^*)}{\mathrm{Ai}'(-y^*)} + \frac{1}{2}\,\mathrm{Ai}(-y_{\text{in}})^2 \biggr\} . \end{aligned} \tag{6.10}$$

PROOF: In order to compute the limits, we write

$$y_{\text{fin}} = y^* - \eta$$

where $\eta > 0$ will be sent to 0. The Taylor expansions

$$\begin{aligned} \mathrm{Ai}(-y_{\text{fin}}) &= \eta\,\mathrm{Ai}'(-y^*)\bigl[1 + \mathcal{O}(\eta^2)\bigr] , \\ \mathrm{Bi}(-y_{\text{fin}}) &= \mathrm{Bi}(-y^*) + \eta\,\mathrm{Bi}'(-y^*) + \mathcal{O}(\eta^2) \\ &= -\frac{1}{\pi\,\mathrm{Ai}'(-y^*)} + \eta\,\mathrm{Bi}'(-y^*) + \mathcal{O}(\eta^2) , \end{aligned}$$

where we have used the Wronskian identity (A.1) in the last line, immediately imply

$$x_{\mathrm{fin}} = \frac{\mathrm{Ai}'(-y_{\mathrm{fin}})}{\mathrm{Ai}(-y_{\mathrm{fin}})} = \frac{1}{\eta}\bigl[1+\mathcal{O}(\eta^2)\bigr]\;, \qquad r_{\mathrm{fin}} = \frac{1}{\eta^2}\bigl[1+\mathcal{O}(\eta^2)\bigr]\;.$$

It thus follows from (6.2) that

$$A = \frac{a_1}{\eta^2}\bigl[1+\mathcal{O}(\eta^2)\bigr] \qquad \text{where} \qquad a_1 = 1 - \frac{c_{\mathrm{in}}}{\mathrm{Ai}'(-y^\star)^2}\;. \tag{6.11}$$

Replacing this in the expression for $\partial_y T$ in (6.1) and taking the limit $\eta \to 0$ yields (6.9). Proceeding similarly for B, we find

$$B = \frac{b_1}{\eta^3} + \frac{b_2}{\eta^2} + \mathcal{O}\biggl(\frac{1}{\eta}\biggr)\;, \tag{6.12}$$

where

$$b_1 = 2 + \frac{2}{\mathrm{Ai}'(-y^\star)^2}\biggl[-2c_{\mathrm{in}} + \frac{c_{\mathrm{in}}^2}{\mathrm{Ai}'(-y^\star)^2}\biggr] = 2\biggl(1 - \frac{c_{\mathrm{in}}}{\mathrm{Ai}'(-y^\star)}\biggr)^2 = 2a_1^2 \tag{6.13}$$

and

$$b_2 = \frac{2}{\mathrm{Ai}'(-y^\star)^2}\biggl\{c_{\mathrm{in}}x_{\mathrm{in}} - \frac{1}{2}\,\mathrm{Ai}'(-y_{\mathrm{in}})^2 + \pi c_{\mathrm{in}}^2\biggl[\frac{\mathrm{Bi}(-y_{\mathrm{in}})}{\mathrm{Ai}(-y_{\mathrm{in}})} - \frac{\mathrm{Bi}'(-y^\star)}{\mathrm{Ai}'(-y^\star)}\biggr]\biggr\}\;.$$

This last expression can be simplified by noting that the second form of c_{in} in (6.3) and the Wronskian identity (A.1) imply

$$x_{\mathrm{in}} + \pi c_{\mathrm{in}}\frac{\mathrm{Bi}(-y_{\mathrm{in}})}{\mathrm{Ai}(-y_{\mathrm{in}})} = \pi\bigl[y_{\mathrm{in}}\,\mathrm{Ai}(-y_{\mathrm{in}})\,\mathrm{Bi}(-y_{\mathrm{in}}) + \mathrm{Ai}'(-y_{\mathrm{in}})\,\mathrm{Bi}'(-y_{\mathrm{in}})\bigr]\;,$$

which leads to

$$\begin{aligned} b_2 = \frac{2}{\mathrm{Ai}'(-y^\star)^2}\biggl[&\pi c_{\mathrm{in}}\bigl[y_{\mathrm{in}}\,\mathrm{Ai}(-y_{\mathrm{in}})\,\mathrm{Bi}(-y_{\mathrm{in}}) + \mathrm{Ai}'(-y_{\mathrm{in}})\,\mathrm{Bi}'(-y_{\mathrm{in}})\bigr] \\ &- \frac{1}{2}\,\mathrm{Ai}(-y_{\mathrm{in}})^2 - \pi c_{\mathrm{in}}^2\frac{\mathrm{Bi}'(-y^\star)}{\mathrm{Ai}'(-y^\star)}\biggr]\;. \end{aligned}$$

Replacing (6.11) and (6.12) in the second expression in (6.1) yields

$$\partial_{yy}T(x_{\mathrm{in}}, y_{\mathrm{in}}; x_{\mathrm{fin}}) = \frac{2a_1^2 - b_1}{\eta} - b_2 + \mathcal{O}(\eta)\;,$$

which yields (6.10), by taking the limit $\eta \to 0$, since the singular term of order η^{-1} vanishes thanks to (6.13). □

We can now complete the proof of Proposition 2.2.

PROOF OF PROPOSITION 2.2. The limiting values of D and V are given by the integrals

$$\begin{aligned} I_a(y_{\mathrm{in}}, y^\star) &= \int_{y_{\mathrm{in}}}^{y^\star} \partial_{yy}T(x_0(y), y; \infty)\,\mathrm{d}y\;, \\ I_b(y_{\mathrm{in}}, y^\star) &= \int_{y_{\mathrm{in}}}^{y^\star} \bigl(1 + \partial_y T(x_0(y), y; \infty)\bigr)^2\,\mathrm{d}y\;. \end{aligned}$$

We start by computing I_b. Note that $\partial_y T(x_0(y), y; \infty)$ is given by (6.9) with c_{in} replaced by

$$c(y) = y\,\mathrm{Ai}(-y)^2 + \mathrm{Ai}'(-y)^2 = f(-y)\;,$$

where f has been introduced in (2.16). Therefore,

$$I_b(y_{\mathrm{in}}, y^\star) = \frac{1}{\mathrm{Ai}'(-y^\star)^4}\int_{-y^\star}^{-y_{\mathrm{in}}} f(z)^2\,\mathrm{d}z\;,$$

where we have used the change of variables $z = -y$ to avoid sign mistakes when computing the integral. The function $\mathcal{F}$ defined in (2.17) satisfies $\mathcal{F}'(z) = f(z)^2$. Thus, the limiting expression for V follows from the fact that $\mathcal{F}(-y^\star) = -\frac{y^\star}{2}\operatorname{Ai}'(-y^\star)^4$. Here the primitive $\mathcal{F}$ of f^2 has been found by using a homogenous polynomial of degree 4 in $\operatorname{Ai}(z)$ and $\operatorname{Ai}'(z)$ with polynomial coefficients in z as an ansatz, and determining the coefficients by plugging the ansatz into the condition $\mathcal{F}'(z) = f(z)^2$.

Proceeding in a similar way for I_a, we obtain

$$I_a(y_{\text{in}}, y^\star) = \frac{1}{\operatorname{Ai}'(-y^\star)^2}\int_{-y^\star}^{-y_{\text{in}}}\left[-2\pi f(z)g(z) + 2\pi\frac{\operatorname{Bi}'(-y^\star)}{\operatorname{Ai}'(-y^\star)}f(z)^2 + \operatorname{Ai}(z)^2\right]\mathrm{d}z\ ,$$

where

$$g(z) = \operatorname{Ai}'(z)\operatorname{Bi}'(z) - z\operatorname{Ai}(z)\operatorname{Bi}(z)\ .$$

The function $\mathcal{G}(z)$ defined in (2.18) is a primitive of $f(z)g(z)$, while we have already seen the $\mathcal{F}(z)$ is a primitive of $f(z)^2$, and $-f(z)$ is a primitive of $\operatorname{Ai}(z)^2$. The limiting expression for D then follows from the fact that

$$\mathcal{G}(-y^\star) = -\frac{y^\star}{2}\operatorname{Ai}'(-y^\star)^3\operatorname{Bi}'(-y^\star) - \frac{1}{8\pi}\operatorname{Ai}'(-y^\star)^2\ .$$

As already stated, the limiting expressions (2.19) follow from the asymptotical behaviour (A.2) of the Airy functions. This concludes the proof of Proposition 2.2, except for the fact that D is positive, which we show in the separate Lemma below. □

Lemma 6.3. *We have*

$$D(x_{\text{in}}, y_{\text{in}}; \infty) := \lim_{x_{\text{fin}}\to\infty} D(x_{\text{in}}, y_{\text{in}}; x_{\text{fin}}) > 0$$

for any $(x_{\text{in}}, y_{\text{in}})$ on the reference slow solution.

PROOF: Since the limit as $y_{\text{in}} \to -\infty$ is equal to $\frac{3}{4}$, and the integral defining D vanishes when $y_{\text{in}} = y^\star$, it suffices to show that D is decreasing in y_{in}. The derivative of D with respect to y_{in} is given by $-F(-y_{\text{in}})/\operatorname{Ai}'(-y^\star)^2$, where

$$F(z) := 2\pi\left(\frac{\operatorname{Bi}'(-y^\star)}{\operatorname{Ai}'(-y^\star)}f(z) - g(z)\right)f(z) + \operatorname{Ai}(z)^2\ .$$

Note in particular that

$$F(-y^\star) = 0\ , \qquad \lim_{z\to\infty} F(z) = 0\ ,$$

A simple way to study the sign of F in the interval $(-y^\star, \infty)$ is to study the behaviour of its derivative. One finds, after a computation,

$$F'(z) = 4\pi\operatorname{Ai}(z)^2 G(z)\ , \qquad G(z) := g(z) - \frac{\operatorname{Bi}'(-y^\star)}{\operatorname{Ai}'(-y^\star)}f(z)\ .$$

Hence the sign of F' is the same as the sign of G for $z > -y^\star$. Furthermore,

$$G(-y^\star) = 0\ , \qquad \lim_{z\to\infty} G(z) = 0\ ,$$

Again, to study the sign of G, we compute its derivative, which is

$$G'(z) = \operatorname{Ai}(z)H(z)\ , \qquad H(z) := \frac{\operatorname{Bi}'(-y^\star)}{\operatorname{Ai}'(-y^\star)}\operatorname{Ai}(z) - \operatorname{Bi}(z)\ .$$

Since $\operatorname{Ai}(z) > 0$ for all $z > -y^\star$, G' has the same sign as H. Thus the proof reduces to the study of H, which is much simpler than F and G. One checks that

$$H(-y^\star) = -\operatorname{Bi}(-y^\star) = \frac{1}{\pi\operatorname{Ai}'(-y^\star)} > 0\ , \qquad \lim_{z\to\infty} H(z) = -\infty\ ,$$

and direct differentiation gives

$$H'(-y^\star) = 0 \ .$$

This means that H starts positive at $z = -y^\star$, with horizontal tangent there, and tends to $-\infty$ as $z \to \infty$. We want to prove that $H'(z) < 0$ for all $z > -y^\star$. Indeed, this would imply that G, and thus F, is first increasing, and then decreasing to zero, and therefore strictly positive on $(-y^\star, \infty)$.

We first prove that H' is strictly negative on $(-y^\star, 0)$. Differentiating H gives after some rearranging

$$H'(z) = \mathrm{Bi}'(z)\frac{\mathrm{Bi}'(-y^\star)}{\mathrm{Ai}'(-y^\star)}(k(z) - k(-y^\star)) \ , \qquad k(z) := \frac{\mathrm{Ai}'(z)}{\mathrm{Bi}'(z)} \ ,$$

Using the Wronskian identity (A.1), one gets

$$k'(z) = \frac{z}{\pi\,\mathrm{Bi}'(z)^2} \ ,$$

which is negative for $-y^\star < z < 0$. It is known that there exists a value $-y_1^\star \in (-y^\star, 0)$ such that $\mathrm{Bi}'(z) < 0$ on $[-y^\star, -y_1^\star)$ and $\mathrm{Bi}'(z) > 0$ on $(-y_1^\star, \infty)$. It follows that $k(z)$ is negative and decreasing on $(-y^\star, -y_1^\star)$, with a vertical asymptote at $-y_1^\star$, and positive and decreasing on $(-y_1^\star, \infty)$. In both cases, $H'(z)$ is strictly negative.

It remains to show that $H'(z)$ is strictly negative on $[0, \infty)$. Since Ai and Bi are independent solutions of the equation $y'' = zy$, any linear combination satisfies the same ODE. Therefore

$$H''(z) = zH(z) \ .$$

This ODE leaves the sector $H < 0, H' < 0$ invariant for $z > 0$. One can check, using properties of Airy functions, that $H(0) < 0$ and $H'(0) < 0$. Therefore, the proof is complete. □

A Airy functions

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are by definition two independent solutions of the linear, non-autonomous ordinary differential equation

$$y''(z) = zy(z) \ .$$

They can be defined by

$$\begin{aligned}\mathrm{Ai}(z) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\xi^3}{3} + z\xi\right) \mathrm{d}\xi \\ \mathrm{Bi}(z) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{\xi^3}{3} + z\xi\right) + \sin\left(\frac{\xi^3}{3} + z\xi\right)\right] \mathrm{d}\xi \ .\end{aligned}$$

See Figure 8 for plots.

Their Wronskian satisfies

$$W(z) := \det\begin{pmatrix}\mathrm{Ai}(z) & \mathrm{Bi}(z) \\ \mathrm{Ai}'(z) & \mathrm{Bi}'(z)\end{pmatrix} = \mathrm{Ai}(z)\,\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\,\mathrm{Bi}(z) = \frac{1}{\pi} \ . \tag{A.1}$$

Their asymptotic behaviour for large positive z (see for instance [20]) is given by

$$\begin{aligned}\mathrm{Ai}(z) &= \frac{z^{-1/4}}{2\sqrt{\pi}}\,\mathrm{e}^{-\xi}\,u(-\xi) \ , & \mathrm{Bi}(z) &= \frac{z^{-1/4}}{\sqrt{\pi}}\,\mathrm{e}^{\xi}\,u(\xi) \ , \\ \mathrm{Ai}'(z) &= -\frac{z^{1/4}}{2\sqrt{\pi}}\,\mathrm{e}^{-\xi}\,v(-\xi) \ , & \mathrm{Bi}'(z) &= \frac{z^{1/4}}{\sqrt{\pi}}\,\mathrm{e}^{\xi}\,v(\xi) \ ,\end{aligned} \tag{A.2}$$

where $\xi = \frac{2}{3}z^{2/3}$ and u and v have asymptotic expansions starting with

$$u(\xi) = 1 + \frac{5}{72\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \ , \qquad v(\xi) = 1 - \frac{7}{72\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \ .$$

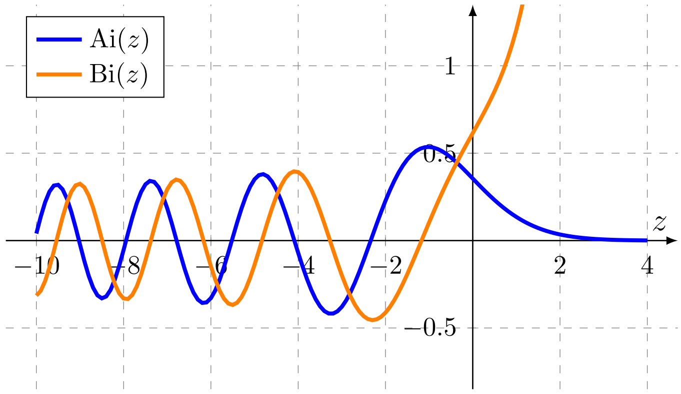

Figure 8: Airy functions.

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