# Spectral theory for random Poincaré maps

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**Abstract**

We consider stochastic differential equations, obtained by adding weak Gaussian white noise to ordinary differential equations admitting $N$ asymptotically stable periodic orbits. We construct a discrete-time, continuous-space Markov chain, called a random Poincaré map, which encodes the metastable behaviour of the system. We show that this process admits exactly $N$ eigenvalues which are exponentially close to 1, and provide expressions for these eigenvalues and their left and right eigenfunctions in terms of committor functions of neighbourhoods of periodic orbits. The eigenvalues and eigenfunctions are well-approximated by principal eigenvalues and quasistationary distributions of processes killed upon hitting some of these neighbourhoods. The proofs rely on Feynman–Kac-type representation formulas for eigenfunctions, Doob's $h$-transform, spectral theory of compact operators, and a recently discovered detailed-balance property satisfied by committor functions.

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## 1 Introduction

A very useful method to analyse the dynamics of ordinary differential equations (ODEs) admitting one or several periodic orbits consists in introducing a submanifold of codimension 1, which is transversal to the flow. Successive intersections of orbits with this submanifold are described by an iterated map, called a *first-return map* or *Poincaré map*. This map has proved extremely useful for a number of reasons. First, it replaces a $d$-dimensional ODE by a $(d-1)$-dimensional map, which is often easier to visualise. Perhaps more importantly, it simplifies the stability analysis of periodic orbits, because it allows to get rid of neutral transversal directions. Furthermore, Poincaré maps simplify the classification of bifurcations of periodic orbits, since the problem is reduced to the easier one of classifying bifurcations of fixed points of maps.

When noise is added to an ODE, it becomes a stochastic differential equation (SDE). SDEs with multiple periodic orbits appear in many applications, such as enzyme reaction models [47, 26], neuron dynamics [28, 39] and related piecewise deterministic Markov processes [29]. A natural analogue of the Poincaré map in this situation was introduced in [37], and further analysed in [38], by Hitczenko and Medvedev who called it *Poincaré map of randomly perturbed periodic motion*, or *random Poincaré map* for short. Random Poincaré maps have already proved useful in several applications: they allowed to study interspike interval statistics in the stochastic FitzHugh–Nagumo equations [12], the first-passage location through an unstable periodic orbit in planar SDEs [9], and mixed-mode oscillation patterns in systems admitting a folded-node singularity [11].

Mathematically, a random Poincaré map is described by a discrete-time, continuous-space Markov chain. If the ODE perturbed by noise admits $N \geqslant 2$ stable periodic orbits, and the noise intensity $\sigma$ is weak, the Markov chain will tend to spend very long time intervals in small neighbourhoods of the periodic orbits, with occasional transitions between these neighbourhoods. This kind of behaviour is known as *metastability*.

The metastable dynamics of SDEs has been studied on the level of exponential asymptotics by Freidlin and Wentzell [32], using the theory of large deviations. In the particular case where the original ODE derives from a potential and the noise is homogeneous and isotropic, the perturbed system's invariant measure is known explicitly, and the dynamics is reversible with respect to this measure. Reversibility greatly simplifies the analysis of the system. In particular, the potential-theoretic approach developed by Bovier, Eckhoff, Gayrard and Klein in [18, 19] for SDEs yields very precise estimates on metastable transition times and small eigenvalues of the generator, which are governed by the so-called Eyring–Kramers formula. See for instance [5] for a recent review, and the monograph [17] for a comprehensive account of the potential-theoretic approach.

A drawback of the potential-theoretic approach to metastability is that it has so far only been developed in the reversible case. Systems admitting periodic orbits are, however, strongly non-reversible. Recently, there have been a few attempts to derive Eyring–Kramers-like formulas for non-reversible systems. For instance, in [45] Lu and Nolen provided expressions for transition times and reactive times in terms of committor functions (that is, probabilities to hit a set $A$ before a set $B$), based on the transition-path theory introduced by E and Vanden–Eijnden [30]. In [16], Bouchet and Reygner formally derived an Eyring–Kramers law for a class of non-reversible systems admitting an isolated saddle, based on WKB asymptotics. Furthermore, in [44], Landim and Seo obtained an Eyring–Kramers formula for certain non-reversible random walks for which the invariant measure is explicitly known, using two variational formulae for the capacity. In [43] Landim, Mariani, and Seo provide a sharp estimate for the transition times between two different wells for a class of non-reversible diffusion processes (again with known invariant measure). Despite these promising results, a full theory providing sharp asymptotics for metastable transition times for general non-reversible systems has yet to be developed.

Fortunately, it turns out that some central ideas in [19], concerning the spectral properties of the generator, do in fact not require any potential-theoretic tools. The key assumption is that the metastable states can be ordered in a particular way, from most stable to least stable, forming a so-called *metastable hierarchy*. Furthermore, it has become apparent that the small eigenvalues of the diffusion and the corresponding eigenfunctions are strongly connected to principal eigenvalues and quasistationary distributions (QSDs) of certain related processes. See for instance [14] for the case of reversible Markovian jump processes, [24] for birth-and-death processes and related population models, and [27] for the case of reversible diffusions. Principal eigenvalues and QSDs are much easier to determine numerically than arbitrary eigenvalues and eigenfunctions.

The aim of the present work is to derive spectral information on random Poincaré maps, associated with non-reversible SDEs obtained by perturbing ODEs admitting $N \geqslant 2$ asymptotically stable periodic orbits. Indeed, discrete-time continuous-space Markov chains are amenable to Fredholm theory, showing that transition probabilities can be represented as sums of projectors on invariant subspaces multiplied by eigenvalues. Our main result, Theorem 3.2, shows that for sufficiently small noise, the random Poincaré map admits exactly $N$ eigenvalues exponentially close to 1, which are all real, while all remaining eigenvalues are bounded away from 1. Theorems 3.4 and 3.8 provide expressions for the

associated right and left eigenfunctions. All these quantities are expressed in terms of committor functions associated with small neighbourhoods of the stable periodic orbits. Furthermore, we show that the eigenvalues and left eigenfunctions are well approximated by principal eigenvalues and QSDs of processes killed upon hitting some of these neighbourhoods. Therefore our results provide links between spectral properties of the random Poincaré map and quantities that are accessible to numerical simulations.

The spectral decomposition that we obtain can be interpreted as showing that on long timescales, the dynamics of the system can be described by an $N$-state Markov chain. The $N$ states correspond to the $N$ stable periodic orbits, and one-step transition probabilities between different states are exponentially small. In particular, the metastable hierarchy assumption implies that there are $N-1$ timescales of the form $\mathrm{e}^{H_i/\sigma^2}$, with $H_1 > H_2 > \cdots > H_{N-1} > 0$. The time needed to reach the union of the $k$ first periodic orbits starting from the $k+1^{\text{st}}$ orbit is of order $\mathrm{e}^{H_k/\sigma^2}$, while the $k^{\text{th}}$ eigenvalue of the random Poincaré map behaves like $1-\mathrm{e}^{-H_k/\sigma^2}$. Note that this is compatible with [32, Theorem 7.3, Chapter 6], which states that the generator of the diffusion admits $N-1$ eigenvalues with exponentially small real parts, of order $-\mathrm{e}^{-H_k/\sigma^2}$. A new feature of our results is that they concern the eigenvalues of the discrete-time Markov chain instead of the continuous-time generator, and that we are able to prove that these eigenvalues are real. Apart from this relation interpretable in terms of metastable transition times, the general link between eigenvalues of the discrete-time and continous-time generators is not yet fully understood (except in trivial cases where the dynamics transversal to periodic orbits is completely decoupled from the phase dynamics).

To obtain these results, we combine various techniques developed in prior works. One of them is the representation of eigenfunctions in terms of Laplace transforms of hitting times of well-chosen sets, already present in [19]. Another key idea is the fact, discovered in [13], that committor functions of not necessarily reversible Markov chains satisfy a kind of detailed balance condition. We also rely on perturbation theory for compact linear operators (see e.g. [41, 35]), Doob's $h$-transform, which is linked to the theory of quasi-stationary distributions as reviewed in [25], as well as sample-path estimates for SDEs which were developed in [7, 8, 9].

The remainder of this work is organised as follows. In Section 2, we define precisely the kind of SDEs that we are going to consider, provide a construction of their random Poincaré maps, and define the spectral decomposition. Section 3 contains the main results of the work. In Section 4, we provide an outline of the main steps of the proofs. Subsequent sections are dedicated to technical details of the proofs. Section 5 contains estimates of the spectral gap and principal eigenfunction of the process killed upon leaving a neighbourhood of a periodic orbit. In Section 6, we show that the random Poincaré map can be approximated by a finite-rank operator by providing estimates for the operator norm of their difference. The spectral properties of the finite-rank operator are described in Section 7. Section 8 provides sample-path estimates needed to apply the bounds on operator norms, while Section 9 contains the proofs of the main results. Appendix A recalls some properties of Doob's $h$-transform, whereas Appendix B recalls some results on Floquet theory.

*Notations:* Unless otherwise specified, $\|\cdot\|$ denotes the supremum norm of a function or a linear operator. The indicator function of an event or set $A$ is denoted $\mathbb{1}_A$ . The symbol id is used for the identity operator as well as the identity matrix.

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# 2 Set-Up

## 2.1 Deterministic system

Let $\mathcal{D}_0 \subset \mathbb{R}^{d+1}$ be an open, connected set and let $f \in \mathcal{C}^2\big(\mathcal{D}_0, \mathbb{R}^{d+1}\big)$. We consider the $(d+1)$-dimensional deterministic ordinary differential equation (ODE) given by

$$\dot{z} = f(z) \; . \tag{2.1}$$

**Assumption 2.1** (Invariant domain)**.** There exists a bounded, open connected set $\mathcal{D} \subset \mathcal{D}_0$ which is positively invariant under the flow of (2.1). ♣

This assumption ensures that for all $z \in \mathcal{D}$ the flow $\varphi_t(z)$ is defined for all $t \geqslant 0$. Recall that the image $\{\varphi_t(z)\colon t \geqslant 0\}$ is called the (positive) *orbit* of $z$. The *$\omega$-limit set* of $z$ is the set of accumulation points of $\varphi_t(z)$ as $t \to \infty$. If $\varphi_t(z)$ is defined for all $t \leqslant 0$, its set of accumulation points as $t \to -\infty$ is called the *$\alpha$-limit set* of $z$. A *heteroclinic connection* between two sets $A, B \subset \mathbb{R}^{d+1}$ invariant under the flow is an orbit admitting $A$ as $\alpha$-limit set and $B$ as $\omega$-limit set.

Recall that $\Gamma$ is a *periodic orbit* of period $T > 0$ of (2.1) if there exists a periodic function $\gamma : \mathbb{R} \to \mathcal{D}$ of minimal period $T$ such that

$$\dot{\gamma}(t) = f(\gamma(t)) \qquad \forall t \in \mathbb{R} \; . \tag{2.2}$$

Then $\Gamma$ is simply the image $\{\gamma(t)\colon t \in [0,T)\}$ of $\gamma$. The periodic orbit is called *linearly asymptotically stable* if all Floquet multipliers of the linearised system $\dot{\xi} = \partial_z f(\gamma(t))\xi$ are strictly smaller than 1 in modulus. A periodic orbit is *linearly unstable* if it admits at least one Floquet multiplier of modulus strictly larger than 1.

**Assumption 2.2** (Limit sets)**.** There are finitely many $\omega$-limit sets in $\mathcal{D}$. They include $N \geqslant 2$ distinct linearly asymptotically stable periodic orbits $\Gamma_1, \dots, \Gamma_N$. All other $\omega$-limit sets in $\mathcal{D}$ are either linearly unstable stationary points, or linearly unstable periodic orbits. Furthermore, there exists a smooth orientable $d$-dimensional manifold $\Sigma \subset \mathcal{D}$ with boundary $\partial\Sigma \subset \partial\mathcal{D}$, such that for all $x \in \Sigma$, $f(x)$ is not tangent to $\Sigma$ (transversality). Each stable periodic orbit $\Gamma_i$ intersects $\Sigma$ at exactly one point $x_i^\star$. In addition, there are no heteroclinic connections between unstable orbits or between unstable orbits and unstable stationary points. ♣

Note that $\mathcal{D}$ is not required to be simply connected: it can have the shape of a solid torus containing all periodic orbits in its interior (cf. Fig. 3). The deterministic *Poincaré map* associated with $\Sigma$ is then the map $\Pi : \Sigma \to \Sigma$ defined by

$$\Pi(x) = \varphi_\tau(x) \qquad \text{where } \tau = \inf\{t > 0\colon \varphi_t(x) \in \Sigma\} \; . \tag{2.3}$$

We will always implicitly assume that $\tau < \infty$ for almost all $x \in \Sigma$. In other words, except perhaps for a set of initial conditions of zero Lebesgue measure, orbits starting on $\Sigma$ always return to $\Sigma$ in a finite time.

We will denote by

$$\mathcal{A}_j = \Big\{x \in \Sigma\colon \lim_{n\to\infty} \Pi^n(x) = x_j^\star\Big\} \tag{2.4}$$

the basin of attraction of the orbit $\Gamma_j$. The $\mathcal{A}_j$ are open, disjoint subsets of $\Sigma$, and the union of their closures is equal to $\Sigma$.

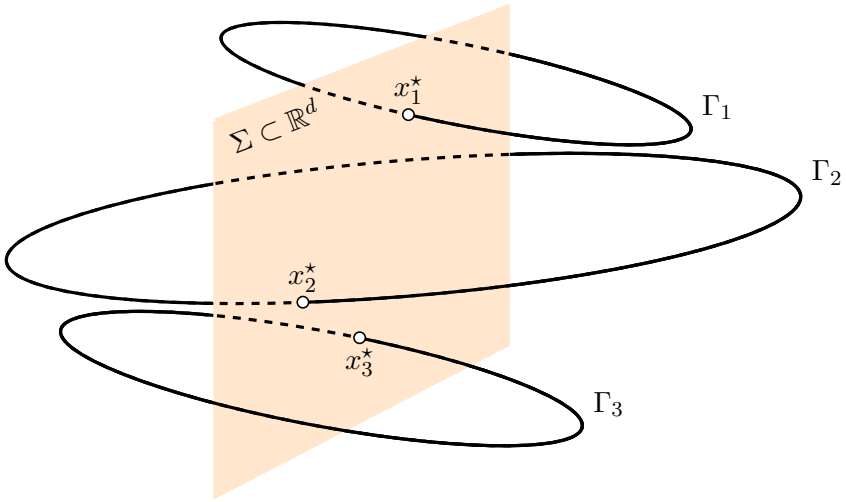


Figure 1: Sketch of Poincaré map for a deterministic system admitting several stables periodic orbits.

**Remark 2.3.** Note that Assumption 2.2 rules out the existence of any other $\omega$-limit sets than periodic orbits and unstable stationary points. We have formulated the assumption in this way for simplicity. In fact, what we really need is that for each $\omega$-limit set other than the $\Gamma_i$, noise added to the system is likely to move sample paths away from these sets in a time which is negligible with respect to typical transition times between the $\Gamma_i$.

Furthermore, we believe that the absence of heteroclinic connections is not required. We only need that a sample path starting near an unstable $\omega$-limit set reaches the neighbourhood of a stable periodic orbit after a negligible time. ♢

## 2.2 Stochastic system

We turn now to random perturbations of the ODE (2.1), given by Itô stochastic differential equations (SDEs) of the form

$$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t\,. \tag{2.5}$$

Here $W_t$ denotes a $k$-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $k \geqslant d+1$, while $g \in \mathcal{C}^1(\mathcal{D}_0, \mathbb{R}^{(d+1)\times k})$, and $\sigma > 0$ is a small parameter. We will denote by $Z_t^z$, or simply $Z_t$, the solution of (2.5) starting in $z$ at time 0. The corresponding probability is written $\mathbb{P}^z\{\cdot\}$, and expectations with respect to $\mathbb{P}^z\{\cdot\}$ are denoted $\mathbb{E}^z\{\cdot\}$. The *infinitesimal generator* of the diffusion process is the second-order differential operator

$$\mathcal{L} = \sum_{i=1}^{d+1} f_i(z)\frac{\partial}{\partial z_i} + \frac{\sigma^2}{2}\sum_{i,j=1}^{d+1} D_{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j} \tag{2.6}$$

where $D(z) = g(z)g(z)^{\mathrm{T}}$ denotes the *diffusion matrix*.

**Assumption 2.4** (Ellipticity)**.** There exist constants $c_+ > c_- > 0$ such that

$$c_-\|\xi\|^2 \leqslant \langle \xi, D(z)\xi\rangle \leqslant c_+\|\xi\|^2 \tag{2.7}$$

for all $z \in \mathcal{D}$ and all $\xi \in \mathbb{R}^{d+1}$. ♣

We recall a few elements from the large-deviation theory for SDEs developed by Freidlin and Wentzell [32]. Given a finite time interval $[0,T]$ and a continuous function $\gamma : [0,T] \to \mathcal{D}$, one defines a *rate function* by

$$I_{[0,T]}(\gamma) = \begin{cases} \frac{1}{2}\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathrm{T}} D(\gamma_s)^{-1} (\dot\gamma_s - f(\gamma_s)) \, \mathrm{d}s & \text{if } \gamma \in H^1, \\ +\infty & \text{otherwise .} \end{cases} \tag{2.8}$$

Roughly speaking, the probability of a sample path of (2.5) tracking $\gamma$ behaves like $\mathrm{e}^{-I_{[0,T]}(\gamma)/\sigma^2}$ in the limit $\sigma \to 0$ (see [32] for details). If $x^\star$ belongs to one of the $\Gamma_i$ and $y \in \mathcal{D}$, we define the *quasipotential*

$$V(x^\star, y) = \inf_{T>0} \inf_{\gamma : x^\star \to y} I_{[0,T]}(\gamma) \, , \tag{2.9}$$

where the second infimum runs over all continuous paths $\gamma$ such that $\gamma_0 = x^\star$ and $\gamma_T = y$. It is easy to see that if $y_1$ and $y_2$ belong to the same periodic orbit, then $V(x^\star, y_1) = V(x^\star, y_2)$. Indeed one can connect $y_1$ to $y_2$ at zero cost by tracking the deterministic flow, so that $V(y_1, y_2) = 0$, and similarly one has $V(y_2, y_1) = 0$. Thus for $1 \leqslant i \neq j \leqslant N$, the quantity

$$H(i,j) = V(x_i^\star, x_j^\star) \tag{2.10}$$

measures the cost of reaching the $j$th periodic orbit from the $i$th periodic orbit in arbitrary time. If $i \notin A \subset \{1, \dots, N\}$ it will be convenient to use the notation

$$H(i,A) = \min_{j \in A} H(i,j) \tag{2.11}$$

for the cost of reaching any of the orbits in $\bigcup_{j \in A} \Gamma_j$. The following non-degeneracy assumption will greatly simplify the spectral analysis.

**Assumption 2.5** (Metastable hierarchy)**.** There exists a constant $\theta > 0$ such that the stable periodic orbits $\Gamma_1, \dots, \Gamma_N$ can be ordered in such a way that if one writes $M_j = \{1, \dots, j\}$, then

$$H(j, M_{j-1}) \leqslant \min_{i<j} H(i, M_j \setminus \{i\}) - \theta \, . \tag{2.12}$$

We say that the orbits are in *metastable order*, and write $\Gamma_1 \prec \Gamma_2 \prec \cdots \prec \Gamma_N$. ♣

The metastable order can be determined in the following way. First one computes, for each $i$, the minimal cost $H(i, M_N \setminus \{i\})$ for reaching another orbit from $\Gamma_i$. If the minimum $\min_{1 \leqslant i \leqslant N} H(i, M_N \setminus \{i\})$ is reached in a unique $i$, then this $i$ will be relabelled $N$. The procedure is then reiterated with the other $N-1$ orbits, discarding the $N$th orbit, until all orbits have been ordered. Figure 2 illustrates the procedure in case the quasipotential derives from a global potential $U$ (i.e., such that $V(x_i^\star, x_j^\star) - V(x_j^\star, x_i^\star) = U(x_j^\star) - U(x_i^\star)$ for all $i, j$), which is not the case for a generic irreversible system.

The metastable hierarchy assumption is related to the concept of $W$-graphs used by Wentzell in [49] in the case of finite matrices, and shown in [32, Theorem 7.3, Chapter 6] to determine the real parts of exponentially small eigenvalues of the generator of a diffusion. The $W$-graphs can be used without the metastable hierarchy assumption to determine the relevant exponential timescales, but if this assumption holds then the $W$-graph algorithm becomes particularly simple, since only the edges from vertex $j$ to one vertex in $M_{j-1}$

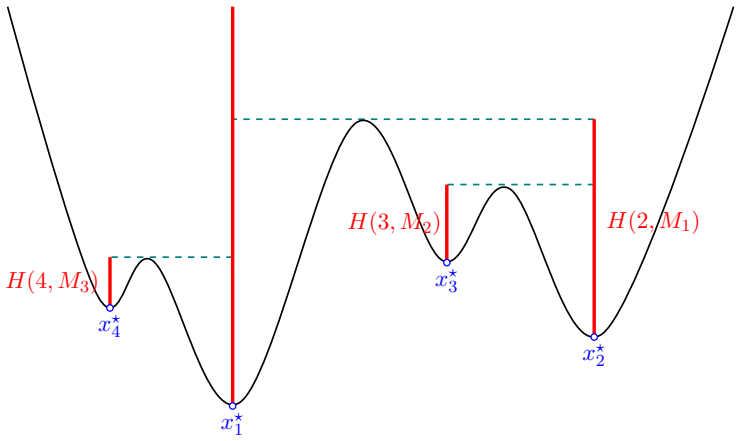


Figure 2: In cases where there exists a global potential $U$, such that $V(x_i^\star, x_j^\star) - V(x_j^\star, x_i^\star) = U(x_j^\star) - U(x_i^\star)$ for all $i, j$, the metastable hierarchy can be determined by a graphical construction. In this example, the $x_i^\star$ have already been labelled in such a way that (2.12) is satisfied, so that $\Gamma_1 \prec \Gamma_2 \prec \Gamma_3 \prec \Gamma_4$.

contribute. See also [22, 21] for recent results based on $W$-graphs on how to determine the metastable hierarchy efficiently in case $N$ is large.

Finally, we will need a type of recurrence assumption, because so far we have not assumed much on the behaviour of the diffusion outside the set $\mathcal{D}$. In particular, the solutions of the SDE (2.5) may not even exist globally in time. In fact, we will consider two slightly different situations, which however can be treated in a uniform way.

**Assumption 2.6** (Confinement property)**.** One of the two following situations holds.

A. Either there exist a *Lyapunov function* $V \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}_+)$ such that $\|V(z)\| \to \infty$ as $z \to \partial\mathcal{D}_0$ (or as $\|z\| \to \infty$ in case $\mathcal{D}_0$ is unbounded) satisfying

$$(\mathcal{L}V)(z) \leqslant -c + d\mathbb{1}_{\{z \in \mathcal{D}\}} \qquad \forall z \in \mathcal{D}_0 \tag{2.13}$$

for some constants $c > 0$ and $d \geqslant 0$.

B. Or

$$\bar{V}(\partial\mathcal{D}) := \min_{1 \leqslant i \leqslant N} \inf_{y \in \partial\mathcal{D}} V(x_i^\star, y) \geqslant \max_{i \neq j} H(i, j) + \theta' \tag{2.14}$$

for a constant $\theta' > 0$. ♣

By [46, Theorem 4.2], variant A implies that the process $\{Z_t\}_{t \geqslant 0}$ is positive Harris recurrent. Recall that a process is Harris recurrent if there exists a $\sigma$-finite measure $\mu$ such that the first-hitting time of a set $A$ is almost surely finite whenever $\mu(A) > 0$. Such a process admits an essentially unique invariant measure $\pi$, and is called positive Harris recurrent if $\pi$ can be normalised to be a probability measure. The ellipticity assumption 2.4 implies that the restriction of $\pi$ to $\mathcal{D}$ is absolutely continuous with respect to Lebesgue measure. Furthermore, [46, Theorem 4.3], applied with $f = 1$, shows that the first-hitting time $\tau_{\mathcal{D}}$ of $\mathcal{D}$ satisfies

$$\mathbb{E}^z\{\tau_{\mathcal{D}}\} \leqslant \frac{1}{c} V(z) \tag{2.15}$$

for all $z \in \mathcal{D}_0$.

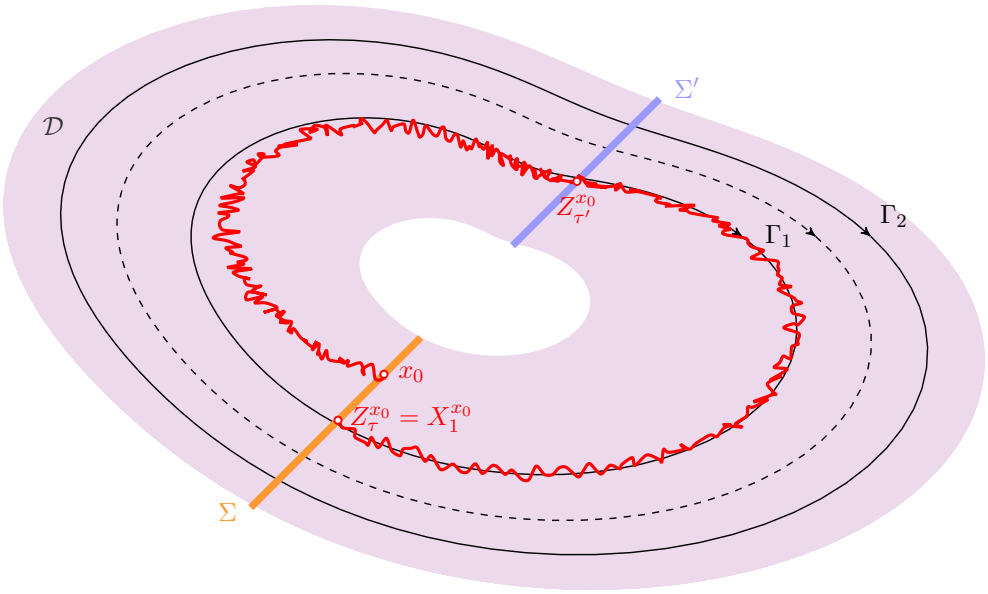


Figure 3: Sketch of a random Poincaré map when the process starts in the basin of attraction of a stable periodic orbit $\Gamma_1$.

**Remark 2.7.** If $V$ is a quadratic form, then we have $\mathcal{L}V = \langle f, \nabla V\rangle + \mathcal{O}(\sigma^2)$. Thus a quadratic deterministic Lyapunov function satisfying $\langle f, \nabla V\rangle \leqslant -cV$ outside $\mathcal{D}$ may already fulfil Condition (2.13) if $\sigma$ is small enough. ◊

Variant B of Assumption 2.6 says that it should be harder to reach the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ than to make any transition between periodic orbits. In that situation, we are going to consider the process conditioned on staying in $\mathcal{D}$. Doob's $h$-transform (cf. Appendix A) will allow us to relate the spectral properties of the conditioned process with those of the process killed upon leaving $\mathcal{D}$. Both processes are not influenced by what happens outside $\mathcal{D}$, so that global existence of solutions is not required.

## 2.3 Random Poincaré map

We now define a discrete-time process recording successive intersections of sample paths with the surface of section $\Sigma$. The following basic estimate shows that solutions starting in $\mathcal{D}$ will hit $\Sigma$ almost surely after a finite time (and thus return to $\Sigma$ infinitely often).

**Proposition 2.8.** *Let $\tau_\Sigma = \inf\{t > 0 \colon Z_t \in \Sigma\}$. There exist constants $\sigma_0, M > 0$ such that for all $\sigma < \sigma_0$,*

$$\sup_{z\in\mathcal{D}} \mathbb{E}^z\big\{\tau_\Sigma\big\} \leqslant M\log(\sigma^{-1})\,. \tag{2.16}$$

Consider first the case where variant A of Assumption 2.6 holds (for variant B, see Section 2.7). Assume we have chosen a parametrisation of $\Sigma$ by a variable $x \in \mathbb{R}^d$. By a slight abuse of notation, we will denote the domain of $x$ by $\Sigma$ as well. For an initial condition $X_0 \in \Sigma$, we would like to study the sequence $(X_1, X_2, \dots)$ of successive intersections of the sample path $(Z_t^{X_0})_{t\geqslant 0}$ with $\Sigma$. We cannot proceed exactly as in the deterministic case, because $\tau$ defined as in (2.3) is equal to 0 almost surely, due to the irregularity of Brownian paths. This problem is cured quite easily, however. One can

for instance introduce a second manifold $\Sigma' \subset \mathcal{D}$ which does not intersect $\Sigma$, such that $\partial\Sigma' \subset \partial\mathcal{D}$ and the vector field $f$ is transversal to $\Sigma'$ as well. Then setting $\tau_0 = 0$, it suffices to set for each $n \in \mathbb{N}_0$

$$\begin{aligned} \tau'_{n+1} &= \inf\{t > \tau_n \colon Z_t^{X_0} \in \Sigma'\} \;, \\ \tau_{n+1} &= \inf\{t > \tau'_{n+1} \colon Z_t^{X_0} \in \Sigma\} \;, \end{aligned} \tag{2.17}$$

and to define $X_{n+1}$ to be the $x$-coordinate of $Z_{\tau_{n+1}}^{X_0} \in \Sigma$, see Figure 3.

The strong Markov property implies that the law of $X_{n+1}$ given $X_n$ is independent of $n$ and of all $X_m$ with $m < n$, that is, $(X_n)_{n\geqslant 0}$ forms a time-homogeneous Markov chain. Since each $X_n$ can be seen as the first-exit location from a bounded set, results from [4] show that the law of $X_n$ has a continuous density. We thus obtain a continuous-space Markov kernel $K$ with continuous density $k$, defined by

$$K(x, A) = \mathbb{P}^x\big\{X_1 \in A\big\} = \int_A k(x, y)\,\mathrm{d}y \tag{2.18}$$

for any $x \in \Sigma$ and any Borel set $A \subset \Sigma$. We will denote $n$-fold iterates of $K$ by

$$K^n(x, A) = \mathbb{P}^x\big\{X_n \in A\big\} = \int_A k^n(x, y)\,\mathrm{d}y \tag{2.19}$$

where the time-$n$ transition densities $k^n$ can be determined recursively by the Chapman–Kolmogorov equation

$$k^{n+1}(x, y) = \int_\Sigma k^n(x, z)k(z, y)\,\mathrm{d}z \;. \tag{2.20}$$

The Markov kernel $K$ induces two Markov semigroups in the standard way. Namely, with any bounded measurable test function $f : \Sigma \to \mathbb{R}$, we associate the function

$$(Kf)(x) = \int_\Sigma k(x, y)f(y)\,\mathrm{d}y = \mathbb{E}^x\big\{f(X_1)\big\} \;. \tag{2.21}$$

Furthermore, with any (signed) Borel measure $\mu$ on $\Sigma$ with density $m$, we associate the measure

$$(\mu K)(\mathrm{d}y) = \left(\int_\Sigma m(x)k(x, y)\,\mathrm{d}x\right)\mathrm{d}y = \mathbb{P}^\mu\{X_1 \in \mathrm{d}y\} \;. \tag{2.22}$$

Since in what follows, all measures will have densities, we will often use the same symbol for a measure and its density, and write $(mK)(y)$ for the integral appearing on the right-hand side of (2.22).

## 2.4 Spectral decomposition

Since $\Sigma$ is bounded and $k$ is continuous, $K$ is a compact operator (cf. [48, Section VI.5]), which implies that the behaviour of its large iterates can be described by Fredholm theory. In particular, the Riesz–Schauder theorem [48, Theorem V1.15] states that $K$ has discrete spectrum, with all eigenvalues except possibly 0 having finite multiplicity. The eigenvalues are roots of the Fredholm determinant, first introduced in [31], which is well-defined since $k$ is bounded.

Let us denote by $(\lambda_j)_{j\in\mathbb{N}_0}$ the eigenvalues of $K$, ordered by decreasing modulus, and by $\pi_j$ and $\phi_j$ the left and right eigenfunctions respectively. That is,

$$(\pi_j K)(x) = \lambda_j \pi_j(x) \qquad \text{and} \qquad (K\phi_j)(x) = \lambda_j \phi_j(x) \tag{2.23}$$

for all $j \in \mathbb{N}_0$. We can normalise the eigenfunctions in such a way that

$$\pi_i \phi_j := \int_\Sigma \pi_i(x)\phi_j(x)\,\mathrm{d}x = \delta_{ij} \; . \tag{2.24}$$

In this way, the kernels $\phi_i(x)\pi_i(y)$ are projectors on invariant subspaces of $K$. If the set of eigenfunctions is complete, and all nonzero eigenvalues have multiplicity 1, then we have the spectral decomposition

$$k(x,y) = \sum_{i \geqslant 0} \lambda_i \phi_i(x)\pi_i(y) \; , \tag{2.25}$$

which entails the very useful property

$$k^n(x,y) = \sum_{i \geqslant 0} \lambda_i^n \phi_i(x)\pi_i(y) \; . \tag{2.26}$$

A similar spectral decomposition holds if there are eigenvalues of multiplicity higher than 1, except that there may be nontrivial Jordan blocks. In what follows, we will show that $K$ is close to a finite-rank operator, defined by a sum with $N$ terms. Therefore, the completeness of the set of all eigenfunctions will not be an issue.

Jentzsch's extension of the Perron–Frobenius theorem [40] shows that $\lambda_0$ is real, positive and simple, and that the eigenfunctions $\pi_0(x)$ and $\phi_0(x)$ can be taken real and positive. Since in our case, $K$ is a stochastic Markov kernel (i.e. $K(x,\Sigma) = 1$ for all $x \in \Sigma$), we have in fact $\lambda_0 = 1$, while $\phi_0(x)$ can be taken identically equal to 1, and $\pi_0(x)$ is an invariant density, which by (2.24) is normalised to be a probability density. Under the spectral-gap condition $|\lambda_1| < 1$, the iterates $k^n(x,y)$ will converge to $\pi_0(y)$ for all $x$.

## 2.5 Process killed upon leaving a subset $A$

Given a Borel set $A \subset \Sigma$, several processes related to $(X_n)_{n\geqslant 0}$ will play an important rôle in what follows. The simplest one is the process $(X_n^A)_{n\geqslant 0}$ killed upon leaving $A$. Its kernel $K_A$ has density

$$k_A(x,y) = k(x,y)\mathbb{1}_{\{x\in A, y\in A\}} \; . \tag{2.27}$$

This is in general a substochastic Markov process ($K_A(x,A) < 1$), which can be turned into a stochastic Markov process by adding to $A$ an absorbing cemetery state $\partial$. We denote its eigenvalues by $\lambda_i^A$ and its left and right eigenfunctions by $\pi_i^A(x)$ and $\phi_i^A(x)$. The largest eigenvalue $\lambda_0^A$ is still real, positive and simple, but in general smaller than 1. It is called the *principal eigenvalue* of the process. Following [19], we call $\phi_0^A$ the (right) *principal eigenfunction* of $(X_n^A)_{n\geqslant 0}$. The normalised left eigenfunction $\pi_0^A$ is called the *quasistationary distribution* (QSD) of the killed process. Under the spectral gap condition $\left|\lambda_1^A\right| < \lambda_0^A$, it satisfies

$$\pi_0^A(B) = \lim_{n\to\infty} \mathbb{P}^x\big\{X_n^A \in B \;\big|\; X_n^A \in A\big\} = \lim_{n\to\infty} \mathbb{P}^x\big\{X_n \in B \;\big|\; \tau_{A^c}^+ > n\big\} \tag{2.28}$$

for any Borel set $B \subset A$, independently of $x \in A$. Here $\tau_{A^c}^+ = \inf\{n \geqslant 1 \colon X_n \notin A\}$ denotes the first-exit time of the original process from $A$. A useful property of the QSD is that for the process $(X_n)_{n\geqslant 0}$ one has

$$\mathbb{P}^{\pi_0^A}\big\{\tau_{A^c}^+ = n\big\} = (\lambda_0^A)^{n-1}(1-\lambda_0^A) \qquad \forall n \geqslant 1 \; , \tag{2.29}$$

that is, the first-exit time from $A$ has a geometric distribution with success probability $(1-\lambda_0^A)$. In particular, we have

$$\mathbb{E}^{\pi_0^A}\{\tau_{A^c}^+\} = \frac{1}{1-\lambda_0^A} \,. \tag{2.30}$$

**Remark 2.9.** Note that if an eigenvalue $\lambda$ of the original process satisfies the lower bound

$$|\lambda| \geqslant \sup_{x\in A} \mathbb{P}^x\{X_1 \in A\} \,, \tag{2.31}$$

then the principal eigenvalue of the process killed upon leaving $A$ satisfies

$$\lambda_0^A \leqslant |\lambda| \,, \tag{2.32}$$

because $\lambda_0^A = \mathbb{P}^{\pi_0^A}\{X_1 \in A\}$. ◊

## 2.6 Trace of the process on $A$

A second important process is the trace $(X_n)|_A$, which describes the process monitored only while it visits $A$. This is still a Markov process, with kernel

$$K|_A(x, \mathrm{d}y) = \mathbb{P}^x\big\{X_{\tau_A^+} \in \mathrm{d}y\big\} \,, \tag{2.33}$$

where $\tau_A^+ = \inf\{n \geqslant 1 \colon X_n \in A\}$ denotes the first-return time to $A$. The density of $K|_A$ is thus given by

$$k|_A(x,y) = \sum_{n\geqslant 1} \mathbb{P}^x\big\{\tau_A^+ = n\big\} k^n(x,y) \mathbb{1}_{\{x\in A, y\in A\}} \,. \tag{2.34}$$

If $R_{A^c}(1; z_1, \mathrm{d}z_2) = [\mathrm{id} - K_{A^c}]^{-1}(z_1, \mathrm{d}z_2)$ denotes the resolvent at 1 of the kernel killed upon leaving $A^c$, then $k|_A$ can also be written

$$k|_A(x,y) = \Big[k(x,y) + \int_{A^c}\int_{A^c} k(x,z_1) R_{A^c}(1; z_1, \mathrm{d}z_2) k(z_2, y)\, \mathrm{d}z_1\Big] \mathbb{1}_{\{x\in A, y\in A\}} \,. \tag{2.35}$$

One of the key points of our analysis will be to characterise the process monitored only while visiting a neighbourhood of a well-chosen subset of the stable periodic orbits.

## 2.7 Process conditioned on staying in $A$

The last important kernel describes the process $(\bar{X}_n^A)_{n\geqslant 0}$ conditioned on remaining in $A$ forever, and is defined by

$$\bar{K}_A(x,B) = \lim_{n\to\infty} \mathbb{P}^x\big\{X_1^A \in B \,\big|\, X_n^A \in A\big\} = \lim_{n\to\infty} \mathbb{P}^x\big\{X_1 \in B \,\big|\, \tau_{A^c}^+ > n\big\} \tag{2.36}$$

for all Borel sets $B \subset A$. It can be constructed using Doob's $h$-transform.

**Proposition 2.10** (Doob $h$-transform)**.** *Assume the spectral gap condition* $\big|\lambda_1^A\big| < \lambda_0^A$. *Then the density of* $\bar{K}_A$ *is given by*

$$\bar{k}_A(x,y) = \frac{1}{\lambda_0^A} \frac{\phi_0^A(y)}{\phi_0^A(x)} k_A(x,y) \,. \tag{2.37}$$

*Furthermore, the eigenvalues and eigenfunctions of* $\bar{K}_A$ *are given by*

$$\bar{\lambda}_n^A = \frac{\lambda_n^A}{\lambda_0^A} \,, \qquad \bar{\pi}_n^A(x) = \pi_n^A(x)\phi_0^A(x) \qquad \textit{and} \qquad \bar{\phi}_n^A(x) = \frac{\phi_n^A(x)}{\phi_0^A(x)} \,. \tag{2.38}$$

This is a standard result, which is closely related to what is called *ground state transformation* in quantum physics. For the reader's convenience, we give a short proof in Appendix A. Integrating (2.37) over $y \in A$, we see immediately that $\bar{K}_A$ is a stochastic Markov kernel. Its principal eigenvalue $\bar{\lambda}_0^A$ is indeed equal to 1, and the corresponding right eigenfunction is identically equal to 1. Proposition 2.10 shows that the spectral properties of $K_A$ and $\bar{K}_A$ determine one another, provided one knows the principal eigenvalue $\lambda_0^A$ and the corresponding right eigenfunction $\phi_0^A$.

We finally discuss the situation where variant B of Assumption 2.6 holds. In that case, we may consider the process $Z_t^{\mathcal{D}}$ killed upon leaving $\mathcal{D}$. Proceeding exactly as above, we can define a continuous-space Markov kernel $K_{\mathcal{D}}$ describing the distribution of first-hitting points of $\Sigma$ after visiting $\Sigma'$. Because of the killing, $K_{\mathcal{D}}$ is a substochastic kernel. However, Doob's $h$-transform allows us to define a stochastic kernel $\bar{K}_{\mathcal{D}}$ of the process conditioned on staying in $\mathcal{D}$ forever. Proposition 2.10 then allows us to deduce spectral properties of $K_{\mathcal{D}}$ from those of $\bar{K}_{\mathcal{D}}$, provided we manage to control the principal eigenvalue and right eigenfunction of $K_{\mathcal{D}}$.

# 3 Results

We can now state the main results of this work. Throughout, we require Assumptions 2.1, 2.2, 2.4, 2.5 and 2.6 to hold. If variant A of the confinement assumption 2.6 holds, all results concern the kernel $K$ defined in (2.18). In case of variant B, they concern the kernel $\bar{K}_{\mathcal{D}}$ of the Doob-transformed process introduced just above.

For $i = 1, \dots, N$, we let $B_i \subset \Sigma$ be the closure of a neighbourhood of $x_i^\star$, contained in a ball centred in $x_i^\star$ and of radius $\delta > 0$. Here $\delta$ is assumed to be small enough for each $B_i$ to be contained in the basin of attraction $\mathcal{A}_i$ of $x_i^\star$ (cf. (2.4)) and such that the deterministic Poincaré map maps $B_i$ strictly into itself (such a ball exists since the orbit $\Gamma_i$ is asymptotically stable). For $1 \leqslant k \leqslant N$ we define the metastable neighbourhood

$$\mathcal{M}_k = \bigcup_{i=1}^{k} B_i \,. \tag{3.1}$$

For a Borel set $A \subset \Sigma$ we denote by $\tau_A = \inf\{n \geqslant 0 \colon X_n \in A\}$ its first-hitting time of $A$ and by $\tau_A^+ = \inf\{n \geqslant 1 \colon X_n \in A\}$ the first-return time of the process to $A$. If $A$ and $B$ are disjoint, an important rôle will be played by the *committor functions*

$$\mathbb{P}^x\big\{\tau_A < \tau_B\big\} \qquad \text{and} \qquad \mathbb{P}^x\big\{\tau_A^+ < \tau_B^+\big\} \tag{3.2}$$

of hitting $A$ before $B$. Note that these functions are identical whenever $x \notin A \cup B$, while $\mathbb{P}^x\{\tau_A < \tau_B\}$ has value 1 in $A$ and 0 in $B$. A rough bound on committor functions can be obtained by large-deviation theory.

**Proposition 3.1.** *For any $\eta > 0$, there exists $\delta_0 > 0$ such that if $\delta < \delta_0$, then for any $1 \leqslant i, j \leqslant N$ with $i \neq j$, any non-empty open set $A \subset \mathcal{A}_j$ and all $x \in B_i$, one has*

$$-H(i,j) - \eta \leqslant \lim_{\sigma \to 0} \sigma^2 \log \mathbb{P}^x\big\{\tau_A^+ < \tau_{B_i}^+\big\} \leqslant -H(i,j) + \eta \,. \tag{3.3}$$

This bound indicates that for $x \in B_i$ and $A \subset \mathcal{A}_j$, the committor $\mathbb{P}^x\{\tau_A^+ < \tau_{B_i}^+\}$ is exponentially small, of order $\mathrm{e}^{-H(i,j)/\sigma^2}$. We will see below that some committor functions can be more precisely estimated in terms of certain principal eigenfunctions.

### 3.1 Eigenvalue estimate

Fix a small constant $\eta \in (0, \theta)$ and set $\theta^- = \theta - \eta$, where $\theta$ is given by the metastable hierarchy assumption 2.5. In all results given below, it is always implicitly understood that there exists a $\sigma_0 > 0$, depending on $\eta$, such that the claims hold for all $\sigma < \sigma_0$. We will not repeat this condition in what follows.

**Theorem 3.2** (Eigenvalue estimates)**.** *The $N$ largest eigenvalues of $K$ are real and positive and satisfy*

$$\begin{aligned} &\lambda_0 = 1 \,, \\ &\lambda_k = 1 - \mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\bigl\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\bigr\}\bigl[1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})\bigr] \qquad \text{for } 1 \leqslant k \leqslant N-1 \,, \end{aligned} \tag{3.4}$$

*where $\mathring{\pi}_0^{B_{k+1}}$ is a probability measure concentrated on $B_{k+1}$ and $\theta_k = H(k+1, M_k)/2 - \eta$. Furthermore, there exists $c > 0$ such that*

$$|\lambda_k| \leqslant \rho := 1 - \frac{c}{\log(\sigma^{-1})} \qquad \text{for all } k \geqslant N \,. \tag{3.5}$$

*Finally, the principal eigenvalue of the process killed upon hitting $\mathcal{M}_k$ satisfies*

$$1 - \lambda_0^{\mathcal{M}_k^c} = (1 - \lambda_k)\bigl[1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})\bigr] \tag{3.6}$$

*for $1 \leqslant k \leqslant N-1$.*

The probability measure $\mathring{\pi}_0^{B_{k+1}}$ has an explicit definition: it is the QSD of the trace process $(X_n^{B_{k+1}})|_{\mathcal{M}_{k+1}}$, monitored only while visiting $\mathcal{M}_{k+1}$ and killed upon hitting $\mathcal{M}_k$ (which is equivalent to the trace process leaving $B_{k+1}$). Note that this process is not the same as (the trace of) the process killed when leaving $B_{k+1}$, meaning that taking the trace and killing do not commute. Regardless of the precise definition of the probability measure $\mathring{\pi}_0^{B_{k+1}}$, Proposition 3.1 shows that

$$\lambda_k = 1 - \mathcal{O}(\mathrm{e}^{-(H(k+1,M_k)-\eta)/\sigma^2}) \qquad \text{for } k = 1, \dots, N-1 \,, \tag{3.7}$$

where $\eta$ can be chosen as small as one likes. The main interest of this estimate is that the spectral decomposition (2.26) becomes

$$k^n(x,y) = \sum_{i=0}^{N-1} \lambda_i^n \phi_i(x) \pi_i(y) + \mathcal{O}(\rho^n) \,, \tag{3.8}$$

which is dominated by the $N$ first terms as soon as $n \gg \log(\rho^{-1})$. Since the $N$ first eigenvalues are exponentially close to 1, the first $N$ terms of the sum decrease very slowly, highlighting the metastable behaviour of the system.

The proof of Theorem 3.2 relies on two main ingredients. In a first step, we show that for each $k \leqslant N-1$, the kernel of the process monitored only while visiting $\mathcal{M}_{k+1}$ can be described by a finite-rank operator, given by a stochastic matrix $P$ with elements

$$P_{ij} = \mathbb{P}^{\mathring{\pi}_0^{B_i}}\{X_{\tau_{\mathcal{M}_{k+1}}} \in B_j\} = \mathbb{P}^{\mathring{\pi}_0^{B_i}}\{\tau_{B_j} < \tau_{\mathcal{M}_{k+1} \setminus B_j}\} \,. \tag{3.9}$$

In a second step we use the metastable hierarchy assumption to show that the largest eigenvalue of $\mathrm{id} - P$ is close to the indicated committor functions.

If variant B of the confinement assumption (2.6) holds, then the following result shows that Theorem 3.2 essentially holds also for the process killed upon leaving $\mathcal{D}$. It can be seen as a generalisation to the case $N > 1$ of the result in [50] by Wentzell, which estimates the principal eigenvalue of the generator of a diffusion killed upon leaving a domain containing a stable equilibrium point as sole attractor.

**Proposition 3.3.** *The principal eigenvalue of the chain killed upon leaving $\mathcal{D}$ satisfies*

$$\lambda_0^{\mathcal{D}} = 1 - \mathbb{P}^{\mathring{\pi}_0^{B_1}}\{\tau_\partial^+ < \tau_{B_1}^+\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta_0/\sigma^2})\big] = 1 - \mathcal{O}(\mathrm{e}^{-\bar{V}(\partial\mathcal{D})/\sigma^2}) \,, \tag{3.10}$$

*where $\partial$ denotes the cemetery state, $\mathring{\pi}_0^{B_1}$ is a probability measure concentrated on $B_1$ and $\theta_0 = \inf_{y\in\partial\mathcal{D}} V(x_1^\star, y)/2 - \eta$.*

Proof: The proof is the same as the proof of Theorem 3.2, except that one adds a fictitious ball $B_0$ with zero boundary conditions, representing the cemetery state $\partial$. □

Indeed, when using Proposition 2.10 to compute the eigenvalues of the killed process, Condition (2.14) ensures that the corrections to the eigenvalues $\bar{\lambda}_k^{\mathcal{D}}$ are negligible for $k = 1, \dots, N-1$.

## 3.2 Right eigenfunctions

For the spectral decomposition (3.8) to be useful, it is desirable to also have a control on the $N$ first right and left eigenfunctions. We start by giving a result on the right eigenfunctions $\phi_k$, which is close in spirit to [19, Theorem 1.3].

**Theorem 3.4** (Right eigenfunctions)**.** *The $N$ first right eigenfunctions of $K$ can be taken real. They satisfy $\phi_0(x) = 1$ for all $x \in \Sigma$, while for $k = 1, \dots, N-1$,*

$$\phi_k(x) = \mathbb{P}^x\big\{\tau_{B_{k+1}} < \tau_{\mathcal{M}_k}\big\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] + \mathcal{O}(\mathrm{e}^{-\theta_k^-/\sigma^2}) \qquad \forall x \in \Sigma \,, \tag{3.11}$$

*where $\theta_k^- = \min\{\theta^-, \theta_k\}$. Furthermore, the right principal eigenfunction of the process killed upon first hitting $\mathcal{M}_k$ satisfies*

$$\phi_0^{\mathcal{M}_k^c}(x) = \mathbb{P}^x\big\{\tau_{B_{k+1}} < \tau_{\mathcal{M}_k}\big\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] + \mathcal{O}(\mathrm{e}^{-\theta_k^-/\sigma^2}) \qquad \forall x \in \mathcal{M}_k^c \tag{3.12}$$

*for $k = 1, \dots, N-1$.*

If $x$ is in the basin of attraction $\mathcal{A}_i$ of $B_i$, then the committor $\mathbb{P}^x\{\tau_{B_i} < \tau_A\}$ is exponentially close to 1 whenever $A$ is not in $\mathcal{A}_i$. This shows that to leading order,

- if $x \in \mathcal{A}_i$ for $1 \leqslant i \leqslant k$, then $\phi_k(x)$ is exponentially small;
- if $x \in \mathcal{A}_{k+1}$, then $\phi_k(x)$ is exponentially close to 1;
- if $x \in \mathcal{A}_j$ for $j > k+1$, then $\phi_k(x)$ is exponentially close to 1 if it is easier to reach $B_{k+1}$ than $\mathcal{M}_k$ from $B_j$, and exponentially small otherwise.

In the case where variant B of Assumption 2.6 holds, the following result together with Proposition 2.10 show that the same expressions for eigenfunctions hold, except perhaps close to the boundary of $\Sigma$.

**Remark 3.5.** The proof actually yields a more precise estimate of the eigenfunctions, of the form

$$\phi_k(x) = \mathbb{P}^x\big\{\tau_{B_{k+1}} < \tau_{\mathcal{M}_k}\big\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] + \sum_{i=1}^{k} \mathbb{P}^x\big\{\tau_{B_i} < \tau_{\mathcal{M}_{k+1}\setminus B_i}\big\}\rho_{ki} \tag{3.13}$$

for $1 \leqslant k \leqslant N-1$, where

$$\rho_{ki} = -\frac{\mathbb{P}^{\mathring{\pi}_0^{B_i}}\big\{\tau^+_{B_{k+1}} < \tau^+_{\mathcal{M}_k}\big\}}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\big\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\big\}} + \mathcal{O}(\mathrm{e}^{-2\theta^-/\sigma^2}) = \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2}) \; . \tag{3.14}$$

Higher-order expansions are also available. This expression may contain more information than (3.11) if the leading term in (3.11) is exponentially small. Note that at least some of the coefficients $\rho_{ki}$ are negative, which is consistent with the orthogonality relation (2.24). ♢

**Proposition 3.6.** *The principal eigenfunction of the chain killed upon leaving $\mathcal{D}$ satisfies*

$$\phi_0^{\mathcal{D}}(x) = \mathbb{P}^x\{\tau_{B_1} < \tau_\partial\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] \qquad \forall x \in \Sigma \; . \tag{3.15}$$

*Thus $\phi_0^{\mathcal{D}}(x) = 1 - \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})$ whenever $x$ is bounded away from $\partial\Sigma$.*

### 3.3 Left eigenfunctions

If the kernel $K$ were reversible, that is, if $\pi_0(x)k(x,y) = \pi_0(y)k(y,x)$ were true for any $x, y \in \Sigma$, then it would be immediate to obtain the left eigenfunctions. Indeed, it is straightforward to check that they would be given by $\pi_k(x) = \pi_0(x)\phi_k(x)$. Since we do not assume reversibility, we have to find another way to determine the left eigenfunctions.

In [13], the authors obtained that first-return times of finite-state space Markov chains satisfy the remarkable identity $\pi_0(x)\mathbb{P}^x\{\tau_y^+ < \tau_x^+\} = \pi_0(y)\mathbb{P}^y\{\tau_x^+ < \tau_y^+\}$, even if the chain is not reversible. The following result shows that a similar property holds in our case. The proof which, arguably, is even more elementary than the one given in [13], is given in Section 4.4.

**Proposition 3.7.** *For any disjoint Borel sets $A_1, A_2 \subset \Sigma$ one has*

$$\int_{A_1} \pi_0(x)\mathbb{P}^x\big\{\tau^+_{A_2} < \tau^+_{A_1}\big\}\,\mathrm{d}x = \int_{A_2} \pi_0(x)\mathbb{P}^x\big\{\tau^+_{A_1} < \tau^+_{A_2}\big\}\,\mathrm{d}x \; . \tag{3.16}$$

*The same relation holds when each $\tau^+_{A_i}$ is replaced by the $n^{th}$ return time $\tau^{+,n}_{A_i}$ to $A_i$.*

Applying this result with $A_1 = \mathcal{M}_N$, $A_2 = \Sigma \setminus \mathcal{M}_N$, and using the fact that the $\Gamma_i$ are the only attractive limit sets, we obtain that $\pi_0$ is concentrated in $\mathcal{M}_N$, in the sense that there exists $\kappa > 0$, depending on the size $\delta$ of the $B_i$, such that

$$\frac{\pi_0(\Sigma \setminus \mathcal{M}_N)}{\pi_0(\mathcal{M}_N)} = \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \; . \tag{3.17}$$

Furthermore, for any compact $D_j$ such that $B_j \subset D_j \subset \mathcal{A}_j$, one has

$$\frac{\pi_0(D_j \setminus B_j)}{\pi_0(D_j)} = \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \tag{3.18}$$

where $\kappa > 0$ may depend on $D_j$. Similar bounds hold for the QSDs $\pi_0^{\mathcal{M}_k^c}$ and the other left eigenfunctions. The essential information is thus contained in the integrals of these measures over the sets $B_j$, which are described by the following result.

**Theorem 3.8** (Left eigenfunctions)**.** *The invariant distribution satisfies*

$$\pi_0(B_1) = 1 - \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \,, \qquad \pi_0(B_j) = \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2}) \quad \text{for } j = 2, \dots, N \,. \tag{3.19}$$

*Similarly, the QSDs $\pi_0^{\mathcal{M}_k^c}$ of the process killed upon first hitting $\mathcal{M}_k$ satisfy*

$$\pi_0^{\mathcal{M}_k^c}(B_{k+1}) = 1 - \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \,, \qquad \pi_0^{\mathcal{M}_k^c}(B_j) = \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \quad \text{for } j = k+2, \dots, N \,. \tag{3.20}$$

*Furthermore, the left eigenfunction $\pi_k$ satisfies*

$$\pi_k(B_j) = \begin{cases} -\dfrac{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau_{B_j}^+ < \tau_{\mathcal{M}_{k+1}\setminus B_j}^+\}}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau_{\mathcal{M}_k}^+ < \tau_{B_{k+1}}^+\}}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2}) & \text{for } 1 \leqslant j \leqslant k \,, \\ \pi_0^{\mathcal{M}_k^c}(B_j)\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] + \mathcal{O}(\mathrm{e}^{-\theta_j/\sigma^2}) & \text{for } j \geqslant k+1 \,. \end{cases} \tag{3.21}$$

This result shows in particular that

- $\pi_k(B_{k+1})$ is exponentially close to 1;
- if $k+1 < j \leqslant N$, then $\pi_k(B_j)$ is exponentially small;
- if $1 \leqslant j \leqslant k$, then $\pi_k(B_j)$ is negative, which is consistent with the orthogonality relation (2.24); it can be close to $-1$ or exponentially small, depending on whether $B_j$ is the easiest ball in $\mathcal{M}_k$ to reach from $B_{k+1}$ or not.

In the case where variant B of Assumption 2.6 holds, combining Propositions 3.6 and 2.10 it is immediate to see that the conclusions of Theorem 3.8 still hold true.

**Remark 3.9.** Using Proposition 3.7, either for the sets $B_1$ and $B_{k+1}$ or for the sets $\mathcal{M}_k$ and $B_{k+1}$, one can obtain more precise estimates for the invariant distribution, namely the relations

$$\begin{aligned} \pi_0(B_{k+1}) &= \frac{\mathbb{P}^{\mathring{\pi}_0^{B_1}}\{\tau_{B_{k+1}}^+ < \tau_{B_1}^+\}}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau_{B_1}^+ < \tau_{B_{k+1}}^+\}}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] \,, \\ \pi_0(B_{k+1}) &= \sum_{j=1}^{k} \pi_0(B_j) \frac{\mathbb{P}^{\mathring{\pi}_0^{B_j}}\{\tau_{B_{k+1}}^+ < \tau_{\mathcal{M}_k}^+\}}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau_{\mathcal{M}_k}^+ < \tau_{B_{k+1}}^+\}}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] \end{aligned} \tag{3.22}$$

which hold for $1 \leqslant k \leqslant N-1$. The second expression, while more complicated, has the merit of making it obvious that $\pi_0(B_{k+1}) = \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})$, as a consequence of Assumption 2.5.

Similar expressions hold for the Doob-conditioned distributions $\bar{\pi}_0^{\mathcal{M}_k^c}$, which immediately imply expressions for the QSDs via Proposition 2.10 and the expression (3.12) of the right principal eigenfunctions. ◊

## 3.4 Link between eigenvalues and expected return times

If the initial condition is distributed according to $\pi_0^{\mathcal{M}_k^c}$, then it follows directly from the properties of QSDs that $\tau_{\mathcal{M}_k}$ has a geometric distribution, with expectation

$$\mathbb{E}^{\pi_0^{\mathcal{M}_k^c}}\{\tau_{\mathcal{M}_k}\} = \frac{1}{1 - \lambda_0^{\mathcal{M}_k^c}} = \frac{1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})}{1 - \lambda_k} = \frac{1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau_{\mathcal{M}_k}^+ < \tau_{B_{k+1}}^+\}} \,. \tag{3.23}$$

Combining this fact with the bounds we obtained on the QSDs $\pi_0^{\mathcal{M}_k^c}$, it is not hard to obtain the following link between expected hitting times and eigenvalues.

**Theorem 3.10** (Expected hitting times)**.** *There exists a constant $\kappa > 0$, depending on the size $\delta$ of the $B_j$, such that for every $k \in \{1, \dots, N-1\}$ one has, for any $x \in B_{k+1}$,*

$$\mathbb{E}^x\{\tau_{\mathcal{M}_k}\} = \frac{1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})}{1 - \lambda_k} = \frac{1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})}{\mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\}} \,. \tag{3.24}$$

### 3.5 Discussion of computational aspects

Our results provide sharp relations between eigenvalues and eigenfunctions of the random Poincaré map, committor functions between and expected first-hitting times of neighbourhoods of periodic orbits, and principal eigenvalues, eigenfunctions and QSDs of processes killed when hitting these sets. One limitation, compared to results in the reversible case, is that we do not have sharp asymptotics for the prefactors of these quantities as in the case of the Eyring–Kramers formula. However, some of them are accessible to numerical methods.

Computing eigenvalues and eigenfunctions of a continuous-space linear operator by discretisation is possible, but costly, especially in high space dimension. By contrast, principal eigenvalues, eigenfunctions and QSDs are much cheaper to compute, since it is sufficient to simulate the process conditioned on survival, starting with an arbitrary initial distribution.

There also exist powerful methods allowing to compute committor functions in certain situations, such as adaptive multilevel splitting, see for instance [23, 2, 20]. The fact that the expressions (3.4) for eigenvalues depend on committors with respect to a QSD is not a problem, since we find that the spectral gap of the associated process is at least logarithmically large in $\sigma$, so that whatever the initial distribution, this QSD can be sampled in a relatively short time.

## 4 Outline of the proof

As described in Section 2.4, in order to quantify transitions between periodic orbits, our main objective is to solve the eigenvalue problem

$$(K\phi)(x) = \mathrm{e}^{-u}\,\phi(x) \tag{4.1}$$

for the discrete-time, continuous-state space kernel $K$. We will start by exhibiting some general properties of this problem.

### 4.1 Continuous-space, discrete-time Markov chains

Let $\Sigma \subset \mathbb{R}^d$ be a bounded set equipped with the Borel $\sigma$-algebra $\mathcal{B}(\Sigma)$. Consider a positive Harris recurrent discrete-time Markov chain $(X_n)_{n\geqslant 0}$ on the continuous state space $\Sigma$ and let $K$ be the associated Markovian kernel having density $k > 0$ with respect to Lebesgue measure, i.e.,

$$K(x, \mathrm{d}y) = k(x, y)\,\mathrm{d}y \,. \tag{4.2}$$

Given a Borel set $A \subset \Sigma$, we introduce the first hitting time and first return time

$$\begin{aligned} \tau_A(x) &= \inf\{n \geqslant 0, X_n \in A\} \,, \\ \tau_A^+(x) &= \inf\{n \geqslant 1, X_n \in A\} \,, \end{aligned} \tag{4.3}$$

where $x$ denotes the initial condition. When the initial condition is clear from the context, then we simply write $\tau_A$, $\tau_A^+$. Note that $\tau_A^+(x) = \tau_A(x)$ for $x \in A^c = \Sigma \backslash A$, whereas $0 = \tau_A(x) < 1 \leqslant \tau_A^+(x)$ if $x \in A$. If $A$ has positive Lebesgue measure, due to the positive Harris recurrence assumption on the Markov chain and the fact that $K$ has positive density, the stopping times $\tau_A$ and $\tau_A^+$ are almost surely finite. To ease notation, we introduce

$$\mathbb{E}^A\{\cdot\} = \sup_{x \in A} \mathbb{E}^x\{\cdot\} \,, \qquad \mathbb{P}^A\{\cdot\} = \sup_{x \in A} \mathbb{P}^x\{\cdot\} \tag{4.4}$$

We also introduce the $n^{\text{th}}$ return time defined inductively by

$$\tau_A^{+,n} = \inf\big\{n > \tau_A^{+,n-1} \colon X_n \in A\big\} \,, \tag{4.5}$$

with $\tau_A^{+,1} = \tau_A^+$.

We recall the following result on existence of Laplace transforms, see e.g. [9, Lemma 5.1].

**Lemma 4.1.** *Consider a positive recurrent Markov chain with state space $\Sigma$. The Laplace transform of the first hitting time $\mathbb{E}^x\big\{\mathrm{e}^{u\tau_A}\big\}$ and the Laplace transform of the first return time $\mathbb{E}^x\big\{\mathrm{e}^{u\tau_A^+}\big\}$ are analytic in $u$ for $u$ such that*

$$\sup_{x \in A^c} \mathbb{P}^x\{X_1 \in A^c\} < \big|\mathrm{e}^{-u}\big| \,. \tag{4.6}$$

Following ideas from the potential-theoretic approach to metastability [18, 19], we are going to study a Dirichlet boundary value problem to solve the eigenvalue problem. Given a set $A \subset \Sigma$, $u \in \mathbb{C}$ and a bounded measurable function $\overline{\phi} : A \to \mathbb{R}$, we want to find a (bounded) function $\phi^u$ which satisfies

$$\begin{aligned} (K\phi^u)(x) &= \mathrm{e}^{-u}\, \phi^u(x), && x \in A^c \,, \\ \phi^u(x) &= \overline{\phi}(x), && x \in A \,. \end{aligned} \tag{4.7}$$

Solutions of such a Dirichlet problem admit a probabilistic representation in terms of Laplace transforms.

**Proposition 4.2** (Feynman–Kac type relation)**.** *For $u$ such that (4.6) is satisfied, the unique solution of the Dirichlet boundary value problem (4.7) is given by*

$$\phi^u(x) = \mathbb{E}^x\big\{\mathrm{e}^{u\tau_A}\, \overline{\phi}(X_{\tau_A})\big\} \,. \tag{4.8}$$

Proof: First, let us check that the proposed function solves the boundary problem. This is obvious for $x \in A$, since in that case $\tau_A = 0$, so that $\mathbb{E}^x\big\{\mathrm{e}^{u\tau_A}\, \overline{\phi}(X_{\tau_A})\big\} = \overline{\phi}(x)$. For $x \in A^c$, splitting the expectation defining $(K\phi^u)(x)$ according to the location of $X_1$, we get

$$\begin{aligned} (K\phi^u)(x) &= \mathbb{E}^x\big\{\mathbb{E}^{X_1}\big\{\mathrm{e}^{u\tau_A}\, \overline{\phi}(X_{\tau_A})\big\}\mathbb{1}_{\{X_1 \in A\}}\big\} + \mathbb{E}^x\big\{\mathbb{E}^{X_1}\big\{\mathrm{e}^{u\tau_A}\, \overline{\phi}(X_{\tau_A})\big\}\mathbb{1}_{\{X_1 \in A^c\}}\big\} \\ &= \mathbb{E}^x\big\{\overline{\phi}(X_1)\mathbb{1}_{\{X_1 \in A\}}\big\} + \mathbb{E}^x\big\{\mathrm{e}^{u(\tau_A - 1)}\, \overline{\phi}(X_{\tau_A})\mathbb{1}_{\{X_1 \in A^c\}}\big\} \\ &= \mathrm{e}^{-u}\, \mathbb{E}^x\big\{\mathrm{e}^{u\tau_A}\, \overline{\phi}(X_{\tau_A})\big\} \,. \end{aligned} \tag{4.9}$$

This shows that $\mathbb{E}^x\{\mathrm{e}^{u\tau_A}\,\overline{\phi}(X_{\tau_A})\}$ is an admissible solution for all $x \in \Sigma$. Uniqueness follows from the Fredholm alternative. Let us assume by contradiction that two functions $f$ and $g$ solve the Dirichlet boundary value problem with $f \neq g$. Then

$$\begin{aligned} ((\mathrm{id} - \mathrm{e}^u\, K)(f-g))(x) &= 0\,, & x &\in A^c\,, \\ (f-g)(x) &= 0\,, & x &\in A\,. \end{aligned} \tag{4.10}$$

The contradiction comes from the fact that under Condition (4.6), $\|\mathrm{e}^u\, K_{A^c}\| < 1$, so that we can apply [35, Theorem 8.1]. In particular $(\mathrm{id} - \mathrm{e}^u\, K_{A^c})$ is invertible and $f \equiv g$. □

The solution of the boundary value problem (4.7) allows us to define a (non Markov) kernel on $A$.

**Corollary 4.3.** *Let $K^u$ be the kernel defined on $A \times \mathcal{B}(A)$ by*

$$K^u(x, \mathrm{d}y) = \mathbb{E}^x\left\{\mathrm{e}^{u\left(\tau_A^+ - 1\right)}\, \mathbb{1}_{\left\{X_{\tau_A^+} \in \mathrm{d}y\right\}}\right\}. \tag{4.11}$$

*For $u$ verifying (4.6), the eigenvalue problem on $\Sigma$*

$$(K\phi^u)(x) = \mathrm{e}^{-u}\, \phi^u(x) \tag{4.12}$$

*is equivalent to the eigenvalue problem on $A$ given by*

$$\left(K^u \overline{\phi}^u\right)(x) = \mathrm{e}^{-u}\, \overline{\phi}^u(x) \tag{4.13}$$

*where $\overline{\phi}^u(x) = \phi^u(x)$ for all $x \in A$.*

Proof: Let $(\mathrm{e}^{-u}, \phi^u)$ be a couple of eigenvalue, eigenfunction for the Markov kernel $K$. Then splitting the integral equation according to $X_1$ and inserting the previous solution in the second term of the right-hand side, we have

$$\begin{aligned} \mathrm{e}^{-u}\, \phi^u(x) = (K\phi^u)(x) &= \mathbb{E}^x\left\{\phi^u(X_1)\mathbb{1}_{\{X_1 \in A\}}\right\} + \mathbb{E}^x\left\{\phi^u(X_1)\mathbb{1}_{\{X_1 \in A^c\}}\right\} \\ &= \mathbb{E}^x\left\{\phi^u(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+ = 1\}}\right\} + \mathbb{E}^x\left\{\mathbb{E}^{X_1}\{\mathrm{e}^{u\tau_A}\,\phi^u(X_{\tau_A})\}\mathbb{1}_{\{\tau_A^+ > 1\}}\right\} \\ &= \mathbb{E}^x\left\{\mathrm{e}^{u(\tau_A^+ - 1)}\,\phi^u(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+ = 1\}}\right\} + \mathbb{E}^x\left\{\mathrm{e}^{u\left(\tau_A^+ - 1\right)}\,\phi^u(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+ > 1\}}\right\} \\ &= \mathbb{E}^x\left\{\mathrm{e}^{u(\tau_A^+ - 1)}\,\phi^u(X_{\tau_A^+})\right\} = (K^u\phi^u)(x)\,. \end{aligned} \tag{4.14}$$

Since for $x \in A$, we have $\phi^u(x) = \overline{\phi}^u(x)$, this proves that (4.13) holds.

On the other hand, if we know a couple $(\mathrm{e}^{-u}, \overline{\phi}^u)$ of eigenvalue and eigenfunction for the kernel $K^u$, we introduce the function

$$\phi^u(x) = \mathbb{E}^x\left\{\mathrm{e}^{u\tau_A}\,\overline{\phi}^u(X_{\tau_A})\right\}. \tag{4.15}$$

Note that $\phi^u(x) = \overline{\phi}^u(x)$ for $x \in A$. By the previous proposition, $\phi^u$ satisfies the eigenvalue equation with eigenvalue $\mathrm{e}^{-u}$. □

In the sequel, we will forget the notation $\overline{\phi}$ since $\overline{\phi}^u = \phi^u|_{x \in A}$.

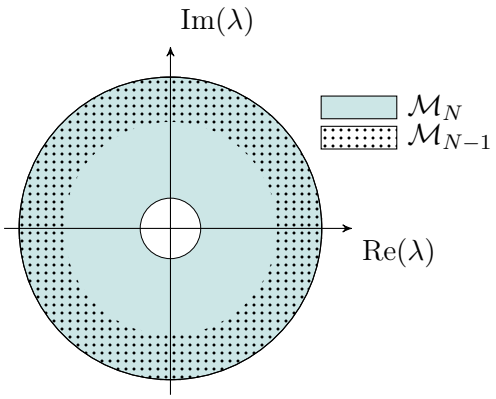


Figure 4: Set of possible eigenvalues $\lambda$ for the eigenvalue problems defined on $\mathcal{M}_N$ and $\mathcal{M}_{N-1}$.

### 4.2 Choice of the set to reduce the eigenvalue problem

Thanks to Theorem 4.3, we have reduced the eigenvalue problem on $\Sigma$ to an eigenvalue problem on a subset $A$ of $\Sigma$, which has yet to be defined. We now discuss the choice of $A$. Under the metastable hierarchy assumption 2.5, we expect that there will be $N$ eigenvalues exponentially close to one, and a gap between the $N^{\text{th}}$ eigenvalue and the remaining part of the spectrum (we recall that eigenvalues are ordered by decreasing modulus).

The general idea of the proof is to first choose a set $A$ which is well suited to estimating $\lambda_{N-1}$, the $N^{\text{th}}$ eigenvalue of the kernel, and also to obtain a rough estimate of the $N-1$ largest eigenvalues. Then we take another set $A$ in order to estimate $\lambda_{N-2}$, and obtain a rough estimate of the $N-2$ largest eigenvalues, and so on up to $\lambda_1$. The way to estimate one of the $N$ largest eigenvalues is based on approximations of the kernel $K^u$ and is explained is the next subsection.

To estimate the $N^{\text{th}}$ eigenvalue, we are going to choose $A := \mathcal{M}_N$. Note that intuitively for such a choice of set, (4.6) is not restrictive. Indeed, due to the attraction of $\mathcal{M}_N$, starting outside the union of the neighbourhoods of the stable periodic orbits, the probability that the first return point is still outside this neighbourhood should be very small (possibly replacing $K$ by a suitable iterate $K^m$). So this should allow us to estimate the $N^{\text{th}}$ eigenvalue of the kernel $K^u$.

Next, to estimate the $(N-1)^{\text{st}}$ eigenvalue $\lambda_{N-2}$, we will study the eigenvalue problem on $A := \mathcal{M}_{N-1}$. It follows that for all $u$ such that

$$\sup_{x \in \mathcal{M}_{N-1}^c} \mathbb{P}^x\big\{X_1 \in \mathcal{M}_{N-1}^c\big\} < \left|\mathrm{e}^{-u}\right| \ , \tag{4.16}$$

the original eigenvalue problem (4.1) is equivalent to an eigenvalue problem on $\mathcal{M}_{N-1}$. Note that the Laplace transform conditions given by (4.6) satisfies

$$\sup_{x \in \mathcal{M}_N^c} \mathbb{P}^x\{X_1 \in \mathcal{M}_N^c\} < \sup_{x \in \mathcal{M}_{N-1}^c} \mathbb{P}^x\big\{X_1 \in \mathcal{M}_{N-1}^c\big\} \ . \tag{4.17}$$

Therefore when solving the eigenvalue equation defined on $\mathcal{M}_{N-1}$ with kernel $K^{u,(N-1)}$ we find eigenvalues which are greater in modulus than the eigenvalues of the kernel $K^{u,(N)}$ defined on $\mathcal{M}_N$.

In general, to estimate the $k^{\text{th}}$ eigenvalue, we will study the eigenvalue problem (4.7) with $A = \mathcal{M}_{k+1}$. Therefore we have to study the spectral properties of the kernel $K^{u,(k+1)}$ defined on $\mathcal{M}_{k+1}$ by

$$K^{u,(k+1)}(x, \mathrm{d}y) = \mathbb{E}^x \left\{ \mathrm{e}^{u(\tau^+_{\mathcal{M}_{k+1}} - 1)} \mathbb{1}_{\left\{ X_{\tau^+_{\mathcal{M}_{k+1}}} \in \mathrm{d}y \right\}} \right\} . \tag{4.18}$$

To ease notation, we will simply write $K^u$ and keep in mind that the kernel depends on $k$ through its domain of definition.

## 4.3 Eigenvalue problem on a union of metastable sets

Reducing our eigenvalue problem on $\mathcal{M}_{k+1}$ is convenient because the kernel is defined where we expect to have information from the deterministic part of the system due to the attraction of the stable periodic orbits. However, the introduced kernel does not have a nice probabilistic interpretation since it depends on the spectral parameter $u$. To circumvent this problem, we are going to introduce a new parameter $v$ and solve the system of two coupled equations

$$\begin{aligned} (K^u \phi^u)(x) &= v \phi^u(x) \\ v &= \mathrm{e}^{-u} \ . \end{aligned} \tag{4.19}$$

In addition, instead of studying the kernel $K^u$ (or its iterate $(K^u)^m$), we are going to approximate it by a kernel having a nicer probabilistic interpretation and for which we can easily obtain the spectrum. The justification for using such an approximation is given by the continuity of eigenvalues of bounded linear operators [35]. Indeed, let $K^\star : Y \to Y$ be a bounded linear operator acting on the Banach space $Y$. The following classical theorem describes what happens to parts of the spectrum $\sigma(K^\star)$ if the operator $K^\star$ is subjected to a small perturbation.

**Proposition 4.4** ([35, Proposition 4.2])**.** *Let $\sigma$ be a finite set of eigenvalues of finite type of $K^\star$, and let $\mathcal{C}$ be a contour around $\sigma$ which separates $\sigma$ from $\sigma(K^\star)\backslash\sigma$. Then there exists $\epsilon > 0$ such that for any operator $K^u$ on $X$ with $\|K^\star - K^u\| < \epsilon$ the following holds true: $\sigma(K^u) \cap \mathcal{C} = \emptyset$, the part of $\sigma(K^u)$ inside $\mathcal{C}$ is a finite set of eigenvalues of finite type and, if we denote $m(\lambda; K)$ the algebraic multiplicity of the eigenvalue $\lambda$ for the operator $K$, then*

$$\sum_{\lambda \ inside \ \mathcal{C}} m(\lambda; K^u) = \sum_{\lambda \ inside \ \mathcal{C}} m(\lambda; K^\star) \ . \tag{4.20}$$

The theory also provides bounds on $\epsilon$, cf. Section 7 for details.

Two approximations are going to be made. Firstly, because we are looking for eigenvalues of $K^u$ that are close to 1, i.e. $u$ close to zero, we can compare the kernels $K^u$ and $K^0$. Note that $K^0$ is a Markov kernel defined on $\mathcal{M}_{k+1} \times \mathcal{B}(\mathcal{M}_{k+1})$ and given by

$$K^0(x, \mathrm{d}y) = \mathbb{P}^x \left\{ X_{\tau^+_{\mathcal{M}_{k+1}}} \in \mathrm{d}y \right\} . \tag{4.21}$$

This is exactly the kernel of the trace process $X|_{\mathcal{M}_{k+1}}$ introduced in Section 2.6. For the second approximation, we introduce the kernel $K^\star$ given by

$$K^\star(x, \mathrm{d}y) = \sum_{i=1}^{k+1} \mathbb{1}_{\{x \in B_i\}} \int_{B_i} \mathring{\pi}_0^{B_i}(z) \ K^0(z, \mathrm{d}y) \, \mathrm{d}z \tag{4.22}$$

where $\mathring{\pi}_0^{B_i}$ is the quasistationary distribution of the process described by the kernel $K_{B_i}^0$ (see Section 2.5). Note that the kernel $K^\star$ is of finite rank, since it is a finite sum of products of two functions, one of which depends on its first argument only.

In order to obtain sharper estimates, instead of considering the Markov chain at each return time to the set $\mathcal{M}_{k+1}$ on the Poincaré map, we will study the diluted chain given by the $m^{\text{th}}$ iterate of the kernel, where $m$ may depend on $\sigma$. It is clear that $\mathrm{e}^{-u}$ is an eigenvalue of $K^u$ if and only if $\mathrm{e}^{-um}$ is an eigenvalue of the kernel $(K^u)^m$. We also introduce the $m$-fold iterates $(K^0)^m$ and $(K^\star)^m$.

We will prove in Section 6.1 the following bound on the norm of the difference between $(K^u)^m$ and $(K^0)^m$.

**Proposition 4.5** (Proposition 6.6)**.** *For all real $u$ verifying the Laplace condition given by (4.6) with $A = \mathcal{M}_{k+1}$, and such that $(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_{k+1}^c}\{\tau^+_{\mathcal{M}_{k+1}}\} < 1$, we have*

$$\|(K^u)^m - \big(K^0\big)^m\| \leqslant \left(1 + \frac{(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_{k+1}}\{\tau^+_{\mathcal{M}_{k+1}} - 1\}}{1-(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_{k+1}^c}\{\tau^+_{\mathcal{M}_{k+1}}\}}\right)^m - 1\,. \tag{4.23}$$

The expectations appearing in this bound will be estimated in Section 8. We will also obtain in Section 6.2 the following bound on the norm of the difference between $(K^0)^m$ and $(K^\star)^m$.

**Proposition 4.6** (Proposition 6.7)**.** *For all $m \in \mathbb{N}$, the norm of the difference between the iterates of $K^0$ and $K^\star$ satisfies the bound*

$$\|\big(K^0\big)^m - (K^\star)^m\| \leqslant \sup_{1\leqslant i\leqslant k+1} R_i\,, \tag{4.24}$$

*where*

$$\begin{aligned} R_i = \|\mathring{\phi}_0^{B_i} - 1\| + 2(\mathring{\lambda}_1^{B_i})^m + 2\frac{1-(\mathring{\lambda}_1^{B_i})^m}{1-\mathring{\lambda}_1^{B_i}}\mathbb{P}^{B_i}\Big\{\tau^+_{\mathcal{M}_{k+1}\setminus B_i} < \tau^+_{B_i}\Big\} \\ + m(m-1)\mathbb{P}^{B_i}\Big\{\tau^+_{\mathcal{M}_{k+1}\setminus B_i} < \tau^+_{B_i}\Big\}\mathbb{P}^{\mathcal{M}_{k+1}\setminus B_i}\Big\{\tau^+_{B_i} < \tau^+_{\mathcal{M}_{k+1}\setminus B_i}\Big\}\,. \end{aligned} \tag{4.25}$$

The quantities $\mathring{\lambda}_k^{B_i}$ and $\mathring{\phi}_0^{B_i}$ appearing in this estimate are related to the trace process $K_{B_i}^0$ killed upon leaving $B_i$. In Section 5, we will derive bounds on the oscillation of the principal eigenfunction $\mathring{\phi}_0^{B_i}$ and the spectral gap $|\mathring{\lambda}_1^{B_i}|/\mathring{\lambda}_0^{B_i}$. Together with the metastable hierarchy assumption, these bounds show that $\|\big(K^0\big)^m - (K^\star)^m\|$ is small for all $m$ such that

$$m\mathbb{P}^{B_{k+1}}\Big\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\Big\} \ll 1\,. \tag{4.26}$$

The difference $(K^u)^m - \big(K^0\big)^m$ is also small under this condition.

Thanks to these approximations, we have reduced our eigenvalue problem to a much simpler one. Since the kernel $(K^\star)^m$ is of finite rank, it admits exactly $N$ eigenvalues. Furthermore, solving the eigenvalue problem for $(K^\star)^m$ is now equivalent to solving a system of linear algebraic equations.

**Proposition 4.7** (Theorem 7.5)**.** *For $0 \leqslant i \leqslant k$, we denote by $\lambda_i^\star$ the eigenvalues of $K^\star$ labelled by decreasing order. The smallest eigenvalue $\lambda_k^\star$ of $K^\star$ is real and simple. It satisfies*

$$\left|\lambda_k^\star - \left(1 - \mathbb{P}^{\mathring{\pi}_0^{B_{k+1}}}\Big\{X_{\tau^+_{\mathcal{M}_{k+1}}} \in \mathcal{M}_k\Big\}\right)\right| \leqslant 2\max_{1\leqslant l\leqslant k} \mathbb{P}^{\mathring{\pi}_0^{B_l}}\Big\{X_{\tau^+_{\mathcal{M}_{k+1}}} \in \mathcal{M}_{k+1}\setminus B_l\Big\}\,. \tag{4.27}$$

*The $k$ remaining eigenvalues satisfy for all $0 \leqslant i < k$*

$$|1 - \lambda_i^\star| \leqslant 4 \max_{1 \leqslant l \leqslant k} \mathbb{P}^{\mathring{\pi}_0^{B_l}} \Big\{ X_{\tau^+_{\mathcal{M}_{k+1}}} \in \mathcal{M}_{k+1} \backslash B_l \Big\} \,. \tag{4.28}$$

Theorem 3.2 then follows essentially by combining the estimate (4.27) with the bound on $\|(K^u)^m - (K^\star)^m\|$ implied by the two previous propositions, for an appropriate choice of $m$. Details are given in Section 9.

## 4.4 Computation of eigenfunctions

Once eigenvalues have been estimated, determining the associated left and right eigenfunctions is relatively easy. The expressions for right eigenfunctions $\phi_k$ are essentially consequences of the Feynman–Kac representation given in Proposition 4.2. As for the left eigenfunctions, a crucial tool is the following result.

**Lemma 4.8.** *For any left eigenfunction $\pi_k$ of the kernel $K$ associated to the eigenvalue $e^{-u_k}$, and for any $B \subset A \subset \Sigma$, we have*

$$\int_A \pi_k(x) K^{u_k}(x, B) \, \mathrm{d}x := \int_A \pi_k(x) \mathbb{E}^x \big\{ \mathrm{e}^{u_k(\tau_A^+ - 1)} \, \mathbb{1}_{\{\tau_B^+ < \tau^+_{A \backslash B}\}} \big\} \, \mathrm{d}x = \mathrm{e}^{-u_k} \, \pi_k(B) \,. \tag{4.29}$$

Proof: Consider the function $h^u(x) = \mathbb{E}^x \big\{ \mathrm{e}^{u\tau_A} \, \mathbb{1}_{\{\tau_B < \tau_{A \backslash B}\}} \big\}$. Note that $h^u(x) = \mathbb{1}_{\{x \in B\}}$ whenever $x \in A$, while a similar argument as in Proposition 4.2 yields

$$(K h^u)(x) = \mathbb{E}^x \big\{ h^u(X_1) \big\} = K^u(x, B) \,. \tag{4.30}$$

It follows that

$$\begin{aligned} \int_A \pi_k(x) K^{u_k}(x, B) \, \mathrm{d}x &= \int_\Sigma \pi_k(x) (K h^{u_k})(x) \, \mathrm{d}x - \int_{\Sigma \backslash A} \pi_k(x) K^{u_k}(x, B) \, \mathrm{d}x \\ &= \mathrm{e}^{-u_k} \int_\Sigma \pi_k(x) h^{u_k}(x) \, \mathrm{d}x - \mathrm{e}^{-u_k} \int_{\Sigma \backslash A} \pi_k(x) h^{u_k}(x) \, \mathrm{d}x \\ &= \mathrm{e}^{-u_k} \int_A \pi_k(x) \mathbb{1}_{\{x \in B\}} \, \mathrm{d}x = \mathrm{e}^{-u_k} \, \pi_k(B) \,. \end{aligned} \tag{4.31}$$

In the second line, we have used the eigenvalue equation $\pi_k K = \mathrm{e}^{-u_k} \, \pi_k$ and the fact that in $\Sigma \backslash A$, $\tau_A = \tau_A^+$ and $\tau_B = \tau_B^+$, and thus $K^u(x, B) = \mathrm{e}^{-u} \, h^u(x)$. □

Proof of Proposition 3.7. Applying (4.29) for the left eigenfunction $\pi_0$ associated to the eigenvalue 1 and with any disjoint $A_1, A_2$ such that $A_1 \cup A_2 = A$, we have

$$\pi_0(A_1) = \int_{A_1 \cup A_2} \pi_0(x) \mathbb{P}^x \big\{ \tau_{A_1}^+ < \tau_{A_2}^+ \big\} \, \mathrm{d}x. \tag{4.32}$$

Decomposing the domain of the integral into $A_1$ and $A_2$, and using the fact that for all $x$, $\mathbb{P}^x \big\{ \tau_{A_1}^+ < \tau_{A_2}^+ \big\} = 1 - \mathbb{P}^x \big\{ \tau_{A_2}^+ < \tau_{A_1}^+ \big\}$, we immediately get the result. □

# 5 Spectral properties of $K^0_{B_i}$

Recall that we have denoted $K^\star$ the kernel on $\mathcal{M}_k \times \mathcal{B}(\mathcal{M}_k)$ defined by

$$K^\star(x, \mathrm{d}y) = \sum_{i=1}^{k} \mathbb{1}_{\{x \in B_i\}} \int_{B_i} \mathring{\pi}_0^{B_i}(z) \mathbb{P}^z\big\{X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}y\big\}\, \mathrm{d}z \;, \tag{5.1}$$

where $\mathring{\pi}_0^{B_i}$ is the quasistationary distribution of the process described by the kernel $K^0_{B_i}$. Also recall that $K^0_{B_i}$ is the kernel associated to the trace process $(X_n)|_{\mathcal{M}_k}$ killed upon leaving $B_i$. To remind us that we are not looking at the process described by the kernel defined on $\Sigma$ but at the process

$$\big(X_n|_{\mathcal{M}_k}\big)_{n\geqslant 0} = \Big(X_{\tau^{+,n}_{\mathcal{M}_k}}\Big)_{n\geqslant 0} \;, \tag{5.2}$$

i.e., the trace of the original process at the return times to $\mathcal{M}_k = \bigcup_{i=1}^k B_i$, we use the symbol $\mathring{}$. Since we study the killed process, we also follow the notations introduced in Section 2.5, by denoting its eigenvalues by $\mathring{\lambda}_j^{B_i}$ and its left and right eigenfunctions by $\mathring{\pi}_j^{B_i}(x)$ and $\mathring{\phi}_j^{B_i}(x)$ respectively. The principal eigenvalue $\mathring{\lambda}_0^{B_i}$ of this kernel is given by

$$\mathring{\lambda}_0^{B_i} = \mathbb{P}^{\mathring{\pi}_0^{B_i}}\Big\{\tau^+_{B_i} < \tau^+_{\mathcal{M}_k \setminus B_i}\Big\} \;. \tag{5.3}$$

Using the spectral decomposition, we can introduce the function $g(x, y)$ such that the density of $K^0_{B_i}$ satisfies

$$k^0_{B_i}(x, y) = \mathring{\lambda}_0^{B_i}\left\{\mathring{\pi}_0^{B_i}(y)\mathring{\phi}_0^{B_i}(x) + \frac{\mathring{\lambda}_1^{B_i}}{\mathring{\lambda}_0^{B_i}} g(x, y)\right\} . \tag{5.4}$$

Note that due to orthogonality of eigenfunctions (see (2.24)),

$$\int_{B_i} g(x, y)\mathring{\phi}_0^{B_i}(y)\, \mathrm{d}y = 0 \;, \qquad \int_{B_i} \mathring{\pi}_0^{B_i}(x) g(x, y)\, \mathrm{d}x = 0 \;. \tag{5.5}$$

It follows that

$$\big(k^0_{B_i}\big)^m(x, y) = \big(\mathring{\lambda}_0^{B_i}\big)^m \left\{\mathring{\pi}_0^{B_i}(y)\mathring{\phi}_0^{B_i}(x) + \left(\frac{\mathring{\lambda}_1^{B_i}}{\mathring{\lambda}_0^{B_i}}\right)^m g^m(x, y)\right\} . \tag{5.6}$$

In addition $g$ has spectral radius 1.

## 5.1 Spectral gap estimate

**Proposition 5.1** (Adapted from [9, Proposition 5.5]). *Assume that for some $n \in \mathbb{N}$, the density of the $n$-fold iterated kernel $\big(k^0_{B_i}\big)^n$ satisfies a uniform positivity condition, i.e., there exists $L(n) > 1$ such that*

$$\inf_{x_0 \in B_i} (k^0_{B_i})^n(x_0, y) \leqslant (k^0_{B_i})^n(x, y) \leqslant L(n) \inf_{x_0 \in B_i} (k^0_{B_i})^n(x_0, y) \qquad \forall x, y \in B_i \;. \tag{5.7}$$

*Then $\theta = |\mathring{\lambda}_1|/\mathring{\lambda}_0$ satisfies*

$$\theta^n \leqslant L(n) - \frac{\inf_{x \in B_i} \mathbb{P}^x\Big\{\tau^{+,n}_{B_i} < \tau^+_{\mathcal{M}_k \setminus B_i}\Big\}}{\big(\mathring{\lambda}_0^{B_i}\big)^n} \;. \tag{5.8}$$

PROOF: To ease notation, we prove the result for $n = 1$, but one can show that it is still true for all $n \geqslant 2$. For any $l \geqslant 1$, the eigenvalue equation for $\mathring{\lambda}_l^{B_i}$ and the orthogonality relation (2.24) of the eigenfunctions $\mathring{\phi}_l^{B_i}$ and $\mathring{\pi}_l^{B_i}$ give

$$\begin{aligned} \mathring{\lambda}_l^{B_i} \mathring{\phi}_l^{B_i}(x) &= \int_{B_i} k_{B_i}^0(x,y) \mathring{\phi}_l^{B_i}(y) \, \mathrm{d}y \, , \\ 0 &= \int_{B_i} \mathring{\pi}_0^{B_i}(y) \mathring{\phi}_l^{B_i}(y) \, \mathrm{d}y \, . \end{aligned} \tag{5.9}$$

For any $\kappa > 0$, we thus obtain

$$\mathring{\lambda}_l^{B_i} \mathring{\phi}_l^{B_i}(x) = \int_{B_i} \left[ k_{B_i}^0(x,y) - \kappa \mathring{\pi}_0^{B_i}(y) \right] \mathring{\phi}_l^{B_i}(y) \, \mathrm{d}y \, . \tag{5.10}$$

Let us denote by $x_0$ the point in $B_i$ where $\mathring{\phi}_l^{B_i}(y)$ reaches its supremum. Evaluating the last equation in $x_0$ we obtain

$$\left| \mathring{\lambda}_l^{B_i} \right| \leqslant \int_{B_i} \left| k_{B_i}^0(x_0,y) - \kappa \mathring{\pi}_0^{B_i}(y) \right| \mathrm{d}y \, . \tag{5.11}$$

Remark that for all $y \in B_i$,

$$\mathring{\lambda}_0^{B_i} \mathring{\pi}_0^{B_i}(y) = \int_{B_i} \mathring{\pi}_0^{B_i}(x) k_{B_i}^0(x,y) \, \mathrm{d}x \geqslant \inf_{x \in B_i} k_{B_i}^0(x,y) \, . \tag{5.12}$$

Taking $\kappa = \mathring{\lambda}_0^{B_i} L(1)$, we can remove the absolute value and write

$$\begin{aligned} \left| \mathring{\lambda}_l^{B_i} \right| &\leqslant \int_{B_i} \left[ \mathring{\lambda}_0^{B_i} L(1) \mathring{\pi}_0^{B_i}(y) - \inf_{x \in B_i} k_{B_i}^0(x,y) \right] \mathrm{d}y \\ &= \mathring{\lambda}_0^{B_i} L(1) - \inf_{x \in B_i} \mathbb{P}^x \left\{ \tau_{B_i}^+ < \tau_{\mathcal{M}_k \setminus B_i}^+ \right\} , \end{aligned} \tag{5.13}$$

which proves (5.8) for $n = 1$. □

The two following results based on Harnack inequalities [33] will enable us to prove that $n$ and $L(n)$ satisfying the uniform positivity condition (5.7) exist.

**Lemma 5.2** ([9, Lemma 5.7])**.** *For any set $\mathcal{D}_1$ such that its closure satisfies $\bar{\mathcal{D}}_1 \subset \mathcal{D}$, there exists a constant $C$, independent of $\sigma$, such that*

$$\frac{\sup_{x \in \mathcal{D}_1} k_{B_i}^0(x,y)}{\inf_{x \in \mathcal{D}_1} k_{B_i}^0(x,y)} \leqslant \mathrm{e}^{C/\sigma^2} \tag{5.14}$$

*for all $y \in \partial D$.*

**Lemma 5.3** ([9, Lemma 5.8])**.** *Let $\mathcal{B}_r(x)$ denote the ball of radius $r$ centred in $x$, and let $\mathcal{D}_1$ be such that its closure satisfies $\bar{\mathcal{D}}_1 \subset \mathcal{D}$. Then for any $x_0 \in \mathcal{D}_1, y \in \partial\mathcal{D}$, and $\eta > 0$, one can find a constant $r = r(y,\eta)$, independent of $\sigma$, such that*

$$\sup_{x \in \mathcal{B}_{r\sigma^2}(x_0)} k_{B_i}^0(x,y) \leqslant (1+\eta) \inf_{x \in \mathcal{B}_{r\sigma^2}(x_0)} k_{B_i}^0(x,y) \, . \tag{5.15}$$

**Proposition 5.4.** *For $x_1, x_2 \in B_i$, define the integer stopping time*

$$N = N(x_1, x_2) = \inf\Big\{ n \geqslant 1 : \Big| \hat{X}_n^{x_2} - \hat{X}_n^{x_1} \Big| \leqslant r_\eta \sigma^2 \Big\} , \tag{5.16}$$

*where $\hat{X}_n^{x_0}$ denotes the Markov chain with transition kernel $K^0_{B_i}(x_0, \mathrm{d}y)/K^0_{B_i}(x_0, B_i)$ (i.e. the Markov chain conditioned to stay in $B_i$) and initial condition $x_0$, and $r_\eta$ is the constant of Lemma 5.3. Let*

$$\rho_n = \sup_{x_1, x_2 \in B_i} \mathbb{P}\{ N(x_1, x_2) > n \} . \tag{5.17}$$

*Then for any $n \geqslant 2$, and any $\eta > 0$, the transition kernel $(K^0_{B_i})^n(x, \mathrm{d}y)$ fulfils a uniform positivity condition with constant $L(n)$ satisfying*

$$L(n) \leqslant \frac{1 + \eta + \rho_{n-1}\, \mathrm{e}^{C/\sigma^2}}{\inf\limits_{x \in B_i} \mathbb{P}^x\Big\{ \tau^{+,n}_{B_i} < \tau^+_{\mathcal{M}_k \setminus B_i} \Big\}} , \tag{5.18}$$

*where $C$ does not depend on $\sigma$.*

Proof: Thanks to [9, Proposition 5.9], we obtain that

$$\sup_{x \in B_i} \frac{(k^0_{B_i})^n(x, y)}{(K^0_{B_i})^n(x, B_i)} \leqslant \inf_{x \in B_i} \frac{(k^0_{B_i})^n(x, y)}{(K^0_{B_i})^n(x, B_i)} \Big( 1 + \eta + \rho_{n-1}\, \mathrm{e}^{C/\sigma^2} \Big) \qquad \forall y \in B_i . \tag{5.19}$$

The result is then immediate. □

## 5.2 Oscillations of the principal right eigenfunction

**Proposition 5.5.** *Assume that $(k^0_{B_i})^n$ satisfies the uniform positivity condition (5.7) for some $n \in \mathbb{N}$. Then there exists $M > 0$, such that the normalised principal right eigenfunction of $K^0_{B_i}$ satisfies*

$$\| \mathring{\phi}_0^{B_i} - 1 \| \leqslant M L(n)^2 \sup_{x \in B_i} \left| 1 - \frac{\mathbb{P}^x\Big\{ \tau^{+,n}_{B_i} < \tau^+_{\mathcal{M}_k \setminus B_i} \Big\}}{\big(\mathring{\lambda}_0^{B_i}\big)^n} \right| . \tag{5.20}$$

Proof: The uniform positivity condition implies that we can apply [15, Theorem 3, Lemma 3], which tells us that for any bounded measurable function $f : B_i \to \mathbb{R}$, there exists a constant $M(f)$ such that for all $m \in \mathbb{N}$,

$$\| (K^0_{B_i})^{nm} f - (\mathring{\lambda}_0^{B_i})^{nm} (\mathring{\pi}_0^{B_i} f) \mathring{\phi}_0^{B_i} \| \leqslant M(f) \varrho^m (\mathring{\lambda}_0^{B_i})^{nm} \| \mathring{\phi}_0^{B_i} \| , \tag{5.21}$$

where $\varrho < 1$. Inspecting the proofs in [15] shows that $\varrho$ satisfies $\varrho \leqslant 1 - 1/L(n)^2$. Taking $f(x) = 1$, it follows that

$$\big| (K^0_{B_i})^{nm}(x, B_i) - (\mathring{\lambda}_0^{B_i})^{nm} \mathring{\phi}_0^{B_i}(x) \big| \leqslant M(1) \varrho^m (\mathring{\lambda}_0^{B_i})^{nm} \| \mathring{\phi}_0^{B_i} \| . \tag{5.22}$$

Dividing by $(\mathring{\lambda}_0^{B_i})^{nm}$ and using the spectral decomposition (5.6), we get

$$\left| \int_{B_i} \left( \frac{\mathring{\lambda}_1^{B_i}}{\mathring{\lambda}_0^{B_i}} \right)^{nm} g^{nm}(x, y)\, \mathrm{d}y \right| = \left| \frac{(K^0_{B_i})^{nm}(x, B_i)}{\big(\mathring{\lambda}_0^{B_i}\big)^{nm}} - \mathring{\phi}_0^{B_i}(x) \right| \leqslant M(1) \varrho^m \| \mathring{\phi}_0^{B_i} \| . \tag{5.23}$$

Since $\varrho < 1$, taking the limit $m \to \infty$, we obtain

$$\mathring{\phi}_0^{B_i}(x) = \lim_{m\to\infty} \frac{(K^0_{B_i})^{nm}(x, B_i)}{\big(\mathring{\lambda}_0^{B_i}\big)^{nm}} = \lim_{m\to\infty} \frac{\mathbb{P}^x\Big\{\tau_{B_i}^{+,nm} < \tau^+_{\mathcal{M}_k\setminus B_i}\Big\}}{\big(\mathring{\lambda}_0^{B_i}\big)^{nm}} \,. \tag{5.24}$$

Let $(h_m)_{m\geqslant 0}$ be the sequence of bounded measurable functions in $B_i$ defined by $h_0 = 1$, and

$$h_{m+1}(x) = \frac{1}{\big(\mathring{\lambda}_0^{B_i}\big)^n} \int_{B_i} (k^0_{B_i})^n(x, y) h_m(y)\, \mathrm{d}y \,, \tag{5.25}$$

so that for all $m$

$$h_m(x) = \frac{(K^0_{B_i})^{nm}(x, B_i)}{\big(\mathring{\lambda}_0^{B_i}\big)^{nm}} \,. \tag{5.26}$$

We can now use a telescopic series to estimate

$$\begin{aligned} 1 - \mathring{\phi}_0^{B_i}(x) &= h_0(x) - \lim_{m\to\infty} h_m(x) \\ &= \sum_{m=0}^{\infty} \big[h_m(x) - h_{m+1}(x)\big] \\ &= \sum_{m=0}^{\infty} \int_{B_i} \frac{(k^0_{B_i})^{nm}(x, y)}{\big(\mathring{\lambda}_0^{B_i}\big)^{nm}} \big[h_0 - h_1(y)\big]\, \mathrm{d}y \,. \end{aligned} \tag{5.27}$$

Since $\displaystyle\int_{B_i} \mathring{\pi}_0^{B_i}(x)\big[h_0 - h_1(x)\big]\, \mathrm{d}x = 0$, the spectral decomposition (5.6) and (5.23) yield

$$\begin{aligned} \|1 - \mathring{\phi}_0^{B_i}\| &\leqslant \sup_{x\in B_i} \sum_{m=0}^{\infty} \left| \int_{B_i} \left(\frac{\mathring{\lambda}_1^{B_i}}{\mathring{\lambda}_0^{B_i}}\right)^{nm} g^{nm}(x, y)\, \mathrm{d}y \right| \|h_0 - h_1\| \\ &\leqslant \sum_{m=0}^{\infty} M(1)\varrho^m \|\mathring{\phi}_0^{B_i}\| \|h_0 - h_1\| \,. \end{aligned} \tag{5.28}$$

Since $\sum_m \varrho^m \leqslant L(n)^2$ and $h_1(x) = (\mathring{\lambda}_0^{B_i})^{-n}(K^0_{B_i})^n(x, B_i)$, the result follows. □

# 6 Estimates on operators norms

The aim of this section is to show that the kernel $K^u$ (or its $m$-fold iterates) defined on $\mathcal{M}_k$ by

$$K^u(x, \mathrm{d}y) = \mathbb{E}^x\left\{ \mathrm{e}^{u(\tau^+_{\mathcal{M}_k} - 1)}\, \mathbb{1}_{\left\{X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}y\right\}} \right\} , \tag{6.1}$$

can be approximated by a finite-rank operator $K^\star$ (or its $m$-fold iterates) given by

$$K^\star(x, \mathrm{d}y) = \sum_{i=1}^{k} \mathbb{1}_{\{x\in B_i\}} \int_{B_i} \mathring{\pi}_0^{B_i}(x_0)\; K^0(x_0, \mathrm{d}y)\, \mathrm{d}x_0 \,. \tag{6.2}$$

We will first compare $K^u$ to $K^0$, and then compare $K^0$ to $K^\star$ (and similarly for their iterates).

## 6.1 Comparison between $K^u$, $K^0$ and their $m$-fold iterates

Note that the difference between $K^u$ and $K^0$ is given by

$$\big(K^u - K^0\big)(x, \mathrm{d}y) = \mathbb{E}^x\left\{ \big(\mathrm{e}^{u(\tau^+_{\mathcal{M}_k}-1)} - 1\big) \mathbb{1}_{\left\{X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}y\right\}} \right\} . \tag{6.3}$$

The following proposition enables us to bound the norm of this difference.

**Proposition 6.1.** *For all real $u$ verifying the Laplace condition given by (4.6) with $A = \mathcal{M}_k$, and such that $(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\{\tau^+_{\mathcal{M}_k}\} < 1$, we have*

$$\|K^u - K^0\| \leqslant \frac{(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k}\{\tau^+_{\mathcal{M}_k} - 1\}}{1-(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\{\tau^+_{\mathcal{M}_k}\}} . \tag{6.4}$$

**Remark 6.2.** Note that for real $u$, the two conditions on $u$ can be summarised as follows:

$$\max\left( \mathbb{P}^{\mathcal{M}_k^c}\{X_1 \in \mathcal{M}_k^c\}, \frac{\mathbb{E}^{\mathcal{M}_k^c}\{\tau^+_{\mathcal{M}_k}\} - 1}{\mathbb{E}^{\mathcal{M}_k^c}\{\tau^+_{\mathcal{M}_k}\}} \right) < \mathrm{e}^{-u} . \tag{6.5}$$

$\Diamond$

To prove this proposition, we will use the following expression for the inverse of $(\mathrm{id} - K_{A^c})$ (which is its resolvent at $z = 1$).

**Lemma 6.3.** *Assume that there is a set $A \subset \Sigma$ such that*

$$\sup_{x \in A^c} \mathbb{P}^x\{X_1 \in A^c\} < 1 . \tag{6.6}$$

*Then the unique solution of the boundary value problem*

$$\begin{aligned} ((\mathrm{id} - K)r)(x) &= g(x), & x &\in A^c , \\ r(x) &= 0, & x &\in A , \end{aligned} \tag{6.7}$$

*is given by*

$$r(x) = \mathbb{E}^x\left\{ \sum_{n=0}^{\tau_A - 1} g(X_n) \right\} \tag{6.8}$$

*where by convention, the empty sum equals zero.*

Proof: First, let us check that the proposed function solves the boundary value problem. This is obvious for $x \in A$, since in that case, with the convention taken for the empty sum, $\mathbb{E}^x\{\sum_{n=0}^{\tau_A-1} g(X_n)\} = 0$. For $x \in A^c$,

$$((\mathrm{id} - K)r)(x) = \mathbb{E}^x\left\{ \sum_{n=0}^{\tau_A - 1} g(X_n) \right\} - \mathbb{E}^x\left\{ \mathbb{E}^{X_1}\left\{ \sum_{n=0}^{\tau_A - 1} g(X_n) \right\} \right\} . \tag{6.9}$$

We can split the expectations according to the location of $X_1$, and use the strong Markov property, to obtain

$$\begin{aligned}
&((\mathrm{id}-K)r)(x)\\
&= \mathbb{E}^x\big\{\mathbb{1}_{\{X_1\in A\}}g(x)\big\} + \mathbb{E}^x\left\{\mathbb{1}_{\{X_1\in A^c\}}\sum_{n=0}^{\tau_A-1} g(X_n)\right\} - \mathbb{E}^x\left\{\mathbb{1}_{\{X_1\in A^c\}}\mathbb{E}^{X_1}\left\{\sum_{n=0}^{\tau_A-1} g(X_n)\right\}\right\}\\
&= \mathbb{E}^x\big\{\mathbb{1}_{\{X_1\in A\}}g(x)\big\} + \mathbb{E}^x\left\{\mathbb{1}_{\{X_1\in A^c\}}\sum_{n=0}^{\tau_A-1} g(X_n)\right\} - \mathbb{E}^x\left\{\mathbb{1}_{\{X_1\in A^c\}}\sum_{n=1}^{\tau_A-1} g(X_n)\right\}\\
&= g(x)\;. \qquad (6.10)
\end{aligned}$$

This shows that we have an admissible solution for all $x \in \Sigma$.

Uniqueness is a consequence of the Fredholm alternative. Indeed, since

$$\|K_{A^c}\| \leqslant \sup_{x\in A^c} \mathbb{P}^x\{X_1 \in A^c\} < 1\;, \qquad (6.11)$$

we can apply [35, Theorem 8.1]. In particular, $(\mathrm{id}-K_{A^c})$ is invertible. □

**Remark 6.4.** For $A = \mathcal{M}_k$, since

$$\|K_{\mathcal{M}_k^c}\| \leqslant \sup_{x\in \mathcal{M}_k^c} \mathbb{P}^x\{X_1 \in \mathcal{M}_k^c\} < 1\;, \qquad (6.12)$$

the assumption of Lemma 6.3 is satisfied. ◊

Proof of Proposition 6.1. Note that

$$\|K^u - K^0\| \leqslant \sup_{x\in \mathcal{M}_k} \mathbb{E}^x\big\{\mathrm{e}^{u(\tau^+_{\mathcal{M}_k}-1)} - 1\big\}\;. \qquad (6.13)$$

Let us assume that this maximum is obtained for $\bar{x} \in \mathcal{M}_k$. Recognizing the sum of terms of a geometric sequence, we obtain

$$\begin{aligned}
\mathbb{E}^{\bar{x}}\big\{\mathrm{e}^{u(\tau^+_{\mathcal{M}_k}-1)} - 1\big\} &= \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^{\bar{x}}\left\{\sum_{n=1}^{\tau^+_{\mathcal{M}_k}-1} \mathrm{e}^{un}\right\}\\
&= \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^{\bar{x}}\left\{\sum_{n=1}^{\tau^+_{\mathcal{M}_k}-1} \mathrm{e}^{u(\tau^+_{\mathcal{M}_k}-n)}\right\}\\
&= \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^{\bar{x}}\left\{\sum_{n=1}^{\tau^+_{\mathcal{M}_k}-1} \mathbb{E}^{X_n}\big\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\big\}\right\}. \qquad (6.14)
\end{aligned}$$

We thus get

$$\|K^u - K^0\| \leqslant \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^{\mathcal{M}_k}\big\{\tau^+_{\mathcal{M}_k} - 1\big\}\mathbb{E}^{\mathcal{M}_k^c}\big\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\big\}\;. \qquad (6.15)$$

Let us now bound the expected value starting from $\mathcal{M}_k^c$. Note that $r(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\} - 1$ solves the boundary value problem

$$\begin{aligned}
((\mathrm{id}-K)r)(x) &= \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^x\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\} && x \in \mathcal{M}_k^c\;,\\
r(x) &= 0 && x \in \mathcal{M}_k\;. \qquad (6.16)
\end{aligned}$$

Thanks to Lemma 6.3, we have

$$r(x) = \mathbb{E}^x\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\} - 1 = \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^x\left\{\sum_{n=0}^{\tau_{\mathcal{M}_k}-1}\mathbb{E}^{X_n}\big\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\big\}\right\}. \tag{6.17}$$

Introducing $M = \mathbb{E}^{\mathcal{M}_k^c}\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\}$, and taking the supremum for $x \in \mathcal{M}_k^c$ in (6.17), we obtain

$$M - 1 \leqslant \big(1-\mathrm{e}^{-u}\big)\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\}M\ . \tag{6.18}$$

Thus if $(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\} < 1$, we have

$$M = \mathbb{E}^{\mathcal{M}_k^c}\big\{\mathrm{e}^{u\tau_{\mathcal{M}_k}}\big\} \leqslant \frac{1}{1-(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\}}\ , \tag{6.19}$$

which gives the result. □

**Remark 6.5.** Note that in the previous proof, we have obtained the bound

$$\mathbb{E}^{\mathcal{M}_k}\left\{\sum_{n=1}^{\tau^+_{\mathcal{M}_k}-1}\mathrm{e}^{un}\right\} \leqslant \frac{\mathbb{E}^{\mathcal{M}_k}\big\{\tau^+_{\mathcal{M}_k}-1\big\}}{1-(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\}}\ . \tag{6.20}$$

◊

We are now going to bound the supremum norm of the difference between the iterates of these two kernels. We recall that we want to prove

**Proposition 6.6.** *For all real $u$ verifying the Laplace condition given by* (4.6) *with $A = \mathcal{M}_k$, and such that $(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\} < 1$, we have*

$$\|(K^u)^m - \big(K^0\big)^m\| \leqslant \left(1 + \frac{(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k}\big\{\tau^+_{\mathcal{M}_k}-1\big\}}{1-(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k^c}\big\{\tau^+_{\mathcal{M}_k}\big\}}\right)^m - 1\ . \tag{6.21}$$

Proof: To ease notation, we introduce $\tau^+_m = \tau^{+,m}_{\mathcal{M}_k}$ for the $m^{\text{th}}$ return time to $\mathcal{M}_k$. Note that the $m^{\text{th}}$ iterated kernel of $K^u$ is given by

$$(K^u)^m(x,\mathrm{d}y) = \mathbb{E}^x\Big\{\mathrm{e}^{u(\tau^+_m-m)}\,\mathbb{1}_{\big\{X_{\tau^+_m}\in \mathrm{d}y\big\}}\Big\}\ . \tag{6.22}$$

Therefore, the norm of the difference between the iterates of $K^u$ and $K^0$ satisfies

$$\|(K^u)^m - \big(K^0\big)^m\| \leqslant \sup_{x\in\mathcal{M}_k}\mathbb{E}^x\Big\{\mathrm{e}^{u(\tau^+_m-m)}-1\Big\}\ . \tag{6.23}$$

As previously, recognizing the sum of terms of a geometric sequence, we can bound the norm by

$$\|(K^u)^m - \big(K^0\big)^m\| \leqslant \big(1-\mathrm{e}^{-u}\big)\sup_{x\in\mathcal{M}_k}\mathbb{E}^x\left\{\sum_{n=1}^{\tau^+_m-m}\mathrm{e}^{un}\right\}. \tag{6.24}$$

We can now split the expected value of the sum as follows:

$$\mathbb{E}^x\left\{\sum_{n=1}^{\tau^+_m-m}\mathrm{e}^{un}\right\} = \mathbb{E}^x\left\{\sum_{n=1}^{\tau^+_1-1}\mathrm{e}^{un}\right\} + \mathbb{E}^x\left\{\sum_{n=\tau^+_1}^{\tau^+_m-m}\mathrm{e}^{un}\right\}. \tag{6.25}$$

Using the strong Markov property for the second term on the right-hand side we get

$$\mathbb{E}^x\left\{\sum_{n=1}^{\tau_m^+-m}\mathrm{e}^{un}\right\}=\mathbb{E}^x\left\{\sum_{n=1}^{\tau_1^+-1}\mathrm{e}^{un}\right\}+\mathbb{E}^x\left\{\mathrm{e}^{u(\tau_1^+-1)}\,\mathbb{E}^{X_{\tau_1^+}}\left\{\sum_{n=1}^{\tau_{m-1}^+-(m-1)}\mathrm{e}^{un}\right\}\right\}. \tag{6.26}$$

Denoting for all $m\in\mathbb{N}$

$$t_m=\mathbb{E}^{\mathcal{M}_k}\left\{\sum_{n=1}^{\tau_m^+-m}\mathrm{e}^{un}\right\}, \tag{6.27}$$

we obtain the induction relation

$$t_m\leqslant t_1+t_{m-1}\big(1+\big(1-\mathrm{e}^{-u}\big)t_1\big)\,. \tag{6.28}$$

Thus, the general term can be bounded by

$$t_m\leqslant\frac{(1+(1-\mathrm{e}^{-u})t_1)^m-1}{1-\mathrm{e}^{-u}}\,. \tag{6.29}$$

Using the bound found in (6.20) for $t_1$, it follows that

$$\mathbb{E}^{\mathcal{M}_k}\left\{\sum_{n=1}^{\tau_m^+-m}\mathrm{e}^{un}\right\}\leqslant\frac{\left(1+(1-\mathrm{e}^{-u})\mathbb{E}^{\mathcal{M}_k}\left\{\sum_{n=1}^{\tau_1^+-1}\mathrm{e}^{un}\right\}\right)^m-1}{1-\mathrm{e}^{-u}}\,, \tag{6.30}$$

which gives the result. □

## 6.2 Comparison between $K^0$, $K^\star$ and their $m$-fold iterates

The aim of this section is to prove the following proposition:

**Proposition 6.7.** *For all $m\in\mathbb{N}$, the norm of the difference between the iterates of $K^0$ and $K^\star$ satisfies the bound*

$$\|\big(K^0\big)^m-(K^\star)^m\|\leqslant\sup_{1\leqslant i\leqslant k}R_i\,, \tag{6.31}$$

*where*

$$\begin{aligned}R_i=\|\mathring{\phi}_0^{B_i}-1\|+2\big|\mathring{\lambda}_1^{B_i}\big|^m+2\frac{1-\big|\mathring{\lambda}_1^{B_i}\big|^m}{1-\big|\mathring{\lambda}_1^{B_i}\big|}\mathbb{P}^{B_i}\Big\{\tau^+_{\mathcal{M}_k\setminus B_i}<\tau^+_{B_i}\Big\}\\+m(m-1)\mathbb{P}^{B_i}\Big\{\tau^+_{\mathcal{M}_k\setminus B_i}<\tau^+_{B_i}\Big\}\mathbb{P}^{\mathcal{M}_k\setminus B_i}\Big\{\tau^+_{B_i}<\tau^+_{\mathcal{M}_k\setminus B_i}\Big\}\,.\end{aligned} \tag{6.32}$$

Proof: Let us first introduce the kernel $(\check{K})^m$ with density

$$(\check{k})^m(x,y)=\sum_{i=1}^k\mathbb{1}_{\{x\in B_i\}}(\check{k}_i)^m(x,y) \tag{6.33}$$

where for any $x\in B_i$

$$\begin{aligned}(\check{k}_i)^m(x,y)&=(k^0_{B_i})^m(x,y)\\&\quad+\sum_{j=0}^{m-1}\int_{\mathcal{M}_k\setminus B_i}\int_{B_i}(k^0_{B_i})^l(x,z_1)k^0(z_1,z_2)(k^0_{\mathcal{M}_k\setminus B_i})^{m-l-1}(z_2,y)\,\mathrm{d}z_1\,\mathrm{d}z_2\,.\end{aligned} \tag{6.34}$$

Note that this kernel describes the process living on $\mathcal{M}_k$ which can only perform one transition, i.e., starting in $B_i$ the Markov chain either stays in $B_i$ or makes an excursion to $\mathcal{M}_k \backslash B_i$ and stays in this set. We introduce the notation

$$\Delta_m = \int_{\mathcal{M}_k} \left[ (k^0)^m(x,y) - (\check{k})^m(x,y) \right] \mathrm{d}y \; . \tag{6.35}$$

We claim that for any $x \in B_i$, for all $m \geqslant 1$

$$\Delta_m \leqslant \frac{1}{2} m(m-1) \mathbb{P}^{B_i}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_i \Big\} \mathbb{P}^{\mathcal{M}_k \backslash B_i}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \in B_i \Big\} \; . \tag{6.36}$$

Let us prove this claim by induction. Since $k^0(x,y) = \check{k}(x,y)$ the base case is verified. The induction step is based on counting the possible ways to make more than one transition when considering the $m+1^{\text{st}}$ iterate. At time $m$, either the process has already made more than two transitions, or the process has made one transition from $B_i$ to $\mathcal{M}_k \backslash B_i$ before time $m$ and made an excursion from $\mathcal{M}_k \backslash B_i$ to $B_i$ at time $m$. Note that in the second case, there are exactly $m$ different ways to perform such transitions (depending on the time of the first excursion). It follows that

$$\Delta_{m+1} \leqslant \Delta_m + m \mathbb{P}^{B_i}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_i \Big\} \mathbb{P}^{\mathcal{M}_k \backslash B_i}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \in B_i \Big\} \; , \tag{6.37}$$

so that the general term indeed satisfies the bound (6.36).

We can now bound, for all $m$, the norm of the difference between the iterates of $K^0$ and $K^\star$, that is

$$\| \big(K^0\big)^m - (K^\star)^m \| \leqslant \max_{1 \leqslant i \leqslant k} \sup_{x \in B_i} \int_{\mathcal{M}_k} \left| (k^0)^m(x,y) - (k^\star)^m(x,y) \right| \mathrm{d}y \; . \tag{6.38}$$

The triangle inequality yields

$$\begin{aligned} \left| (k^0)^m(x,y) - (k^\star)^m(x,y) \right| \leqslant \; & \left| (k^0)^m(x,y) - \check{k}^m(x,y) \right| \\ & + \left| \check{k}^m(x,y) - \int_{B_i} \mathring{\pi}_0^{B_i}(z) \check{k}^m(z,y) \, \mathrm{d}z \right| \\ & + \left| \int_{B_i} \mathring{\pi}_0^{B_i}(z) \big( \check{k}^m(z,y) - (k^0)^m(z,y) \big) \, \mathrm{d}z \right| \; . \end{aligned} \tag{6.39}$$

Integrating over $\mathcal{M}_k$, the first and the last term in the right-hand side can be bounded using (6.36). Using the spectral decomposition (5.4) of $k^0_{B_i}$, we obtain

$$\begin{aligned} & \int_{B_i} \left| \check{k}^m(x,y) - \int_{B_i} \mathring{\pi}_0^{B_i}(z) \check{k}^m(z,y) \, \mathrm{d}z \right| \mathrm{d}y \\ & \qquad \leqslant \big( \mathring{\lambda}_0^{B_i} \big)^m \left| \mathring{\phi}_0^{B_i}(x) - 1 \right| + 2 \left| \mathring{\lambda}_1^{B_i} \right|^m \sup_{z \in B_i} \left| \int_{B_i} g^m(z,y) \right| \mathrm{d}y \; , \end{aligned} \tag{6.40}$$

(since $\check{k}^m(x,y) = (k^0_{B_i})^m(x,y)$ if $x, y \in B_i$) and

$$\begin{aligned} & \int_{\mathcal{M}_k \backslash B_i} \left| \check{k}^m(x,y) - \int_{B_i} \mathring{\pi}_0^{B_i}(z) \check{k}^m(z,y) \, \mathrm{d}z \right| \mathrm{d}y \\ & \leqslant \sum_{l=0}^{m-1} \big( \mathring{\lambda}_0^{B_i} \big)^l \left| \mathring{\phi}_0^{B_i}(x) - 1 \right| \Big( 1 - \mathring{\lambda}_0^{B_i} \Big) + 2 \mathbb{P}^{B_i}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_i \Big\} \left| \mathring{\lambda}_1^{B_i} \right|^l \sup_{z \in B_i} \left| \int_{B_i} g^l(z,y) \, \mathrm{d}y \right| \; . \end{aligned} \tag{6.41}$$

Regrouping the different terms, we obtain the result. □

# 7 Perturbation theory for bounded linear operators

In the previous section, we have shown that the kernel $K^u$ (or its $m$-fold iterates) defined on $\mathcal{M}_k$ by

$$K^u(x, \mathrm{d}y) = \mathbb{E}^x \left\{ \mathrm{e}^{u(\tau^+_{\mathcal{M}_k} - 1)} \mathbb{1}_{\left\{ X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}y \right\}} \right\} , \tag{7.1}$$

can be approximated by a finite-rank operator $K^\star$ (or its iterates). We now study the spectral properties of $K^\star$ (and its iterates) to deduce the spectral properties of $K^u$.

In the following, to ease notation, we will consider the case $m = 1$, but the results remain true for all $m$ considering the $m^{\text{th}}$ return time to $\mathcal{M}_k$.

## 7.1 General idea

Let $\sigma(K^\star)$ denote the spectrum of the operator $K^\star$. For $\sigma$ an isolated part of $\sigma(K^\star)$, we define the Riesz projection $\Pi_\sigma(K^\star)$ by

$$\Pi_\sigma(K^\star) = \frac{1}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} (z\,\mathrm{id} - K^\star)^{-1}\,\mathrm{d}z \ , \tag{7.2}$$

where we assume that $\mathcal{C} \subset \mathbb{C}$ is a Cauchy contour (in the resolvent set of $K^\star$) around $\sigma$. Recall that $\Pi_\sigma$ is a projection [35, Lemma 2.1] and that

$$\Pi_{\sigma(K^\star)}(K^\star) = \mathrm{id} \ . \tag{7.3}$$

We want to know what happens to the spectrum $K^u$ when $K^u$ can be seen as a perturbation of $K^\star$. We choose a Cauchy contour $\mathcal{C}$ in $\mathbb{C}$ surrounding $\sigma(K^\star)$ and first give a condition that ensures that $\mathcal{C}$ does not contain any eigenvalue of $K^u$. This amounts to checking that $(z\,\mathrm{id} - K^u)$ is invertible for all $z \in \mathcal{C}$.

**Proposition 7.1** ([34, Corollary 8.2])**.** *If $(z\,\mathrm{id} - K^\star)$ is invertible and $\|K^u - K^\star\| = \|(z\,\mathrm{id} - K^\star) - (z\,\mathrm{id} - K^u)\| < \|(z\,\mathrm{id} - K^\star)^{-1}\|^{-1}$, then $(z\,\mathrm{id} - K^u)$ is invertible and*

$$\|(z\,\mathrm{id} - K^\star)^{-1} - (z\,\mathrm{id} - K^u)^{-1}\| \leqslant \frac{\|(z\,\mathrm{id} - K^\star)^{-1}\|^2 \|K^u - K^\star\|}{1 - \|(z\,\mathrm{id} - K^\star)^{-1}\| \|K^u - K^\star\|} \ . \tag{7.4}$$

To ensure that $(z\,\mathrm{id} - K^u)$ is invertible for all $z \in \mathcal{C}$, it is rather natural to require

$$\|K^u - K^\star\| \leqslant \frac{1}{2}\gamma := \frac{1}{2} \min\Big\{ \|(z\,\mathrm{id} - K^\star)^{-1}\|^{-1} | z \in \mathcal{C} \Big\} \ . \tag{7.5}$$

Under this assumption, (7.4) shows that $(z\,\mathrm{id} - K^u)$ is invertible for all $z \in \mathcal{C}$ and that

$$\|(z\,\mathrm{id} - K^\star)^{-1} - (z\,\mathrm{id} - K^u)^{-1}\| \leqslant 2\|(z\,\mathrm{id} - K^\star)^{-1}\|^2 \|K^u - K^\star\| \ . \tag{7.6}$$

Thus the Riesz projection on the part of $\sigma(K^u)$ inside $\mathcal{C}$, given by

$$\Pi = \frac{1}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} (z\,\mathrm{id} - K^u)^{-1}\,\mathrm{d}z \ , \tag{7.7}$$

is well-defined. Using (7.3) and (7.6), we obtain

$$\begin{aligned}\|\mathrm{id} - \Pi\| &= \left\| \frac{1}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} (z\,\mathrm{id} - K^\star)^{-1} - (z\,\mathrm{id} - K^u)^{-1}\,\mathrm{d}z \right\| \\ &\leqslant \frac{1}{2\pi} \int_{\mathcal{C}} \|(z\,\mathrm{id} - K^\star)^{-1} - (z\,\mathrm{id} - K^u)^{-1}\|\,\mathrm{d}z \\ &\leqslant \frac{1}{\pi} \int_{\mathcal{C}} \|(z\,\mathrm{id} - K^\star)^{-1}\|^2\,\mathrm{d}z\,\|K^u - K^\star\| \,. \end{aligned} \tag{7.8}$$

If $\|\mathrm{id} - \Pi\| < 1$, since $\Pi$ is a projection, it follows that $\mathrm{id} - \Pi = 0$, and therefore $\sigma(K^u)$ is inside $\mathcal{C}$. Thus a second natural assumption to make in order to control the spectrum of $K^u$ is that

$$C := \frac{1}{\pi} \int_{\mathcal{C}} \|(z\,\mathrm{id} - K^\star)^{-1}\|^2\,\mathrm{d}z < \frac{1}{\|K^\star - K^u\|} \,. \tag{7.9}$$

This yields the following proposition.

**Proposition 7.2** ([35, Proposition 4.2])**.** *Let $\Omega$ be an open neighbourhood of $\sigma(K^\star)$. Then there exists $\epsilon > 0$ such that $\sigma(K^u) \subset \Omega$ for any operator $K^u$ with $\|K^\star - K^u\| < \epsilon$.*

More precisely, the above discussion shows that if

$$\|K^u - K^\star\| < \min\left\{ \frac{1}{2}\gamma, (C+1)^{-1} \right\} , \tag{7.10}$$

where $\gamma$ and $C$ are the quantities introduced in (7.5) and (7.9) and $\mathcal{C}$ is a Cauchy contour that separates a simple eigenvalue of $K^\star$ from the remaining part of its spectrum, then $\mathcal{C}$ also contains a simple eigenvalue for $K^u$. Before estimating the quantities $\gamma$ and $\mathcal{C}$, we need to study the spectrum of $K^\star$.

## 7.2 Estimation of the eigenvalues of $K^\star$

We are interested in the eigenvalues of the finite-rank kernel $K^\star$ given by

$$K^\star(x, \mathrm{d}y) = \sum_{i=1}^{k} \mathbb{1}_{\{x \in B_i\}} \int_{B_i} \mathring{\pi}_0^{B_i}(x_0)\; K^0(x_0, \mathrm{d}y)\,\mathrm{d}x_0 \,. \tag{7.11}$$

A kernel has finite rank whenever it can be written as a sum of a finite number of products of functions of its first argument alone by functions of its second argument alone. Because $K^\star$ has finite rank, we can associate to it a $k \times k$ matrix whose non-zero eigenvalues correspond to the non-zero eigenvalues of $K^\star$. Indeed, non-zero eigenvalues of $K^\star$ are solutions of the homogeneous Fredholm equation of the second kind

$$\lambda\phi(x) = \int_{\mathcal{M}_k} K^\star(x, \mathrm{d}y)\phi(y) \,. \tag{7.12}$$

Let us introduce the unknown constants

$$c_i = \int_{\mathcal{M}_k} \mathbb{P}^{\mathring{\pi}_0^{B_i}}\left\{ X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}x \right\} \phi(x) \tag{7.13}$$

which depend on the eigenfunction $\phi(x)$. It follows that

$$\lambda\phi(x) = \sum_{i=1}^{k} c_i \mathbb{1}_{\{x \in B_i\}} \,. \tag{7.14}$$

For $\lambda \neq 0$, inserting this expression in (7.12) we obtain

$$\sum_{i=1}^{k} \mathbb{1}_{\{x\in B_i\}} \left[ c_i - \frac{1}{\lambda} \int_{\mathcal{M}_k} \mathbb{P}^{\mathring{\pi}_0^{B_i}} \left\{ X_{\tau^+_{\mathcal{M}_k}} \in \mathrm{d}y \right\} \sum_{j=1}^{k} c_j \mathbb{1}_{\{y\in B_j\}} \right] = 0 \,. \tag{7.15}$$

Writing

$$P_{ij} = \mathbb{P}^{\mathring{\pi}_0^{B_i}} \left\{ X_{\tau^+_{\mathcal{M}_k}} \in B_j \right\} \tag{7.16}$$

and since $\left(\mathbb{1}_{\{x\in B_i\}}\right)_{1\leqslant i\leqslant k}$ is a set of linearly independent functions, we obtain the system of linear algebraic equations

$$\lambda c_i = \sum_{j=1}^{k} P_{ij} c_j, \qquad 1 \leqslant i \leqslant k \,. \tag{7.17}$$

It follows that non-zero eigenvalues of $K^\star$ correspond to the non-zero eigenvalues of the matrix $P$. For $0 \leqslant i \leqslant k-1$, we denote these eigenvalues $\lambda_i$.

Note that the matrix $P$ is a stochastic matrix and due to the Laplace transform condition (4.6), these eigenvalues should satisfy

$$\sup_{x\in\mathcal{M}_k^c} \mathbb{P}^x\{X_1 \in \mathcal{M}_k^c\} < |\lambda_i| \leqslant 1 \tag{7.18}$$

for all $0 \leqslant i \leqslant k-1$.

Let us examine the structure of the matrix $P$. Thanks to the large-deviations estimates of Proposition 3.1, elements on the main diagonal of $P$ are close to one, whereas off-diagonal elements are close to zero. In order to study a matrix where all elements are small we introduce $\hat{P} = \mathrm{id} - P$. Its diagonal elements are given by

$$\hat{P}_{ii} = \mathbb{P}^{\mathring{\pi}_0^{B_i}} \left\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_i \right\} = \mathbb{P}^{\mathring{\pi}_0^{B_i}} \left\{ X_{\tau^+_{\mathcal{M}_k}} \in \mathcal{M}_k \backslash B_i \right\} . \tag{7.19}$$

Our aim is now to derive spectral properties of the matrix $\hat{P}$. Such a problem has been studied by Wentzell [49] using $W$-graphs. Here we use a different approach based on block-triangularisation [6, Section 6.1], which also gives direct access to eigenfunctions. We write $\hat{P}$ in the form

$$\hat{P} = \begin{pmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{a} \end{pmatrix} \tag{7.20}$$

where $\hat{P}_{11} \in \mathbb{R}^{(k-1)\times(k-1)}$, $\hat{P}_{12} \in \mathbb{R}^{k-1}$, $\hat{P}_{21}^\top \in \mathbb{R}^{k-1}$ and $\hat{a} \in \mathbb{R}$. We want to prove that there exist matrices $S, T$ in $\mathbb{R}^{k\times k}$ of the form

$$S = \begin{pmatrix} \mathrm{id} & S_{12} \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & \alpha \end{pmatrix} \tag{7.21}$$

with the submatrices having the same dimensions as those of $\hat{P}$ and verifying

$$\hat{P}S = ST \,. \tag{7.22}$$

Following the argument of [6, Section 6.1], if we manage to prove that

$$\hat{P}_{11}S_{12} - S_{12}\hat{a} - S_{12}\hat{P}_{21}S_{12} + \hat{P}_{12} = 0 \tag{7.23}$$

admits a unique solution, it will follow that $\hat{P}$ is similar to the block-diagonal matrix $T$, and the eigenvalues of $\hat{P}$ are $\alpha$ and those of $T_{11}$. Note that $T_{11}, T_{21}$ and $\alpha$ are then given by

$$T_{11} = \hat{P}_{12} - S_{12}\hat{P}_{12} \ , \quad T_{21} = \hat{P}_{21} \ , \quad \alpha = \hat{a} + \hat{P}_{21}S_{12} \ . \tag{7.24}$$

The fact that (7.23) admits a unique solution will be proven using the Banach fixed point theorem. In the sequel, the matrix norm used is the sup-norm.

**Proposition 7.3.** *Introduce the notations*

$$b = \max_{1 \leqslant l \leqslant k-1} \mathbb{P}^{\mathring{\pi}_0^{B_l}}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_l \Big\} \ , \tag{7.25}$$

$$\hat{a} = \mathbb{P}^{\mathring{\pi}_0^{B_k}}\Big\{ X_{\tau^+_{\mathcal{M}_k}} \notin B_k \Big\} \neq 0 \ . \tag{7.26}$$

*For fixed blocks $\hat{P}_{11}$, $\hat{P}_{12}$, $\hat{P}_{21}$ and $\hat{a}$, if*

$$\frac{b}{\hat{a}} < \frac{1}{8} \tag{7.27}$$

*then* (7.23) *admits a unique solution. Moreover, this solution satisfies*

$$\|S^\star_{12}\| \leqslant 2\frac{\|\hat{P}_{12}\|}{\hat{a}} \ . \tag{7.28}$$

Proof: Let $\mathcal{B}$ be the ball $\mathcal{B} = \{\Xi \in \mathbb{R}^{k-1}, \|\Xi\| \leqslant 2\frac{\|\hat{P}_{12}\|}{\hat{a}}\} \subset \mathbb{R}^{k-1}$. We equip the Banach space $\mathbb{R}^{k-1}$ with the supremum norm, and define a map $\Phi : \mathcal{B} \to \mathcal{B}$ by

$$\Phi(\Xi) = \frac{1}{\hat{a}}\Big(\hat{P}_{12} + \hat{P}_{11}\Xi - \Xi\hat{P}_{21}\Xi\Big) \ . \tag{7.29}$$

Note that

$$\|\hat{P}_{11}\| \leqslant 2b, \ \|\hat{P}_{12}\| \leqslant b, \ \|\hat{P}_{21}\| = \hat{a} \ . \tag{7.30}$$

It is then straightforward to check that $\Phi$ is a contraction on $\mathcal{B}$. □

**Remark 7.4.** It follows from (7.23) that $S^\star_{12}$ satisfies

$$\begin{aligned} S^\star_{12} &= \left( \mathrm{id} - \frac{\hat{P}_{11}}{\hat{a}} + \frac{\hat{P}_{21}S^\star_{12}}{\hat{a}}\,\mathrm{id} \right)^{-1} \frac{\hat{P}_{12}}{\hat{a}} \\ &= \sum_{k \geqslant 0} \left( \frac{\hat{P}_{11}}{\hat{a}} - \frac{\hat{P}_{21}S^\star_{12}}{\hat{a}}\,\mathrm{id} \right)^k \frac{\hat{P}_{12}}{\hat{a}} \ . \end{aligned} \tag{7.31}$$

This yields a more precise estimate than the a priori estimate (7.28). In particular, at the first order, it follows

$$\left\| S^\star_{12} - \frac{\hat{P}_{12}}{\hat{a}} \right\| \leqslant \frac{4b/\hat{a}}{1 - 4b/\hat{a}} \frac{\|\hat{P}_{12}\|}{\hat{a}} \tag{7.32}$$

◊

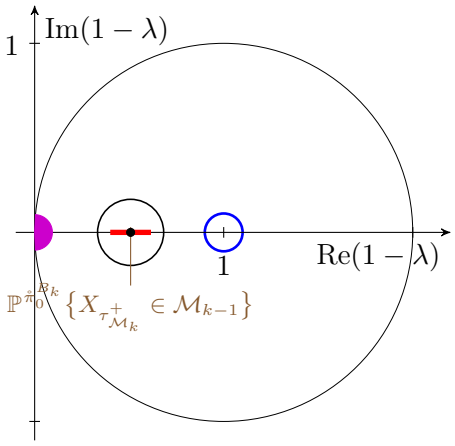


Figure 5: Sketch of the location of eigenvalues of $\hat{P} = \mathrm{id} - P$.

**Corollary 7.5.** *For $0 \leqslant i \leqslant k-1$, we denote $\lambda^\star_i$ the eigenvalues of $K^\star$ labelled by decreasing order. Then the smallest in modulus non-zero eigenvalue $\lambda^\star_{k-1}$ of $K^\star$ is real, simple and satisfies*

$$\left|\lambda^\star_{k-1} - \left(1 - \mathbb{P}^{\mathring{\pi}_0^{B_k}}\left\{X_{\tau^+_{\mathcal{M}_k}} \notin B_k\right\}\right)\right| \leqslant 2\|\hat{P}_{12}\| \leqslant 2b\,, \tag{7.33}$$

*Furthermore, for $0 \leqslant i \leqslant k-2$, the other non-zero eigenvalues of $K^\star$ satisfy*

$$|1 - \lambda^\star_i| \leqslant \|T_{11}\| \leqslant 4\|\hat{P}_{12}\| \leqslant 4b\,. \tag{7.34}$$

Proof: To each non-zero eigenvalue of $P$ corresponds a non-zero eigenvalue of $K^\star$. From Proposition 7.3, the biggest non-zero eigenvalue of $\mathrm{id} - P$, thus the smallest non-zero of $K^\star$, is real and positive and satisfies

$$\left|(1 - \lambda^\star_{k-1}) - \mathbb{P}^{\mathring{\pi}_0^{B_k}}\left\{X_{\tau^+_{B_k}} \notin B_k\right\}\right| \leqslant 2\|\hat{P}_{12}\| \leqslant 2b\,. \tag{7.35}$$

□

**Remark 7.6.** Note that $1 - \mathbb{P}^{\mathring{\pi}_0^{B_k}}\left\{X_{\tau^+_{B_k}} \notin B_k\right\}$ is the principal eigenvalue of kernel $K^\star_{B_k}$, (the process with kernel $K^\star$ killed upon leaving $B_k$). ◊

Thanks to the block-triangularisation, we also get an explicit expression for the eigenfunction associated to the smallest eigenvalue of $K^\star$.

**Lemma 7.7.** *Up to a multiplicative constant, the eigenfunction of $K^\star$ corresponding to the eigenvalue $\lambda^\star_{k-1}$ is given by*

$$\phi^\star_{k-1}(x) = \sum_{i=1}^{k-1} \mathbb{1}_{\{x \in B_i\}}(S^\star_{12})_i + \mathbb{1}_{\{x \in B_k\}} \tag{7.36}$$

*where $S^\star_{12}$ is a unique solution of* (7.23).

The proof is immediate since

$$\hat{P}\begin{pmatrix} S^\star_{12} \\ 1 \end{pmatrix} = (1 - \lambda^\star_{k-1})\begin{pmatrix} S^\star_{12} \\ 1 \end{pmatrix}\,. \tag{7.37}$$

### 7.3 Resolvent estimate of $K^\star$

We now want to estimate the quantities $C$ and $\gamma$ associated to the real and simple eigenvalue $\lambda^\star_{k-1}$ of $K^\star$, cf. (7.9) and (7.5). Thus we need an upper bound on the norm of the resolvent $\|(z\,\mathrm{id}-K^\star)^{-1}\|$ when $z$ is close to $\lambda^\star_{k-1}$. Note that for $z\neq 0$, the resolvent operator satisfies the resolvent equation

$$(z\,\mathrm{id}-K^\star)^{-1}=\frac{1}{z}\big(\mathrm{id}+K^\star(z\,\mathrm{id}-K^\star)^{-1}\big)\,. \tag{7.38}$$

Solving the Fredholm linear integral equation of the second kind,

$$\begin{aligned} z\phi(x)&=\varphi(x)+(K^\star\phi)(x)\\ &=\varphi(x)+\int_{\mathcal{M}_k}\sum_{i=1}^k \mathbb{1}_{\{x\in B_i\}}\mathbb{P}^{\mathring{\pi}_0^{B_i}}\Big\{X_{\tau^+_{\mathcal{M}_k}}\in \mathrm{d}y\Big\}\phi(y)\\ &=\varphi(x)+\sum_{i=1}^k \mathbb{1}_{\{x\in B_i\}}c_i \end{aligned} \tag{7.39}$$

and following the same procedure as for the homogeneous equation (7.12), we obtain the system of linear algebraic equations given for all $1\leqslant i\leqslant k$ by

$$zc_i-\sum_{i=1}^k P_{ij}c_j=\varphi_i:=\int_{\mathcal{M}_k}\mathbb{P}^{\mathring{\pi}_0^{B_i}}\Big\{X_{\tau^+_{\mathcal{M}_k}}\in \mathrm{d}y\Big\}\varphi(y)\,. \tag{7.40}$$

If $z$ is such that $\det(z\,\mathrm{id}-K^\star)\neq 0$, i.e. not an eigenvalue of $K^\star$, the system has the unique solution given by

$$c_i=\sum_{j=1}^k (z\,\mathrm{id}-P)^{-1}_{ij}\varphi_j,\qquad 1\leqslant i\leqslant k\,. \tag{7.41}$$

Inserting the previous result in (7.39), we see that

$$z\phi(x)=\varphi(x)+\int_{\mathcal{M}_k}\sum_{i=1}^k\sum_{j=1}^k \mathbb{1}_{\{x\in B_i\}}(z\,\mathrm{id}-P)^{-1}_{ij}\mathbb{P}^{\mathring{\pi}_0^{B_j}}\Big\{X_{\tau^+_{\mathcal{M}_k}}\in \mathrm{d}y\Big\}\varphi(y)\,, \tag{7.42}$$

from which it follows that the resolvent operator $(z\,\mathrm{id}-K^\star)^{-1}$ admits a resolvent kernel $R(z;x,\mathrm{d}y)$ given by

$$R(z;x,\mathrm{d}y)=\frac{1}{z}\Bigg[\mathrm{id}+\sum_{i=1}^k\sum_{j=1}^k \mathbb{1}_{\{x\in B_i\}}(z\,\mathrm{id}-P)^{-1}_{ij}\mathbb{P}^{\mathring{\pi}_0^{B_j}}\Big\{X_{\tau^+_{\mathcal{M}_k}}\in \mathrm{d}y\Big\}\Bigg]\,. \tag{7.43}$$

Since $R=\frac{1}{z}\{\mathrm{id}+(RK^\star)\}$ and thanks to (7.38), it follows that the resolvent kernel of the resolvent operator $(z\,\mathrm{id}-K^\star)^{-1}$ is also given by

$$R(z;x,\mathrm{d}y)=\sum_{i=1}^k\sum_{j=1}^k \mathbb{1}_{\{x\in B_i\}}(z\,\mathrm{id}-P)^{-1}_{ij}\mathbb{P}^{\mathring{\pi}_0^{B_j}}\Big\{X_{\tau^+_{\mathcal{M}_k}}\in \mathrm{d}y\Big\}\,. \tag{7.44}$$

We are now able to bound the resolvent of $K^\star$. Let $\mathcal{C}$ be the contour defined by

$$\big\{z\in\mathbb{C}\colon\ |z-\lambda_k|=r\big\}\,, \tag{7.45}$$

and let us assume that

$$r < \frac{\hat{a}}{4} < \frac{\hat{a} - 6b}{2} \ . \tag{7.46}$$

Then have the following resolvent estimate.

**Proposition 7.8.** *Recall that we denote $\lambda^\star_{k-1}$ the smallest non-zero eigenvalue of the kernel $K^\star$. There exists a numerical constant $c_1 > 0$, independent of $\sigma$, such that for all $z \in \mathcal{C}$*

$$\|(z\,\mathrm{id} - K^\star)^{-1}\| < c_1 \big(z - \lambda^\star_{k-1}\big)^{-1} \ . \tag{7.47}$$

Proof: Note that we have equality between the supremum norms of the resolvent of the operator $K^\star$ and of the matrix $P$, i.e.

$$\|(z\,\mathrm{id} - K^\star)^{-1}\| = \|(z\,\mathrm{id} - P)^{-1}\| \ . \tag{7.48}$$

Let us now derive an upper bound on $\|(z\,\mathrm{id} - P)^{-1}\|$ for $z \in \mathcal{C}$. Thanks to the block-triangularisation (7.22), we get

$$\begin{aligned} \|(z\,\mathrm{id} - P)^{-1}\| &= \left\| \left( (1-z)\,\mathrm{id} - \hat{P} \right)^{-1} \right\| \\ &\leqslant \|S\| \|S^{-1}\| \|((1-z)\,\mathrm{id} - T)^{-1}\| \ . \end{aligned} \tag{7.49}$$

Since $\|S\| = \|S^{-1}\| = 1 + \|S_{12}\|$, we can bound $\|S\|\|S^{-1}\|$ thanks to (7.28). We also have an explicit expression for $((1-z)\,\mathrm{id} - T)^{-1}$, given by

$$\begin{aligned} &\big[(1-z)\,\mathrm{id} - T\big]^{-1} \\ &\quad = \begin{pmatrix} \mathrm{id} & 0 \\ T_{21}(1 - \lambda^\star_{k-1})^{-1} & 1 \end{pmatrix} \begin{pmatrix} \big[(1-z)\,\mathrm{id} - T_{11}\big]^{-1} & 0 \\ 0 & \big[(1-z) - (1 - \lambda^\star_{k-1})\big]^{-1} \end{pmatrix} \ . \end{aligned} \tag{7.50}$$

Since $|1 - z| > \|T_{11}\|$ for all $z \in \mathcal{C}$, we have the classical bound

$$\|((1-z) - T_{11})^{-1}\| \leqslant \frac{1}{|1-z| - \|T_{11}\|} \ . \tag{7.51}$$

For all $r < \frac{\hat{a} - 6b}{2}$, we finally get

$$\|(z\,\mathrm{id} - P)^{-1}\| \leqslant \frac{1}{\big|z - \lambda^\star_{k-1}\big|} \left(1 + 2\frac{b}{\hat{a}}\right)^2 \left(1 + \frac{\hat{a}}{\hat{a} - 6b - r}\right) . \tag{7.52}$$

The result is then immediate since

$$\|(z\,\mathrm{id} - P)^{-1}\| \leqslant 9 \left(1 + \frac{1}{4}\right)^2 \big|z - \lambda^\star_{k-1}\big|^{-1} \ . \tag{7.53}$$

□

We can now estimate $C$. Since the resolvent kernel is bounded, the M-L inequality yields the upper bound

$$C = \frac{1}{\pi} \int_{\mathcal{C}} \|(z\,\mathrm{id} - K^\star)^{-1}\|^2 \,\mathrm{d}z \leqslant 2r \|(z\,\mathrm{id} - K^\star)^{-1}\|^2 \ . \tag{7.54}$$

It follows that

$$\epsilon = \min\left\{\frac{1}{2}\gamma, (C+1)^{-1}\right\} \geqslant \frac{r}{396} \,. \tag{7.55}$$

Thus, if

$$\|K^u - K^\star\| < \frac{r}{396} \,, \tag{7.56}$$

we have the desired result for the approximation of the eigenvalues of $K^u$, where $K^u$ is seen as an approximation of $K^\star$. In Section 9, we will check that this inequality and (7.46) are indeed satisfied.

# 8 Sample path estimates

## 8.1 Stochastic differential equations with coexistence of periodic orbits

In this section, if $x \in \mathbb{R}^d$ then $\|x\|$ denotes the Euclidean norm of $x$. We assume that the deterministic system (2.1) admits $N$ stable periodic orbits and $N^+$ unstable periodic orbits, i.e., there are periodic functions $\gamma_i^- : \mathbb{R} \to \mathcal{D}$ of respective periods $T_i$ such that

$$\dot{\gamma}_i(t) = f(\gamma_i(t)) \qquad \forall t \in \mathbb{R} \tag{8.1}$$

for $1 \leqslant i \leqslant N$, and there are periodic functions $\gamma_i^+ : \mathbb{R} \to \mathcal{D}$ of respective periods $T_i^+$ such that

$$\dot{\gamma}_i^+(t) = f\big(\gamma_i^+(t)\big) \qquad \forall t \in \mathbb{R} \,. \tag{8.2}$$

In what follows, we study the behaviour of the system in the neighbourhood of a stable or unstable periodic orbit. Adapting [9, Proposition 2.1] and [9, Proposition 3.3] to the multidimensional case, it follows that the SDE (2.5) can be written in polar-type coordinates as shown in the following proposition.

**Proposition 8.1.** *There exists a change of coordinates such that in a neighbourhood of a stable periodic orbit (i.e., for $(x, \varphi)$ such that $\|x\|$ is small enough), the SDE takes the form*

$$\begin{aligned} \mathrm{d}x_t &= (-\Lambda x_t + b_x(x_t, \varphi_t))\mathrm{d}t + \sigma g_x(x_t, \varphi_t)\mathrm{d}W_t \,, \\ \mathrm{d}\varphi_t &= \left(\frac{1}{T_i} + b_\varphi(x_t, \varphi_t)\right)\mathrm{d}t + \sigma g_\varphi(x_t, \varphi_t)\mathrm{d}W_t \,, \end{aligned} \tag{8.3}$$

*where $\Lambda$ is a triangular matrix with positive diagonal elements corresponding to the Lyapunov exponents of the stable orbit, $b_x$, $b_\varphi$, $g_x$, $g_\varphi$ are periodic in $\varphi$ with period 1, and the nonlinear drift terms satisfy $\|b_x(x,\varphi)\|, |b_\varphi(x,\varphi)| = \mathcal{O}(\|x\|^2)$.*

Note that we can choose $\Lambda$ to be in Jordan canonical form, and that in these variables the $B_i$ can be taken to be balls $\{x \colon \|x\| \leqslant \delta\}$. In the neighbourhood of an unstable periodic orbit, we have the following similar result.

**Proposition 8.2.** *There exists a change of coordinates such that in a neighbourhood of an unstable periodic orbit (i.e., for $(x, \varphi)$ such that$\|x\|$ is small enough), the SDE takes the form*

$$\begin{aligned} \mathrm{d}x_t &= \left(\begin{pmatrix} -\Lambda^- & 0 \\ 0 & \Lambda^+ \end{pmatrix} x_t + b_x(x_t, \varphi_t)\right)\mathrm{d}t + \sigma g_x(x_t, \varphi_t)\mathrm{d}W_t \,, \\ \mathrm{d}\varphi_t &= \left(\frac{1}{T_i} + b_\varphi(x_t, \varphi_t)\right)\mathrm{d}t + \sigma g_\varphi(x_t, \varphi_t)\mathrm{d}W_t \,, \end{aligned} \tag{8.4}$$

*where $\Lambda^-$ is a triangular matrix with positive diagonal elements, $\Lambda^+$ is a triangular matrix with non-negative diagonal elements and at least one strictly positive diagonal element, corresponding to the Lyapunov exponents, $b_x$, $b_\varphi$, $g_x$, $g_\varphi$ are periodic in $\varphi$ with period 1, and the nonlinear drift terms satisfy $\|b_x(x,\varphi)\|, |b_\varphi(x,\varphi)| = \mathcal{O}(\|x\|^2)$.*

Proof of Propositions 8.1 and 8.2. Using the time parametrisation proposed in [9], i.e., setting $\Gamma_i(\varphi) = \gamma_i(T_i\varphi)$ so that $\varphi \in \mathbb{R}/\mathbb{Z}$ ($\varphi$ parametrises time), and Itô's formula, the stochastic differential equation (2.5) is equivalent, thanks to the transformation of Proposition B.1, to a system of the form

$$\begin{aligned} \mathrm{d}x_t &= f_x(x_t, \varphi_t, \sigma)\mathrm{d}t + \sigma g_x(x_t, \varphi_t)\mathrm{d}W_t \\ \mathrm{d}\varphi_t &= f_\varphi(x_t, \varphi_t, \sigma)\mathrm{d}t + \sigma g_\varphi(x_t, \varphi_t)\mathrm{d}W_t \, . \end{aligned} \tag{8.5}$$

As noted in [9], a drawback of this system is that the drift term $f_x$ does not vanish in $x = 0$. We use a similar argument as in [9, Proposition 3.3] to obtain the desired form. □

## 8.2 General estimates

We consider the system in continuous time describing the dynamics near a periodic orbit in the polar-type coordinates (8.3) or (8.4). We first recall a result proved in [9] which shows that $\varphi_t$ does not differ much from $t/T_i$ on rather long timescales. Given $T, H > 0$, we introduce two stopping times by

$$\begin{aligned} \tilde\tau_H &= \inf\{t > 0 : \|x_t\| \geqslant H\} \, , \\ \tilde\tau_\varphi &= \inf\left\{ t > 0 : \left| \varphi_t - \frac{t}{T_i} \right| \geqslant M\Big( H^2 t + \sqrt{H^3 T} \Big) \right\} . \end{aligned} \tag{8.6}$$

Then [9, Proposition 6.3] gives us the following result.

**Proposition 8.3** (Control of the diffusion along $\varphi$)**.** *There is a constant $C_1$, depending only on the ellipticity constants of the diffusion terms, such that*

$$\mathbb{P}^{(x,0)}\{\tilde\tau_\varphi < \tilde\tau_H \wedge T\} \leqslant \mathrm{e}^{-H/(C_1\sigma^2)} \tag{8.7}$$

*holds for all $T, \sigma, H > 0$ and all $x$ with $\|x\| < H$.*

The following result bounds the probability to escape from one of the metastable neighbourhood $B_i$.

**Proposition 8.4.** *There exist $C > 0$ and $\kappa > 0$ such that for all $x \in B_i \subset \mathcal{M}_k$,*

$$\mathbb{P}^x\{X_1 \notin \mathcal{M}_k\} \leqslant C\, \mathrm{e}^{-\kappa |B_i|/\sigma^2} \, , \tag{8.8}$$

*where $|B_i|$ denotes the radius of the ball $B_i$.*

Proof: We introduce the continuous stopping time $\tilde\tau_\Sigma = \inf\{t > 0 \colon \varphi_t > 1\}$ corresponding to the first return time to the Poincaré map. Then for any initial condition $(x_0, 0)$ with $x_0 \in B_i$,

$$\mathbb{P}^{x_0}\{X_1 \notin \mathcal{M}_k\} = \mathbb{P}\big\{ \|x_{\tilde\tau_\Sigma}^{(x_0,0)}\| > |B_i| \big\} \, . \tag{8.9}$$

Introducing a second sample path starting on the $i^{\text{th}}$ stable periodic at time 0, i.e. started in $(0,0)$, and driven by the same Brownian motion, we can use an upper bound on the

probability that the two sample paths do not approach each other exponentially fast to bound (8.9). Indeed, for $\varrho \in (0,1)$,

$$\begin{aligned}\mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(x_0,0)}\| > |B_i|\big\} &\leqslant \mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(0,0)}\| > (1-\varrho)\,|B_i|\big\} \\ &\quad + \mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(x_0,0)} - x_{\tilde\tau_\Sigma}^{(0,0)}\| > \varrho\,|B_i|\big\}\;. \end{aligned} \tag{8.10}$$

Adapting [9, Proposition 6.12], we obtain the existence of $c_0 > 0$ and $\varrho < 1$ such that the second term on the right-hand side, corresponding to the difference of the two sample paths, is bounded by

$$\mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(x_0,0)} - x_{\tilde\tau_\Sigma}^{(0,0)}\| > \varrho\,|B_i|\big\} \leqslant \mathrm{e}^{-c_0|B_i|/\sigma^2}\;. \tag{8.11}$$

In order to apply Proposition 8.3 to bound the first term on the right-hand side of (8.10), we decompose

$$\begin{aligned}\mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(0,0)}\| > (1-\varrho)\,|B_i|\big\} &\leqslant \mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(0,0)}\| > (1-\varrho)\,|B_i|\,, \tilde\tau_\varphi > \tilde\tau_H \wedge T\big\} \\ &\quad + \mathbb{P}^{(0,0)}\{\tilde\tau_\varphi < \tilde\tau_H \wedge T\}\;, \end{aligned} \tag{8.12}$$

where we will choose $T = 2T_i$ and $H = (1-\varrho)\,|B_i|$. Note that the solution of (8.3) with initial condition $(0,0)$ can be written as

$$x_t^{(0,0)} = \int_0^t \mathrm{e}^{-\Lambda(t-s)}\, b_x\big(x_s^{(0,0)}, \varphi_s^{(0,0)}\big)\,\mathrm{d}s + \sigma\int_0^t \mathrm{e}^{-\Lambda(t-s)}\, g_x\big(x_s^{(0,0)}, \varphi_s^{(0,0)}\big)\,\mathrm{d}W_s\;. \tag{8.13}$$

To bound the first term on the right-hand side of (8.12), observe that on $\{\tilde\tau_\varphi > \tilde\tau_H \wedge T\}$ we have $\tilde\tau_\Sigma < 2T_i$ and thus

$$\mathbb{P}\big\{\|x_{\tilde\tau_\Sigma}^{(0,0)}\| > (1-\varrho)\,|B_i|\,, \tilde\tau_\varphi > \tilde\tau_H \wedge T\big\} \leqslant \mathbb{P}\Big\{\sup_{0\leqslant s\leqslant 2T_i}\|x_s^{(0,0)}\| > (1-\varrho)\,|B_i|\Big\}\;. \tag{8.14}$$

Using a Bernstein inequality and a partition of the interval $[0, 2T_i]$, as in [8, Theorem 5.1.18] or [11, Proposition 3.3], we can show that there exist $C_0, \kappa_0 > 0$ such that

$$\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant 2T_i}\|x_s^{(0,0)}\| \geqslant (1-\varrho)\,|B_i|\Big\} \leqslant C_0\,\mathrm{e}^{-\kappa_0|B_i|/\sigma^2}\;. \tag{8.15}$$

Using Proposition 8.3 to bound the second term on the right-hand side of (8.12), we obtain the result. □

We also need to bound the probability of staying close to an unstable periodic orbit. Let $\mathcal{U} \subset \Sigma$ be a union of neighbourhoods of size $\delta$ of the unstable periodic orbits, with $\delta$ of order 1, and let $\mathcal{S} \subset \mathcal{U}$ be a union of neighbourhoods of size $h = \sigma^{3/4}$ of the unstable periodic orbits on the Poincaré section.

**Proposition 8.5.** *Let $h = \sigma^{3/4}$ and $\tilde\tau_{\mathcal{S}^c} = \inf\{t > 0 : \|x_t\| = h\}$. There exists a constant $C_2$ such that for any $x$ such that $\|x\| < h$ and $0 < T \leqslant 1/h$,*

$$\mathbb{P}^{(x,0)}\{\tilde\tau_{\mathcal{S}^c} > T, \tilde\tau_\varphi > \tilde\tau_{\mathcal{S}^c} \wedge T\} \leqslant C_2\sigma^{1/2}\;. \tag{8.16}$$

PROOF: We introduce the stopping time

$$\tilde{\tau}_{h^+} = \inf\bigl\{t > 0 : \|x_t^+\| = h\bigr\} \tag{8.17}$$

where $x^+$ corresponds to the coordinates with positive Lyapunov exponents. Note that

$$\mathbb{P}^{(x,0)}\{\tilde{\tau}_{\mathcal{S}^c} > T, \tilde{\tau}_\varphi > \tilde{\tau}_{\mathcal{S}^c} \wedge T\} \leqslant \mathbb{P}^{(x,0)}\{\tilde{\tau}_{h^+} > T, \tilde{\tau}_\varphi > \tilde{\tau}_{\mathcal{S}^c} \wedge T\} \; . \tag{8.18}$$

On $\{\tilde{\tau}_\varphi > \tilde{\tau}_{\mathcal{S}^c} \wedge T\}$, $\varphi_t$ is close to $t/T_i$, hence the equation for $x_t^+$ can be written

$$\mathrm{d}x_t^+ = \bigl(\Lambda^+ x_t^+ + b_{x^+}(x_t, \varphi_t)\bigr)\,\mathrm{d}t + \sigma(g_0(t) + g_1(x_t, \varphi_t, t))\,\mathrm{d}W_t \tag{8.19}$$

where $g_0(t) = g_{x^+}(0, t/T_i)$ and $g_1 = \mathcal{O}(\|x\| + h)$. The solution can be expressed as

$$x_t^+ = \mathrm{e}^{\Lambda^+ t}\biggl\{\sigma\int_0^t \mathrm{e}^{-\Lambda^+ s}\, g_0(s)\,\mathrm{d}W_s + \sigma\int_0^t \mathrm{e}^{-\Lambda^+ s}\, g_1(x_s, \varphi_s, s)\,\mathrm{d}W_s + \int_0^t \mathrm{e}^{-\Lambda^+ s}\, b_{x^+}(x_s, \varphi_s)\,\mathrm{d}s\biggr\} \; . \tag{8.20}$$

The proof is then similar to [8, Theorem 3.2.2]. □

The following proposition will allow us to extend the previous estimate to an exit from the larger set $\mathcal{U}$. We denote $\mathcal{U}\backslash\mathcal{S}$ by $\mathcal{K}$.

**Proposition 8.6.** *Let $\tilde{\tau}_{\mathcal{K}^c} = \inf\{t > 0 \colon x_t \notin \mathcal{K}\}$. There exists a constant $\kappa_2 > 0$ such that for any initial condition $(x, 0) \in \mathcal{K}$,*

$$\mathbb{P}^{(x,0)}\{\tilde{\tau}_{\mathcal{K}^c} > t\} \leqslant \mathrm{e}^{-\kappa_2 t/\log(\sigma^{-1})} \; . \tag{8.21}$$

*Furthermore, if $\tilde{\tau}_{\mathcal{U}^c} = \inf\{t > 0 \colon x_t \notin \mathcal{U}\}$ and $\tilde{\tau}_{\mathcal{S}} = \inf\{t > 0 \colon x_t \in \mathcal{S}\}$, for any $T_0 > 0$ there exists a constant $\kappa_3 > 0$ such that*

$$\mathbb{P}^{(x,0)}\{T_0 \leqslant \tilde{\tau}_{\mathcal{S}} < \tilde{\tau}_{\mathcal{U}^c}\} \leqslant \mathrm{e}^{-\kappa_3/\sigma^{1/2}} \; . \tag{8.22}$$

PROOF: First, note that $\{\tilde{\tau}_{\mathcal{K}^c} > t\} \subset \bigl\{\|x_T^+\| < \delta\bigr\}$. Assume that the unstable periodic orbit admits $m^+$ positive Lyapunov exponents. We introduce the Lyapunov function

$$U_t = \sum_{i=1}^{m^+} (x_{t,i}^+)^2 \; . \tag{8.23}$$

Applying Itô's formula we obtain

$$\mathrm{d}U_t = \Bigl\{\sum_{i=1}^{m^+} \lambda_i^+ (x_{t,i}^+)^2 + \beta(x_t, \varphi_t)\Bigr\}\,\mathrm{d}t + \sigma\sum_{i=1}^{m^+} g_{x,i}(x_t, \varphi_t)\,\mathrm{d}W_t^i \tag{8.24}$$

where $\beta(x_t, \varphi_t) \leqslant M\bigl((U_t)^{3/2} + \sigma^2\bigr)$. The proof is then similar to the proof of [10, Proposition D.4]. Indeed, the drift term is bounded below by a constant times $U_t$, and $\bigl\{\|x_T^+\| < \delta\bigr\} \subset \bigl\{U_T < m^+\delta^2/2\bigr\}$. Using an endpoint estimate and the Markov property to restart the process at times which are multiples of $\log(\sigma^{-1})$, we obtain (8.21). The estimate (8.22) is obtained by bounding the probability that $U_t$ leaves a neighbourhood of size $\sigma^{3/4}$ around an exponentially growing term, similarly to [10, Proposition D.7]. □

**Proposition 8.7.** *For all $x \in (\mathcal{U} \cup \mathcal{M}_N)^c$, there exist constants $C_1, \kappa_1 > 0$ such that*

$$\mathbb{P}^x\Big\{\tau_{\mathcal{U}}^+ < \tau_{\mathcal{M}_N}^+\Big\} \leqslant C_1\, \mathrm{e}^{-\kappa_1/\sigma^2} \ . \tag{8.25}$$

Proof: Consider a deterministic solution $z_t^{\mathrm{det}} = \big(x_t^{\mathrm{det}}, \varphi_t^{\mathrm{det}}\big)$ with initial condition $z_0 = (x, 0)$. Since $\partial\mathcal{U}$ is at distance of order 1 of any unstable periodic orbit, and because the stable periodic orbits are the only attractive limit sets (Assumption 2.2), $z_t^{\mathrm{det}}$ will reach a neighbourhood of a stable periodic orbit in a time $T$ of order 1. Using [8, Theorem 5.1.18], it follows that for $t \geqslant 0$,

$$\mathbb{P}^{(x,0)}\Big\{\sup_{0\leqslant s\leqslant t} \|z_s - z_s^{\mathrm{det}}\| > h_0\Big\} \leqslant C_0(1+t)\, \mathrm{e}^{-\kappa_0 h_0^2/\sigma^2} \tag{8.26}$$

for some constants $C_0, \kappa_0 > 0$. Note that the estimate holds for $h_0 \leqslant h_1/\chi(t)$, where $h_1$ is another constant and $\chi(t)$ is related to the local Lyapunov exponent of $z_t^{\mathrm{det}}$. Since $z_t^{\mathrm{det}}$ is attracted by the stable orbit, there exists $M_0 > 0$ such that $\chi(T) \leqslant 1 + M_0 T$. Applying (8.26) with $h_0 = |B_i|/2$, we find that any sample path which does not reach $\mathcal{U}$ before time $T$ will hit $B_i$ with high probability. □

## 8.3 Mean return time estimates

The following two lemmas are useful to bound expectations of first return times.

**Lemma 8.8.** *For any $A \subset \Sigma$, $n_0 \in \mathbb{N}$ and $x \in \Sigma$, the expectation of the first return time to $A$ satisfies*

$$\mathbb{E}^x\big\{\tau_A^+\big\} \leqslant \frac{n_0 \mathbb{P}^x\big\{\tau_A^+ \geqslant n_0\big\}}{1 - \mathbb{P}^{A^c}\big\{\tau_A^+ \geqslant n_0\big\}} \ . \tag{8.27}$$

Proof: Using the Markov property, we decompose the expectation as

$$\begin{aligned}
\mathbb{E}^x\big\{\tau_A^+\big\} &= \sum_{i\geqslant 0} \sum_{n=1}^{n_0} \mathbb{P}^x\big\{\tau_A^+ \geqslant i n_0 + n\big\} \\
&\leqslant n_0 \sum_{i\geqslant 0} \mathbb{P}^x\big\{\tau_A^+ \geqslant (i+1) n_0\big\} \\
&\leqslant n_0 \sum_{i\geqslant 0} \mathbb{P}^x\big\{\tau_A^+ \geqslant n_0\big\}\big(\mathbb{P}^{A^c}\big\{\tau_A^+ \geqslant n_0\big\}\big)^i
\end{aligned} \tag{8.28}$$

which gives the result by summing a geometric series. □

The next lemma is inspired by results in [13].

**Lemma 8.9.** *For any $A, B, C \subset \Sigma$,*

$$\mathbb{E}^A\big\{\tau_B^+\big\} \leqslant \mathbb{E}^A\big\{\tau_{B\cup C}^+\big\} + \mathbb{P}^A\big\{\tau_C^+ < \tau_B^+\big\}\mathbb{E}^C\big\{\tau_B^+\big\} \ . \tag{8.29}$$

Proof: Splitting the expectation according to the event $\big\{\tau_B^+ < \tau_C^+\big\}$ or $\big\{\tau_C^+ < \tau_B^+\big\}$ and then using the strong Markov property, we obtain

$$\begin{aligned}
\mathbb{E}^x\big\{\tau_B^+\big\} &= \mathbb{E}^x\Big\{\tau_B^+ \mathbb{1}_{\{\tau_B^+ < \tau_C^+\}}\Big\} + \mathbb{E}^x\Big\{\tau_B^+ \mathbb{1}_{\{\tau_C^+ < \tau_B^+\}}\Big\} \\
&= \mathbb{E}^x\Big\{\tau_B^+ \mathbb{1}_{\{\tau_B^+ < \tau_C^+\}}\Big\} + \mathbb{E}^x\Big\{\big[(\tau_B^+ - \tau_C^+) + \tau_C^+\big]\mathbb{1}_{\{\tau_C^+ < \tau_B^+\}}\Big\} \\
&= \mathbb{E}^x\big\{\tau_{B\cup C}^+\big\} + \mathbb{E}^x\Big\{(\tau_B^+ - \tau_C^+)\mathbb{1}_{\{\tau_C^+ < \tau_B^+\}}\Big\} \\
&\leqslant \mathbb{E}^x\big\{\tau_{B\cup C}^+\big\} + \mathbb{P}^A\big\{\tau_C^+ < \tau_B^+\big\}\mathbb{E}^C\big\{\tau_B^+\big\} \ ,
\end{aligned} \tag{8.30}$$

which gives the result by taking the supremum for $x \in A$. □

**Corollary 8.10.** *For $\mathcal{U}$ as defined in Section 8.2,*

$$\mathbb{E}^{\mathcal{M}_N^c}\{\tau^+_{\mathcal{M}_N}\} \leqslant \frac{\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{U}^c}\} + \mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{U}\cup\mathcal{M}_N}\}}{1 - \mathbb{P}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{U}} < \tau^+_{\mathcal{M}_N}\}} \,. \tag{8.31}$$

Proof: For all $x \in \mathcal{M}_N^c$,

$$\mathbb{E}^{x}\{\tau^+_{\mathcal{M}_N}\} \leqslant \max\Big\{\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{M}_N}\}, \mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{M}_N}\}\Big\} \,. \tag{8.32}$$

Applying Lemma 8.9 with $A = \mathcal{U}$, $B = \mathcal{M}_N$ and $C = (\mathcal{U}\cup\mathcal{M}_N)^c$, we obtain

$$\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{M}_N}\} \leqslant \mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{U}^c}\} + \mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{M}_N}\} \,, \tag{8.33}$$

whereas taking $A = (\mathcal{U}\cup\mathcal{M}_N)^c$, $B = \mathcal{M}_N$ and $C = \mathcal{U}$, we get

$$\mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{M}_N}\} \leqslant \mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{M}_N\cup\,\mathcal{U}}\} + \mathbb{P}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{U}} < \tau^+_{\mathcal{M}_N}\}\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{M}_N}\} \,. \tag{8.34}$$

Combining these two bounds, we obtain

$$\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{M}_N}\} \leqslant \mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{U}^c}\} + \mathbb{E}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{M}_N\cup\,\mathcal{U}}\} + \mathbb{P}^{(\mathcal{U}\cup\mathcal{M}_N)^c}\{\tau^+_{\mathcal{U}} < \tau^+_{\mathcal{M}_N}\}\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{M}_N}\} \,, \tag{8.35}$$

which yields (8.31). □

In order to bound the expected value $\mathbb{E}^{\mathcal{U}}\{\tau^+_{\mathcal{U}^c}\}$, we will again use Lemma 8.9 with two neighbourhoods of an unstable periodic orbit. First, we show that the sample paths are likely to leave the small neighbourhood $\mathcal{S}$ of the unstable periodic orbit (of size $h = \sigma^{3/4}$), then as soon as paths have left $\mathcal{S}$, the drift term will make it easier to escape from the larger neighbourhood $\mathcal{U}$. Using similar arguments as in the proof of Corollary 8.10, we obtain the following result.

**Lemma 8.11.** *For $\mathcal{S} \subset \mathcal{U}$ and $\mathcal{K} = \mathcal{U} \setminus \mathcal{S}$, as defined in Section 8.2, and all $x \in \mathcal{U}$,*

$$\mathbb{E}^{x}\{\tau^+_{\mathcal{U}^c}\} \leqslant \frac{\mathbb{E}^{\mathcal{K}}\{\tau^+_{\mathcal{K}^c}\} + \mathbb{E}^{\mathcal{S}}\{\tau^+_{\mathcal{S}^c}\}}{1 - \mathbb{P}^{\mathcal{K}}\{\tau^+_{\mathcal{S}} < \tau^+_{\mathcal{U}^c}\}} \,. \tag{8.36}$$

The different expected values involved in (8.36) will be bounded using Lemma 8.8 and results from Section 8.2.

**Proposition 8.12.** *There exist constants $M_1, \kappa > 0$ such that*

$$\begin{aligned} \mathbb{E}^{\mathcal{S}}\{\tau^+_{\mathcal{S}^c}\} &\leqslant M_1\sigma^{1/2} \,, \\ \mathbb{E}^{\mathcal{K}}\{\tau^+_{\mathcal{K}^c}\} &\leqslant M_1 \log(\sigma^{-1}) \,, \\ \mathbb{P}^{\mathcal{K}}\{\tau^+_{\mathcal{S}} < \tau^+_{\mathcal{U}^c}\} &\leqslant \mathrm{e}^{-\kappa/\sigma^{1/2}} \,. \end{aligned} \tag{8.37}$$

Proof: Recall that $\tilde\tau_{\mathcal{S}^c} = \{\inf t > 0 : \|x_t\| \geqslant h\}$, and let $n_0 > T/T_i + \sigma^{3/4}$, with $0 < T \leqslant 1/\sigma^{3/4}$. For all $x \in \mathcal{S}$,

$$\begin{aligned} \mathbb{P}^{x}\{\tau^+_{\mathcal{S}^c} > n_0\} &\leqslant \mathbb{P}^{(x,0)}\{\tilde\tau_\varphi < \tilde\tau_{\mathcal{S}^c} \wedge T\} + \mathbb{P}^{(x,0)}\{\tilde\tau_\varphi > \tilde\tau_{\mathcal{S}^c} \wedge T, \tau^+_{\mathcal{S}^c} > n_0\} \\ &\leqslant \mathbb{P}^{(x,0)}\{\tilde\tau_\varphi < \tilde\tau_{\mathcal{S}^c} \wedge T\} + \mathbb{P}^{(x,0)}\{\tilde\tau_\varphi > \tilde\tau_{\mathcal{S}^c} \wedge T, \tau^+_{\mathcal{S}^c} > n_0, \tilde\tau_{\mathcal{S}^c} < T\} \\ &\quad + \mathbb{P}^{(x,0)}\{\tilde\tau_\varphi > \tilde\tau_{\mathcal{S}^c} \wedge T, \tilde\tau_{\mathcal{S}^c} > T\} \,. \end{aligned} \tag{8.38}$$

However since $n_0 > T/T_i + \sigma^{3/4}$,

$$\mathbb{P}^{(x,0)}\big\{\tilde{\tau}_\varphi > \tilde{\tau}_{\mathcal{S}^c} \wedge T, \tau^+_{\mathcal{S}^c} > n_0, \tilde{\tau}_{\mathcal{S}^c} < T\big\} = 0 \; . \tag{8.39}$$

We obtain the bound on $\mathbb{E}^{\mathcal{S}}\big\{\tau^+_{\mathcal{S}^c}\big\}$ using (8.16) and applying Lemma 8.8. The two other bounds follow in a similar way, using Proposition 8.6. □

Combining the last three results with Proposition 8.7, we immediately get:

**Corollary 8.13.** *There exists a constant $M_2 > 0$ such that*

$$\mathbb{E}^{\mathcal{M}_N^c}\big\{\tau^+_{\mathcal{M}_N}\big\} \leqslant M_2 \log(\sigma^{-1}) \; . \tag{8.40}$$

We are now going to estimate the mean return time $\mathbb{E}^x\big\{\tau^+_{\mathcal{M}_k}\big\}$ for $x \in \mathcal{M}_k$. This estimate is needed to bound the norm of the difference between $K^u$ and $K^\star$ in Proposition 4.5. By decreasing induction on $k$ ($1 \leqslant k \leqslant N$), we can prove that for all $x \in \mathcal{M}_k$, the expectation $\mathbb{E}^x\big\{\tau^+_{\mathcal{M}_k}\big\}$ is exponentially close to one. We start by estimating the expectation of the first return time to $\mathcal{M}_N$.

**Lemma 8.14.** *For all $x \in \mathcal{M}_N$,*

$$\mathbb{E}^x\big\{\tau^+_{\mathcal{M}_N} - 1\big\} \leqslant \mathbb{P}^x\{X_1 \notin \mathcal{M}_N\}\mathbb{E}^{\mathcal{M}_N^c}\big\{\tau^+_{\mathcal{M}_N}\big\} \; . \tag{8.41}$$

Proof: Splitting the expectation according to the location of $X_1$, we have

$$\mathbb{E}^x\big\{\tau^+_{\mathcal{M}_N}\big\} \leqslant 1 + \mathbb{P}^x\{X_1 \notin \mathcal{M}_N\}\mathbb{E}^{\mathcal{M}_N^c}\big\{\tau^+_{\mathcal{M}_N}\big\} \; . \tag{8.42}$$

□

**Lemma 8.15.** *For all $k < N$, for all $x \in B_i \subset \mathcal{M}_k$,*

$$\mathbb{E}^x\big\{\tau^+_{\mathcal{M}_k}\big\} \leqslant \mathbb{E}^x\big\{\tau^+_{\mathcal{M}_{k+1}}\big\} + \mathbb{P}^x\big\{\tau^+_{B_{k+1}} < \tau^+_{\mathcal{M}_k}\big\}\mathbb{E}^{B_{k+1}}\big\{\tau^+_{\mathcal{M}_k}\big\} \; . \tag{8.43}$$

Proof: The proof is a direct application of Lemma 8.9 with $A = B_i$, $B = \mathcal{M}_k$ and $C = B_{k+1}$. □

Combining the last two lemmas with Corollary 8.13 and Proposition 8.4 shows that, as announced, $\mathbb{E}^{\mathcal{M}_k}\big\{\tau^+_{\mathcal{M}_k}\big\} = 1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})$ for all $k$, where $\kappa > 0$ is proportional to the size of the neighbourhood $B_i$.

## 8.4 Coupling argument

In order to apply the coupling argument in Proposition 5.4, we need to estimate the probability that two trajectories $(X_n^{x_1})_n$ and $(X_n^{x_2})_n$ driven by the same realization of the Brownian motion drift apart, i.e., their difference leaves a contracting "layer".

**Proposition 8.16** ([9, Proposition 6.12])**.** *There exist $C, \kappa > 0$ and $\varrho < 1$, independent of $\sigma$ such that for $x_1, x_2 \in B_i$,*

$$\mathbb{P}\{\|X_n^{x_1} - X_n^{x_2}\| > \varrho^n \|x_1 - x_2\|\} \leqslant C\,\mathrm{e}^{-\kappa/\sigma^2} \; . \tag{8.44}$$

The proof is a straightforward generalisation of the proof of [9, Proposition 6.12] to the multidimensional case. As explained in [9, Section 6.3], it follows that the stopping time $N$ introduced in (5.16) satisfies $\mathbb{P}\{N > n_0\} \leqslant n_0\,\mathrm{e}^{-\kappa/\sigma^2}$ for an $n_0$ of order $\log(\sigma^{-1})$. Using the Markov property at multiple times of $n_0$, if follows that

$$\rho_{kn_0} = \mathbb{P}\{N > kn_0\} \leqslant \big(M \log(\sigma^{-1})\,\mathrm{e}^{-\kappa/\sigma^2}\big)^k \; . \tag{8.45}$$

Choosing $k$ such that $k\kappa > C + 1$ in (5.18), we obtain a constant $L(n)$ close to 1.

### 8.5 Miscellaneous a priori bounds

Proof of Proposition 2.8. If the initial condition $z$ lies within the basin of attraction of one of the stable periodic orbits, the same argument as in Proposition 8.7 shows that $Z_t$ will reach $\Sigma$ in a time of order 1 with high probability, so that $\mathbb{P}^z\{\tau_\Sigma > 2T\}$ is exponentially small. If $z$ belongs to the neighbourhood of an unstable periodic orbit, the results from Section 8.3 show that $Z_t$ will leave this neighbourhood in a mean time of order $\log(\sigma^{-1})$. A similar result holds if $z$ belongs to the neighbourhood of an unstable equilibrium point, as shown in [42, 3, 1]. Combining this with the strong Markov property and (2.15) yields the result. □

Proof of Proposition 3.1. For two points $x, y \in \Sigma$, the continuous-time large-deviation principle naturally induces a discrete-time large-deviation principle with rate function

$$J(x,y) = \inf_{T>0} \inf_{\gamma:(x,0)\to(y,1)} I_{[0,T]}(\gamma) \;, \tag{8.46}$$

where the notation $\gamma : (x,0) \to (y,1)$ implies that we consider trajectories visiting $\Sigma'$ between the points $x$ and $y$ (this can be viewed as an instance of the contraction principle). More generally, for any sequence $(x_0, \dots, x_n)$ of points in $\Sigma$, the rate function is given by $J(x_0, \dots, x_n) = \sum_{j=0}^{n-1} J(x_j, x_{j+1})$. The fact that $V(x_i^\star, x_j^\star) = H(i,j)$ implies that for any $\eta > 0$, there exists a $T > 0$ and a continuous-time trajectory $\gamma$ connecting the two periodic orbits in time $T$ such that

$$I_{[0,T]}(\gamma) \leqslant H(i,j) + \frac{\eta}{2} \;. \tag{8.47}$$

Enlarging $T$ if needed, one can assume that $\gamma$ starts and ends on $\Sigma$, since one can follow the deterministic flow at zero cost. Furthermore, there exists $\delta > 0$ such that if the neighbourhood $B_i$, $B_j$ have radius $\delta$, they can be connected by a trajectory $\gamma$ such that $I_{[0,T]}(\gamma) \leqslant H(i,j) + \eta$. We may assume that $\gamma$ intersects $B_i \cup B_j$ only at its endpoints, for otherwise there would exist a cheaper way to connect the neighbourhoods. Therefore, there exists $n \geqslant 1$ and points $x_0 \in B_i, x_1, \dots, x_{n-1} \notin B_i \cup B_j, x_n \in B_j$, defined by the successive intersections of $\gamma$ with $\Sigma$, such that

$$J(x_0, \dots, x_n) \leqslant H(i,j) + \eta \;. \tag{8.48}$$

On the other hand, for any $\eta > 0$, there exists a neighbourhood of radius $\delta > 0$ such that for any $x \in B_i$ and $y \in B_j$, $V(x,y) \geqslant H(i,j) - \eta$. A similar argument as above shows that any discrete-time trajectory connecting the neighbourhoods must also have a cost larger than $H(i,j) - \eta$. □

## 9 Last steps of the proofs

### 9.1 Proof of Theorem 3.2

Fix a small constant $\eta > 0$. We start by estimating the $k^{\text{th}}$ eigenvalue $\lambda_{k-1}$ of $K$, by showing that it is close to the $k^{\text{th}}$ eigenvalue $\lambda_{k-1}^\star$ of the finite rank kernel $K^\star$, estimated in Corollary 7.5.

As discussed in Section 8.4, we can find an $n$ of order $\log(\sigma^{-1})$ such that each kernel $K_{B_i}^0$ satisfies the uniform positivity condition (5.7), with $L(n) - 1$ an arbitrary positive constant of order 1. Then Proposition 5.1 shows the existence of a constant $c_0 > 0$ such that

$$\left|\mathring{\lambda}_1^{B_i}\right| \leqslant \mathrm{e}^{-c_0/\log(\sigma^{-1})} \;. \tag{9.1}$$

Proposition 5.5, (5.3) and the large-deviation estimate in Proposition 3.1 yield the bound

$$\|\mathring{\phi}_0^{B_i} - 1\| \leqslant M_0 \log(\sigma^{-1})\, \mathrm{e}^{-[H(i,M_k\setminus\{i\})-\eta]/\sigma^2} \tag{9.2}$$

on the oscillation of the principal eigenfunction. Plugging this into Proposition 6.7 and using Assumption 2.5 to compare the various $H(i,j)$ yields

$$\|(K^0)^m - (K^\star)^m\| \leqslant 2\,\mathrm{e}^{-mc_0/\log(\sigma^{-1})} + \bigl[M_0 \log(\sigma^{-1}) + m^2\,\mathrm{e}^{-H_k'/\sigma^2}\bigr]\,\mathrm{e}^{-H_k'/\sigma^2} \tag{9.3}$$

where $H_k' = H(k, M_{k-1}) - \eta$. Combining this with Proposition 6.6 and the mean return time estimates in Section 8.3 shows that $\|(K^u)^m - (K^\star)^m\|$ is bounded by

$$\begin{aligned}\Delta_m = {} & 2\,\mathrm{e}^{-mc_0/\log(\sigma^{-1})} + \bigl[M_0 \log(\sigma^{-1}) + m^2\,\mathrm{e}^{-H_k'/\sigma^2}\bigr]\,\mathrm{e}^{-H_k'/\sigma^2} \\ & + \bigl(1 + 2(1-\mathrm{e}^{-u})\,\mathrm{e}^{-\theta'/\sigma^2}\bigr)^m - 1\,,\end{aligned} \tag{9.4}$$

provided $(1-\mathrm{e}^{-u})\,\mathrm{e}^{[H(k+1,M_k)+\eta]/\sigma^2} \leqslant 1/2$. The argument given in Section 7.1 shows that $(K^u)^m$ admits a unique eigenvalue $\lambda_{k-1}^m$ inside the contour $\mathcal{C}$ of radius $c_2\Delta_m$ centred in $(\lambda_{k-1}^\star)^m$ (for a $c_2$ of order 1), and that

$$\frac{1-\lambda_{k-1}^m}{1-(\lambda_{k-1}^\star)^m} = 1 + \mathcal{O}\biggl(\frac{\Delta_m}{1-(\lambda_{k-1}^\star)^m}\biggr)\,. \tag{9.5}$$

Note that this eigenvalue is necessarily real, since $(K^u)^m$ is real and has exactly one eigenvalue inside $\mathcal{C}$. Using the fact that for any $x \in (0,1)$ such that $m(1-x) < 2$, one has

$$(1-x)\Bigl[1 - \frac{1}{2}m(1-x)\Bigr] \leqslant \frac{1-x^m}{m} \leqslant 1-x\,, \tag{9.6}$$

we obtain

$$\frac{1-\lambda_{k-1}}{1-\lambda_{k-1}^\star} = 1 + \mathcal{O}\bigl(m(1-\lambda_{k-1}^\star)\bigr) + \mathcal{O}\biggl(\frac{\Delta_m}{1-(\lambda_{k-1}^\star)^m}\biggr)\,. \tag{9.7}$$

The optimal error term is obtained for $m = \log(\sigma^{-1})\,\mathrm{e}^{(2\eta+\delta)/\sigma^2}$, with $\delta = H(k, M_{k-1})/2$. Together with Corollary 7.5, this shows that $\lambda_{k-1}$ satisfies (3.4).

As discussed in Section 4.2, applying this argument to the kernels $K^{u,(k)}$ for $k = 1, \dots N$ shows that $K^{u,(N)}$ has exactly $N$ eigenvalues outside some disc centred in the origin. The system (4.19) can then be used to show that the original kernel $K$ also has exactly $N$ eigenvalues outside this disc, satisfying the same asymptotics.

**Remark 9.1.** Strictly speaking, to justify this argument, we have to make sure that the eigenvalues of the $K^{u,(k)}$ vary sufficiently slowly as functions of $u$. This, however, is easy to obtain. Indeed, a standard perturbation argument shows that if $K(u)$ is a family of linear operators depending differentiably on $u$, and $\lambda$ is an isolated simple eigenvalue of $K(u_0)$ with left and right eigenfunctions $\pi$ and $\phi$, then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}u}(u_0) = \pi \frac{\mathrm{d}K}{\mathrm{d}u}(u_0)\phi\,. \tag{9.8}$$

In our case, the relevant derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}u}K^u(x,\mathrm{d}y) = \mathbb{E}^x\biggl\{\bigl(\tau_{\mathcal{M}_k}^+ - 1\bigr)\,\mathrm{e}^{u(\tau_{\mathcal{M}_k}^+ - 1)}\,\mathbb{1}_{\{X_{\tau_{\mathcal{M}_k}^+} \in \mathrm{d}y\}}\biggr\}\,. \tag{9.9}$$

Proceeding as in Proposition 6.1, it is not hard to check that the norm of this operator is of order $\mathbb{E}^{\mathcal{M}_k}\{\tau_{\mathcal{M}_k}^+ - 1\}$ for $u$ as in the above computation. $\Diamond$

To prove the spectral gap estimate (3.5), one can use the fact that

$$\mathbb{P}^{\mathcal{M}_N^c}\big\{X_m \in \mathcal{M}_N^c\big\} \leqslant \frac{1}{2} \tag{9.10}$$

for $m$ of order $\log(\sigma^{-1})$, as a consequence of (8.40), Proposition 8.4 and Markov's inequality. If $(\tilde{X}_n)_{n\geqslant 0} = (X_{mn})_{n\geqslant 0}$ denotes the process diluted by a factor $m$, then the Laplace transform of the first time $\tilde{X}_n$ hits $\mathcal{M}_N$ exists for all $u$ such that $|\mathrm{e}^{-u}| \geqslant 1/2$. Therefore, by the above argument, $K^m$ has exactly $N$ eigenvalues outside a disc of radius $1/2$, which implies that $K$ has exactly $N$ eigenvalues outside a disc of radius $\mathrm{e}^{-c_0/\log(\sigma)^{-1}}$.

Finally, the result (3.6) on the principal eigenvalue follows from the fact that the principal eigenfunction of the process killed when hitting $\mathcal{M}_{k-1}$ satisfies

$$\phi_0^{\mathcal{M}_{k-1}^c}(x) = \mathbb{E}^x\Big\{\mathrm{e}^{u\tau_{B_k}}\,\phi_0^{\mathcal{M}_{k-1}^c}\big(X_{\tau_{B_k}}\big)\Big\}\,. \tag{9.11}$$

Therefore, it is also an eigenfunction of the kernel

$$K_{B_k}^u(x,\mathrm{d}y) = \mathbb{E}^x\left\{\mathrm{e}^{u(\tau_{B_k}^+ - 1)}\,\mathbb{1}_{\left\{X_{\tau_{B_k}^+}\in \mathrm{d}y, \tau_{B_k}^+ < \tau_{\mathcal{M}_{k-1}}^+\right\}}\right\}. \tag{9.12}$$

This kernel can be approximated by

$$K_{B_k}^\star(x,\mathrm{d}y) = \int_{B_k} \mathring{\pi}_0^{B_i}(z)\; K_{B_k}^0(z,\mathrm{d}y)\,\mathrm{d}z = \mathbb{P}^{\mathring{\pi}_0^{B_k}}\big\{X_{\tau_{B_k}^+} \in \mathrm{d}y, \tau_{B_k}^+ < \tau_{\mathcal{M}_{k-1}}^+\big\}\,, \tag{9.13}$$

which is a rank 1 operator, whose single nonzero eigenvalue is $\mathbb{P}^{\mathring{\pi}_0^{B_k}}\big\{\tau_{B_k}^+ < \tau_{\mathcal{M}_{k-1}}^+\big\}$. The approximation arguments applied to $K^u$ and $K^\star$ apply in this case as well, because the norm of the difference $K_{B_k}^u - K_{B_k}^\star$ is trivially bounded above by the norm of the difference $K^u - K^\star$. □

## 9.2 Proof of Theorem 3.4

Recall that the $k^{\text{th}}$ eigenfunction $\phi_{k-1}^\star$ of $K^\star$ has been obtained in Lemma 7.7, and that $\|\phi_{k-1}^\star\| = 1$. In order to bound the difference between $\phi_{k-1}$ and $\phi_{k-1}^\star$, we choose a countour $\mathcal{C}$ around $\lambda_{k-1}$ and consider the associated Riesz projector $\Pi_\sigma(K^u)$ (cf. (7.2)). Since $\Pi_\sigma(K^u)$ projects on the subspace associated with $\lambda_{k-1}$, $\phi_{k-1}$ is given, up to multiplication by a constant, by

$$\phi_{k-1} = \Pi_\sigma(K^u)\phi_{k-1}^\star\,. \tag{9.14}$$

We also have the relation

$$\phi_{k-1}^\star = \Pi_\sigma(K^\star)\phi_{k-1}^\star\,, \tag{9.15}$$

where the Riesz projector $\Pi_\sigma(K^\star)$ is defined with the same contour $\mathcal{C}$. Taking the difference, it follows from Proposition 7.1 that

$$\|\phi_{k-1} - \phi_{k-1}^\star\| \leqslant C\|K^u - K^\star\|\,, \tag{9.16}$$

where $C$ is defined in (7.9), provided $\|K^u - K^\star\| < \gamma/2$, cf. (7.5). An analogous bound holds for the iterates $(K^u)^m$ and $(K^\star)^m$, with a coutour around $\lambda_{k-1}^m$. Choosing $m$ as in the previous section, and a circular contour of radius $(1-\lambda_{k-1}^m)/2$, one obtains

$$\|\phi_{k-1} - \phi_{k-1}^\star\| = \mathcal{O}(\mathrm{e}^{-\theta_{k-1}/\sigma^2})\,, \tag{9.17}$$

where $\theta_{k-1}$ is $\eta$-close to $H(k, M_{k-1})/2$.

Applying the Feynman–Kac relation of Proposition 4.2 with $\mathrm{e}^{-u} = \lambda_{k-1}$, we obtain

$$\mathrm{e}^{-u}\,\phi_{k-1}(x) = \mathbb{E}^x\big\{\phi_{k-1}(X_{\tau_{\mathcal{M}_k}})\big\} + \mathbb{E}^x\big\{(\mathrm{e}^{u(\tau_{\mathcal{M}_k}-1)}-1)\phi_{k-1}(X_{\tau_{\mathcal{M}_k}})\big\}\;. \tag{9.18}$$

By Proposition 6.1, the second term on the right-hand side has order $\mathrm{e}^{-(H(k,M_{k-1})+\theta'-\eta)/\sigma^2}$. As for the first term, it can be rewritten (recall that $\phi^\star_{k-1}$ is constant on each $B_j$)

$$\sum_{j=1}^{k}\mathbb{E}^x\Big\{\mathbb{1}_{\{X_{\tau_{\mathcal{M}_k}}\in B_j\}}\phi_{k-1}\big(X_{\tau_{\mathcal{M}_k}}\big)\Big\} = \sum_{j=1}^{k}\mathbb{P}^x\big\{\tau_{B_j} < \tau_{\mathcal{M}_k\setminus B_j}\big\}\phi^\star_{k-1}(x^\star_j) + \mathcal{O}(\mathrm{e}^{-\theta_{k-1}/\sigma^2})\;. \tag{9.19}$$

To lowest order, using Lemma 7.7 and Remark 7.4, we have $\phi^\star_{k-1}(x^\star_j) = \delta_{jk} + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})$, which yields (3.11). The more precise expression (3.13) is based on the fact that

$$\phi^\star(x^\star_j) = -\frac{\mathbb{P}^{\hat\pi_0^{B_j}}\big\{\tau^+_{B_k} < \tau^+_{\mathcal{M}_{k-1}}\big\}}{\mathbb{P}^{\hat\pi_0^{B_k}}\big\{\tau^+_{\mathcal{M}_{k-1}} < \tau^+_{B_k}\big\}} + \mathcal{O}(\mathrm{e}^{-2\theta^-/\sigma^2})\;, \tag{9.20}$$

as a consequence of Remark 7.4. As for the principal eigenfunction $\phi_0^{\mathcal{M}^c_{k-1}}$, it satisfies

$$\begin{aligned}\mathrm{e}^{-u}\,\phi_0^{\mathcal{M}^c_{k-1}}(x) = {}&\mathbb{E}^x\Big\{\phi_0^{\mathcal{M}^c_{k-1}}\big(X_{\tau_{B_k}}\big)\mathbb{1}_{\{\tau_{B_k}<\tau_{\mathcal{M}_{k-1}}\}}\Big\}\\ &+\mathbb{E}^x\Big\{\big(\mathrm{e}^{u(\tau_{\mathcal{M}_k}-1)}-1\big)\phi_0^{\mathcal{M}^c_{k-1}}\big(X_{\tau_{B_k}}\big)\mathbb{1}_{\{\tau_{B_k}<\tau_{\mathcal{M}_{k-1}}\}}\Big\}\;,\end{aligned} \tag{9.21}$$

where $\mathrm{e}^{-u} = \lambda_0^{\mathcal{M}^c_{k-1}}$. The first term on the right-hand side is equal to

$$\mathbb{P}^x\big\{\tau_{B_k} < \tau_{\mathcal{M}_{k-1}}\big\}\big(1+\mathcal{O}(\mathrm{e}^{-\theta_{k-1}/\sigma^2})\big)\;, \tag{9.22}$$

while the second one can be bounded as above by $\mathcal{O}(\mathrm{e}^{-(H(k,M_{k-1})+\theta'-\eta)/\sigma^2})$.

## 9.3 Proof of Theorem 3.8

Using Proposition 3.7 with $A_1 = B_1$ and $A_2 = \mathcal{M}_N \setminus B_1$ and the large-deviation a priori bounds of Proposition 3.1 shows that $\pi_0(\mathcal{M}_N\setminus B_1) \leqslant \mathrm{e}^{-\theta^-/\sigma^2}\,\pi_0(B_1)$. Together with (3.17), this proves (3.19).

The bound (3.20) can be proved by reasoning on the stationary distribution of the Doob-transformed process $\bar X_{\mathcal{M}_k^c}$ and using the relation (2.38) between the left eigenfunctions of both processes.

In order to prove the first relation in (3.21), we use Lemma 4.8, showing that $\pi_{k-1}$ is a left eigenfunction of the kernel $K^u$, cf. (6.1). Therefore we expect $\pi_{k-1}$ to be close to the left eigenfunction $\pi^\star_{k-1}$ of $K^\star$. Using the block-triangularisation of Section 7.2, one easily obtains that

$$\pi^\star_{k-1} = (\hat\pi, 1-\hat\pi S^\star_{12}) \qquad \text{where} \qquad \hat\pi = (\alpha\,\mathrm{id} - T_{11})^{-1}\hat P_{21}\;, \tag{9.23}$$

which implies

$$\begin{aligned}&\pi^\star_{k-1}(B_k) = 1 + \mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\;,\\ &\pi^\star_{k-1}(B_j) = -\frac{\mathbb{P}^{\hat\pi_0^{B_k}}\big\{\tau^+_{B_j} < \tau^+_{\mathcal{M}_k\setminus B_j}\big\}}{\mathbb{P}^{\hat\pi_0^{B_k}}\big\{\tau^+_{\mathcal{M}_{k-1}} < \tau^+_{B_k}\big\}}\big[1+\mathcal{O}(\mathrm{e}^{-\theta^-/\sigma^2})\big] \qquad \text{for } 1\leqslant j\leqslant k-1\;.\end{aligned} \tag{9.24}$$

To compare $\pi_{k-1}$ and $\pi^\star_{k-1}$, it suffices to realise that the $L^1$-operator norm of a kernel $K$, acting on signed measures, can be bounded by $\sup_{x\in\mathcal{M}_k} K(x,\mathcal{M}_k)$. Therefore, the same bounds on $\|K^u - K^\star\|$ and their iterates apply for the action of these operators on signed measures, so that one can repeat the argument of the previous section showing that

$$\bigl|\pi_{k-1}(B_j) - \pi^\star_{k-1}(B_j)\bigr| = \mathcal{O}(\mathrm{e}^{-\theta_{k-1}/\sigma^2})\;. \tag{9.25}$$

Finally, the second relation in (3.21) is obained by comparing the original and killed process monitored while visiting $\mathcal{M}_j$. The kernel of the original process can be approximated by a kernel $K^\star$ of rank $j$, while the killed process is described by the restriction of this kernel to $B_k \cup \cdots \cup B_j$. Using a similar block-triangularisation as in Section 7.2, with blocks of size $k-1$ and $j-k+1$, the result follows easily. □

## 9.4 Proof of Theorem 3.10

The result will be proved if we manage to control the oscillation of $\mathbb{E}^x\bigl\{\tau^+_{\mathcal{M}_{k-1}}\bigr\}$ when $x$ varies in $B_k$. To this end, consider the process $(\hat X_n)_n$, killed when hitting $\mathcal{M}_{k-1}$ and monitored only while visiting $\mathcal{M}_k$, whose kernel is $K^0_{B_k}$. If $\hat\tau_{\mathcal{M}_{k-1}}$ denotes the killing time of $\hat X_n$, then we have

$$\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\} \leqslant \mathbb{E}^x\bigl\{\tau^+_{\mathcal{M}_{k-1}}\bigr\} = \mathbb{E}^x\biggl\{\sum_{n=0}^{\hat\tau-1}\mathbb{E}^{\hat X_n}\bigl\{\tau_{\mathcal{M}_k}\bigr\}\biggr\} \leqslant \mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\}\mathbb{E}^{B_k}\bigl\{\tau_{\mathcal{M}_k}\bigr\}\;, \tag{9.26}$$

so that

$$1 \leqslant \frac{\mathbb{E}^x\bigl\{\tau^+_{\mathcal{M}_{k-1}}\bigr\}}{\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\}} \leqslant \mathbb{E}^{B_k}\bigl\{\tau_{\mathcal{M}_k}\bigr\}\;. \tag{9.27}$$

It follows that

$$\frac{\mathbb{E}^{B_k}\bigl\{\tau^+_{\mathcal{M}_{k-1}}\bigr\}}{\inf\limits_{x\in B_k}\mathbb{E}^x\bigl\{\tau^+_{\mathcal{M}_{k-1}}\bigr\}} \leqslant \frac{\mathbb{E}^{B_k}\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\}}{\inf\limits_{x\in B_k}\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\}}\mathbb{E}^{B_k}\bigl\{\tau_{\mathcal{M}_k}\bigr\}\;. \tag{9.28}$$

To control the oscillation of $\hat\tau_{\mathcal{M}_{k-1}}$, we note that the spectral decomposition (5.4) yields

$$\begin{aligned}\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\} &= \sum_{n\geqslant 0}\bigl(K^0_{B_k}\bigr)^n(x,B_k)\\ &= \sum_{n\geqslant 0}\bigl(\mathring\lambda_0^{B_k}\bigr)^n\biggl\{\mathring\phi_0^{B_k}(x) + \biggl(\frac{\mathring\lambda_1^{B_k}}{\mathring\lambda_0^{B_k}}\biggr)^n g^n(x,B_k)\biggr\}\;.\end{aligned} \tag{9.29}$$

We know that the kernel $K^0_{B_k}$ satisfies the uniform positivity condition (5.7) with an $n_0$ of order $\log(\sigma^{-1})$. It follows that

$$\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\} = \frac{1}{1-\mathring\lambda_0^{B_k}}\mathring\phi_0^{B_k}(x) + \mathcal{O}\biggl(\frac{1}{1-\varrho^{1/n_0}\mathring\lambda_0^{B_k}}\biggr)\;. \tag{9.30}$$

Together with Proposition 5.5, this shows that the oscillation of $\mathbb{E}^x\bigl\{\hat\tau_{\mathcal{M}_{k-1}}\bigr\}$ is bounded by a term of order $\log(\sigma^{-1})\,\mathrm{e}^{-(H(k,M_{k-1})-\eta)/\sigma^2}$. Combined with (9.28), this completes the proof. □

# A Doob's $h$-transform

Consider a Markov process $(X_n)_{n\geqslant 0}$ with state space $\Sigma$ and transition kernel having density $k(x,y)$. Given a subset $A \subset \Sigma$, the process conditioned on remaining in $A$ can be constructed using the functions

$$h_n(x) = \mathbb{P}^x\{\tau_{A^c} > n\} \,, \tag{A.1}$$

where $\tau_{A^c} = \inf\{n > 0 \colon X_n \in A^c\}$ denotes the first-exit time from $A$. Indeed, assuming $h_n(x) > 0$ for all $x \in A$, then for $y \in A$ we have

$$\mathbb{P}^x\big\{X_1 \in \mathrm{d}y \;\big|\; \tau_{A^c} > n\big\} = \frac{1}{h_n(x)}\mathbb{E}^x\Big\{\mathbb{1}_{\{X_1\in \mathrm{d}y\}}\mathbb{P}^y\{\tau_{A^c} > n-1\}\Big\} = \frac{h_{n-1}(y)}{h_n(x)}\mathbb{P}^x\{X_1 \in \mathrm{d}y\} \,. \tag{A.2}$$

This shows that the kernel

$$\bar{k}_A(x,y;n) = \frac{h_{n-1}(y)}{h_n(x)}k(x,y)\mathbb{1}_{\{x\in A, y\in A\}} \tag{A.3}$$

describes the process conditioned to stay in $A$ up to time $n$. Thus if

$$\bar{k}_A(x,y) = \lim_{n\to\infty} \bar{k}_A(x,y;n) \tag{A.4}$$

exists, it will describe the process conditioned on staying in $A$ forever.

Let $k_A(x,y) = k(x,y)\mathbb{1}_{\{x\in A, y\in A\}}$ denote the kernel of the process killed upon leaving $A$, and write $\lambda_i^A$ for its eigenvalues ordered by decreasing module, $\pi_i^A$ for its left eigenfunctions and $\phi_i^A$ for its right eigenfunctions. Recall that the principal eigenvalue $\lambda_0^A$ is real and positive, and that $\pi_0^A(x)$ and $\phi_0^A(x)$ can be chosen real and positive as well. We also choose to normalise the eigenfunctions in such a way that

$$\int_A \pi_i^A(x)\phi_j^A(x)\,\mathrm{d}x = \delta_{ij} \,. \tag{A.5}$$

**Lemma A.1.** *Under the spectral gap condition $\big|\lambda_1^A\big| < \lambda_0^A$, we have*

$$\lim_{n\to\infty} \frac{h_{n-1}(y)}{h_n(x)} = \frac{1}{\lambda_0^A}\frac{\phi_0^A(y)}{\phi_0^A(x)} \,. \tag{A.6}$$

Proof: We can write

$$k_A(x,y) = \lambda_0^A\Pi_0(x,y) + k_\perp(x,y) \,, \tag{A.7}$$

where $\Pi_0(x,y) = \phi_0^A(x)\pi_0^A(y)$ is the projector on the subspace of $\lambda_0^A$, and the remainder $k_\perp$ satisfies $\Pi_0 k_\perp = 0$, $k_\perp\Pi_0 = 0$. Furthermore, $k_\perp$ has spectral radius $\big|\lambda_1^A\big|$. Therefore

$$k_A^n(x,y) = (\lambda_0^A)^n\Pi_0(x,y) + k_\perp(x,y)^n \,, \tag{A.8}$$

and thus

$$h_n(x) = \int_A k_A^n(x,y)\,\mathrm{d}y = (\lambda_0^A)^n\phi_0^A(x) + \mathcal{O}\big(\big|\lambda_1^A\big|^n\big) \,. \tag{A.9}$$

The result follows at once from the spectral-gap assumption. □

We have thus obtained

$$\bar{k}_A(x,y) = \frac{1}{\lambda_0^A}\frac{\phi_0^A(y)}{\phi_0^A(x)}k_A(x,y) \,. \tag{A.10}$$

**Corollary A.2.** *The eigenvalues and eigenfunctions of $\bar{K}_A$ are given by*

$$\bar{\lambda}_n^A = \frac{\lambda_n^A}{\lambda_0^A} , \qquad \bar{\pi}_n^A(x) = \pi_n^A(x)\phi_0^A(x) \qquad \textit{and} \qquad \bar{\phi}_n^A(x) = \frac{\phi_n^A(x)}{\phi_0^A(x)} . \tag{A.11}$$

PROOF: A direct computation shows that $K_A\phi_n^A = \lambda_n^A\phi_n^A \Leftrightarrow \bar{K}_A\bar{\phi}_n^A = \bar{\lambda}_n^A\bar{\phi}_n^A$, and similarly for the left eigenfunctions. □

# B Floquet theory

Floquet theory and its application to the stability of periodic orbits is explained in many standard text books, such as [36, Chapters III and VI]. Here we briefly recall some important facts and notations used in the present work.

Consider a $d+1$-dimensional deterministic ODE

$$\dot{z} = f(z) , \tag{B.1}$$

where $f \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}^{d+1})$. We assume that this system admits a periodic solution $\gamma$ of period $T$ with associated orbit $\Gamma$. We introduce the variable $\varphi \in \mathbb{R}/\mathbb{Z}$ and set $\Gamma(\varphi) = \gamma(T\varphi)$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\Gamma(\varphi) = Tf(\Gamma(\varphi)) , \tag{B.2}$$

so that $\dot{\varphi} = 1/T$ is constant on the periodic orbit. In order to analyse the dynamics near $\Gamma$, we start by linearising the equation. Let $A(\varphi) = \partial_z f(\Gamma(\varphi))$ be the Jacobian matrix of $f$ at $\Gamma(\varphi)$. The linearisation around the periodic orbit is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\zeta = TA(\varphi)\zeta . \tag{B.3}$$

Therefore $\zeta(\varphi) = U(\varphi, \varphi_0)\zeta(\varphi_0)$, where the principal solution $U(\varphi, \varphi_0)$ satisfies

$$\partial_\varphi U(\varphi, \varphi_0) = TA(\varphi)U(\varphi, \varphi_0), \qquad U(\varphi_0, \varphi_0) = \mathrm{id} . \tag{B.4}$$

Since $A(\varphi) = A(\varphi + 1)$ for all $\varphi$, Floquet's theorem allows us to decompose the principal solution as

$$U(\varphi, \varphi_0) = P(\varphi, \varphi_0)\, \mathrm{e}^{T(\varphi-\varphi_0)B(\varphi_0)} , \tag{B.5}$$

where $P(\cdot, \varphi_0)$ is periodic with same period as $A(\cdot)$, i.e. 1, and $P$ satisfies $P(\varphi_0, \varphi_0) = \mathrm{id}$, and $B(\varphi_0)$ is a constant matrix which can always be chosen to be real even if it means taking $P(\cdot, \varphi_0)$ to be $2-$periodic. Note that $P$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}P(\varphi, \varphi_0) = T\big[A(\varphi)P(\varphi, \varphi_0) - P(\varphi, \varphi_0)B(\varphi_0)\big] . \tag{B.6}$$

The asymptotic behaviour of $\Gamma(\varphi)$ only depends on the eigenvalues of $TB(\varphi_0)$, which are called characteristic exponents (or Floquet exponents) of $\Gamma$. The matrix $U(1 + \varphi_0, \varphi_0) = \exp(TB(\varphi_0))$ is called the monodromy matrix in $\varphi_0$, and its eigenvalues are called the characteristic multipliers. Note that Floquet multipliers do not depend on $\varphi_0$. Indeed, one can show that all monodromy matrices are similar and thus have the same eigenvalues. Differentiating (B.2) with respect to $\varphi$, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\Gamma'(\varphi) = T\frac{\mathrm{d}}{\mathrm{d}\varphi}f(\Gamma(\varphi)) = TA(\varphi)\Gamma'(\varphi) . \tag{B.7}$$

Thus, owing to periodicity, we have

$$\Gamma'(\varphi) = \Gamma'(\varphi + 1) = U(\varphi + 1, \varphi)\Gamma'(\varphi) \ , \tag{B.8}$$

showing that 1 is an eigenvalue of $U(1 + \varphi, \varphi)$ with eigenvector $\Gamma'(\varphi)$.

**Proposition B.1.** *There exist $L > 0$ and a $d \times d$ triangular matrix $\Lambda$ such that system* (B.1) *is equivalent for $\|x\| < L$ to*

$$\begin{aligned} \dot{x} &= \Lambda x + \mathcal{O}(\|x\|^2) \\ \dot{\varphi} &= \frac{1}{T} + \mathcal{O}(\|x\|^2) \ . \end{aligned} \tag{B.9}$$

Proof: We are going to define the change of coordinates explicitly but we first introduce some notations. Let $\hat{\Lambda} = S^{-1}BS = \mathrm{diag}(0, \Lambda)$ be the Jordan canonical form of the constant matrix $B$ defined in (B.5), where $\Lambda \in \mathbb{R}^{d\times d}$. We also write $P(\varphi, \varphi_0)S = [u(\varphi), R(\varphi)]$, where $u$ is a column vector of dimension $d + 1$ and $R$ is a matrix of dimension $(d + 1) \times d$. It follows from (B.6) that the vector $u$ and the matrix $R$ satisfy the equations

$$\begin{aligned} u'(\varphi) &= TA(\varphi)u(\varphi) \ , \\ R'(\varphi) &= T(A(\varphi)R(\varphi) - R(\varphi)\Lambda) \ . \end{aligned} \tag{B.10}$$

Note that we can choose the matrix $S$ such that $u(\varphi) = \Gamma'(\varphi)$.

We now introduce the transformation

$$z = \Gamma(\varphi) + R(\varphi)x \ . \tag{B.11}$$

We can first check that this transformation is well defined in a neighbourhood of $\Gamma$. Indeed, if $F(z, x, \varphi) = \Gamma(\varphi) + R(\varphi)x - z$, the partial derivatives of $F$ with respect of $x$ and $\varphi$ are

$$\begin{aligned} \frac{\partial F}{\partial \varphi} &= \Gamma'(\varphi) + R'(\varphi)x \ , \\ \frac{\partial F}{\partial x} &= R(\varphi) \ . \end{aligned} \tag{B.12}$$

For $x = 0$, we have $\det[\partial_\varphi F, \partial_x F] \neq 0$ for all $\varphi$, since $[\Gamma'(\varphi), R(\varphi)]$ is the matrix $P(\varphi, \varphi_0)S$ which is invertible.

If $z(t) = \Gamma(\varphi(t)) + R(\varphi(t))x(t)$ satisfies $\dot{z} = f(z)$ then

$$f(\Gamma(\varphi) + R(\varphi)x) = \dot{\varphi}\Gamma'(\varphi) + \dot{\varphi}R'(\varphi)x + R(\varphi)\dot{x} \ . \tag{B.13}$$

Performing a Taylor expansion of the left-hand side and using (B.10), we obtain

$$\mathcal{O}(\|x\|^2) = \big(\dot{\varphi} - \frac{1}{T}\big)\big[\Gamma'(\varphi) + TA(\varphi)R(\varphi)x\big] + R(\varphi)(\dot{x} - \dot{\varphi}T\Lambda x) \ . \tag{B.14}$$

The result follows by projecting on a normal vector to the space generated by the column vectors of $R$. □

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