Adiabatic Dynamical Systems and Hysteresis

Thesis presented by

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"Combien j'aimerais à passer huit jours à Vevey! Je louerais une chambre à la montagne, à une grande lieue de la ville. Je suis touché, à ce voyage–ci, de ce point admirable, où les montagnes sévères et couvertes de sapins se rapprochent du lac, remplacent l'ignoble champ cultivé et donnent au paysage un si grand caractère.

Les industriels me le pardonneront–ils? Pour les gens un peu au–dessus du vulgaire, la perspective du gain annuel *qui récompense les travaux du gentilhomme campagnard s'oppose net aux sensations sublimes que les sonnets de Pétrarque ou la musique de Mozart donnent à certaines âmes; à la vérité, ces âmes–là ne sont pas destinées à avoir dans le monde un avancement rapide et déplaisent souverainement aux députés épais ou aux commis avides qui disposent de ce même avancement."*

Stendhal, "Mémoires d'un touriste"

Meinen Eltern

Version abrégée

Ce travail est dédié à l'étude de Systèmes Dynamiques dépendant d'un paramètre lentement variable. Il contient en particulier une analyse détaillée de certains effets de mémoire, tels que l'hystérèse, qui apparaissent fréquemment dans les systèmes faisant intervenir plusieurs échelles de temps.

Dans une première partie de cet exposé, nous développons un cadre mathématique ayant pour but de résoudre les équations différentielles adiabatiques. Pour ce faire, nous favorisons dans la mesure du possible l'approche géométrique de la théorie, ce qui permet de dériver des propriétés qualitatives de la dynamique, telles que l'existence de cycles d'hystérèse et les lois d'échelle, avec un minimum de calculs analytiques.

Nous commençons par analyser des systèmes adiabatiques unidimensionnels de la forme $\varepsilon\dot{x} = f(x,\tau)$. Nous montrons d'abord l'existence de solutions adiabatiques, qui restent proches de branches d'équilibre du système, et admettent des séries asymptotiques dans la paramètre adiabatique ε. Ensuite, nous fournissons une méthode permettant d'analyser les solutions près de points de bifurcation, et montrons qu'elles suivent des lois d'échelle non triviales en fonction d'ε, avec un exposant qui peut être aisément calculé. Cette analyse est conclue en examinant des propriétés globales du flot et, en particulier, l'existence de cycles d'hystérèse.

Ces résultats sont ensuite étendus au cas à n dimensions. La discussion des solutions adiabatiques se transpose de manière immédiate. La dynamique au voisinage de ces solutions est, par contre, plus difficile à analyser. Nous fournissons d'abord une méthode de diagonalisation dynamique des équations linéarisées, et nous montrons que les croisements de valeurs propres conduisent à des comportements similaires que les bifurcations. Nous introduisons ensuite quelques méthodes permettant de contrôler les termes non–linéaires, en particulier des variétés adiabatiques et des formes normales dynamiques.

Dans une seconde partie de ce travail, nous appliquons les méthodes développées précédemment à quelques exemples choisis. Nous discutons d'abord la dynamique de certains oscillateurs non–linéaires de basse dimension. En particulier, nous présentons l'exemple d'un pendule amorti, monté sur une table tournant à fréquence angulaire lentement oscillante. Ce système adopte des mouvements chaotiques même pour un paramètre adiabatique arbitrairement petit. Ce phénomène est expliqué en calculant une expression asymptotique de l'application de Poincaré.

Comme seconde application, nous analysons quelques modèles de ferromagnétisme. En partant d'un modèle sur réseau avec dynamique stochastique, nous montrons comment dériver une équation du mouvement déterministe du genre Ginzburg–Landau, dans le cas d'une interaction de portée infinie et dans la limite thermodynamique. Nous analysons l'influence de la dimensionnalité et de l'anisotropie de l'interaction sur la forme et les propriétés d'échelle des cycles d'hystérèse. Quelques approximations simples de la dynamique

du modèle d'Ising sont également présentées.

Nous concluons ce travail en étendant quelques propriétés des équations différentielles adiabatiques aux applications itérées. Nous donnons quelques résultats sur l'existence d'invariants adiabatiques pour les applications lentes–rapides intégrables perturbées, et les appliquons aux billards.

Abstract

This work is dedicated to the study of Dynamical Systems, depending on a slowly varying parameter. It contains, in particular, a detailed analysis of memory effects, such as hysteresis, which frequently appear in systems involving several time scales.

In a first part of this dissertation, we develop a mathematical framework to deal with adiabatic differential equations. We do this, whenever possible, by favouring the geometrical approach to the theory, which allows to derive qualitative properties of the dynamics, such as existence of hysteresis cycles and scaling laws, with a minimum of analytic calculations.

We begin by analysing one–dimensional adiabatic systems of the form $\varepsilon\dot{x} = f(x,\tau)$. We first show existence of adiabatic solutions, which remain close to equilibrium branches of the system, and admit asymptotic series in the adiabatic parameter ε. We then provide a method to analyse solutions near bifurcation points, and show that they scale in a nontrivial way with ε, with an exponent that can be easily computed. The analysis is concluded by examining global properties of the flow, in particular existence of hysteresis cycles.

These results are then extended to the n–dimensional case. The discussion of adiabatic solutions carries over in a natural way. The dynamics of neighboring solutions is, however, more difficult to analyse. We first provide a method to diagonalize linear equations dynamically, and show that eigenvalue crossings lead to similar behaviours than bifurcations. We then introduce some methods to deal with nonlinear terms, in particular adiabatic manifolds and dynamic normal forms.

In a second part of this work, we apply the previously developed methods to some selected examples. We first discuss the dynamics of some low–dimensional nonlinear oscillators. In particular, we present the example of a damped pendulum, on a table rotating with a slowly oscillating angular frequency. This system displays chaotic motion even for arbitrarily small adiabatic parameter. This phenomenon is explained by computing an asymptotic expression of the Poincaré map.

As a second application, we analyse a few models of ferromagnetism. Starting from a lattice model with stochastic spin flip dynamics, we show how to derive a deterministic equation of motion of Ginzburg–Landau type, in the case of infinite range interactions and in the thermodynamic limit. We analyse the influence of dimensionality and interaction anisotropy on shape and scaling properties of hysteresis cycles. A few simple approximations to the dynamics of an Ising model are also discussed.

We conclude this work by extending some properties of adiabatic differential equations to iterated maps. We give some results on existence of adiabatic invariants for near–integrable slow–fast maps, and apply them to billiards.

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This work would not have been possible without the help of many people, to whom I would like to express my gratitude at this place.

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My last thanks go to all my friends who helped me to remember, especially during the last phase of my writing, that there still exists a world "out there".

Note

The present text uses the following notational conventions. Chapters, sections and subsections are numbered respectively by one, two or three figures separated by a dot (e.g. Section 5.3). Figures, equations, definitions, theorems and similar environments are numbered independently, by two figures, the first of which is the corresponding chapter number (e.g. Theorem 2.4, equation (6.54)). Special fonts are used for words where they are **defined**, and for *emphasized* words. In citations, we distinguish between [**books**] and [articles].

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Chapter 1

Introduction

"For those who like this sort of thing, this is the sort of thing they like."
Abraham Lincoln

"Try not to have a good time ... This is supposed to be educational."
Charles Schulz

1.1 Non–Technical Description

1.1.1 Dynamic Variables and Parameters

Since the discovery of Newton's equation and its application to the study of the Solar System, it has become apparent that an important number of physical problems could be modeled, more or less accurately, by ordinary differential equations (ODEs). Sometimes, these equations are direct consequences of the fundamental laws of Physics, like Newton's equation (for classical mechanical systems) or Maxwell's equations (for electromagnetic problems). Macroscopic systems, for which we cannot neglect the fact that they are composed of a very large number of atoms or molecules, may sometimes be modeled by somewhat more phenomenological laws, taking into account the interaction of a small number of effective degrees of freedom. This applies to the equations of thermodynamics (applicable for instance to kinetics of chemical reactions), master equations (lasers) or mean field equations (phase transitions). There also exist a number of systems, which are not directly related to Physics, but are nevertheless modeled, on a very phenomenological level, by ODEs: this is the case, for instance, for population dynamics in ecology.

When we consider some specific examples, like those given in Table 1.1, we realize that such differential equations will depend on two kinds of variables: **dynamic variables** and **parameters**. As far as the mathematical model is concerned, the distinction between these two types of variables is clear:

- dynamic variables define the state of the system; their role is twofold: on one hand, they evolve in time, specifying the state of the system at each instant; on the other hand, they determine the future evolution of the system;
- parameters also influence the future evolution, but their value remains fixed; in fact, a different dynamical system is obtained for each value of the parameters.

System	**Dynamic variables**	**Parameters**
Mechanical system	Positions and momenta	External driving force
Electric device	Charges and currents	Power supply, tunable resistance
Chemical reaction	Concentration of reacting substances	Supply flux, temperature
Laser	Level population, internal field	External field
Magnet	Order parameter (magnetization)	Magnetic field, temperature
Population dynamics	Number of individuals of each species	Climate, reproduction rate

TABLE 1.1. Examples of systems which can be modeled by ODEs, with associated dynamic variables and parameters.

Are the "parameters" of Table 1.1 really always fixed? Let us examine more closely different kinds of parameters which may appear in a physical experiment. We may distinguish the following types:

- parameters which are related to physical constants or technical specifications of the experimental set–up, and are, therefore, fixed during the experiment; this applies to masses and coupling constants of particles, and dimensions of a cavity or reactor;
- **control parameters**, which can be accurately tuned, say by turning a knob of the experimental device; this may be the case for the supply voltage of an electric device, an applied external field, or the temperature difference between two sides of a cavity;
- parameters that one would like to maintain fixed, but which are not so easy to control in a real experiment, like a supply flux of chemicals, or the temperature in a reactor.

One usually characterizes a dynamical system by its **bifurcation diagram**, representing the asymptotic state (which may be stationary, periodic or more complicated) against the control parameter (Fig. 1.1). What do we mean when we say that the bifurcation diagram is determined experimentally by varying the control parameter?

According to the mathematical modeling, the bifurcation diagram should be determined as follows. Fix the control parameter and choose an initial state for the system. Let the system evolve until it has reached an asymptotic state. Repeat this procedure for different initial conditions, in order to find other possible asymptotic states. Then increase the control parameter, reset the initial state, and repeat the whole experiment. Apply this procedure for the desired set of parameter values, and plot the asymptotic state(s) against the control parameter.

In practice, it is not always possible to carry out this rather elaborate program. We may not have the time to wait for the system relaxing to equilibrium for each parameter value, or we may not be able to reset the initial condition. In fact, it is very tempting[1] to turn slowly the knob controlling the parameter during the experiment, in the hope that if this parameter variation is sufficiently slow, it will not affect the bifurcation diagram very much.

[1]Every person who has ever seen an experimental device with a knob for the control parameter knows that it is indeed *very* difficult not to turn this knob during the experiment.

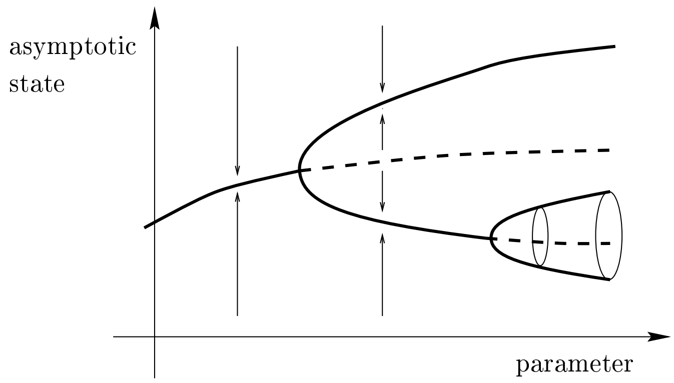

FIGURE 1.1. Example of a bifurcation diagram. For each value of the parameter, one plots the asymptotic state of the system. In this example, there is a unique stable equilibrium state for small parameter (thick full line). At some parameter value, this equilibrium becomes unstable (dotted line), while two new stable equilibria are formed. One of them is then replaced by a limit cycle, i.e., a stable periodic orbit. To determine this diagram experimentally, one should fix a value of the parameter and an initial condition, and wait for the system to relax to equilibrium (vertical arrows). This procedure should be repeated for several parameter values.

Is this hope justified? The answer to this question is not immediate at all. It requires a precise understanding of the relation that exists between, on one hand, a one–parameter family of autonomous Dynamical Systems and, on the other hand, the system with slowly time–dependent parameter. This relation is by no means trivial in all cases, since memory effects, in particular hysteresis, may show up in such systems. An understanding of this relation would allow us, for example, to solve the following problems:

- If the control parameter is swept slowly in time, do we obtain a trustworthy representation of the bifurcation diagram?
- How do parameters, which cannot be controlled completely, but are subject to slow fluctuations, affect our modeling of the system?
- Consider a system subject to a slowly time–dependent driving force. Can we use the static bifurcation diagram (which is analytically more tractable) to gain some information on the time–dependent system?

To deal with this kind of questions, we should begin by understanding the role of time scales in Physics.

1.1.2 Slow–Fast Systems and Hysteresis

Physical systems are often characterized by one or several time scales. A characteristic time might be the period of a typical periodic solution, or the relaxation time to equilibrium. Let us consider a dynamical system with characteristic time T_1, called the **fast system**, and couple it to another system with much larger characteristic time $T_2 \gg T_1$, called the **slow system**.

Two particular situations are of interest:

1. The evolution of the slow system is imposed from outside, and acts on the fast system as a slowly time–dependent parameter. For this purpose, it need not be governed by a differential equation. We call this coupled system an **adiabatic system**.

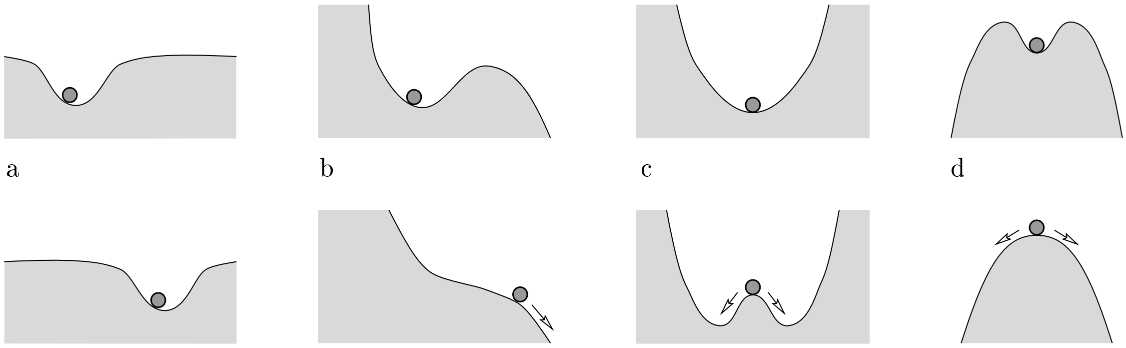

FIGURE 1.2. The damped motion of a particle in a slowly varying potential provides a simple example of adiabatic system. If the potential admits an isolated, slowly moving minimum, the particle will follow this well adiabatically (a). Bifurcations correspond to situations where this minimum interacts with other equilibrium points. For instance, the minimum may annihilate with a maximum (saddle–node bifurcation), and the particle leaves the vicinity of the bifurcation point (b). We may also have creation of two new equilibria (direct pitchfork bifurcation), so that the particle has to choose between its current, unstable position, and two potential wells (c). Or the minimum may disappear in favor of a maximum (indirect pitchfork bifurcation) (d).

2. The slow system is also a dynamical system, which is coupled to and influenced by the fast one. In this situation, we speak of a **slow–fast system**.

As an illustration, let us imagine the following population model. In some relatively small ecosystem, predators and prey reproduce, say, a couple of times a year. Their populations have attained a cyclic regime, with a period of a few years. Now the climate begins to change slowly, due for instance to human impact, modifying the reproduction rate of the predator. This would be an example of an adiabatic system, since the climate change is imposed from outside. Another situation appears when, due to continual food consumption by the prey, vegetation and *micro-climate* are slowly modified, changing the reproduction rates in turn. This would be an example of a slow–fast system.

In this work, we are mainly interested in adiabatic systems. We believe, however, that most results can be transposed to slow–fast systems (see Section 1.2).

What do we expect from the behaviour of an adiabatic system? To fix the ideas, we can keep in mind the example of the motion of a damped particle, in a slowly time–dependent potential. Let us first examine the case when the static system (obtained by freezing the potential) admits a stable stationary state (a potential minimum), depending smoothly on the parameter. When the parameter is fixed, orbits starting in its neighborhood will relax to this equilibrium. When the parameter is swept slowly in time, it is generally believed that the orbit will follow the equilibrium curve adiabatically, i.e., the particle will remain close (in a sense to be made precise later) to the potential minimum.

This behaviour has the following physical interpretation: in the adiabatic limit, the asymptotic state will be identical with the static equilibrium curve. In other words, the fast system is enslaved by the slow one, its state being entirely determined by the value of the slow variables (i.e., the parameters).[2]

New phenomena arise when the equilibrium loses stability, a situation known as **bifur-**

[2]To avoid a confusion due to terminology, we point out that in thermodynamics, such a motion will be called **quasistatic** rather than adiabatic.

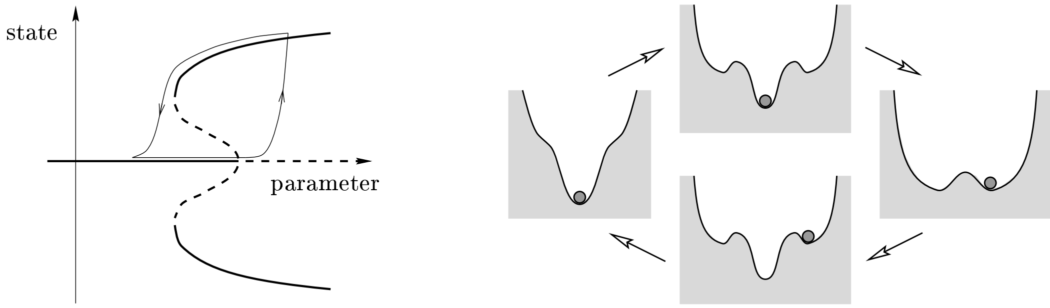

FIGURE 1.3. Example of a bifurcation diagram leading to hysteresis (a similar diagram is found in [**Wi**]). It can be seen as a combination of the bifurcations in Fig. 1.2b and d. For increasing parameter, the solution follows the stable origin at least until the bifurcation (in fact, we will see that it may even follow the unstable origin for some time). When it finally reaches the new stable branch, and the parameter is decreased again, is stays close to this branch down to a smaller value of the parameter, hence describing a hysteresis cycle.

cation. Different scenarios are possible (Fig. 1.2): the equilibrium may simply disappear, or it may become unstable after interacting with one or several other equilibria. The particle's motion depends a lot on the local structure of the bifurcation. In some cases, it leaves the vicinity of the bifurcation point, until reaching some other equilibrium or limit cycle. It may also follow a new equilibrium branch created in the bifurcation, or even remain close to an *unstable* equilibrium for some time, a phenomenon known as **bifurcation delay**, which can be interpreted as metastability. These problems belong to the field of **dynamic bifurcations**, which has received much attention in recent years.

These local features of dynamics have a strong influence on global properties. Let us focus on the situation when the parameter is varied periodically in time. Without bifurcations, the solution will merely follow the periodic motion of a stable equilibrium, independently of whether the parameter is increasing or decreasing. The situation changes in presence of bifurcations. It may happen, for instance, that the fast system follows a different equilibrium branch for increasing or decreasing parameter. This phenomenon is known as **hysteresis**: the asymptotic state depends not only on the present value of the parameter, but also on its history (Fig. 1.3).

Hysteresis can be interpreted as the non–commutation of two limits, the asymptotic and the adiabatic one. Mathematically, it is easier to take the adiabatic limit first, which amounts to freezing the slow system. The motion of the fast system is then governed by an autonomous (closed) equation, and taking the asymptotic limit merely corresponds to analysing its equilibria (or other attractors).

This is, however, not the physically interesting information. We would like instead to fix some small, but positive frequency of the parameter variation, and study the asymptotic motion of the time–dependent fast system (whatever this motion may be). *Then* we would like to determine how this asymptotic motion behaves in the adiabatic limit, i.e., when the frequency of parameter variation goes to zero.

Without bifurcation, the two limits can be taken in either order: the asymptotic motion will approach a simple function of the parameter in the adiabatic limit (see Example 1.1 below). In presence of bifurcations, hysteresis may occur. This is the central topic of this work. To understand how hysteresis arises in adiabatic Dynamical Systems, we first need to develop methods which enable us to determine solutions for small, but positive parameter sweeping rates. In particular, we have to understand if, for a periodically varied parameter, these solutions tend asymptotically to periodic ones, or if more complicated dynamics are possible. Then we will be able to study their behaviour in the adiabatic limit.

1.1.3 Historical Account

We have no intention of giving here an exhaustive historical account of the theory of Dynamical Systems with multiple time scales. Besides the fact that such an exposition would take many pages, we do not feel sufficiently well acquainted with the multiple aspects of this large domain to be able to cite correctly the numerous researchers who contributed to one (or several) of its facets. Instead, we would like to mention at this place the major sources of inspiration of this work.

Research on adiabatic systems, hysteresis and related subjects appears to have been pursued almost independently by mathematicians and physicists. The former have been mostly interested in slow–fast systems, adiabatic invariants and, more recently, in dynamic bifurcations and bifurcation delay. The latter have rediscovered several times during this century the importance of adiabatic systems. Recently, there has been renewed interest in hysteresis appearing in lasers and magnets. Different models have been considered, and studied mainly by numerical methods.

Mathematics: slow–fast systems and bifurcation delay

Slow–fast systems have been studied almost since the beginning of differential equations theory itself. They appear naturally in perturbed integrable systems, where angle variables define the fast system, and action variables the slow one. For instance, in the Solar System, fast variables describe the motion of planets in their orbits, while slow variables describe the spatial orientation of these orbits.[3] When the interaction between planets is neglected, these orbits are frozen in space, whereas they begin to deform slowly in time when their interaction is taken into account.

Research on these systems has mainly focused on the dynamics of *slow* variables. The method of averaging, for instance, aims at replacing the dynamics of the slow variables by an effective equation, where the fast variables have been averaged out [**Ar2**]. One often tries to construct **adiabatic invariants**, which are functions on phase space remaining (almost) constant in time. A highlight of this line of research is the celebrated Kolmogorov–Arnol'd–Moser (KAM) theorem, which proves the existence of exact adiabatic invariants for some initial conditions.

Adiabatic dynamics have been, for a long time, mainly studied in relation with quantum mechanics [Berry]. The **quantum adiabatic theorem** states that solutions of the slowly time–dependent Schrödinger equation will adiabatically follow the eigenspaces of the instantaneous Hamiltonian. Although this problem is relatively old, rigorous proofs have been given only very recently [JKP]. Classical adiabatic systems (mostly linear ones) have been studied in some detail by Wasow [**Wa**].

An early result on nonlinear slow–fast systems is due to Pontryagin and Rodygin [PR] in 1960. They showed that orbits of the fast system, which start sufficiently close to a

[3] See for instance Laskar's article in [**DD**] for a non–technical discussion.

stable equilibrium or limit cycle, will follow this attractor adiabatically. Problems involving bifurcations seem to have been studied for the first time by Lebovitz and Schaar [LS] in 1977. They considered problems where two equilibrium branches exchange stability, and showed that under some generic conditions, the orbit will follow a stable branch after the bifurcation.

In 1979, Haberman [Hab] considered a class of one and two–dimensional problems. He introduced the notion of slowly varying states, computed as series in the adiabatic parameter, and studied in particular jump phenomena (also known as catastrophes) occurring near saddle–node bifurcations.

The topic which would soon be given the name of **dynamic bifurcations** developed rapidly in the second half of the eighties. The importance of the **bifurcation delay** phenomenon in various physical situations (lasers, neurons) was emphasized by Mandel, Erneux and co–workers [ME1, ME2, BER], who derived an approximate formula for the delay time using slowly varying states. This phenomenon, and the related problem of **ducks** (also called **canards**) were then studied by several mathematicians, using non–standard analysis (see [**Ben**] for a summary of these works and a more detailed history).

A common feature of most of these works, including Wasow's, is that the authors try to construct particular solutions as series in the adiabatic parameter. The problem is, however, that these series are in general *not* convergent. A naive treatment of such equations may therefore yield, in some cases, incorrect results. In order to obtain the right answers with these methods (for instance the fact that there exists a maximal value for the bifurcation delay), one has to use rather elaborate techniques, as resummation of divergent series (see [**Ben**], in particular the articles by Diener and Diener, and by Canalis–Durand).

An entirely new direction to treat these problems was initiated by Neishtadt [Ne1, Ne2]. Returning to the old technique of successive changes of variables, but combined with estimations inspired by Nekhoroshev, he was able to prove rigorously the existence of a bifurcation delay. Moreover, with the help of a technique involving deformation of an integration path into the complex plane, he could give an explicit lower bound to the delay time. Diener and Diener [**Ben**] have examined under which generic conditions this formula gives an upper bound as well.

Recently, these results have been generalized to the case of a periodic orbit undergoing Hopf bifurcation [NST].

Physics: hysteresis and scaling laws

Research on hysteresis has been pursued by physicists, almost independently of mathematicians, and mostly with numerical methods. For a long time, the standard model for hysteretic phenomena has been the Preisach model [**May**, MNZ]. This model, however, is artificial and provides no derivation of hysteresis from microscopic principles.

Interest in microscopic models of magnetic hysteresis was renewed in 1990, by an important article by Rao, Krishnamurthy and Pandit [RKP]. They analyse numerically two models, an Ising model with Monte–Carlo dynamics, and a continuous model with $\mathrm{O}(N)$ symmetry in the large N limit. They proposed in particular that the area $\mathcal{A}$ enclosed by the hysteresis cycle should scale with the amplitude H_0 of the magnetic field and its frequency Ω according to the power law $\mathcal{A} \sim H_0^\alpha \Omega^\beta$, where $\alpha \sim 0.66$ and $\beta \sim 0.33$ for small frequency and amplitude.

	System	**Scaling of $\mathcal{A}$**
Experiments	Iron [Stei]	$H_0^{1.6}$
	Fe/Au film [HW]	$H_0^{0.59}\Omega^{0.31}$
	Co/Cu film [JYW]	$\mathcal{A}_0 + (H_0^2 - H_c^2)^{0.67/2}\Omega^{0.66}$
	Fe/W film [SE]	$H_0^{0.25}\Omega^{0.03}$
Numerical simulations	$(\Phi^2)^2$ model [RKP]	$H_0^{0.66}\Omega^{0.33}$
	$(\Phi^2)^2$ model [ZZS]	$\Omega^{0.5}$
	$(\Phi^2)^3$ model [ZZS]	$\mathcal{A}_0 + \Omega^{0.7}$
	Ising 2D Monte–Carlo [LP]	$H_0^{0.46}\Omega^{0.36}$
	Ising 2D Monte–Carlo [ZZL]	$\mathcal{A}_0 + \Omega^{0.36}$
	Ising 2D Monte–Carlo [AC1]	$H_0^{0.7}\Omega^{0.36}$
	Ising 3D Monte–Carlo [AC1]	$H_0^{0.67}\Omega^{0.45}$
	Cell–dynamical system [ZZL]	$\mathcal{A}_0 + \Omega^{0.66}$
	Mean field [LZ]	$\mathcal{A}_0 + H_0^{0.66}\Omega^{0.66}$
Analytical arguments	Mean field [JGRM]	$\mathcal{A}_0 + (H_0^2 - H_c^2)^{1/3}\Omega^{2/3}$
	$(\Phi^2)^2$ model [DT]	$H_0^{1/2}\Omega^{1/2}$
	$(\Phi^2)^2$ model [SD]	$H_0^{1/2}\Omega^{1/2}$
	$(\Phi^2)^2$ model [ZZ]	$\Omega^{1/2}$
	Ising dD [SRN]	$\|\ln\Omega\|^{-1/(d-1)}$

TABLE 1.2. Some results on the scaling behaviour of the area $\mathcal{A}$ enclosed by a hysteresis cycle, as a function of magnetic field amplitude H_0 and frequency Ω. Recent experiments were made with ultrathin films. Numerical Monte–Carlo simulations have been carried out on the two–dimensional (2D) and three-dimensional (3D) Ising model with Glauber dynamics. Other numerical experiments concern the Langevin equation in a Ginzburg–Landau $(\Phi^2)^2$ or $(\Phi^2)^3$ potential with $O(N)$ symmetry. In the large N limit, the noise can be eliminated from the equation, and one obtains deterministic ODE. The proposed exponents differ a lot from one experiment to another. In particular, it is not clear whether the area should go to zero, or to a finite limit $\mathcal{A}_0$ when $\Omega \to 0$. It is amusing to note that results of one experiment [JYW] could be fitted on the mean field result [JGRM], while another one [HW] was fitted on results of the $(\Phi^2)^2$ model studied in [RKP]. Although the mean–field studies in [JGRM] and [LZ] predict the same Ω–dependence, they do not agree on the H_0–dependence. In fact, we will show that both laws are incorrect.

This work inspired a large number of articles trying to exhibit scaling laws for hysteresis cycles. In the case of a laser system [JGRM], analytical arguments showed that a one–dimensional model equation admits a hysteresis cycle with area $\mathcal{A}(\Omega) \sim \mathcal{A}(0) + \Omega^{2/3}$. The discrepancy between this result and the one in [RKP] lead in following years to some controversy [Ra].

Still in the year 1990, a numerical study of a mean field approximation of the Ising model introduced the concept of a **dynamic phase transition** [TO]: regions with zero and non–zero average magnetization by cycle are separated by a transition line in the temperature–magnetic-field-amplitude plane.

These papers were followed by various numerical simulations (on lattice models and continuous ones) and experiments, which proposed new sets of exponents. We show some of them in Table 1.2. The trouble is that even for one and the same model, these exponents differ widely from one experiment to the other.

There have been several attempts to derive these exponents analytically. Relatively simple systems, like lasers, seem to be described satisfactorily by one–dimensional equations, as shown in [HL&, GBS] which extend results in [JGRM]. However, for magnetic systems, no satisfactory explanation has been obtained. Some analytical arguments, using rescaling [SD] or renormalization [ZZ] seem to indicate that the area should scale as $\mathcal{A} \sim H_0^{1/2}\Omega^{1/2}$. Various explanations have been proposed for these discrepancies, for instance logarithmic corrections [DT].

In fact, it is not clear at all whether the area should really follow a power law [SRN]. It depends probably in a crucial way on the detailed dynamics of droplets during magnetization reversal. At any rate, understanding how these scaling laws may appear in the model equations would be a good criterion to test their adequacy against real physical systems. Recently, several authors have introduced other models, including quantum effects [BDS]; they have also become interested in other indicators, like pulse susceptibility [AC2].

1.2 Mathematical Formulation

1.2.1 Adiabatic Systems and Slow–Fast Systems

We will consider dynamical systems described by ordinary differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda), \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the vector of dynamic variables, and $\lambda \in \mathbb{R}^p$ is a set of parameters. We shall assume that f is a function of class $\mathcal{C}^2$ at least.

The slow variation of parameters is described by a function $G(\varepsilon t)$, where $0 < \varepsilon \ll 1$ is the **adiabatic parameter**:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, G(\varepsilon t)). \tag{1.2}$$

This formulation should be interpreted as follows: $f(x, \lambda)$ and $G(\tau)$ are given functions, fixed once and for all,[4] and we would like to understand the behaviour of (1.2) in the

[4]One may, in fact, allow for an ε–dependence of f, provided f behaves smoothly (in some sense) in the limit $\varepsilon \to 0$, see Section 4.1.

adiabatic limit $\varepsilon \to 0$. For instance, $G(\varepsilon t) = \sin(\varepsilon t)$ would describe a periodic variation of the parameter, with small frequency ε.

The adiabatic limit should be taken with some care. If we naively replace ε by 0 in (1.2), we obtain the autonomous system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, G(0))$. This is due to the fact that with respect to the slow time scale, we have zoomed on a particular instant. This is not what we are interested in: it is more natural, for our purpose, to study the system on the slow time scale of parameter variation. We do that by introducing a **slow time** $\tau = \varepsilon t$, so that (1.2) can be rewritten

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} =: \varepsilon \dot{x} = f(x, G(\tau)). \tag{1.3}$$

We call this equation an **adiabatic system**. In the adiabatic limit, it reduces to the algebraic equation $f(x, G(\tau)) = 0$. We will see that although this limit is singular, it is less problematic to analyse than for (1.2).

By contrast, a **slow–fast system** is described by a set of coupled ODE of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y). \end{aligned} \tag{1.4}$$

In some circumstances, adiabatic and slow–fast systems are equivalent and may be transformed into one another. For instance, if $G(\tau)$ is the solution of a differential equation $\dot{y} = g(y)$, the adiabatic system (1.3) can be transformed into a slow–fast system. If $\lambda \in \mathbb{R}$, this transformation is only possible for monotonous $G(\tau)$. There are other ways to write (1.3) as a vector field, for instance by considering the slow time τ as a dynamic variable (see next subsection). In some particular cases, it may be helpful to introduce additional variables, for instance $G(\tau) = \sin \tau$ is a solution of $\dot{y} = z$, $\dot{z} = -y$.

On the other hand, if $g(x, y)$ depends only on y, the slow–fast system (1.4) is equivalent to the adiabatic system (1.3), with $G(\tau)$ given by the solution of $\dot{y} = g(y)$. If g depends on x as well, this reduction is not possible, but one can sometimes construct a solution in the following way: in first approximation, x is related to y by the algebraic equation $f(x, y) = 0$. If this equation admits a unique solution $x = x^\star(y)$, $y(\tau)$ may be approximated by a solution of the equation $\dot{y} = g(x^\star(y), y)$, which can be used in turn to estimate corrections to the solution $x(\tau) \simeq x^\star(y(\tau))$.

1.2.2 Adiabatic Systems and Vector Fields

We can exploit the similarities with slow–fast systems to obtain valuable informations on the solutions of the adiabatic system (1.3) without any analytical calculation. This is done by using geometric properties of vector fields. For simplicity, we consider the case of a scalar parameter $\lambda \in \mathbb{R}$.

It is always possible to write (1.3) as a vector field by considering the slow time τ as a dynamic variable:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x, G(\tau)) \\ \frac{\mathrm{d}\tau}{\mathrm{d}t} &= \varepsilon. \end{aligned} \tag{1.5}$$

A major drawback is that this vector field has no singular points. One can however deduce some general properties of the flow. When $f(x, G(\tau)) \neq 0$, orbits have a large slope of

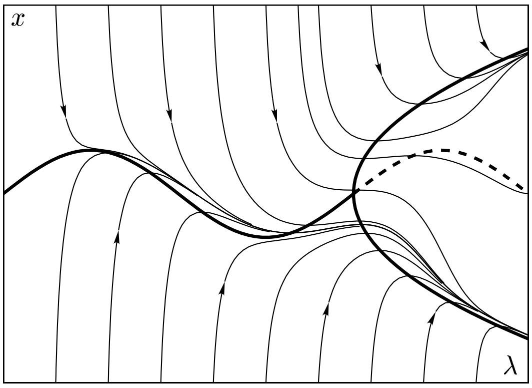

FIGURE 1.4. Solutions of the equation $\varepsilon\dot{x} = f(x,\lambda)$, for $\lambda(\tau) = \tau$ [here, the function f is given by $f(x,\lambda) = (x - \sin(\pi\lambda)/2)(\lambda - x^2)$]. The curves on which $f(x,\lambda) = 0$ (thick lines) delimit regions where the vector field has positive or negative slope. This imposes geometrical constraints on the solutions. In the left half of the picture, there exists a stable equilibrium branch. Solutions lying above this branch are decreasing, while those lying below are increasing. From this construction, one can already deduce existence of adiabatic solutions, remaining close to the equilibrium. For a special parameter value, there is a bifurcation: the equilibrium becomes unstable, and new stable branches are created. In this case, adiabatic solutions coming from the left follow the lower branch.

order ε^{-1}, due to the short characteristic time of the fast variable x. On the other hand, when $f(x, G(\tau)) = 0$, the vector field is parallel to the τ–axis. In fact, in a neighborhood of order ε of an equilibrium branch, the motion of the fast variable becomes slow: it is the region where **adiabatic solutions** (also known as slowly varying states) may exist.

In the case $x \in \mathbb{R}$ $(n = 1)$, the form of the vector field (1.5) imposes strong constraints on the solutions. It is possible to show, using only geometric arguments, that some solutions will remain in the neighborhood of equilibrium branches of f (Fig. 1.4). We will see that this property can be generalized to the n–dimensional case.

There are two particular classes of functions $G(\tau)$ for which it is possible to say more:

Monotonous case

If $G(\tau)$ is strictly monotonous, it admits an inverse function G^{-1}. We may thus use $\lambda = G(\tau)$ as a dynamic variable, giving

$$\begin{aligned} \varepsilon\dot{x} &= f(x,\lambda) \\ \dot{\lambda} &= g(\lambda) := G'(G^{-1}(\lambda)). \end{aligned} \tag{1.6}$$

If $G'(\tau)$ goes to zero in some limit, fixed points may appear in the vector field.

Consider for instance the case $G(\tau) = \operatorname{th}\tau$. Then $g(\lambda) = 1 - \lambda^2$ vanishes at $\lambda = \pm 1$. As τ goes to infinity $(\lambda \to 1)$, trajectories will be attracted by stable fixed points of $f(x, 1)$. We conclude that if λ moves infinitely smoothly from an initial to a final value, we can construct a smooth transformation which compactifies phase space, and in this way the asymptotic limit $\tau \to \infty$ can be properly defined (Fig. 1.5a).

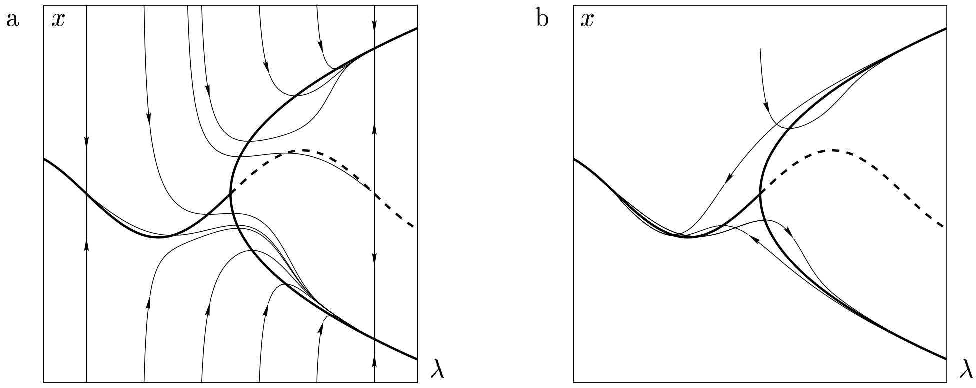

FIGURE 1.5. Same equation as in Fig. 1.4, but with (a) $\lambda(\tau) = \mathrm{th}(\tau)$ and (b) $\lambda(\tau) = \sin(\tau)$. In (a), the system admits hyperbolic fixed points at $(\pm 1, 0)$, and stable nodes at $(1, \pm 1)$, which define the asymptotic states. The stable manifold of $(1, 0)$ delimits the basins of attraction. In this case, all trajectories reach the lower equilibrium. In (b), during the first cycle the solution follows the upper branch, which is still a transient motion. From the next cycle on, it is attracted by a periodic orbit following the lower branches.

Periodic case

Assume $G(\tau)$ is periodic, say $G(\tau) = \sin\tau$. We can write the adiabatic system in the form (1.5), with the particularity that τ can be considered as a periodic variable (i.e., the phase space has the topology of a cylinder). Since the flow is transverse to every plane $\tau =$ constant, dynamics can be characterized by the **Poincaré section** at $\tau = 0$ (say), and its **Poincaré map** $T : x(0) \mapsto x(1)$. In particular, periodic orbits correspond to fixed points of T. In the one–dimensional case, this fact can be used to prove that every orbit is either periodic, or attracted by a periodic orbit.

Of course, to study hysteresis properties, we would like to go back to (λ, x)–variables, which is done by "wrapping" the (τ, x)–space (Fig. 1.5b). Some information can also be gained by using a representation of the form (1.5) on each interval in which $G(\tau)$ is monotonous. One should, however, pay attention to the fact that this transformation introduces artificial singularities in the vector field at those points where $G'(\tau)$ vanishes.

1.2.3 Some Simple Examples

Let us return to the example of the damped motion of a particle in a potential, which is described by an equation of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \Phi}{\partial x}(x, \lambda) = 0. \tag{1.7}$$

We will show in Chapter 3 that for sufficiently large friction, this system is governed by the one–dimensional equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \simeq -\frac{\partial \Phi}{\partial x}(x, \lambda). \tag{1.8}$$

Thus, any one–dimensional system of the form

$$\varepsilon \dot{x} = f(x, \lambda(\tau)) \tag{1.9}$$

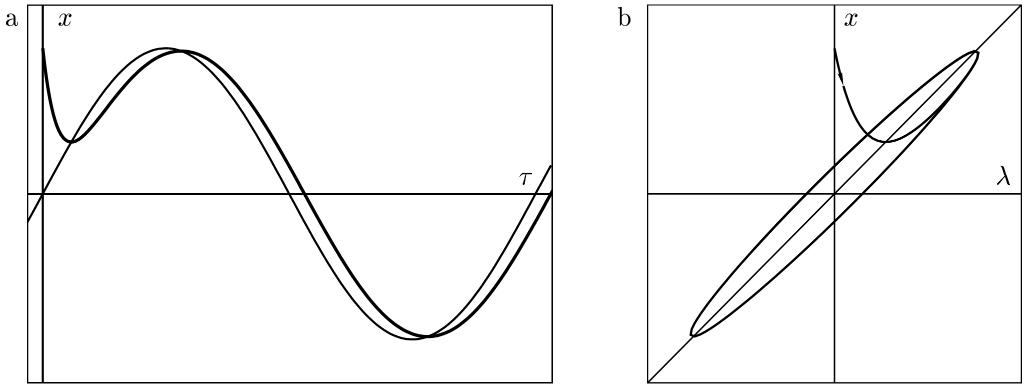

FIGURE 1.6. Solutions of the equation $\varepsilon\dot{x} = -x + \sin\tau$ of Example 1.1. (a) After a short transient, the solution $x(\tau)$ (thick line) follows adiabatically the forcing $\sin\tau$ (thin line), with a phase shift of order ε. (b) The "Lissajous plot" of this solution in the (λ, x)–plane is attracted by an ellipse, at a distance of order ε from the line $x = \lambda$. This cycle encloses an area of order ε which vanishes in the adiabatic limit, thus we do not consider it as a hysteresis cycle.

may be interpreted as describing the overdamped motion of a particle in a slowly varying potential $\Phi(x, \lambda(\tau))$, with $f \simeq -\partial_x\Phi$. Let us examine some particular cases, in order to illustrate the previously discussed concepts.

Example 1.1. Consider the equation

$$\varepsilon\dot{x} = -x + \lambda(\tau), \qquad \lambda(\tau) = \sin\tau, \tag{1.10}$$

which describes the motion of an overdamped, adiabatically forced harmonic oscillator. It can be solved explicitly, with the result

$$x(\tau) = \left[x(0) + \frac{\varepsilon}{1+\varepsilon^2}\right]\mathrm{e}^{-\tau/\varepsilon} + \frac{1}{1+\varepsilon^2}(\sin\tau - \varepsilon\cos\tau). \tag{1.11}$$

The second term is a periodic particular solution of (1.10). It follows the forcing $\lambda(\tau)$ with a phase shift of order ε (Fig. 1.6a). This is precisely what we call an **adiabatic solution**, since it remains in a neighborhood of order ε of the static equilibrium $x = \lambda(\tau)$. In the (λ, x)–plane, it is represented by an ellipse with width of order ε (which can be interpreted as a Lissajous plot of the solution), see Fig. 1.6b.

The first term in (1.11) is a transient one, which decreases exponentially fast. In fact, it is of order ε as soon as $\tau \geqslant \tau_1(\varepsilon) = \varepsilon|\ln\varepsilon|$. Since $\lim_{\varepsilon\to 0}\tau_1(\varepsilon) = 0$, we may write

$$\lim_{\varepsilon\to 0} x(\tau;\varepsilon) = \lambda(\tau) \qquad \text{for } \tau > 0. \tag{1.12}$$

The state of the system is thus determined entirely by the slow variable λ. According to the discussion of Subsection 1.1.2, we are in a situation without hysteresis, since the adiabatic and asymptotic limit commute. Indeed, the physically meaningful procedure is to take the asymptotic limit first: we find that trajectories converge to the periodic solution $\bar{x}(\tau, \varepsilon) = (\sin\tau - \varepsilon\cos\tau)/(1 + \varepsilon^2)$. Then we see that $\bar{x}(\tau, \varepsilon)$ tends to $\lambda(\tau)$ in the adiabatic limit $\varepsilon \to 0$. On the other hand, taking the adiabatic limit directly in (1.10) yields the correct result $x = \lambda(\tau)$.

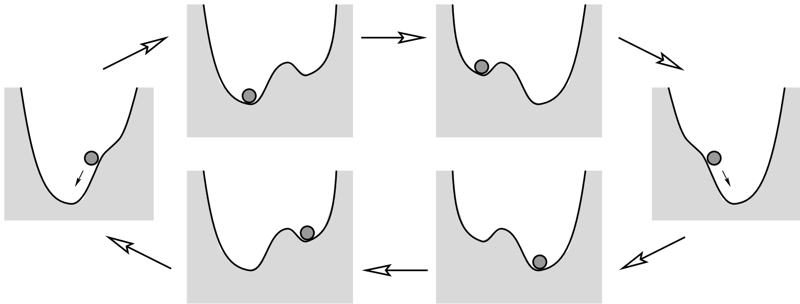

FIGURE 1.7. Equation (1.14) can be interpreted as describing the overdamped motion of a particle in a slowly varying potential of the form shown here. When $\lambda < -\lambda_c$, the particle joins the equilibrium $x^\star_-(\lambda)$. For $-\lambda_c < \lambda < \lambda_c$, a new minimum $x^\star_+(\lambda)$ has appeared, but the particle still remains in the left well. Only at $\lambda = \lambda_c$, when the left equilibrium disappears, will the particle join $x^\star_+(\lambda)$, which it follows as long as $\lambda > -\lambda_c$. For intermediate values of λ, the position of the particle depends not only on $\lambda(\tau)$, but also on $\dot\lambda(\tau)$: the system displays hysteresis.

The fact that all orbits are attracted by a periodic one can also be seen on the Poincaré map (taken at $\tau = 0 \equiv 2\pi$), which reads

$$T : x \mapsto \left[x + \frac{\varepsilon}{1+\varepsilon^2}\right] \mathrm{e}^{-2\pi/\varepsilon} - \frac{\varepsilon}{1+\varepsilon^2}, \tag{1.13}$$

and admits a stable fixed point at $x^\star = -\varepsilon/(1+\varepsilon^2)$. Let us finally point out that the fact that the periodic solution $\bar x(\tau;\varepsilon)$ admits a convergent series in ε is rather exceptional, in general we will only be able to obtain asymptotic series.

Example 1.2. The equation

$$\varepsilon\dot x = x - x^3 + \lambda(\tau), \qquad \lambda(\tau) = \sin\tau, \tag{1.14}$$

describes the overdamped motion of a particle in a Ginzburg–Landau type double–well potential $\Phi(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, with an external field λ (Fig. 1.7). This is the most common example for hysteresis in ODE found in textbooks [**MR**, **MK&**].

Taking the adiabatic limit $\varepsilon \to 0$ in (1.14), we obtain the algebraic equation of a cubic $\lambda = -x + x^3$, admitting stationary points $\pm(x_c, -\lambda_c)$, where $x_c = 1/\sqrt{3}$ and $\lambda_c = 2/3\sqrt{3}$. When $|\lambda| > \lambda_c$, there is a single solution $x^\star(\lambda)$, which corresponds to a stable equilibrium of the static system. But when $|\lambda| < \lambda_c$, there are *three* equilibrium curves (two stable and one unstable), and it is not clear from this analysis which one the trajectory will follow. Let us denote by $x^\star_+(\lambda)$ the upper stable equilibrium, and $x^\star_-(\lambda)$ the lower one.

Despite its simplicity, equation (1.14) admits no exact solution. But the qualitative behaviour of orbits can be easily understood by drawing the vector field in the (τ, x)–plane (Fig. 1.8a). Starting for instance at $(\tau, x) = (0, 0)$, the orbit will be attracted by the upper branch $x^\star_+$, and follow it until it disappears when λ becomes smaller than $-\lambda_c$. If ε is small enough, the trajectory will quickly reach the lower branch $x^\star_-$, and follow it until λ becomes larger than λ_c again. This behaviour will repeat itself periodically, and it can be checked (using only geometric properties of the Poincaré map) that the trajectory is attracted by a periodic solution $\bar x(\tau;\varepsilon)$. We thus obtain an asymptotic cycle, characterized by alternating phases with slow and fast motion. Such a solution is called a **relaxation oscillation**.

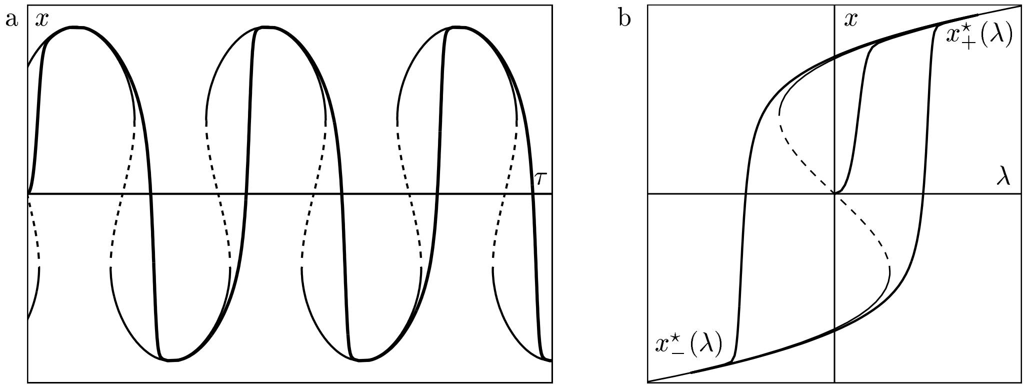

FIGURE 1.8. Solutions of equation (1.14), (a) in the (τ, x)–plane and (b) in the (λ, x)–plane. Thin full lines indicate stable equilibria of the static system, dashed lines indicate unstable equilibria. These curves are solutions of the equation $x - x^3 + \lambda = x - x^3 + \sin\tau = 0$. The first representation (a) is useful to draw the vector field. One easily understands that the solution follows stable branches until the next saddle–node bifurcation, and then moves rapidly to the other branch. We obtain a periodic solution with alternating slow and fast motions, called a **relaxation oscillation**. When this solution is wrapped to the (λ, x)–plane, we obtain a familiar–looking hysteresis cycle. In the limit $\varepsilon \to 0$, this cycle approaches a curve delimited by the equilibrium branches $x_\pm^\star(\lambda)$ and two verticals.

When wrapping this solution to the (λ, x)–plane, we obtain that

$$\lim_{\varepsilon\to 0} \bar{x}(\tau;\varepsilon) = \begin{cases} x_+^\star(\lambda) & \text{if } \lambda > \lambda_{\mathrm{c}} \text{ or } \lambda > -\lambda_{\mathrm{c}} \text{ and } \dot{\lambda} < 0, \\ x_-^\star(\lambda) & \text{if } \lambda < -\lambda_{\mathrm{c}} \text{ or } \lambda < \lambda_{\mathrm{c}} \text{ and } \dot{\lambda} > 0. \end{cases} \tag{1.15}$$

This solution displays the most familiar type of hysteresis. When $|\lambda| < \lambda_{\mathrm{c}}$, the asymptotic state (in the adiabatic limit) depends not only on λ, but also on its derivative.[5]

The limiting hysteresis cycle (Fig. 1.8b) has a well–defined area, given by the geometric formula

$$\mathcal{A}(0) = 2\int_{-\lambda_{\mathrm{c}}}^{\lambda_{\mathrm{c}}} x_+^\star(\lambda)\,\mathrm{d}\lambda. \tag{1.16}$$

It is clear from the vector field analysis that $\mathcal{A}(\varepsilon)$ increases with ε. In fact, it has been shown in [JGRM] that

$$\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{2/3}. \tag{1.17}$$

We will show in Chapter 4 that this exponent $2/3$ can be computed in a very simple way, using only local properties of the bifurcation points. We point out that in this example, we have assumed the amplitude of $\lambda(\tau)$ to be larger than λ_{c}, so that $x(\tau)$ necessarily changes sign. We will examine in Chapter 7 what happens when the amplitude approaches λ_{c}.

[5] If $\lambda(\tau)$ is a more complicated function than $\sin\tau$, admitting several different maxima and minima, it may require more information than $\lambda(\tau)$ and $\dot{\lambda}(\tau)$ to compute the asymptotic state at time τ. In fact, this state will depend on the velocity of the last passage of $\lambda(\tau)$ through $\pm\lambda_{\mathrm{c}}$.

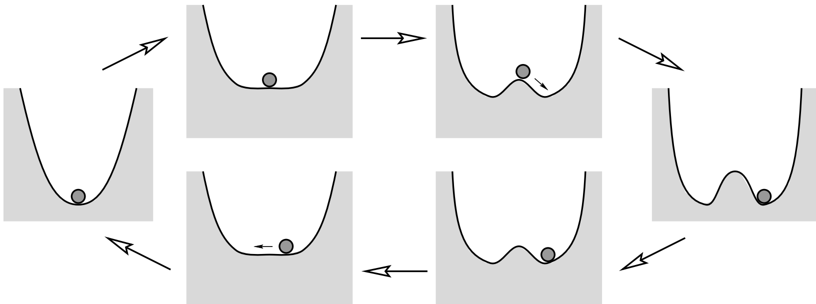

FIGURE 1.9. The potential corresponding to equation (1.18). For negative λ, the particle joins the single well at the origin. When this equilibrium becomes unstable, and two new wells are formed, the particle does nor react immediately to the bifurcation: it remains for some time in unstable equilibrium near the origin. This situation is called **delayed bifurcation**. Finally, the particle chooses a potential minimum, and follows it until the minima merge to form a single well again. This system displays hysteresis.

Example 1.3. The equation

$$\varepsilon \dot{x} = \lambda(\tau)x - x^3, \qquad \lambda(\tau) = \sin\tau, \tag{1.18}$$

describes the overdamped motion in a potential $\Phi(x,\lambda) = -\frac{1}{2}\lambda x^2 + \frac{1}{4}x^4$. In the Ginzburg–Landau analogy, the parameter λ controls the temperature. The potential has a single well at the origin if $\lambda < 0$ ($T > T_c$, T_c being the critical temperature of a phase transition), and a double well if $\lambda > 0$ ($T < T_c$). In the language of dynamical systems, we have a pitchfork bifurcation at $\lambda = 0$. For positive λ, the adiabatic system has to choose between two stable equilibria $\pm\sqrt{\lambda}$ and the unstable origin.

Equation (1.18) admits the explicit solution

$$x(\tau) = \frac{x(\tau_0)\,\mathrm{e}^{\alpha(\tau)/\varepsilon}}{\sqrt{1 + \dfrac{2}{\varepsilon}x(\tau_0)^2 \displaystyle\int_{\tau_0}^{\tau} \mathrm{e}^{2\alpha(s)/\varepsilon}\,\mathrm{d}s}}, \qquad \alpha(\tau) := \int_{\tau_0}^{\tau} \lambda(s)\,\mathrm{d}s = \cos\tau_0 - \cos\tau. \tag{1.19}$$

It is not straightforward to analyse this solution analytically. Let us consider the special case $\tau_0 = -\pi/2$, $\xi(\tau_0) = 1$. Then

$$x(\tau) = \frac{\mathrm{e}^{-\cos\tau/\varepsilon}}{\sqrt{1 + \dfrac{2}{\varepsilon}\displaystyle\int_{-\pi/2}^{\tau} \mathrm{e}^{-2\cos s/\varepsilon}\,\mathrm{d}s}}. \tag{1.20}$$

For $-\pi/2 < \tau < \pi/2$, $-\cos\tau$ is negative, and the behaviour is governed by the numerator $\mathrm{e}^{-\cos\tau/\varepsilon}$, which is exponentially small. Thus, the solution remains exponentially close to the origin until $\tau = \pi/2$. For negative τ, this is not surprising, since the origin is stable. Although the origin becomes unstable at $\tau = 0$, the trajectory still remains close to it until $\tau = \pi/2$. This is a simple example of **bifurcation delay**: the effective bifurcation takes place at $\tau = \pi/2$ rather than at $\tau = 0$. In the Ginzburg–Landau analogy, this phenomenon may be interpreted as **metastability**.

When $\tau > \pi/2$, the solution leaves the origin, and, in fact, settles near the equilibrium position at $x = \sqrt{\sin\tau}$ until $\tau = \pi$, when this branch merges with the origin again.

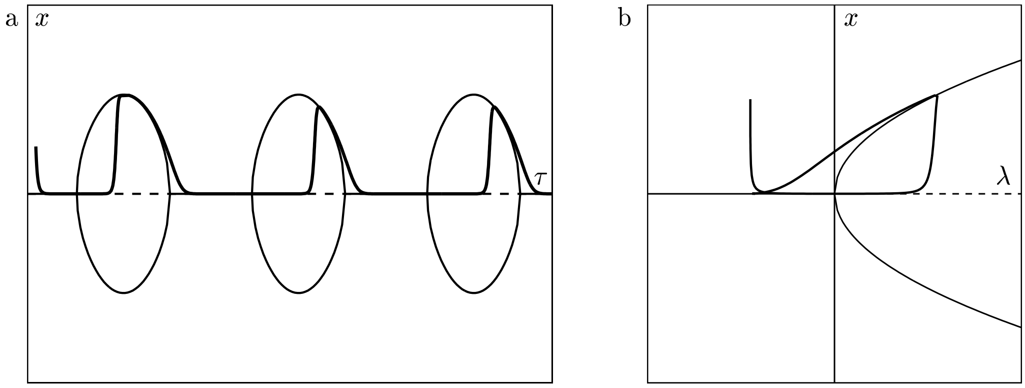

FIGURE 1.10. Solutions of equation (1.18), (a) in the (τ, x)–plane and (b) in the (λ, x)–plane, but for the function $\lambda(\tau) = \sin\tau + 0.1$. Thin full lines indicate stable equilibria of the static system, dashed lines indicate unstable equilibria. These curves are solutions of the equation $\lambda x - x^3 = 0$. When the origin is stable, solutions reach it after a short time. They follow the origin for some macroscopic time after it has become unstable, a phenomenon known as **bifurcation delay**. If the solution finally jumps on another equilibrium, we obtain a hysteresis cycle.

One can show (using for instance the saddle point method to estimate the integral in (1.20)) that $x(\pi) = \mathcal{O}(\varepsilon^{1/4})$. If we plot this solution in the (λ, x)–plane, we find that the bifurcation delay leads to **hysteresis**, since the trajectory always follows a stable branch for decreasing λ, but sometimes follows an unstable one for increasing λ.

In fact, the solution analysed here is still a transient one. During the next cycle of λ, the bifurcation delay is so large that the trajectory ends up by always following the origin. But it is sufficient to add an offset to $\lambda(\tau)$, of the form $\lambda(\tau) = \sin\tau + c$, to obtain an asymptotic hysteresis cycle as in Fig. 1.10b. We will show that its area scales as

$$\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{3/4}. \tag{1.21}$$

Considering the one–dimensional equations studied in these three examples, we observe that

- solutions of a periodically forced system are always attracted by periodic ones;
- without bifurcations, the periodic solution encloses an area of order ε, and does not display hysteresis in the adiabatic limit;
- when bifurcations are present, the periodic solution displays hysteresis, and encloses an area which follows a scaling relation of the form $\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{\mu}$, where μ is a nontrivial, fractional exponent.

One of the goals of this work will be to find out if these properties remain valid for more general equations. We will see that asymptotic solutions are not necessarily periodic. However, if such a periodic solution exists, its area will usually follow a scaling law of the above mentioned form, with an exponent μ which can be computed in a relatively simple way.

1.3 About this Thesis

1.3.1 Objectives

We pursue two major objectives in this work:

1. Establish a coherent mathematical framework in order to deal with adiabatic systems of the form (1.3). In particular, we would like to understand the relation between an adiabatic system, and the corresponding family of autonomous equations. We are also interested in developing some practical tools allowing to establish existence of periodic orbits and hysteresis cycles, and to determine their scaling behaviour as a function of the adiabatic parameter.
2. Apply these methods to some concrete examples. This should allow to check their efficiency to deal with a given equation, and to detect aspects of the theory which need further development. Since many authors, after spending much effort to derive equations describing magnetic hysteresis, analyse them by numerical simulations, we would like to show how the theory of Dynamical Systems can be used to obtain valuable information on such equations with relatively small effort.

As we discussed in Subsection 1.1.3, much work has already been done on adiabatic systems, in particular on bifurcation delay. We feel, however, that this work is worth extending in two directions. Firstly, results obtained by mathematicians are often formulated in a rather abstract language, which is not easily accessible to the average (even theoretical) physicist. Thus it is certainly useful to translate them into a language facilitating their application to concrete problems. Secondly, several aspects of the fundamental theory still need to be clarified. For instance, hysteresis itself, and the associated scaling behaviour, have almost not been studied by mathematicians. We also discovered, when analysing particular examples, that several basic concepts still needed to be developed, for instance adiabatic manifolds.

We have chosen two types of applications. The first one, which we call nonlinear oscillators, concerns various situations where a damped particle is placed in a slowly varying force field. Such low-dimensional Dynamical Systems are interesting for several reasons: we have some physical intuition for their behaviour; they are sufficiently simple to be analysed in great detail, so that we have a better chance to understand fundamental mechanisms of hysteresis; still, some of these systems are known to exhibit chaotic motion when forced periodically, and it is important to understand what happens when this forcing becomes adiabatic.

As a second application, we will consider a few models of magnetic hysteresis. This program appears to be much more ambitious, since magnets are so complicated systems that it is not clear at all whether they may be modeled by finite dimensional equations. We think, however, that such an attempt is justified by the mere fact that it will reveal both the power and limits of such a kind of modeling. It may give some hints as to what characteristics a realistic model should include, and in what directions the theory should be extended in order to give more reliable predictions.

1.3.2 Philosophy

In this work, we adopt the point of view of Mathematical Physics. This implies that physicists may regard it as an unnecessarily pedantic way of establishing evidences, while mathematicians may consider it as a pedestrian approach to a problem, which might be described much more nicely using non–standard analysis and Borel series.

To the former, we would like to point out that there exist numerous examples of problems, for which it was considered as evident that their solutions behave in some special way, until this evidence was proved wrong by a serious analysis. The precise mathematical understanding of a problem is always desirable when it reveals the power and limits of an empirical approach. For the latter, we would like to underline that our work aims at providing a method of practical use, allowing the physicist to obtain useful information on a concrete adiabatic system, with a minimum of technical tools.

There are different approaches to Mathematical Physics. One of them relies on exact solutions. We believe that this approach is useful as far as it provides very precise information on a particular model equation, which is assumed to be generic. There are, however, two major drawbacks: Firstly, differential equation which can be solved exactly are very scarce, so that only very few model equations are likely to be analysable in this way. Secondly, even when a system has been solved exactly, the interesting features are not immediately apparent, and it may require a lot of hard analysis to derive them. This approach does not favour the physical intuition, and often yields incorrect interpretations.[6]

A good illustration of these difficulties is provided by Example 1.3: this system is still relatively simple to solve, if one knows about Bernouilli's equation. But it turns out that the important phenomenon, namely bifurcation delay, can also be obtained in a much simpler way, by studying the linearized equation $\varepsilon\dot{x} = \lambda(\tau)x$. The behaviour of solutions far from the origin can be analysed by different methods, that do not depend on the detailed form of the nonlinear term, which is necessary for the equation to be exactly solvable. We will show that even the scaling law (1.21) can be obtained using only a local analysis around the bifurcation point.

We will thus prefer those methods which favour the physical intuition. To analyse some complicated equation, one has to understand first which terms are important, and which terms have a negligible influence. Then one starts by solving the simplified equation containing only the important terms. Perturbative methods are often well adapted so such a procedure.

But one has to be careful not to confuse perturbation and approximation. It is very tempting (and often done) to assume that a solution can be written as power series of some small parameter, to insert this series into the equation, and to solve it for the first few orders. This procedure is often dangerous, since these power series do usually *not* converge.[7] In fact, it is better to apply perturbation theory to the equation than to its solution.

We will often proceed in two steps. Firstly, we will derive an iterative scheme that allows to decrease the order of some remainder in the equation, which prevents us from solving it. Secondly, we have to prove in an independent way that the influence of this small remainder on the solution can be *bounded*. Thus, if we write that a solution contains a remainder $R = \mathcal{O}(\varepsilon)$, we mean something very precise: namely there exist positive constants c and ε_0 such that $|R| \leqslant c\varepsilon$ for $0 < \varepsilon \leqslant \varepsilon_0$. These constants are independent of ε and could be computed explicitly (although this computation may turn out to be quite

[6] A startling example of such a misunderstanding is found in [AC1], who analyse an equation linearized around an *unstable* equilibrium, which is never reached by the solution.

[7] For instance, the perturbative analysis of a Hopf bifurcation in [ME1, BER] fails to reveal the phenomenon of maximal delay.

cumbersome).

Once such a bound on the remainder is known, when applying the theory to a concrete example one can forget about the proof of the second part, and use the iterative scheme to determine the behaviour of the solution at leading order in the small parameter. It appears that the bounds c and ε_0 are often far too pessimistic, and that, at least for finite dimensional systems, the asymptotic theory provides rather accurate information for fairly "large" values of the "small" parameter.

1.3.3 Reader's Guide

There are different ways to write a Ph.D. dissertation. One can choose to present a summary of the major results, or one can give a more detailed account with background information on the subject and complete proofs. We preferred the second possibility. We believe that such a detailed presentation is justified for the kind of subject we have been working on, which lies on the boundary of Mathematics and Physics, provided the structure of the text is sufficiently apparent.

Although the chapters of this dissertation are not self–contained, we tried to write them as far as possible in an independent way. Thus it is certainly not necessary, in order to understand the contents of a given chapter, to have read all the preceding ones.

Roughly speaking, Chapters 2 and 3 present some aspects of mathematical and physical theory which are already well known. Chapters 4 and 5 are dedicated to the abstract, mathematical theory which we have developed to deal with adiabatic Dynamical Systems. Chapters 6 and 7 provide applications of this theory to some concrete examples. Chapter 8 contains extensions of some results to iterated maps.

Chapter 2: Mathematical Tools

We present some important notions from the theory of Dynamical Systems (as equilibrium points, stability and Lyapunov functions, invariant manifolds, bifurcations and normal forms), and some elements of analysis which are used in the proofs (Banach spaces, Fréchet derivatives, asymptotic series and differential equations). The notations we use do in general not differ from standard ones. This chapter should be considered as a reference chapter, and the reader who is already well acquainted with the theory, as well as the reader who is not interested in detailed mathematics, can safely skip it.

Chapter 3: Physical Models

We describe some physical concepts on which rely the examples discussed in Chapters 6 and 7. The damped motion of a particle in a potential serves as paradigm for a wide class of Dynamical Systems. We discuss the most common models for ferromagnets at equilibrium and out of equilibrium, and show how to derive a deterministic evolution equation from the governing master equation. Finally, we briefly present some existing phenomenological models of hysteresis.

Chapter 4: One–Dimensional Systems

We present a detailed mathematical framework to deal with one–dimensional adiabatic equations of the form $\varepsilon\dot{x} = f(x,\tau)$. We first discuss the properties of adiabatic solutions,

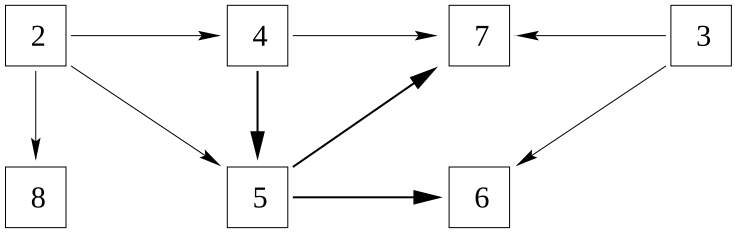

FIGURE 1.11. Logical organization of the chapters. Arrows indicate that a chapter relies on the contents of another one, in an essential way if the arrow is thick.

which are particular solutions remaining close to non–bifurcating equilibria, and admitting asymptotic series in the adiabatic parameter ε. We then analyse in detail the dynamics near bifurcation points, in particular the way how they scale with ε. We provide a simple geometrical method, based on the Newton polygon, to determine the scaling exponents. Finally, we discuss some global aspects of the flow, in particular how to determine periodic orbits, and the ε–dependence of hysteresis cycles.

Chapter 5: n–Dimensional Systems

We extend results of the previous chapter to n–dimensional equations. The discussion of adiabatic solutions is quite similar to the 1D case. We then examine the linear equation $\varepsilon\dot{y} = A(\tau)y$, which describes the linearized motion around an adiabatic solution. This is a rather lengthy task, but we show that the problem of diagonalizing such an equation can be transformed into the problem of finding adiabatic solutions of an auxiliary equation. This transformation allows to treat eigenvalue crossings and bifurcations in a unified way. Next we develop some tools to deal with nonlinear terms, in particular adiabatic manifolds and dynamic normal forms. Finally, we examine some global properties of the flow.

Chapter 6: Nonlinear Oscillators

We consider different examples involving the damped motion of a particle in a slowly varying potential. The most important one is equivalent to a simple physical system, namely a pendulum on a table rotating with slowly modulated angular frequency. This system displays two important phenomena: a bifurcation delay leading to hysteresis, and the possibility of a chaotic motion, even for arbitrarily small adiabatic parameter. We use the methods developed in Chapters 4 and 5 to compute an asymptotic expression of the Poincaré map, which allows to delineate precisely the parameter regions where hysteresis and chaos occur. The other two examples discussed in this chapter illustrate the effect of eigenvalue crossings.

Chapter 7: Magnetic Hysteresis

We discuss a few simple models of ferromagnets in a slowly varying magnetic field. When the interaction between spins has infinite range (i.e., in a Curie–Weiss type model), the dynamics can be described in the thermodynamic limit by a low–dimensional differential equation, of Ginzburg–Landau type. We examine the phenomenon of dynamic phase transition for 1D spins, and the effect of anisotropy on the dynamics of 2D spins. Finally, we explain why models with short range interaction are much harder to analyse, and present a few simple approximations.

Chapter 8: Iterated Maps

We extend some of the previous results to adiabatic iterated maps. We start by showing that some basic properties of adiabatic ODEs, such as existence of adiabatic solutions and the behaviour of linear systems, can be extended to maps depending on a slowly varying parameter. We conclude by presenting some results on existence of adiabatic invariants for slow–fast maps, and illustrate them on a few billiard problems.

Chapter 2

Mathematical Tools

"J'ai lu une fois un article d'un professeur de l'EPFL qui disait que les mathématiques ne servent qu'à faire un peu de physique et de comptabilité. J'aimerais répondre par un coup de pied au derrière ... Au–dessous d'un certain niveau, il n'y a plus de réponse rationnelle possible."

Prof. M. Ojanguren, Université de Lausanne

"Basically, a tool is an object that enables you to take advantage of the laws of physics and mechanics in such a way that you can seriously injure yourself."

Dave Barry, "The Taming of the Screw"

In this Chapter, we introduce the basic mathematical tools used throughout this work. Our purpose is to collect at the same place a number of concepts and methods from the theory of Dynamical Systems that we will need, together with the major notations, definitions and mathematical results which are necessary for their quantitative analysis. We try to expose the theory in a consistent way, even though we do not aim at giving an exhaustive account of the considered subjects. We give only a limited number of illustrative proofs, referring to relevant literature at the beginning of each section.

In this exposition, we voluntarily distinguish between the analytical and geometrical aspects of the theory of Dynamical Systems.

- **Section 2.1** We start by recalling a few notions from elementary functional analysis, linear algebra and complex analysis, stressing in particular the differences between differentiability and analyticity, convergent and asymptotic series.
- **Section 2.2** We state the basic results on existence, unicity and regularity of solutions. We also present the very limited number of exact solutions that we will use, and discuss some properties of linear differential equations.
- **Section 2.3** We explain the basic concepts of the geometric theory of Dynamical Systems, which originated in the remarkable work of H. Poincaré one century ago. We discuss some properties of the orbits of flows and iterated maps, in particular the linear and nonlinear stability of singular points and periodic orbits.
- **Section 2.4** We push further the analysis of the effect of nonlinear terms near singular points. We present the main theories of reduction to linear or lower dimensional systems, and some topics of bifurcation theory.

2.1 Basic Analysis

We begin by introducing some function spaces in which the orbits of our Dynamical Systems will live, and provide them with the necessary structure for applying the methods of analysis. We then recall a few properties of matrices. The reason is that the average physicist, because of the strong influence of Quantum Mechanics, is used to working with self–adjoint linear operators, whereas in Dynamical Systems we are usually confronted with non–normal and even non–diagonalizable matrices.

In the next subsections, we discuss the notions of differentiability and power series. It is, indeed, very tempting to try to expand solutions of differential equations as power series. Unfortunately, in the singular perturbation problems that we will consider, these series usually do not converge, and we have to use the concept of asymptotic series.

The most important theorems in this section are

1. the Banach fixed point theorem (Theorem 2.1),
2. the Jordan decomposition of matrices (Theorem 2.4),
3. the implicit function theorem (Theorem 2.7),
4. and Cauchy's formula (Theorem 2.8).

We follow mostly the books of Hale [**Hal**] and Wasow [**Wa**]. For basic analysis, see [**Sch**]. Properties of matrices are discussed in [**Bel**].

2.1.1 Banach Spaces

Notation 2.1. In this section, $i, j, m, n \in \mathbb{N}$ will denote positive integers. $\delta_{ij} = 1$ if $i = j$, 0 otherwise, is the Kronecker symbol. $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. $|x|$ denotes the absolute value of $x \in \mathbb{R}$, $|z|$ the module of $z \in \mathbb{C}$, z^* its conjugate and $\arg z$ its argument, while $\operatorname{Re} z$ and $\operatorname{Im} z$ denote its real and imaginary part.

Definition 2.1.

1. A $\mathbb{K}$–**vector space** is a commutative group $(\mathcal{E}, +)$ with an action of $\mathbb{K}$, that is a map $\mathbb{K} \times \mathcal{E} \to \mathcal{E}$, $(a, x) \mapsto ax$, such that $(ab)x = a(bx)$, $a(x+y) = ax+ay$, $(a+b)x = ax+bx$ and $1x = x$ for all $a, b \in \mathbb{K}$ and $x, y \in \mathcal{E}$.
2. A **norm** $\|\cdot\|$ on a $\mathbb{K}$–vector space $\mathcal{E}$ is a map $\|\cdot\| : \mathcal{E} \to \mathbb{R}$ such that $\|x\| \geqslant 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{K}$ and $x, y \in \mathcal{E}$. The pair $(\mathcal{E}, \|\cdot\|)$ is called a **normed vector space**.

The norm allows us to define a **topology**, and thus the usual notions of open and closed sets, convergence of a sequence and continuity of a function. In particular, we have the following notions of convergence.

Definition 2.2. Let $(\mathcal{E}, \|\cdot\|)$ be a normed vector space. A sequence $(x_n)_{n \geqslant 0}$ of elements in $\mathcal{E}$ **converges towards** $x \in \mathcal{E}$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. A sequence $(x_n)_{n \geqslant 0}$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there is an $N(\varepsilon) > 0$ such that $\|x_n - x_m\| < \varepsilon$ if $n, m \geqslant N(\varepsilon)$. The space $\mathcal{E}$ is **complete** if every Cauchy sequence converges towards an $x \in \mathcal{E}$. A complete normed vector space is called a **Banach space**.

For instance, the set $\mathbb{Q}$ of rational numbers is not complete since the sequence of rationals $\{x_n \in \mathbb{Q} \mid x_0 = 1, x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\}$ converges towards $\sqrt{2}$ which is not in $\mathbb{Q}$. The smallest complete space (for the norm $|\cdot|$) containing $\mathbb{Q}$ is the set of real numbers $\mathbb{R}$.

Notation 2.2.

1. Let $\mathbb{K}^n = \{x = (x[1], \ldots, x[n]) \,|\, x[i] \in \mathbb{K}, i = 1, \ldots, n\}$ with sum and action of $\mathbb{K}$ defined component by component. We introduce the following norms on $\mathbb{K}^n$:
$$\|x\|_1 := \sum_{i=1}^{n} |x[i]|, \quad \|x\|_2 := \left[\sum_{i=1}^{n} |x[i]|^2\right]^{1/2}, \quad |x| := \max_{1 \leqslant i \leqslant n} |x[i]|. \tag{2.1}$$
$\|x\|_2$ is called the **Euclidean norm** and $|x|$ the **sup norm**, while $\|x\|_1$ is sometimes called **Manhattan norm**.
2. Let $\mathcal{D}$ be a compact subset of $\mathbb{K}^n$, we denote by $\mathcal{C}^0(\mathcal{D}, \mathbb{K}^m)$ the set of continuous functions $f : \mathcal{D} \to \mathbb{K}^m$. We introduce the norms
$$\|f\|_1 := \int_{\mathcal{D}} |f(x)| \,\mathrm{d}x, \quad \|f\|_2 := \left[\int_{\mathcal{D}} |f(x)|^2 \,\mathrm{d}x\right]^{1/2}, \quad |f| := \sup_{x \in \mathcal{D}} |f(x)|. \tag{2.2}$$

Proposition 2.1. *The following spaces are Banach spaces:*

1. $(\mathbb{K}^n, \|\cdot\|)$*, for any of the norms* (2.1).
In fact we have $|x| \leqslant \|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{n}\,\|x\|_2 \leqslant n|x|$. *Thus, a sequence converging in one of these norms will converge in all others.*
2. $(\mathcal{C}^0(\mathcal{D}, \mathbb{K}^m), |\cdot|)$. *A sequence of functions converging with this norm is said to* **converge uniformly** *on* $\mathcal{D}$.

$\mathcal{C}^0(\mathcal{D}, \mathbb{K}^m)$ is *not* a Banach space for the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. For instance, a discontinuous function on $\mathcal{D}$ admits a Fourier series, the terms of which are continuous (see Example 2.1 below).

Banach spaces are useful because it is much easier to show that a sequence is a Cauchy sequence than to show its convergence (since the limit is often unknown). An important application is the *fixed point theorem*:

Definition 2.3. Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be Banach spaces, $\mathcal{D} \subset \mathcal{E}$ and $T : \mathcal{D} \to \mathcal{F}$. T is said to be a **contraction** if there exists $\lambda \in [0, 1)$ (called **contraction constant**) such that $\|Tx - Ty\|_{\mathcal{F}} \leqslant \lambda \|x - y\|_{\mathcal{E}}$ $\forall x, y \in \mathcal{D}$.

Theorem 2.1 (Banach–Cacciopoli). *If* $\mathcal{D}$ *is a closed subset of a Banach space* $(\mathcal{E}, \|\cdot\|)$ *and* $T : \mathcal{D} \to \mathcal{D}$ *is a contraction, there is a unique* $x^\star$ *in* $\mathcal{D}$ *such that* $Tx^\star = x^\star$; $x^\star$ *is called a* **fixed point**. *For any* $x_0 \in \mathcal{D}$, *the sequence* $\{x_n \,|\, x_{n+1} = Tx_n\}$ *converges to* $x^\star$ *with* $\|x_n - x^\star\| \leqslant \lambda^n \|x_1 - x_0\|/(1 - \lambda)$ *where* λ *is the contraction constant of* T.

Notation 2.3. Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ and $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ be Banach spaces and $f : \mathcal{E} \to \mathcal{F}$ $g : \mathcal{E} \to \mathcal{G}$ be continuous in a neighborhood of $x = 0$ (except possibly at $x = 0$). We write
$$f(x) = \mathcal{O}(g(x)) \qquad \text{if} \qquad \lim_{\|x\|_{\mathcal{E}} \to 0} \frac{\|f(x)\|_{\mathcal{F}}}{\|g(x)\|_{\mathcal{G}}} < \infty, \tag{2.3a}$$
$$f(x) = o(g(x)) \qquad \text{if} \qquad \lim_{\|x\|_{\mathcal{E}} \to 0} \frac{\|f(x)\|_{\mathcal{F}}}{\|g(x)\|_{\mathcal{G}}} = 0. \tag{2.3b}$$

2.1.2 Hilbert Spaces

Definition 2.4. Let $\mathcal{E}$ be a $\mathbb{K}$–vector space. A **scalar product** (or **inner product**) on $\mathcal{E}$ is a map $\langle\cdot|\cdot\rangle : \mathcal{E} \times \mathcal{E} \to \mathbb{K}$ such that $\langle x|x\rangle \geqslant 0$, $\langle x|x\rangle = 0 \Leftrightarrow x = 0$, $\langle x|y\rangle^* = \langle y|x\rangle$ and $\langle x|ay + bz\rangle = a\langle x|y\rangle + b\langle x|z\rangle$ for all $x, y, z \in \mathcal{E}$ and $a, b \in \mathbb{K}$. The pair $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ is called an **inner product space**.

Proposition 2.2. *Let* $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ *be an inner product space. For all* $x, y \in \mathcal{E}$*,*
- $|\langle x|y\rangle| \leqslant \langle x|x\rangle^{1/2}\langle y|y\rangle^{1/2}$ *(Cauchy–Schwarz),*
- $\langle x+y|x+y\rangle^{1/2} \leqslant \langle x|x\rangle^{1/2} + \langle y|y\rangle^{1/2}$ *(Minkowski).*

As a consequence, $\|x\| := \langle x|x\rangle^{1/2}$ *is a norm on* $\mathcal{E}$*.*

Definition 2.5. An inner product space $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ is **complete** if it is a Banach space for the norm $\|x\| = \langle x|x\rangle^{1/2}$. A complete inner product space is called a **Hilbert space**.

Notation 2.4.
1. On $\mathbb{K}^n$, we define the scalar product[1]

$$\langle x|y\rangle := \sum_{i=1}^{n} x[i]^* y[i]. \tag{2.4}$$

2. On $\mathcal{C}^0(\mathcal{D}, \mathbb{K}^m)$ we introduce the scalar product

$$\langle f|g\rangle := \int_{\mathcal{D}} f(x)^*\, g(x)\,\mathrm{d}x. \tag{2.5}$$

We see that the associated norms are $\langle x|x\rangle^{1/2} = \|x\|_2$ and $\langle f|f\rangle^{1/2} = \|f\|_2$. By Proposition 2.1, we immediately have that $(\mathbb{K}^n, \langle\cdot|\cdot\rangle)$ is a Hilbert space.

However, $\mathcal{C}^0(\mathcal{D}, \mathbb{K}^m)$ is too small to be a Hilbert space. The standard completion procedure works as follows. If $\mathcal{D}$ is any subset of $\mathbb{R}^n$, we define a set $\mathcal{L}^2(\mathcal{D})$ of **measurable** functions f on $\mathcal{D}$, for which $\|f\|_2$ is defined. Two functions f, g on $\mathcal{L}^2(\mathcal{D})$ are considered as equivalent if they differ on a set of zero measure. The space $L^2(\mathcal{D})$ is the set of equivalence classes in $\mathcal{L}^2(\mathcal{D})$ with respect to this equivalence relation.

A major interest of the scalar product is the possibility of decomposition on an orthogonal basis. While the theory is trivial in finite dimensional spaces, the situation is more subtle in the infinite dimensional case. We summarize some of the important notions below.

Definition 2.6. Let $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ be an inner product space. Two elements $x, y \in \mathcal{E}$ are **orthogonal** if $\langle x|y\rangle = 0$. A sequence $(x_n \in \mathcal{E})_{n \geqslant 1}$ is **orthonormal** if $\langle x_i|x_j\rangle = \delta_{ij}$. It is **total** if the set of all finite linear combinations of the x_n is dense in $\mathcal{E}$. $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ is **separable** if it admits a total sequence. The sequence $(x_n)_{n \geqslant 1}$ is an **orthonormal basis** if every $x \in \mathcal{E}$ can be written $x = \sum_{n \geqslant 1} \langle x_n|x\rangle x_n$. It is **complete** (or **maximal**) if $\langle x_n|x\rangle = 0\ \forall n$ implies $x = 0$.

Theorem 2.2. *Let* $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ *be a separable inner product space, and* $(x_n)_{n \geqslant 1}$ *an orthonormal sequence. Then*
- $\mathcal{E}$ *admits an orthonormal basis.*
- *The four conditions below satisfy* $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$*:*
  1. (x_n) *is a basis;*
  2. (x_n) *is total;*
  3. $\forall x \in \mathcal{E}$, $\|x\|_2^2 = \sum_{n \geqslant 1} |\langle x_n|x\rangle|^2$ *(Parseval relation);*
  4. (x_n) *is complete.*
- *If* $(\mathcal{E}, \langle\cdot|\cdot\rangle)$ *is a Hilbert space, then* $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$*.*

[1] We use here the physicists' convention. The mathematician's convention defines $\langle x|y\rangle$ as $\sum_i x[i] y[i]^*$.

Example 2.1. On the space $L^2([0,1])$ with the scalar product (2.5), we consider the sequence $(f_p(x))_{p\in\mathbb{Z}}$ defined by $f_p(x) = \mathrm{e}^{\mathrm{i}\,2\pi px}$. One shows that this is a complete orthonormal sequence. Hence we can decompose $f \in L^2([0,1])$ as its **Fourier series**

$$f(x) = \sum_{p=-\infty}^{\infty} \hat{f}(p)\,\mathrm{e}^{\mathrm{i}\,2\pi px}, \qquad \hat{f}(p) := \langle f_p | f\rangle = \int_0^1 \mathrm{e}^{-\mathrm{i}\,2\pi px}\, f(x)\,\mathrm{d}x. \tag{2.6}$$

2.1.3 Linear Operators and Matrices

Definition 2.7. If $(\mathcal{E}, \|\cdot\|_{\mathcal{E}}), (\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ are Banach spaces on $\mathbb{K}$, the map $L : \mathcal{E} \to \mathcal{F}$ is **linear** if $L(ax + by) = aLx + bLy$ for all $x, y \in \mathcal{E}$ and $a, b \in \mathbb{K}$. L is **bounded** if there exists $K > 0$ such that $\|Lx\|_{\mathcal{F}} \leqslant K\|x\|_{\mathcal{E}}$ for all $x \in \mathcal{E}$. We denote by $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the set of all bounded linear maps from $\mathcal{E}$ to $\mathcal{F}$, and introduce on $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the **operator norm**

$$\|L\| := \sup_{x\neq 0} \frac{\|Lx\|_{\mathcal{F}}}{\|x\|_{\mathcal{E}}} = \sup_{\|x\|_{\mathcal{E}}=1} \|Lx\|_{\mathcal{F}}. \tag{2.7}$$

Proposition 2.3.
- *A linear map is bounded if and only if it is continuous.*
- *$\mathcal{L}(\mathcal{E}, \mathcal{F})$ is a Banach space for the operator norm.*
- *This norm satisfies $\|Lx\|_{\mathcal{F}} \leqslant \|L\|\,\|x\|_{\mathcal{E}}$, and if $M \in \mathcal{L}(\mathcal{F}, \mathcal{G})$, $\|ML\| \leqslant \|M\|\,\|L\|$.*

Notation 2.5. If $\mathcal{E} = \mathbb{K}^m$ and $\mathcal{F} = \mathbb{K}^n$, the linear transformation can be represented by a **matrix**. We denote by $\mathbb{M}_{n\times m}(\mathbb{K})$, or simply $\mathbb{M}_{n\times m}$ the set of matrices with n rows and m columns. To simplify the notation, we will identify $\mathbb{M}_{n\times 1}$ with $\mathbb{M}_{1\times n}$ and $\mathbb{K}^n$.

Let $A \in \mathbb{M}_{n\times m}(\mathbb{K})$. We denote its components by $A[ij]$. We denote by
- $A^{\mathrm{T}} \in \mathbb{M}_{m\times n}(\mathbb{K})$ the **transpose** of A, $A^{\mathrm{T}}[ij] := A[ji]$.
- $A^* \in \mathbb{M}_{m\times n}(\mathbb{K})$ the **adjoint** of A, $A^*[ij] := (A[ji])^*$.

We now show that with respect to the norms $\|\cdot\|_1$ and $|\cdot|$, there is a very simple relationship between the operator norm and the elements of a matrix (for the norm $\|\cdot\|_2$, see Corollary 2.1).

Proposition 2.4. *Using in Definition 2.7 the norms of Notation 2.2, we have for any $A \in \mathbb{M}_{n\times m}(\mathbb{K})$*

$$\|A\|_1 := \sup_{\|x\|_1=1} \|Ax\|_1 = \max_{1\,\leqslant\, j\,\leqslant\, m} \sum_{i=1}^{n} |A[ij]|; \tag{2.8}$$

$$|A| := \sup_{|x|=1} |Ax| = \max_{1\,\leqslant\, i\,\leqslant\, n} \sum_{j=1}^{m} |A[ij]|. \tag{2.9}$$

Proof: To show the first equality, we first note that

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} A[ij]x[j] \right| \leqslant \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |A[ij]| \right) |x[j]| \leqslant \max_{1\,\leqslant\, j\,\leqslant\, m} \left(\sum_{i=1}^{n} |A[ij]| \right) \|x\|_1. \tag{2.10}$$

If $\hat{\jmath}$ is any value of j where the maximum is reached, equality holds for the vector x such that $x[j] = \delta_{j\hat{\jmath}}$. To show the second equality, we note that

$$|Ax| = \max_{1 \leqslant i \leqslant n} \left| \sum_{j=1}^{m} A[ij] x[j] \right| \leqslant \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{m} |A[ij]| |x[j]| \leqslant \max_{1 \leqslant i \leqslant n} \left(\sum_{j=1}^{m} |A[ij]| \right) |x|. \tag{2.11}$$

If $\hat{\imath}$ is any value of i where the maximum is reached, equality holds for the vector x such that $x[j] = \operatorname{sign}(A[\hat{\imath}j])$. □

We now discuss some properties of applications in $\mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)$, called **endomorphisms**, which are represented by square matrices.

Notation 2.6. We write $\mathbb{M}_n$ instead of $\mathbb{M}_{n\times n}$; this set defines a noncommutative algebra[2] with the usual sum and matrix product. We denote by

- $[A, B] := AB - BA$ the **commutator** of $A, B \in \mathbb{M}_n$,
- $\operatorname{diag}(a_1, \ldots, a_n)$ the **diagonal** matrix $A \in \mathbb{M}_n$ such that $A[ij] = a_i \delta_{ij}$,
- $\mathbb{1}_n$, or simply $\mathbb{1}$, the **unit matrix** $\operatorname{diag}(1, 1, \ldots, 1) \in \mathbb{M}_n$,
- $J(n, a)$ the **Jordan bloc** $J \in \mathbb{M}_n$ such that $J[ij] = a$ if $j = i$, 1 if $j = i + 1$ and 0 otherwise.

For $A \in \mathbb{M}_n$, we denote by

- $\det A := \sum_{\sigma\in\mathfrak{S}^n} (-1)^\sigma \prod_{i=1}^n A[i\sigma(i)]$ the **determinant** of A, where $\mathfrak{S}^n$ denotes the set of permutations of $\{1, \ldots, n\}$ and $(-1)^\sigma$ is the **signature** of σ;
- $\operatorname{Tr} A := \sum_{i=1}^n A[ii]$ the **trace** of A;
- $c_A(t) := \det(t\mathbb{1}_n - A) = t^n - \operatorname{Tr} A\, t^{n-1} + \cdots + (-1)^n \det A$ the **characteristic polynomial** of A;
- $m_A(t)$ the **minimal polynomial**, which is the unitary polynomial[3] of smallest degree such that $m_A(A) = 0$.

Proposition 2.5. *Let $A, B \in \mathbb{M}_n(\mathbb{K})$. Then*

- $\langle A^*x|y\rangle = \langle x|Ay\rangle$, $(AB)^* = B^*A^*$;
- $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$;
- $\det(AB) = \det A \det B$, $\det A = \det A^{\mathrm{T}}$;
- *If and only if $\det A \neq 0$, there is a unique matrix A^{-1}, called the* **inverse** *of A, such that $AA^{-1} = A^{-1}A = \mathbb{1}$.*

Definition 2.8. The matrix $A \in \mathbb{M}_n(\mathbb{K})$ is said to be

- **upper triangular** if $A[ij] = 0$ for $i > j$; **lower triangular** if A^{T} is upper triangular;
- **invertible** or **non–singular**, $A \in \mathrm{GL}(n, \mathbb{K})$, if $\det A \neq 0$;
- **special**, $A \in \mathrm{SL}(n, \mathbb{K})$, if $\det A = 1$;
- **normal** if $AA^* = A^*A$;
- **symmetric** if $A^{\mathrm{T}} = A$; **antisymmetric** if $A^{\mathrm{T}} = -A$;
- **hermitian** if $A^* = A$; **antihermitian** if $A^* = -A$;
- **positive definite** if $\langle x|Ax\rangle > 0$ for all $x \neq 0$ in $\mathbb{K}^n$.
- **orthogonal**, $A \in \mathrm{O}(n)$, if $\mathbb{K} = \mathbb{R}$ and $AA^{\mathrm{T}} = \mathbb{1}$; we write $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$;
- **unitary**, $A \in \mathrm{U}(n)$, if $\mathbb{K} = \mathbb{C}$ and $AA^* = \mathbb{1}$; we write $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$;

[2] We call (noncommutative) algebra on $\mathbb{K}$ a $\mathbb{K}$–vector space $\mathbb{A}$ with a (noncommutative) multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, such that $(\mathbb{A}, +, \cdot)$ is a ring and $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$ for all $a \in \mathbb{K}$, $x, y \in \mathbb{A}$.

[3] A polynomial is unitary if its term with largest degree has coefficient 1.

- **nilpotent** if there exists $k \in \mathbb{N}$ such that $A^k = 0$;
- a **projector** if $A^2 = A$;

Remark 2.1.

1. The sets $\mathrm{GL}(n,\mathbb{K})$, $\mathrm{SL}(n,\mathbb{K})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ are **groups** with respect to matrix multiplication. They play an important role in the classification of symmetry transformations in physics. For instance, Lorentz transformations in $\mathbb{R}^4$ can be mapped to $\mathrm{SL}(2,\mathbb{C})$.
2. If A is invertible, we have $\|x\| = \|A^{-1}Ax\| \leqslant \|A^{-1}\|\|Ax\|$, so that $\|Ax\| \geqslant \|x\|/\|A^{-1}\|$.

Definition 2.9. A number $a \in \mathbb{K}$ is an **eigenvalue** of $A \in \mathbb{M}_n(\mathbb{K})$ if there exists $x \neq 0$ in $\mathbb{K}^n$ such that $Ax = ax$; x is called a (right) **eigenvector** associated with a (a left eigenvector y is defined by $yA = ay$). The **geometric multiplicity** of a, $\mathrm{m_g}(a, A)$, is the number of independent eigenvectors associated with a. Since $Ax = ax$ has a nontrivial solution if and only if $(A - a\mathbb{1})$ is not invertible, a is an eigenvalue of A if and only if $c_A(a) = 0$. The **algebraic multiplicity** of a $\mathrm{m_a}(a, A)$ is the multiplicity of a as a root of c_A. We denote by $\sigma(A) = \{a_1, \ldots, a_m\}$ ($m \leqslant n$) the set of all eigenvalues, or **spectrum** of A. If $\mathbb{K} = \mathbb{C}$, we have therefore $c_A(t) = \prod_{i=1}^m (t - a_i)^{\mathrm{m_a}(a_i,A)}$.

Since $A \in \mathbb{M}_n$ represents the linear transformation $x \mapsto Ax$, it is of interest to consider the change of variables $y = Sx$, which transforms the map into $y \mapsto SAS^{-1}y$. This motivates the following definitions:

Definition 2.10. Two matrices $A, B \in \mathbb{M}_n(\mathbb{K})$ are **similar** if there exists $S \in \mathrm{GL}(n,\mathbb{K})$ such that $SAS^{-1} = B$. A matrix $A \in \mathbb{M}_n(\mathbb{K})$ is **diagonalizable** if it is similar to a diagonal matrix, **triangularizable** if it is similar to a triangular one. In the same way one defines notions like unitarily diagonalizable. A function f defined on $\mathbb{M}_n$ is a **similarity invariant** if $f(A) = f(B)$ whenever A and B are similar.

Proposition 2.6. *The characteristic polynomial $c_A(t)$ and the minimal polynomial $m_A(t)$ are similarity invariants. In particular,* $\det A$ *and* $\operatorname{Tr} A$ *are similarity invariants.*

Let us now list a few results on similarity properties of matrices. Some of the important basic properties are summarized in the following theorem:

Theorem 2.3. *Let $A \in \mathbb{M}_n(\mathbb{K})$ and $\sigma(A) = \{a_1, \ldots, a_m\}$. Then*

- $\mathrm{m_g}(a_i, A) \leqslant \mathrm{m_a}(a_i, A)\ \forall i$.
- *A is triangularizable $\Leftrightarrow$ $c_A(t)$ can be completely factorized, $c_A(t) = \prod_{i=1}^m (t - a_i)^{\mathrm{m_a}(a_i,A)}$.*
- *A is diagonalizable $\Leftrightarrow$ $\sum_{i=1}^m \mathrm{m_g}(a_i, A) = n$ $\Leftrightarrow$ $c_A(t)$ can be completely factorized and $\mathrm{m_g}(a_i, A) = \mathrm{m_a}(a_i, A)\ \forall i$.*
- *A is nilpotent $\Leftrightarrow$ $c_A(t) = t^n$.*
- *For a Jordan bloc, $m_{J(n,a)}(t) = c_{J(n,a)}(t) = (t - a)^n$.*
- *$m_A(t)$ divides $c_A(t)$, or, equivalently, $c_A(A) = 0$ (Cayley–Hamilton).*
- *$m_A(t)$ and $c_A(t)$ have the same irreducible factors (Frobenius).*

A projector P has eigenvalues 0 or 1, and minimal polynomial t, $t - 1$ or $t(t - 1)$.

Clearly, the answer to the question whether A is triangularizable or diagonalizable will depend on which field we are working in. We will usually consider complex matrices, even though originally, many physical problems are modelled with real matrices. The most important result for us will be:

Theorem 2.4 (Jordan decomposition). *Every matrix $A \in \mathbb{M}_n(\mathbb{C})$, having distinct eigenvalues $\{a_1, \dots, a_m\}$, can be decomposed as $A = D + N$ where*

- $D = \sum_{i=1}^m a_i P_i$, *the P_i are projectors satisfying $P_i P_j = \delta_{ij} P_i$, $\sum_{i=1}^m P_i = \mathbb{1}$ and $\mathrm{m_a}(a_i, A) = \dim(P_i \mathbb{K}^n)$;*
- $N = \sum_{i=1}^m N_i$, *where the N_i are nilpotent, with $N_i^{\mathrm{m_a}(a_i,A)} = 0$, satisfying $N_i N_j = 0$ when $i \neq j$ and $P_i N_j = N_j P_i = \delta_{ij} N_i$;*
- *Each N_i is similar to a block–diagonal matrix composed of $\mathrm{m_g}(a_i, A)$ Jordan blocks of the form $J(k_{ij}, 0)$.*

Equivalently, one can say that there is a matrix $S \in \mathrm{GL}(n, \mathbb{C})$ such that SAS^{-1} is block–diagonal, with $\sum_{i=1}^m \mathrm{m_g}(a_i, A)$ Jordan blocks of the form $J(k_{ij}, a_i)$, $j = 1, \dots, \mathrm{m_g}(a_i, A)$. For each i, $\sum_{j=1}^{\mathrm{m_g}(a_i,A)} k_{ij} = \mathrm{m_a}(a_i, A)$. The minimal polynomial of A is given by $m_A(t) = \prod_{i=1}^m (t - a_i)^{\max_j k_{ij}}$. The matrix $S^{-1}AS$ is called the **Jordan reduced** *of A.*

Example 2.2. Assume that $A \in \mathbb{M}_4(\mathbb{C})$ and $c_A(t) = (t - a)^4$.

1. If $m_A(t) = (t - a)$, then $\mathrm{m_g}(a, A) = 4$ and $A = a\mathbb{1}_4$.
2. If $m_A(t) = (t - a)^2$, then either $\mathrm{m_g}(a, A) = 3$ and the Jordan reduced of A contains the blocks $J(2, a)$ and $\mathbb{1}_2$; or $\mathrm{m_g}(a, A) = 2$ and the Jordan reduced of A contains two blocks $J(2, a)$.
3. If $m_A(t) = (t - a)^3$, then $\mathrm{m_g}(a, A) = 2$ and the Jordan reduced of A contains the blocks $J(3, a)$ and $J(1, a) = (a)$.
4. If $m_A(t) = (t - a)^4$, then $\mathrm{m_g}(a, A) = 1$ and the Jordan reduced of A is $J(4, a)$.

We give here one useful application of Theorem 2.4:

Lemma 2.1. *Assume that the eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$ have real parts contained in the interval (a_-, a_+). Then there exists $S \in \mathrm{GL}(n, \mathbb{C})$ such that*

$$a_- \|x\|_2^2 \leqslant \mathrm{Re}\langle x | SAS^{-1} x \rangle \leqslant a_+ \|x\|_2^2. \tag{2.12}$$

Proof: Let S_0 be a matrix such that $S_0 A S_0^{-1} = J$ is the Jordan reduced of A, and let $\varepsilon > 0$. If $J(k, a)$ is a Jordan block of J and $T(k, \varepsilon) := \mathrm{diag}(1, \varepsilon^{-1}, \dots, \varepsilon^{1-k})$, then $TJ(k, a)T^{-1} = \varepsilon J(k, \frac{a}{\varepsilon})$. We can thus construct a diagonal matrix S_1 such that $S_1 J S_1^{-1}$ has off–diagonal terms of order ε. Therefore, if $S = S_1 S_0$, $\langle x | SAS^{-1} x \rangle = \sum_j a_j |x[j]|^2 + \mathcal{O}(\varepsilon)$, where the a_j are eigenvalues of A. Since ε can be made arbitrarily small, the result follows. $\square$

When additional symmetries are present, we have some stronger results, that we summarize in the following way:

Theorem 2.5.

- *Any $A \in \mathbb{M}_n(\mathbb{C})$ is unitarily triangularizable (Schur);*
- *$A \in \mathbb{M}_n(\mathbb{C})$ is unitarily diagonalizable $\Leftrightarrow$ A is normal; if, moreover, A is hermitian, then its eigenvalues are real;*
- *$A \in \mathbb{M}_n(\mathbb{R})$ is orthogonaly diagonalizable $\Leftrightarrow$ A is symmetric.*

Corollary 2.1. *Since $\|Ax\|_2^2 = \langle x | A^* A x \rangle$, $\|A\|_2^2$ is the largest eigenvalue of A^*A.*

Theorem 2.6 (Polar decomposition). *For any $A \in \mathrm{GL}(n, \mathbb{C})$, there exist unique hermitian matrices R, S and unique unitary matrices U, V such that $A = RU = VS$. In fact, R is the unique solution of $R^2 = AA^*$, S is the unique solution of $S^2 = A^*A$, $U = R^{-1}A$ and $V = AS^{-1}$.*

2.1.4 Derivatives

Definition 2.11. Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be Banach spaces, $\mathcal{D}$ be a subset of $\mathcal{E}$, let $T : \mathcal{D} \mapsto \mathcal{F}$ and $t \in \mathbb{R}$.

- h is $\mathcal{D}$–**admissible** at $x \in \mathcal{D}$ if there exists $\varepsilon > 0$ such that $x + th \in \mathcal{D}$ if $|t| \leqslant \varepsilon$ (if $\mathcal{D}$ is open, any h is admissible).
- T is said to be **Fréchet differentiable** at $x \in \mathcal{D}$ if there exists a bounded linear operator $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ such that for any $\mathcal{D}$–admissible h,

$$T(x+h) - T(x) - L(x)h = o(h). \tag{2.13}$$

  $L =: \frac{\partial T}{\partial x}$ is called the **derivative** of T at x, and $L(x)h$ is the **differential** of T at x.
- The **directional derivative** (or **Gâteaux derivative**) of T at x in the direction h is given by the following limit, if it exists:

$$\delta T(x, h) := \lim_{t \to 0} \frac{T(x+th) - T(x)}{t}. \tag{2.14}$$

Lemma 2.2. *Assume that the map $h \mapsto \delta T(x,h)$ is linear and continuous, and that the map $x \mapsto \delta T(x,h)$ is continuous at x. Then $T(x)$ is Fréchet differentiable at x with differential $L(x)h = \delta T(x,h)$.*

We say that the map $T : \mathcal{D} \subset \mathcal{E} \to \mathcal{F}$ is of class $\mathcal{C}^1$, and write $T \in \mathcal{C}^1(\mathcal{D}, \mathcal{F})$ if $T(x)$ is Fréchet differentiable at every $x \in \mathcal{D}$.

If $T : \mathcal{E} \to \mathcal{F}$ is differentiable in $\mathcal{D}$, it may admit a **second derivative** at x, defined as a bilinear map $B : \mathcal{E} \times \mathcal{E} \to \mathcal{F}$ such that $\delta[\delta f(\cdot, h_1)](x, h_2) = B(h_1, h_2)$. The k^{th} derivative is a k–linear map from $\mathcal{E} \times \cdots \times \mathcal{E} \to \mathcal{F}$. The map T is of class $\mathcal{C}^k$ if the map $x \mapsto \delta T(x,h)$ is of class $\mathcal{C}^{k-1}$ (see [**EDM**], p. 1075).

We can now state one of the most important theorems for our future calculations. It gives a simple answer to the problem of solving the equation $\Phi(x,y) = 0$ with respect to x, in the neighborhood of a particular solution.

Theorem 2.7 (Implicit function theorem). *Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be Banach spaces. Let $\mathcal{D}$ be a neighborhood of $(x_0, y_0) \in \mathcal{E} \times \mathcal{F}$, and let $\Phi \in \mathcal{C}^k(\mathcal{D}, \mathcal{G})$ be a function satisfying*

- $\Phi(x_0, y_0) = 0$,
- *If $A := \partial\Phi/\partial x(x_0, y_0) \in \mathcal{L}(\mathcal{E}, \mathcal{G})$, there exists a bounded linear operator $B \in \mathcal{L}(\mathcal{G}, \mathcal{E})$ such that $AB = \mathbb{1}$.*

Then there exists a neighborhood $\mathcal{U}$ of y_0 in $\mathcal{F}$ and a function $\varphi \in \mathcal{C}^k(\mathcal{U}, \mathcal{E})$ such that $\varphi(y)$ is the unique solution of $\Phi(\varphi(y), y) = 0$ in $\mathcal{U}$ with $\varphi(y_0) = x_0$.

Moreover, let $\varphi^{(0)}(y) \equiv x_0$ and $(\varphi^{(N)})_{N \geqslant 1}$ be the sequence of functions defined recursively by **Newton's method***:*

$$\varphi^{(N+1)}(y) = T\varphi^{(N)}(y), \qquad T(\varphi) := \varphi - B\Phi(\varphi, y). \tag{2.15}$$

Then $\lim_{N\to\infty} \varphi^{(N)}(y) = \varphi(y)$. If $k \geqslant 2$, there exists a positive constant C such that $\|\Phi(\varphi^{(N)}(y), y)\| \leqslant C\|y - y_0\|^{N+1}$.

Let us particularize the discussion to the case where $\mathcal{E} = \mathbb{K}^m$ and $\mathcal{F} = \mathbb{K}^n$.

Notation 2.7. Let $f : \mathcal{D} \subset \mathbb{K}^m \to \mathbb{K}^n$. We denote by $\partial f/\partial x_{[j]}(x_0)$ the derivative of f at x_0 in the direction of the j^{th} basis vector. Then the Fréchet derivative is represented by the **Jacobian matrix** $\partial f/\partial x(x_0) \in \mathbb{M}_{n\times m}(\mathbb{K})$, where $\partial f/\partial x(x_0)_{[ij]} = \partial f_{[i]}/\partial x_{[j]}(x_0)$. We will sometimes write $\partial_x f(x_0)$ instead of $\partial f/\partial x(x_0)$. We denote by $\mathcal{C}^1(\mathcal{D}, \mathbb{K}^n)$ the set of functions $f : \mathcal{D} \to \mathbb{K}^m$ which have a continuous Jacobian matrix at every point in $\mathcal{D}$. We also say that f is **of class** $\mathcal{C}^1$. If $\mathcal{D}$ is compact, we introduce the norm

$$\|f\|_{\mathcal{C}^1} := \sup_{\substack{x\in\mathcal{D}\\ i,j}} \left\{ |f(x)|, \left| \frac{\partial f_{[i]}}{\partial x_{[j]}}(x) \right| \right\}. \tag{2.16}$$

If $f \in \mathcal{C}^1(\mathcal{D}, \mathcal{D})$ is invertible, and its inverse f^{-1} is also in $\mathcal{C}^1(\mathcal{D}, \mathcal{D})$, f is called a **diffeomorphism**. If f and f^{-1} are only in $\mathcal{C}^0(\mathcal{D}, \mathcal{D})$, f is called a **homeomorphism**.

If, in the hypotheses of the implicit function theorem, $\mathcal{E} = \mathcal{G} = \mathbb{R}^n$ and $\mathcal{F} = \mathbb{R}^m$, then A is a matrix in $\mathbb{M}_n$. The condition on A reduces to $\det A \neq 0$, and $B = A^{-1}$.

Proposition 2.7. *If $\mathcal{D} \subset \mathbb{K}^m$ is compact, $(\mathcal{C}^1(\mathcal{D}, \mathbb{K}^m), \|\cdot\|_{\mathcal{C}^1})$ is a Banach space.*

To keep the notations for higher order derivatives simple, we consider the case of a function $f : \mathbb{K}^n \to \mathbb{K}$. It is then straightforward to generalize the derivatives to vector–valued functions by considering each component individually.

Notation 2.8. Let $f : \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}$. Let $x_0 \in \mathcal{D}$ and $y = (y_{[1]}, \dots, y_{[n]})$ such that $x_0 + y \in \mathcal{D}$. We introduce the vectorial index p and the notations

$$\begin{aligned}
\mathrm{p} &:= (\mathrm{p}_{[1]}, \dots, \mathrm{p}_{[n]}) \in \mathbb{N}^n \\
|\mathrm{p}| &:= \mathrm{p}_{[1]} + \dots + \mathrm{p}_{[n]} \\
\mathrm{p}! &:= \mathrm{p}_{[1]}! \dots \mathrm{p}_{[n]}! \\
\partial_\mathrm{p} &:= \left(\frac{\partial}{\partial x_{[1]}}\right)^{\mathrm{p}[1]} \cdots \left(\frac{\partial}{\partial x_{[n]}}\right)^{\mathrm{p}[n]} \\
y^\mathrm{p} &:= y_{[1]}{}^{\mathrm{p}[1]} \dots y_{[n]}{}^{\mathrm{p}[n]}.
\end{aligned} \tag{2.17}$$

We denote by $\mathcal{C}^k(\mathcal{D}, \mathbb{K})$ the set of functions such that $\partial_\mathrm{p} f(x)$ is continuous in $\mathcal{D}$ for $|\mathrm{p}| \leqslant k$. For compact $\mathcal{D}$, $\mathcal{C}^k(\mathcal{D}, \mathbb{K})$ is a Banach space for the norm

$$\|f\|_{\mathcal{C}^k} := \sup_{\substack{x\in\mathcal{D}\\ |\mathrm{p}|\leqslant k}} |\partial_\mathrm{p} f|. \tag{2.18}$$

Invertible functions in $\mathcal{C}^k(\mathcal{D}, \mathcal{D})$, the inverse of which is also in $\mathcal{C}^k(\mathcal{D}, \mathcal{D})$, are called $\mathcal{C}^k$–**diffeomorphisms**.

Proposition 2.8 (Taylor series). *Let $f \in \mathcal{C}^k(\mathcal{D}, \mathbb{K})$, $\mathcal{D} \subset \mathbb{K}^n$, $x_0 \in \mathcal{D}$. Then, for all y such that $x_0 + y \in \mathcal{D}$,*

$$f(x_0 + y) = \sum_{|\mathrm{p}|<k} \frac{1}{\mathrm{p}!} \partial_\mathrm{p} f(x_0) y^\mathrm{p} + \sum_{|\mathrm{p}|=k} y^\mathrm{p} R_\mathrm{p}(y) \tag{2.19}$$

where

$$R_{\mathrm{p}}(y) := \frac{k!}{\mathrm{p}!}\int_0^1 \mathrm{d}t_1\, t_1^{k-1}\int_0^1 \mathrm{d}t_2\, t_2^{k-2}\cdots\int_0^1 \mathrm{d}t_k\, \partial_{\mathrm{p}} f(x_0 + t_1t_2\cdots t_k y), \tag{2.20}$$

$$|R_{\mathrm{p}}(y)| \leqslant \frac{1}{\mathrm{p}!}\|f\|_{\mathcal{C}^k}, \qquad \sum_{|\mathrm{p}|=k}|R_{\mathrm{p}}(y)| \leqslant \frac{n^k}{k!}\|f\|_{\mathcal{C}^k}. \tag{2.21}$$

PROOF: The result is obtained by writing

$$f(x_0+y) - f(x_0) = \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}t} f(x_0+ty)\right]\mathrm{d}t = \sum_{|\mathrm{p}|=1}\int_0^1 \partial_{\mathrm{p}} f(x_0+ty)\,\mathrm{d}t\, y^{\mathrm{p}}, \tag{2.22}$$

and iterating the procedure until $|\mathrm{p}| = k$, using straightforward combinatorics. □

2.1.5 Complex Analysis

Proposition 2.8 tells us that an indefinitely differentiable function $f \in \mathcal{C}^\infty(\mathcal{D},\mathbb{K})$ can be locally described by a series of the form $f(x_0 + y) = \sum_{|\mathrm{p}|\leqslant k} c_{\mathrm{p}} y^{\mathrm{p}} + \mathcal{O}(y^k)$, for any k. This means that f admits an asymptotic series in y, to be discussed in Subsection 2.1.6. However, this tells us nothing on the convergence of the series when $k \to \infty$. In fact, the Taylor series of f may diverge for all x, or converge to another function than f (as in the case of the function $\mathrm{e}^{-1/x}$, discussed in Appendix 2.A).

The situation is rather different for analytic functions, where the mere existence of the first derivative in some domain has very strong (and sometimes surprising[4]) consequences on the global behaviour of f in $\mathcal{D}$. We consider below the case of a function $f : \mathbb{C} \to \mathbb{C}$. Generalization to functions on $\mathbb{C}^n$ is straightforward, following the formalism of Notation 2.8.

Definition 2.12. Let $\mathcal{D}$ be an open connected non–empty subset of $\mathbb{C}$. The derivative of f at $z \in \mathcal{D}$ is defined by means of the directional derivative, $f'(z) \equiv \mathrm{d}f/\mathrm{d}z(z) := \delta f(z,h)/h$, if the limit exists independently of $h \in \mathbb{C}$. $f(z)$ is **holomorphic** or **analytic** in $\mathcal{D}$ if $f'(z)$ is continuous at every z in $\mathcal{D}$. We denote by $\mathcal{C}^\omega(\mathcal{D},\mathbb{C})$ the set of analytic functions from $\mathcal{D}$ to $\mathbb{C}$. The k^{th} derivative of f is written $(\mathrm{d}/\mathrm{d}z)^k f(z) =: f^{(k)}(z)$.

Proposition 2.9 (Cauchy–Riemann conditions). *Let $z = x + \mathrm{i}\,y$, $x,y \in \mathbb{R}$, and $f(z) = u(x,y) + \mathrm{i}\,v(x,y)$ where $u,v \in \mathcal{C}^1(\mathcal{D},\mathbb{R})$. Then f is analytic in $\{z = x+\mathrm{i}\,y \,|\, (x,y) \in \mathcal{D}\}$ if and only if*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \qquad \forall\ (x,y)\ \text{in}\ \mathcal{D}. \tag{2.23}$$

Proposition 2.10. *Let S be the* **power series** $\sum_{k=0}^\infty a_k(z-z_0)^k$.

- *There exists a unique R, $0 \leqslant R \leqslant \infty$, called the* **radius of convergence** *of S, such that the power series converges absolutely if $|z-z_0| < R$ and diverges if $|z-z_0| > R$.*
- *If $L = \limsup_k |a_k|^{1/k}$ exists, then $R = 1/L$ (Cauchy criterion).*
- *If $L = \lim_{n\to\infty}|a_{k+1}/a_k|$ exists, then $R = 1/L$ (d'Alembert criterion).*

[4]The strong influence of local properties on the global behaviour of the function motivates the name "holomorphic" (without any postmodern conotation). See, for instance, [Man] for a discussion of these issues.

- *If $R > 0$, then $f(z)$ is holomorphic in the open disc $|z - z_0| < R$ and its derivative is given by $f'(z) = \sum_k k a_k (z - z_0)^{k-1}$.*

Theorem 2.8 (Cauchy formula). *Let $f \in \mathcal{C}^\omega(\mathcal{D}, \mathbb{C})$, and let $\mathcal{C}$ be a simple closed piecewise $\mathcal{C}^1$ curve in $\mathcal{D}$, whose interior is contained in $\mathcal{D}$. Then*

$$\int_{\mathcal{C}} f(z)\,\mathrm{d}z = 0; \tag{2.24}$$

Furthermore, $f^{(k)}$ exists in all of $\mathcal{D}$, and for any z_0 in the interior of $\mathcal{C}$,

$$f^{(k)}(z_0) = \frac{k!}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{k+1}}\,\mathrm{d}z. \tag{2.25}$$

The second formula is very useful to estimate derivatives. Indeed, we have immediately

Corollary 2.2. *If $f(z)$ is analytic and $|f(z)| \leqslant M$ in the disc $|z - z_0| \leqslant R$, then its derivatives are bounded by $|f^{(k)}(z_0)| \leqslant k! M R^{-k}$.*

Example 2.3. If $f(z + 1) = f(z)$ is periodic, analytic and bounded by M in the strip $|\mathrm{Im}\, z| \leqslant D$, we obtain by (2.6) that its Fourier coefficients decrease exponentially as $|\hat{f}(p)| \leqslant \mathrm{e}^{-2\pi D|p|}$. It is sufficient to shift the integration path by a distance $\pm D\,\mathrm{i}$.

We also have an analog of the Taylor series, but this time we do control the convergence.

Proposition 2.11 (Laurent series). *Let f be analytic in the annulus $\mathcal{D} = \{z \,|\, r_1 < |z - z_0| < r_2\}$ and let $\mathcal{C}$ be a simple closed curve in $\mathcal{D}$ whose interior contains z_0. Then, for all $z \in \mathcal{D}$, $f(z)$ is equal to its* **Laurent series**

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k, \qquad c_k := \frac{1}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} \frac{f(w)}{(w - z_0)^{k+1}}\,\mathrm{d}w. \tag{2.26}$$

If $f(z)$ is analytic in the whole disc $|z| < r_2$, then $c_k = 0$ for $k < 0$. If $c_{-m} \neq 0$, $m > 0$ and $c_k = 0$ for $k < -m$ then $f(z)$ is said to have a **pole of order** *m at $z = z_0$. If there are arbitrarily large k such that $c_{-k} \neq 0$, z_0 is called an* **essential pole**.

There is, in fact, a complete zoology of the singularities of analytic functions, which can be classified into poles, isolated and non–isolated critical points (e.g. branch cuts) [Man]. Let us finally state a few interesting properties of analytic functions.

Proposition 2.12. *Let $\mathcal{D}$ be an open subset of $\mathbb{C}$ and $f \in \mathcal{C}^\omega(\mathcal{D}, \mathbb{C})$.*

- *If $\mathcal{D} = \mathbb{C}$ and f is bounded, then $f(z)$ is constant.*
- *If $(z_n \in \mathcal{D})_{n \geqslant 1}$ is a sequence of distinct complex numbers converging towards $z_0 \in \mathcal{D}$ and $f(z_n) = 0$ $\forall n$ then $f(z) \equiv 0$ in $\mathcal{D}$.*
- *If $z_0 \in \mathcal{D}$ and $f^{(k)}(z_0) = 0$ $\forall k$, then $f(z) \equiv 0$ in $\mathcal{D}$.*
- *If f is non–constant in the connected set $\mathcal{D}$ then $|f|$ has no relative minima in $\mathcal{D}$.*

2.1.6 Asymptotic Series

Proposition 2.11 shows that a function may be equal to its power series in some open domain. Unfortunately, this is in general not the case for many solutions of differential equations that we will encounter, although these solutions admit in general Taylor series up to a given order. The relevant concept in this situation is the **asymptotic development**. Let us start with a classic example.

Example 2.4. Let us study the behaviour of the **exponential integral**

$$\mathrm{Ei}(x) := \int_{-\infty}^{x} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t \tag{2.27}$$

for large negative x. If we set $\mathrm{Ei}(1/\varepsilon) =: \varepsilon\,\mathrm{e}^{1/\varepsilon} f(\varepsilon)$, successive integrations by part yield

$$f(\varepsilon) = 1 + \varepsilon + 2!\varepsilon^2 + \dots + k!\varepsilon^k + R_{k+1}(\varepsilon), \tag{2.28}$$

where (we assume ε to be negative)

$$R_k(\varepsilon) = \frac{1}{\varepsilon}\,\mathrm{e}^{-1/\varepsilon}\,k! \int_{-\infty}^{1/\varepsilon} \frac{\mathrm{e}^t}{t^{k+1}}\,\mathrm{d}t, \qquad |R_k(\varepsilon)| \leqslant |\varepsilon|^k k!, \tag{2.29}$$

where the last inequality is obtained by bounding the integrand by $\varepsilon^{k+1}\,\mathrm{e}^t$.

If we apply d'Alembert's criterion (Proposition 2.10), we see immediately that (2.28) does not define a convergent series.[5] Nevertheless, for any k we have $R_k(\varepsilon) = \mathcal{O}(\varepsilon^k)$, so that the remainder is small for sufficiently small ε. In fact, for given ε, we can easily determine the optimal value of k such that R_k is as small as possible. Using Stirling's formula (2.116), we have, for large k, $\ln(|\varepsilon|^k k!) \simeq k\ln(k|\varepsilon|\,\mathrm{e}^{-1})$, whose derivative with respect to k vanishes at $k = 1/|\varepsilon|$. Thus, the minimum of $k!|\varepsilon|^k$ is of order $\exp(|\varepsilon|^{-1}\ln(\mathrm{e}^{-1})) = \mathrm{e}^{-1/|\varepsilon|}$. We may thus write

$$f(\varepsilon) = \sum_{k=0}^{N(\varepsilon)} k!\varepsilon^k + \mathcal{O}(\mathrm{e}^{-1/|\varepsilon|}), \qquad N(\varepsilon) := \left[\frac{1}{|\varepsilon|}\right]. \tag{2.30}$$

This example motivates the following definition:

Definition 2.13. Let $f : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subset \mathbb{C}$ contains $z = 0$ as an accumulation point. Assume that there exists a sequence $(c_k \in \mathbb{C})_{k\,\geqslant\,0}$ such that

$$f(z) = \sum_{j=0}^{k} c_j z^j + o(z^k) \qquad \forall\, z \in \mathcal{D},\ \forall\, k \geqslant 0. \tag{2.31}$$

Then we say that $\sum_{k=0}^{\infty} c_k z^k$ is an **asymptotic series** of $f(z)$ in $\mathcal{D}$ as $z \to 0$ and write

$$f(z) \asymp \sum_{k=0}^{\infty} c_k z^k. \tag{2.32}$$

We state below a few results on asymptotic series which are proved in [**Wa**].

[5] If the k^{th} term grows like $k!$, the series is said to be **Gevrey–1**.

Proposition 2.13. *Let $f : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$.*

- *$f(z)$ can have at most one asymptotic series $\sum_{k=0}^{\infty} c_k z^k$ in $\mathcal{D}$, where the c_k are defined recursively by*

$$c_0 = \lim_{z \to 0} f(z), \qquad c_k = \lim_{z \to 0} z^{-k} \left[f(z) - \sum_{j=0}^{k-1} c_j z^j \right]. \tag{2.33}$$

- *Assume that the functions f and g admit asymptotic expansions and that one of the expressions $f(z)+g(z)$, $f(z)g(z)$, $f(g(z))$ or $1/g(z)$ is defined for $z \in \mathcal{D}$. Then this expression admits an asymptotic series in $\mathcal{D}$, obtained by formal algebraic manipulations on the series of f and g.*

Theorem 2.9. *Let $f : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$.*

- *If $f(z)$ is analytic in the annulus $0 < |z| \leqslant R$ and $f(z) \asymp \sum_{k=0}^{\infty} c_k z^k$, then the asymptotic series converges for $0 < |z| \leqslant R$.*
- *If $f(z)$ is analytic in the sector $\mathcal{S} = \{z \,|\, 0 < |z| \leqslant R, \theta_1 \leqslant \arg z \leqslant \theta_2\}$, $\theta_2 > \theta_1$, and $f(z) \asymp \sum_{k=0}^{\infty} c_k z^k$ in $\mathcal{S}$, then*

$$f'(z) \asymp \sum_{k=0}^{\infty} k \, c_k \, z^{k-1} \qquad \textit{and} \qquad \int_0^z f(w) \, \mathrm{d}w \asymp \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} \tag{2.34}$$

for any path of integration is $\mathcal{S}$.

- *For any sequence $(c_k \in \mathbb{C})_{k \geqslant 0}$, there exists a function $f(z)$ analytic in the sector S such that $f(z) \asymp \sum_{k=0}^{\infty} c_k z^k$.*

Remark 2.2. This theorem stresses the difference between analyticity in a disc or a sector. Assume, for instance, that $f(z)$ has a pole at a distance R from the origin and is analytic for $|z| < R$, so that its Taylor series converges to $f(z)$ inside the disc. Then the series must diverge for $|z| > R$. If the pole moves towards the origin, the function will admit no convergent series around 0. However, it may happen that $f(z)$ is still analytic in a sector S, and even admits a convergent asymptotic series in S, only the series will not converge *to* $f(z)$ (see the example $\mathrm{e}^{-1/z}$ in Appendix 2.A).

Further examples of asymptotic series can be found in Appendix 2.A, where we discuss some special functions which will appear in future developments.

2.2 Ordinary Differential Equations

Differential equations play a central role in this work. We will consider the following situation. Let $\mathcal{D}$ be an open set in $\mathbb{R}^{n+1}$ and $f \in \mathcal{C}^0(\mathcal{D}, \mathbb{R}^n)$. We denote elements of $\mathcal{D}$ by $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. A **first order differential equation** is a relation of the form

$$\dot{x} = f(x,t), \qquad \text{where } \dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{2.35}$$

The problem of solving this equation may be considered in two ways:

1. **Analytic approach:** find, in some interval I, a function $x(t) \in \mathcal{C}^1(I, \mathbb{R}^n)$, such that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t),t)\ \forall\, t \in I$.
2. **Geometric approach:** find a curve $x(t) \in \mathcal{D}$ which is everywhere tangent to the vector field $f(x,t)$.

The principal use of the analytical theory is to give precise characterizations of existence, unicity and regularity properties of the solutions, which we discuss below. There are very few differential equations that can be solved exactly, but we do not consider this as a major drawback, since we are mainly looking for qualitative (yet precisely delineated) properties of the solutions. Many informations can be obtained at low cost from the geometric theory, which we expose in the next section. In the remainder of this section, we present a few useful solvable one–dimensional equations, and discuss the properties of linear equations.

Good reference books on the analytic theory of ordinary differential equations are those of Hartman [**Har**] and Hale [**Hal**]. Hirsch and Smale [**HS**] give a slightly more geometrical approach, with a few practical applications. For a more elementary introduction, oriented towards numerical applications, see [**HW**].

2.2.1 Existence, Unicity and Regularity of Solutions

A solution of the ordinary differential equation (2.35) with **initial condition** x_0 **at** t_0 is a function $x(t)$ satisfying (2.35) such that $x(t_0) = x_0$, or, equivalently, a solution curve passing through the point (x_0, t_0). Note that (2.35) is equivalent to the integral equation

$$x(t) = Tx(t) := x_0 + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}s. \tag{2.36}$$

This observation is at the base of Picard's method for constructing a solution by iteration of T. Variants of this method also allow to prove the basic theorems below[6].

Theorem 2.10 (Peano–Cauchy). *If $f(x,t)$ is continuous in $\mathcal{D}$, there exists at least one solution of* (2.35) *passing through any $(x_0, t_0) \in \mathcal{D}$.*

Next we can wonder about the domain in which this solution exists.

Theorem 2.11. *Under the hypotheses of Theorem 2.10, any solution can be continued to a maximal interval of existence. If (t_1, t_2) is such an interval, then $x(t)$ tends to the boundary of $\mathcal{D}$ as $t \to t_{1,2}$.*

[6]Although the proofs relying on Picard–like iteration schemes are constructive, this does not mean that the rate of convergence is optimal. There exists a huge number of iterative methods for solving ordinary differential equations, and alternative proofs of the fundamental theorems have been proposed (see [**HW**]).

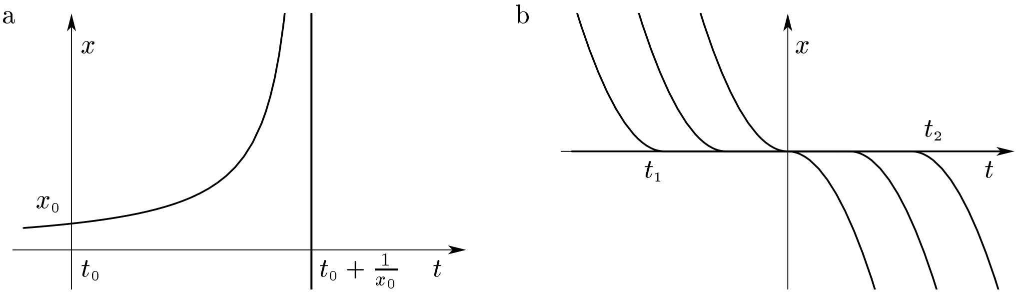

FIGURE 2.1. (a) The solution of $\dot{x} = x^2$ with initial condition $x(t_0) = x_0$ diverges at the time $t = t_0 + \frac{1}{x_0}$. (b) A family of solutions of $\dot{x} = -\sqrt{|x|}$ such that $x(0) = 0$. The times $t_1 \leqslant 0 \leqslant t_2$ are arbitrary.

This result means that the solution cannot cease to exist at some point inside the domain $\mathcal{D}$. In particular, if $\mathcal{D} = \mathbb{R}^{n+1}$ and $t_{1,2}$ are finite, then it must diverge at $t = t_{1,2}$. The **domain of definition** of the ordinary differential equation is the set of the maximal intervals of existence for all initial conditions.

To obtain uniqueness of solutions, we need to impose further restrictions on the function f.

Definition 2.14. $f(x,t)$ defined on $\mathcal{D}$ is said to be **locally Lipschitzian** with respect to x if for any compact $\mathcal{U} \subset \mathcal{D}$, there exists a constant $K_{\mathcal{U}} > 0$ such that $|f(x,t) - f(y,t)| \leqslant K_{\mathcal{U}}|x-y|$ for any $(x,t),(y,t)$ in $\mathcal{U}$.

In particular, if $f \in \mathcal{C}^1(\mathcal{D},\mathbb{R}^n)$, it is locally Lipschitzian because of Proposition 2.8. A contraction is a locally Lipschitzian function with $K_{\mathcal{U}} < 1\ \forall \mathcal{U}$.

Theorem 2.12 (Picard–Lindelöf). *If $f(x,t)$ is continuous and locally Lipschitzian in $\mathcal{D}$, then for any $(x_0,t_0) \in \mathcal{D}$ there is a unique solution $x(t)$ of* (2.35) *passing through (x_0,t_0). The domain of definition is open and $x(t)$ depends continuously on x_0, t_0.*

Example 2.5. We will usually assume that the physical system modelled by the ordinary differential equation must admit unique solutions existing for all times. One can however give very simple examples of equations which do not have these properties. Consider for instance

$$\dot{x} = x^2 \quad \Rightarrow \quad x(t) = \frac{1}{\frac{1}{x_0} - (t - t_0)}. \tag{2.37}$$

The solution explodes at $t = t_0 + \frac{1}{x_0}$. On the other hand, consider the **leacky bucket equation**[7]

$$\dot{x} = -\sqrt{|x|} \quad \Rightarrow \quad x(t) = \begin{cases} \frac{1}{4}(t-t_1)^2 & \text{for } t \leqslant t_1 \\ 0 & \text{for } t_1 < t < t_2 \\ -\frac{1}{4}(t-t_2)^2 & \text{for } t \geqslant t_2 \end{cases} \tag{2.38}$$

for arbitrary $-\infty \leqslant t_1 \leqslant t_2 \leqslant \infty$. Physically, we have to require $t_2 = \infty$, since x is positive. There is no unicity because, if the bucket is empty at time $t > t_1$, we cannot determine when it has been full.

[7] Applying the Bernouilli law, one obtains that the velocity squared of the water leaving the bucket is proportional to the height of water in the bucket [**HW**].

We finally state some results on the smoothness of the solutions, and on their dependence on initial conditions and parameters. The smoothness properties are important if we wish to compute Taylor series of solutions. Let us remark that the problems of dependence on initial conditions and parameters are related. Indeed, the system $\dot{x} = f(x,t)$, $x(t_0) = x_0$ is equivalent to $\dot{x} = f(x - x_0, t - t_0)$, $x(0) = 0$, so that x_0 and t_0 can be considered as parameters. On the other hand, the system $\dot{x} = f(x,t,\lambda)$, $x(t_0) = x_0$ is equivalent to $\dot{x} = f(x,t,y), \dot{y} = 0$ with initial conditions $(x(t_0), y(t_0)) = (x_0, \lambda)$.

Theorem 2.13. *Let $f(x,t,\lambda) \in \mathcal{C}^k(\mathcal{D}, \mathbb{R}^n)$, $k \geqslant 1$, for an open set $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$. Then the solution of $\dot{x} = f(x,t,\lambda)$ with initial condition $x(t_0) = x_0$ is a $\mathcal{C}^k$ function of x_0, t_0, t and λ on its domain of existence. Moreover, if $y(t)$ is a solution of the same equation with initial condition $y(t_0) = y_0$, then*

$$|x(t) - y(t)| \leqslant |x_0 - y_0| \,\mathrm{e}^{K(t-t_0)}, \tag{2.39}$$

provided there exists a uniform Lipschitz constant K for f on $\mathcal{D}$.

Remark 2.3.
- If $f(x,t,\lambda)$ is an analytic function on an open set of complex numbers, then the solution is an analytic function of x_0, t_0, t and λ, but we do not know the radius of convergence of its Taylor series.
- Analyticity of the solutions does not imply analyticity of the curves on which they exist. For instance the system $\dot{\xi} = \frac{1}{2}\xi^3, \dot{\eta} = -\eta + \frac{1}{2}\xi^2$ admits analytic solutions for all times, but they define curves of the form

$$\eta(\xi) = \mathrm{e}^{-1/\xi^2} \int_c^{1/\xi^2} \frac{e^s}{s} \,\mathrm{d}s, \tag{2.40}$$

which are $\mathcal{C}^\infty$ but not analytic at the origin.
- Equation (2.39) implies uniform convergence of solutions on *compact* sets, and thus stability on finite times. Indeed, if the initial conditions are sufficiently close, the solutions will remain as close as we like after any given *finite* time. They may, nevertheless, diverge exponentially. We give below a technically useful related result.

Lemma 2.3. *Suppose $(f_n)_{n \geqslant 1}$ is a sequence of continuous functions on $\mathcal{D}$, converging uniformly to f_∞ on compact subsets of $\mathcal{D}$. Let $((x_n^0, t_n^0))_{n \geqslant 1}$ be a sequence of points converging to (x_∞^0, t_∞^0) and let $x_n(t)$ be the solutions of $\dot{x} = f_n(x,t)$ with initial condition x_n^0 at t_n^0. If $x_\infty(t)$ exists and is unique on the interval $[t_1, t_2]$, then for large enough n, $x_n(t)$ exists on $[t_1, t_2]$ and converges uniformly to $x_\infty(t)$ on $[t_1, t_2]$.*

2.2.2 One–Dimensional Equations

We will need only a very limited number of elementary exactly solvable equations. We list below some one–dimensional equations. In the next section, we will consider n–dimensional linear equations.

A **separable equation** is an ordinary differential equation of the form

$$\dot{x} = f(x)g(t) \qquad \Rightarrow \qquad \int_{x_0}^{x(t)} \frac{\mathrm{d}x}{f(x)} = \int_{t_0}^{t} g(s)\,\mathrm{d}s. \tag{2.41}$$

Let us note two interesting special cases. The **autonomous equation**

$$\dot{x} = f(x) \qquad \Rightarrow \qquad \int_{x_0}^{x(t)} \frac{\mathrm{d}x}{f(x)} = t - t_0, \tag{2.42}$$

and the **linear equation**

$$\dot{x} = a(t)x \qquad \Rightarrow \qquad x(t) = \mathrm{e}^{\alpha(t)}\, x_0, \quad \alpha(t) := \int_{t_0}^{t} a(s)\,\mathrm{d}s. \tag{2.43}$$

Note that the autonomous equation has a solution for all times if and only if $1/f(x)$ is not integrable. Otherwise, the solution diverges at the time $t_0 + \int_{x_0}^{\infty} \frac{\mathrm{d}x}{f(x)}$, as in Example 2.5.

Some generalizations of the linear equation can be solved using the method of **variation of constants**, setting $x(t) = c(t)x_0(t)$, where $x_0(t)$ is the solution of the linear equation (2.43). In particular, we have the **inhomogeneous linear equation**[8]

$$\dot{x} = a(t)x + h(t) \quad \Rightarrow \quad x(t) = \mathrm{e}^{\alpha(t)} \left[x_0 + \int_{t_0}^{t} \mathrm{e}^{-\alpha(s)}\, h(s)\,\mathrm{d}s \right] \tag{2.44}$$

which is a special case of the **Bernouilli equation**

$$\dot{x} = a(t)x + b(t)x^n \quad \Rightarrow \quad x(t) = \frac{x_0\, \mathrm{e}^{\alpha(t)}}{\left[1 + (1-n)x_0^{n-1} \displaystyle\int_{t_0}^{t} \mathrm{e}^{(n-1)\alpha(s)}\, b(s)\,\mathrm{d}s \right]^{\frac{1}{n-1}}}. \tag{2.45}$$

Finally, let us also note that the **Ricatti equation**

$$\dot{x} = a(t)x + b(t)x^2 + h(t) \tag{2.46}$$

can be transformed, by the change of variables $x = -\frac{1}{b}\frac{\dot{\psi}}{\psi}$, into the linear equation

$$\ddot{\psi} - \left(a(t) + \frac{\dot{b}(t)}{b(t)} \right) \dot{\psi} + h(t)\, b(t)\, \psi = 0, \tag{2.47}$$

provided $b(t) \neq 0$. It can be transformed into a first order equation by putting $y = (\psi, \dot{\psi})$.

2.2.3 Linear Equations

Linear equations, of the form $\dot{x} = A(t)x$, play in important role in the theory of ordinary differential equations, because the local behaviour of more general equations can sometimes be reduced to the study of a linear one. Although the solutions of linear equations are better understood than those of nonlinear ones, they can only be solved in a limited number of cases. One of these is the autonomous case that we consider below. Then we will present some properties of general and time–periodic linear equations.

[8] An intuitive interpretation for economists of (2.44) is given in [**HW**].

Autonomous Case

Let us first consider the equation

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ A \in \mathbb{M}_n(\mathbb{C}). \tag{2.48}$$

An application of Picard's iterative method to this equation yields the formal series of the exponential function. Thus we make the following definition:

Definition 2.15. For $A \in \mathbb{M}_n(\mathbb{C})$ and $t \in \mathbb{R}$, the **exponential** of At is defined by the series

$$\exp(At) \equiv \mathrm{e}^{At} := \mathbb{1} + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots \tag{2.49}$$

Proposition 2.14. *The series* (2.49) *converges for all A uniformly for t in compact subsets of $\mathbb{R}$. Moreover, it has the following properties:*

1. *Equation* (2.48) *with initial condition $x(0) = x_0$ has the unique solution $x(t) = \mathrm{e}^{At}x_0$.*
2. $\mathrm{e}^{At}\,\mathrm{e}^{Bt} = \mathrm{e}^{(A+B)t}$ $\forall t$ *if and only if* $[A,B] = 0$.
3. *The exponential of At can be explicitly computed using the Jordan decomposition of Theorem 2.4:*

$$\mathrm{e}^{At} = \sum_{i=1}^{m} \mathrm{e}^{a_it}P_i\biggl(\mathbb{1} + N_it + \frac{1}{2!}N_i^2t^2 + \cdots + \frac{1}{(m_i-1)!}N_i^{m_i-1}t^{m_i-1}\biggr), \tag{2.50}$$

where $m_i := \mathrm{m_a}(a_i, A)$.

4. *If $a_+ := \max_i \operatorname{Re} a_i$, there exists a continuous function $K(\delta)$ such that*

$$\|\mathrm{e}^{At}\| \leqslant K(\delta)\,\mathrm{e}^{a_+t+\delta|t|} \qquad \textit{for all } \delta > 0. \tag{2.51}$$

Proof: The convergence follows from the fact that the series is absolutely convergent, with a norm bounded by $\mathrm{e}^{\|A\|t}$. The commutation relations of Theorem 2.4 imply that $\mathrm{e}^{At} = \mathrm{e}^{Dt}\,\mathrm{e}^{Nt}$ where

$$\begin{aligned}
\mathrm{e}^{Dt} &= \prod_{i=1}^{m}\mathrm{e}^{a_iP_it} = \prod_{i=1}^{m}\bigl(\mathbb{1} + (\mathrm{e}^{a_it}-1)P_i\bigr) = \mathbb{1} + \sum_{i=1}^{m}(\mathrm{e}^{a_it}-1)P_i = \sum_{i=1}^{m}\mathrm{e}^{a_it}P_i,\\
\mathrm{e}^{Nt} &= \prod_{i=1}^{m}\mathrm{e}^{N_it} = \mathbb{1} + \sum_{i=1}^{m}(\mathrm{e}^{N_it}-\mathbb{1}).
\end{aligned} \tag{2.52}$$

The result follows from $P_i(\mathrm{e}^{N_jt}-\mathbb{1}) = \delta_{ij}(\mathrm{e}^{N_jt}-\mathbb{1})$, and the fact that the series of e^{Nt} contains only a finite number of terms since N is nilpotent. Finally, the bound on the norm follows from (2.50), using the fact that $|t|^k \leqslant k!\delta^{-k}\mathrm{e}^{\delta|t|}$ for $\delta > 0$. □

Lemma 2.4. *If $\det A \neq 0$, there exists a matrix B such that $\mathrm{e}^B = A$. In terms of the Jordan decomposition of A, it is given by*

$$B = \sum_{i=1}^{m}\biggl(\ln(a_i)P_i - \sum_{j=1}^{m_i}\frac{(-N_i)^j}{ja_i^j}\biggr). \tag{2.53}$$

Proof: We can repeat the proof of Proposition 2.14 with D replaced by $\sum_i \ln(a_i)P_i$, and N replaced by the nilpotent matrix obtained using the power series of $\ln(1+N)$. □

General Case

Let us now consider the general case of a linear ordinary differential equation. Given a continuous matrix function $A(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{M}_n(\mathbb{C}))$ and a continuous **forcing function** $h(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{C}^n)$, we define the **linear inhomogeneous equation**

$$\dot{x} = A(t)x + h(t), \tag{2.54}$$

and the **linear homogeneous equation**

$$\dot{x} = A(t)x. \tag{2.55}$$

Theorem 2.14. *Equation* (2.54) *has a unique solution for all times, satisfying the* **superposition principle**. *Namely, if $\dot{x} = A(t)x + h_1(t)$, $x(t_0) = x_0$ and $\dot{y} = A(t)y + h_2(t)$, $y(t_0) = y_0$, then the linear combination $z = \alpha x + \beta y$ satisfies the equation $\dot{z} = A(t)z + \alpha h_1(t) + \beta h_2(t)$, with $z(t_0) = \alpha x_0 + \beta y_0$.*

In particular, if $h_1 = h_2 \equiv 0$, it is sufficient to know the solutions for a set of linearly independent initial conditions. This leads to the following definitions.

Notation 2.9. If $U(t)$ is any invertible matrix satisfying $\dot{U} = A(t)U$, it is called a **fundamental solution** of the linear equation. We denote by $U(t, t_0)$ the fundamental solution such that $U(t_0, t_0) = \mathbb{1}$. It is called **principal solution** with initial time t_0 (by mathematicians) or **propagator** or **evolution operator** (by physicists). It can be obtained from any fundamental solution by $U(t, t_0) = U(t)U(t_0)^{-1}$. The solution of (2.55) with initial condition x_0 at t_0 is then given by $x(t) = U(t, t_0)x_0$.

Proposition 2.15. *The principal solution has the properties*
1. *$U(t, t_0) = U(t_0, t)^{-1}$ and $U(t, t_0) = U(t, s)U(s, t_0)$;*
2. *$\det U(t, t_0) = \exp \int_{t_0}^t \operatorname{Tr} A(s)\,\mathrm{d}s$; hence, $U(t)$ is invertible for all $t \Leftrightarrow U(t_0)$ is.*
3. *$\|U(t, t_0)\| \leqslant \exp\bigl(\bigl|\int_{t_0}^t \|A(s)\|\,\mathrm{d}s\bigr|\bigr)$.*
4. *If $A(t)$ is anti-hermitian, then $U(t, t_0)$ is unitary.*
5. *The solution of the inhomogeneous equation* (2.54) *is given by*

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^t U(t, s)h(s)\,\mathrm{d}s. \tag{2.56}$$

Proof: The first property follows from unicity of the solution. The second is due to the fact that $\det U$ satisfies the equation $\dot{z} = \operatorname{Tr} A(t)z$. The third property follows from Picard's iterative method. The fourth one is due to the fact that $\mathrm{d}_t\langle Ux|Ux\rangle = \langle UAx|Ux\rangle + \langle Ux|UAx\rangle = 0$. The last one is a straightforward generalization of (2.44). □

Periodic Case

Let us now specialize to the case of a periodic matrix:

$$\dot{x} = A(t)x, \qquad A(t+T) = A(t)\ \forall t, \quad T > 0. \tag{2.57}$$

Theorem 2.15 (Floquet). *Every fundamental solution of* (2.57) *can be written*

$$U(t) = P(t)\,\mathrm{e}^{Bt}, \tag{2.58}$$

where $P(t+T) = P(t)$ $\forall t$ and B is a constant matrix.

PROOF: Since, by periodicity, $U(t)$ and $U(t+T)$ are fundamental solutions of (2.57), unicity of solutions implies that there exists an invertible matrix C such that $U(t+T) = U(t)C$. By Lemma 2.4, there is a matrix B such that $C = \mathrm{e}^{BT}$. If we define $P(t) := U(t)\,\mathrm{e}^{-Bt}$, we have $P(t+T) = U(t+T)\,\mathrm{e}^{-B(t+T)} = U(t)\,\mathrm{e}^{BT}\,\mathrm{e}^{-B(t+T)} = P(t)$. □

Corollary 2.3. *There exists a linear non–singular change of variables $x = P(t)y$, transforming* (2.57) *into the autonomous system $\dot{y} = By$.*

Remark 2.4. If $U(t,0)$ is the principal solution with initial time 0, then $P(0) = \mathbb{1}$. We may also introduce a periodic matrix $Q(t)$ such that $U(t,0) = \mathrm{e}^{B't}\,Q(t)$, defined by the relation $U(t+T,0) =: \mathrm{e}^{B'T}\,U(t,0)$. Then we have $\mathbb{1} = U(0,0) = P(0) = Q(0)$ and $U(T,0) = \mathrm{e}^{BT} = \mathrm{e}^{B'T}$, which implies that we may choose $B' = B$. Furthermore, we have by definition $U(t+T,0) = U(T,0)U(t,0) = U(t,0)U(T,0)$, which means that $P(t)$ and $Q(t)$ commute with e^{BT}. Finally, we also have $U(t+T,t) = U(t+T,0)U(t,0)^{-1} = U(T,0)$.

The matrix e^{BT} is called the **monodromy matrix** associated with $U(t)$. Its eigenvalues are called **characteristic multipliers** and the eigenvalues of B are called **characteristic exponents** of (2.57). The difficulty is that in general, there is no obvious relation between the characteristic exponents and the eigenvalues of $A(t)$.[9] However, we will be mainly interested in the case of a matrix A depending adiabatically on time, and we will show that such a relation may exist in that case.

Let us finally consider the inhomogeneous periodic equation

$$\dot{x} = A(t)x + h(t), \qquad A(t+T) = A(t), \quad h(t+T) = h(t). \tag{2.59}$$

Theorem 2.16. *The inhomogeneous system* (2.59) *has a periodic solution for every periodic forcing $h(t)$ if and only if the homogeneous equation* (2.57) *has no periodic solution except the identically zero one. If it exists, the periodic solution is given by*

$$x(t) = \big[U(T,0)^{-1} - \mathbb{1}\big]^{-1} \int_0^T U(t,t+s)h(t+s)\,\mathrm{d}s. \tag{2.60}$$

PROOF: It follows from Proposition 2.15 that if $x(t)$ is a periodic solution, then it must satisfy

$$\begin{aligned}
0 &= x(t) - x(t+T) \\
&= \big[U(t,0) - U(t+T,0)\big]x(0) + \int_0^t U(t,s)h(s)\,\mathrm{d}s - \int_0^{t+T} U(t+T,s)h(s)\,\mathrm{d}s \\
&= \big[\mathbb{1} - U(T,0)\big]U(t,0)\Big[x(0) + \int_0^t U(0,s)h(s)\,\mathrm{d}s\Big] - U(T,0)\int_t^{t+T} U(t,s)h(s)\,\mathrm{d}s.
\end{aligned}$$

where we have used $U(t+T,0) = U(T,0)U(t,0)$ (see Remark 2.4). Multiplying by the inverse of $[\mathbb{1} - U(T,0)]$, the general solution (2.56) yields the conclusion. □

If the homogeneous equation admits a periodic orbit, then the system admits 1 as a characteristic multiplier,[10] and the expression (2.60) is not defined. This is nothing but the phenomenon of **resonance** of a system, which is being forced with an eigenfrequency. In fact, Theorem 2.16 is related to the concept of **non–criticality** [**Hal**], and can be extended to quasiperiodic and bounded forcings.

[9] Except that, because of Proposition 2.15, $\det \mathrm{e}^{TB} = \exp \int_0^T \operatorname{Tr} A(s)\,\mathrm{d}s$.
[10] See Proposition 2.16.

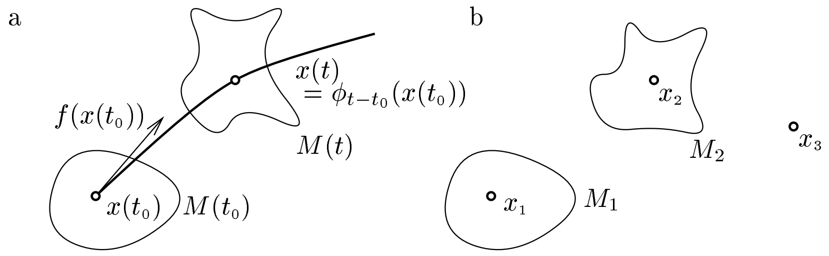

FIGURE 2.2. (a) Evolution of a volume with the flow, (b) evolution with an iterated map.

2.3 Dynamical Systems

We now come to the central theme of this work, the notion of **Dynamical Systems**. An abstract dynamical system is defined by a **phase space** and a **dynamics** on this space. To be specific, we will parametrize the phase space by an open subset $\mathcal{D}$ of $\mathbb{R}^n$.[11] Given functions $f, F \in \mathcal{C}^k(\mathcal{D}, \mathbb{R}^n)$, $k \geqslant 1$, we may define two kinds of dynamics:

1. A **continuous time** dynamical system is given by the flow of the autonomous ordinary differential equation
$$\dot{x} = f(x), \qquad t \in \mathbb{R}. \tag{2.61}$$
2. A **discrete time** dynamical system is defined by the iterated map
$$x_{p+1} = F(x_p), \qquad p \in \mathbb{Z}. \tag{2.62}$$

Understanding the dynamical system amounts to understanding the geometric structure of its **orbits** in phase space. This is in general a difficult task, but there exist a few simple methods which can provide useful informations. We discuss some general properties, such as the evolution of volumes with time. Then we examine in more detail the behaviour near singular points and periodic orbits.

We follow mainly the books of Guckenheimer and Holmes [**GH**], and Hirsch and Smale [**HS**]. See also Arnold [**Ar2**], Hale and Koçak [**HK**], Wiggins [**Wi**] and Hale [**Hal**] for the theory of Lyapunov functions.

2.3.1 General Properties of Flows and Orbits

Definition 2.16.
1. The **flow** of (2.61) is the map $\phi_t : \mathcal{D} \to \mathcal{D}$ defined by the relation $x(t) =: \phi_{t-t_0}\bigl(x(t_0)\bigr)$. It satisfies $\phi_0(x) = x$ and the **group property** $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$. The **orbit** of (2.61) through the point x is the curve $\{\phi_t(x) \mid t \in I\}$ where $I \subset \mathbb{R}$ is the interval of existence of the solution with initial condition x at some $t_0 \in I$. Unicity of the solution implies that there is only one orbit through each point in phase space.
2. An **orbit** of (2.62) is a set of points $\{x_p = F^p(x) \mid p \in J\}$ where $J \subset \mathbb{Z}$. If $F(\mathcal{D}) \subset \mathcal{D}$, the orbit exists for all $p \geqslant p_0$. If $F(x)$ is invertible, it is unique. In particular, if $F(x)$ is a diffeomorphism, every $x \in \mathcal{D}$ admits a unique doubly infinite orbit.

[11] More generally, one can define Dynamical Systems on manifolds [**Ar2**], but in the cases we consider, we can always assume that this manifold has a global parametrization.

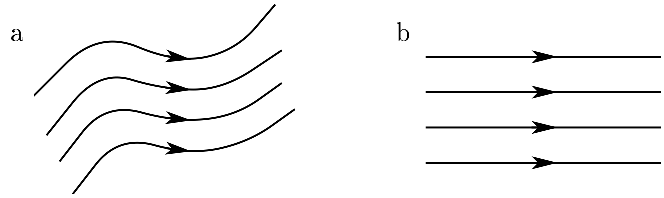

FIGURE 2.3. Illustration of Theorem 2.18: (a) the flow in the original variables x, (b) the "lifted" flow in the new variables y.

Theorem 2.17 (Jacobi–Liouville).
1. Let $M \subset \mathcal{D}$ and $V(t)$ be the volume of $\phi_t(M)$. Then

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \int_{\phi_t(M)} \nabla \cdot f(x) \, \mathrm{d}x. \tag{2.63}$$

2. Let $M \subset \mathcal{D}$ and V_p be the volume of $F^p(M)$. Then

$$V_{p+1} = \int_{F^p(M)} \big|\det \partial_x F(x)\big| \, \mathrm{d}x. \tag{2.64}$$

PROOF: (2.64) is simply the formula for a change of variables in an integral. Similarly, (2.63) is obtained by computing the evolution of the Jacobian of the transformation $y = \phi_t(x)$. The Jacobian matrix $J := \partial_x y$ obeys the linear equation $\dot{J} = \mathrm{d}_x f(\phi_t(x)) = \partial_x f(y) J$. Proposition 2.15 then implies that $\mathrm{d}_t \det J(t) = \mathrm{Tr}(\partial_x f(y)) \det J(t) = \nabla \cdot f(y) \det J(t)$, so that $\dot{V}(t) = \int_M \mathrm{d}_t \det J(t) \, \mathrm{d}x = \int_{\phi_t(M)} \nabla \cdot f(y) \, \mathrm{d}y$. □

Definition 2.17.
1. The system (2.61) is **conservative** if $\nabla \cdot f = 0 \ \forall x$, **dissipative** if $\nabla \cdot f < 0 \ \forall x$.
2. The system (2.62) is **conservative** if $|\det \partial_x F| = 1 \ \forall x$, **dissipative** if $|\det \partial_x F| < 1 \ \forall x$.

Theorem 2.17 states that a conservative system will preserve volumes in phase space, while a dissipative system will contract them. They are the most common in physical models, although there also exist systems which are neither conservative nor dissipative.

Definition 2.18.
1. Let $x^\star \in \mathcal{D}$ be a solution of $f(x^\star) = 0$. Then $x^\star$ is called a **singular point** of the vector field f, or a **fixed point**, **stationary point** or **equilibrium point** of the flow.
2. A **fixed point** of (2.62) is a point $x^\star \in \mathcal{D}$ such that $F(x^\star) = x^\star$.

The importance of this definition comes from the theorem below, which states that away from the fixed points, the orbits are "locally parallel" (see Fig. 2.3).

Theorem 2.18. *In a neighborhood of a point which is not singular, there exists a $\mathcal{C}^k$–diffeomorphism $y = h(x)$ transforming the system (2.61) into $\dot{y}_{[i]} = \delta_{i1}$.*

2.3.2 Fixed Points and Stability

Let $x^\star$ be a singular point of the system $\dot{x} = f(x)$. If we write $x = x^\star + y$, we obtain

$$\dot{y} = Ay + b(y), \tag{2.65}$$

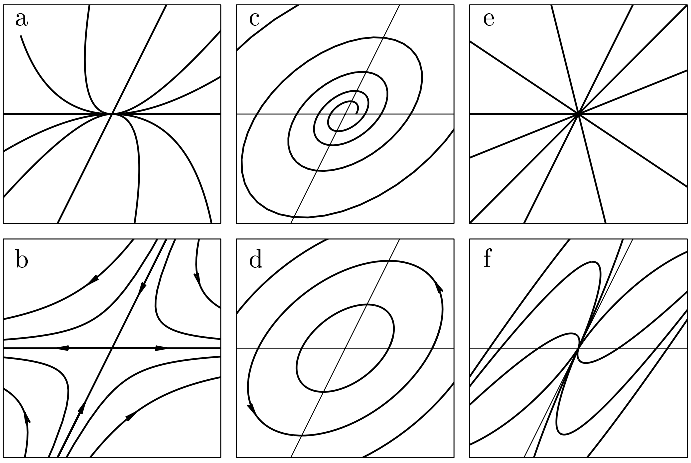

FIGURE 2.4. Phase portraits of a linear two–dimensional system: (a) node, (b) saddle, (c) focus, (d) center, (e) degenerate node, (f) improper node.

where $A := \partial_x f(x^\star)$ and $b(y) = \mathcal{O}(y)$. The **linearization** of this equation is the system

$$\dot{y} = Ay \qquad \Rightarrow y(t) = \mathrm{e}^{At}\, y(0). \tag{2.66}$$

We have studied the solutions in Proposition 2.14. Formula (2.50) tells us that their long–time behaviour is dictated by the eigenvalues of A, although the polynomial terms due to the nilpotent part may interfere on a short timescale. This leads to

Definition 2.19. The **unstable**, **stable** and **center subspace** of the fixed point are defined, respectively, by

$$E^+ := P^+\mathbb{R}^n = \big\{y \,\big|\, \lim_{t\to-\infty} \mathrm{e}^{At}\, y = 0\big\}, \qquad P^+ := \sum_{i:\mathrm{Re}\, a_i > 0} P_i, \qquad \mathrm{e}^{At}\, E^+ \subset E^+; \tag{2.67a}$$

$$E^- := P^-\mathbb{R}^n = \big\{y \,\big|\, \lim_{t\to+\infty} \mathrm{e}^{At}\, y = 0\big\}, \qquad P^- := \sum_{i:\mathrm{Re}\, a_i < 0} P_i, \qquad \mathrm{e}^{At}\, E^- \subset E^-; \tag{2.67b}$$

$$E^0 := P^0\mathbb{R}^n, \qquad P^0 := \sum_{i:\mathrm{Re}\, a_i = 0} P_i, \qquad \mathrm{e}^{At}\, E^0 \subset E^0. \tag{2.67c}$$

The fixed point is called

- a **sink** if $E^+ = E^0 = \{0\}$,
- a **source** if $E^- = E^0 = \{0\}$,
- a **hyperbolic point** if $E^0 = \{0\}$,
- an **elliptic point** if $E^+ = E^- = \{0\}$.

Example 2.6. Let $A \in \mathbb{M}_2(\mathbb{R})$, $\det A \neq 0$, and $J = S^{-1}AS$ be its Jordan reduced. Then $\mathrm{e}^{At} = S\,\mathrm{e}^{Jt}\,S^{-1}$ and we can distinguish between the following behaviours, depending on the eigenvalues a_1, a_2 of A (see Fig. 2.4).

1. $a_1 \neq a_2$
   (a) If $a_1, a_2 \in \mathbb{R}$, then $\mathrm{e}^{Jt} = \mathrm{diag}(\mathrm{e}^{a_1 t}, \mathrm{e}^{a_2 t})$; $x^\star$ is called
      i. a **node** if $a_1 a_2 > 0$,
      ii. a **saddle** if $a_1 a_2 < 0$.

   (b) If $a_1 = a_2^* = a + \mathrm{i}\,\omega \in \mathbb{C}$, then $\mathrm{e}^{Jt} = \mathrm{e}^{at}\,\mathrm{diag}(\mathrm{e}^{\mathrm{i}\,\omega t}, \mathrm{e}^{-\mathrm{i}\,\omega t})$; $x^\star$ is called
      i. a **focus** if $a \neq 0$,
      ii. a **center** if $a = 0$.
2. $a_1 = a_2 =: a$
   (a) If $m_A(t) = (t-a)$, $A = J = a\mathbb{1}$ and $\mathrm{e}^{Jt} = \mathrm{e}^{at}\,\mathbb{1}$; $x^\star$ is called a **degenerate node**.
   (b) If $m_A(t) = (t-a)^2$, $J = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ and $\mathrm{e}^{Jt} = \mathrm{e}^{at}\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$; $x^\star$ is called an **improper node**.

In the case of the iterated map (2.62), the situation is similar. The linearization around a fixed point is

$$y_{p+1} = By_p \qquad \Rightarrow y_p = B^p y_0. \tag{2.68}$$

where $B := \partial_x F(x^\star)$. A similar computation as in Proposition 2.14 gives, in terms of the Jordan decomposition of B,

$$B^p = \sum_{i=1}^{m} P_i \sum_{j=0}^{\min\{p, m_i - 1\}} \binom{p}{j} a_i^{p-j} N_i^j. \tag{2.69}$$

We can thus make similar definitions as (2.67), replacing the condition $\mathrm{Re}\, a_i \gtreqless 0$ with $|a_i| \gtreqless 1$.

Once the behaviour of the linearization is understood, the natural question is what relation exists between the orbits of the linearized system (2.66) and the original nonlinear system (2.65). A first important concept is stability.

Definition 2.20. The fixed point $x^\star$ of the system $\dot{x} = f(x)$ is **stable** if for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$, such that $\|x(t_0) - x^\star\| < \delta$ implies $\|x(t) - x^\star\| < \varepsilon\ \forall t \geqslant t_0$. The fixed point $x^\star$ is **asymptotically stable** if, moreover, there is a $\delta_0 > 0$ such that $\lim_{t\to\infty} x(t) = x^\star$ if $\|x(t_0) - x^\star\| < \delta_0$. It is **unstable** if it is not stable.

Definition 2.19 implies that the origin of the linearized system (2.66) is stable if $E^+ = \{0\}$, asymptotically stable if $E^+ = E^0 = \{0\}$, and unstable if $E^+ \neq \{0\}$. A useful method to extend these properties to the nonlinear system (2.65) relies on **Lyapunov functions**.

Theorem 2.19 (Lyapunov, Četaev). *Let $x^\star$ be a fixed point of the system $\dot{x} = f(x)$, $\mathcal{U}$ a neighborhood of $x^\star$ and $\mathcal{U}_0 := \mathcal{U} \setminus \{x^\star\}$.*
1. *Assume there exists a function $V(x) \in \mathcal{C}^1(\mathcal{U}_0, \mathbb{R})$ such that, $\forall x \in \mathcal{U}_0$,*
   (a) $V(x) > V(x^\star)$, and
   (b) $\dot{V}(x) := \partial_x V f(x) \leqslant 0$.
   Then $x^\star$ is stable. Moreover, if
   (c) $\dot{V}(x) < 0$,
   then $x^\star$ is asymptotically stable.
2. *Assume there exists an open set $\mathcal{D}$ containing $x^\star$ in its closure and a function $V(x) \in \mathcal{C}^1(\mathcal{U}_0, \mathbb{R})$ such that*
   (a) $V(x) > 0$ and $\dot{V}(x) \geqslant 0$ on $\mathcal{U}_0 \cap \mathcal{D}$ and
   (b) $V(x) = 0$ on $\mathcal{U}_0 \cap \partial\mathcal{D}$.
   Then $x^\star$ is unstable.

The proofs, which have an intuitive geometric interpretation (see Fig. 2.5) are found in [**HS**] and [**Hal**, **HK**]. We will prove extensions of these results in Section 5.2.

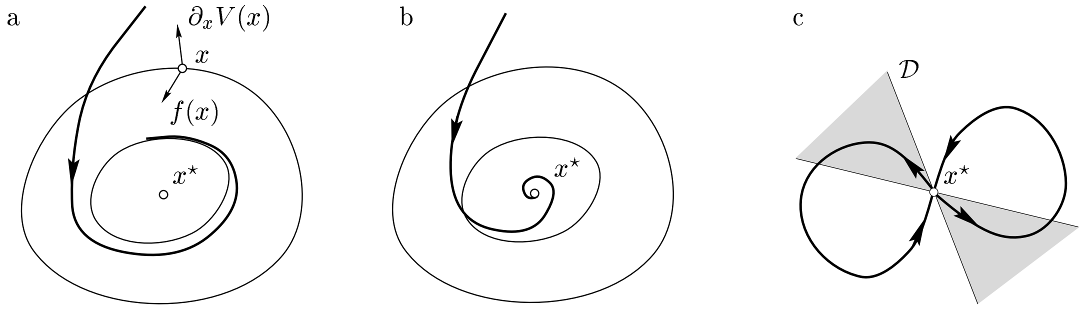

FIGURE 2.5. (a) Stable fixed point with level curves of a Lyapunov function. (b) Asymptotically stable fixed point, here the vector field must cross all level curves in the same direction. (c) Unstable fixed point with the set $\mathcal{D}$ of Corollary 2.4.

Corollary 2.4. *If all eigenvalues of $A := \partial_x f(x^\star)$ have negative real part, then $x^\star$ is asymptotically stable. If one eigenvalue of A has a positive real part, then $x^\star$ is unstable.*

PROOF: Assume all eigenvalues have negative real part, and take $x^\star = 0$ for simplicity. Let S be the matrix of Lemma 2.1, $y = Sx$ and $V(x) = \langle y|y\rangle$. Then $\dot V = 2\operatorname{Re}\langle Sx|S\dot x\rangle = 2\operatorname{Re}\langle y|SAS^{-1}y\rangle + \mathcal{O}(y^2)$. Lemma 2.1 implies that $\dot V$ is negative for sufficiently small y, and thus V is a Lyapunov function implying asymptotic stability. Assume now one eigenvalue has positive real part, and let x_1 be a corresponding eigenvector. Let $V_\pm(x) = \langle SP^\pm x|SP^\pm x\rangle$, $V_0(x) = \langle SP^0 x|SP^0 x\rangle$ (where $P^{\pm,0}$ are the projectors of Definition 2.19), and take $V = V_+ + V_0 - V_-$. By the lemma, $\dot V > 0$ for small y. We can apply Theorem 2.19, where $\mathcal{D}$ is the cone $\{x \mid V(x) > 0\}$. $\mathcal{D}$ is not empty because it contains the eigenvector x_1. □

In other words, a hyperbolic point is asymptotically stable if it is a sink, otherwise it is unstable. In fact, there are even stronger similarities between the orbits of a nonlinear system near a hyperbolic point, and those of its linearization. The following two theorems describe some of them.

Theorem 2.20 (Hartman–Grobman). *If $x^\star$ is a hyperbolic fixed point of the system $\dot x = f(x)$, then there exists, in a neighborhood of $x^\star$, a homeomorphism h taking the orbits of* (2.65) *to those of the flow* e^{At} *of* (2.66). *The homeomorphism preserves the sense of orbits and may be chosen to preserve parametrization by time.*

The orbits are said to be **topologically equivalent**. A more difficult question is the existence of a **diffeomorphism** $h(x)$. In fact, we will see in the next section that such a diffeomorphism exists under more restrictive conditions on the eigenvalues.

Definition 2.21. Let $x^\star$ be a fixed point of the system $\dot x = f(x)$, and let $\mathcal{U}$ be a neighborhood of $x^\star$. The **local stable and unstable manifolds** are defined, respectively, by

$$W^{\mathrm{s}}_{\mathrm{loc}}(x^\star) := \bigl\{x \in \mathcal{U} \bigm| \lim_{t\to\infty} \phi_t(x) = x^\star \text{ and } \phi_t(x) \in \mathcal{U}\ \forall t \geqslant 0\bigr\} \tag{2.70a}$$

$$W^{\mathrm{u}}_{\mathrm{loc}}(x^\star) := \bigl\{x \in \mathcal{U} \bigm| \lim_{t\to-\infty} \phi_t(x) = x^\star \text{ and } \phi_t(x) \in \mathcal{U}\ \forall t \leqslant 0\bigr\}. \tag{2.70b}$$

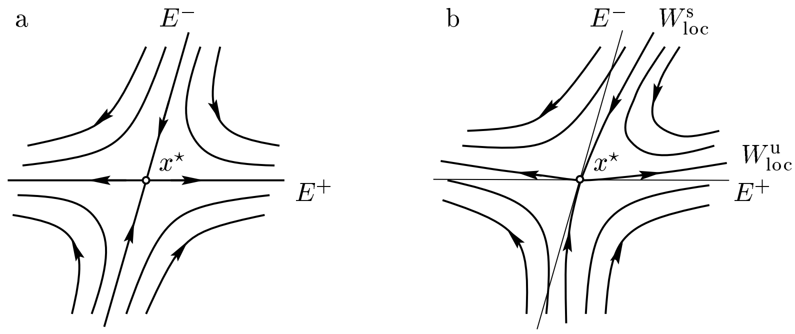

FIGURE 2.6. Orbits near a hyperbolic fixed point: (a) orbits of the linearized system, (b) orbits of the nonlinear system with local stable and unstable manifolds.

Moreover, we define **global stable and unstable manifolds** by

$$W^{\mathrm{s}}(x^\star) = \bigcup_{t \leqslant 0} \phi_t(W^{\mathrm{s}}_{\mathrm{loc}}(x^\star)), \tag{2.70c}$$

$$W^{\mathrm{u}}(x^\star) = \bigcup_{t \geqslant 0} \phi_t(W^{\mathrm{u}}_{\mathrm{loc}}(x^\star)). \tag{2.70d}$$

Theorem 2.21 (Stable Manifold Theorem). *If $x^\star$ is a hyperbolic fixed point of the system $\dot{x} = f(x)$, there exist local stable and unstable manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(x^\star)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(x^\star)$, of the same dimension as E^- and E^+, which are tangent to $E^\pm$ at $x^\star$ and as smooth as $f(x)$.*

Similar, but slightly weaker properties hold for the center manifold. We will examine that case in the next section. There also exist equivalent results for maps, the only difference being that the orbits on the invariant manifolds are collections of points instead of curves.

2.3.3 Periodic Orbits and Stability

A **periodic orbit** of period N of an iterated map is an orbit $(x_p)_{p\in\mathbb{Z}}$ such that $x_{p+N} = x_p$ $\forall p$. In other words, it is the orbit of a fixed point of the map $F^N(x)$. Thus, its linear stability is determined by the matrix

$$\partial_x F^N(x^\star) = \partial_x F\big(F^{N-1}(x^\star)\big)\partial_x F\big(F^{N-2}(x^\star)\big)\dots\partial_x F(x^\star). \tag{2.71}$$

A **periodic orbit** of period T of the ordinary differential equation $\dot{x} = f(x)$ is a solution $\gamma(t)$ satisfying $\gamma(t+T) = \gamma(t)$ $\forall t$. It corresponds to a closed curve in phase space. If we write $x = \gamma(t) + y$, we obtain for y the differential equation

$$\dot{y} = A(t)y + b(y,t), \tag{2.72}$$

where $A(t) := \partial_x f(\gamma(t))$ and $b(y,t) = \mathcal{O}(y)$. The linearization of this equation, $\dot{y} = A(t)y$, has been studied in the Floquet theorem (Theorem 2.15). The solutions being of the form $P(t)\,\mathrm{e}^{BT}$, the stability is determined by the characteristic exponents. The geometric interpretation of this fact proceeds as follows:

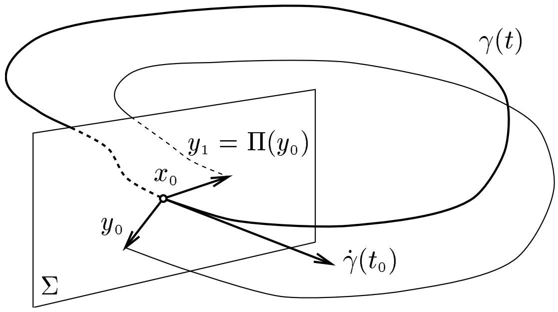

FIGURE 2.7. Definition of the Poincaré map associated with the periodic orbit $\gamma(t)$.

Definition 2.22. Let $\gamma(t)$ be a periodic orbit, $x_0 = \gamma(t_0)$ and Σ a hyperplane transverse to the orbit at x_0 (see Fig. 2.7). There is a neighborhood $\mathcal{U}$ of x_0 in Σ such that for all $x = x_0 + y \in \mathcal{U}$, we can define a smooth map $\tau(y)$, $\tau(0) = T$, such that the trajectory returns for the first time to Σ in a vicinity of x_0 at $t = \tau(y)$. The **Poincaré map** Π associated with the periodic orbit is defined by

$$x_0 + \Pi(y) := \phi_{\tau(y)}(x_0 + y). \tag{2.73}$$

Proposition 2.16. *The Poincaré map is as smooth as the vector field in a neighborhood of the origin. The characteristic multipliers of the periodic orbit are given by* 1 *and the* $n-1$ *eigenvalues of the Jacobian matrix* $\partial_y\Pi(0)$.

PROOF: The smoothness of the Poincaré map is a direct consequence of Theorem 2.13 and the implicit function theorem. Next we note that $\mathrm{d}_t\dot\gamma(t) = \mathrm{d}_t f(\gamma(t)) = \partial_x f(\gamma(t))\dot\gamma(t) = A(t)\dot\gamma(t)$, which implies, by the Floquet theorem, that $\dot\gamma(t) = P(t)\,\mathrm{e}^{Bt}\,\dot\gamma(t_0)$ for some periodic $P(t)$, and thus $\dot\gamma(t_0) = \dot\gamma(t_0+T) = \mathrm{e}^{BT}\,\dot\gamma(t_0)$. Hence $\dot\gamma(t_0)$ is an eigenvector of the monodromy matrix e^{BT} with eigenvalue 1. Let $(e_1,\dots,e_{n-1},\dot\gamma(t_0))$, $e_1,\dots,e_{n-1} \in \Sigma$, be a basis of $\mathbb{R}^n$. In this basis,

$$\mathrm{e}^{BT} = \begin{pmatrix} \mathrm{e}^{B_\Sigma T} & 0 \\ \dots & 1 \end{pmatrix}, \tag{2.74}$$

where B_Σ is the restriction of B to Σ, and the dots denote arbitrary entries. Now, if we consider momentarily y as a vector in $\mathbb{R}^n$ instead of Σ, linearization of (2.73) gives $\partial_y\Pi(0) = \dot\phi_T(x_0)\partial_y\tau(0) + \partial_y\phi_T(x_0) = \dot\gamma(t_0)\partial_y\tau(0) + \mathrm{e}^{BT}$. But the component of $\partial_y\tau(0)$ along $\dot\gamma(t_0)$ is 0, so that $\partial_y\Pi(0)$ has the same representation as e^{BT}, save for the entries marked by dots in (2.74). In particular, when y is restricted to Σ, $\partial_y\Pi(0)$ has the same eigenvalues than $\mathrm{e}^{B_\Sigma T}$. $\square$

The consequence of this result is that, by studying the stability and invariant manifolds of the Poincaré map, we obtain a complete characterization of the stability of the periodic orbit, for which we can also define stable and unstable manifolds.

2.4 Normal Forms and Bifurcations

In the preceding section, we have stated two results on the relation between the nonlinear equation $\dot{y} = Ay + b(y)$ and its linearization $\dot{y} = Ay$, when the origin is a hyperbolic point. The Hartman–Grobman theorem (Theorem 2.20) tells us that the orbits of both systems are topologically equivalent near the origin. The stable manifold theorem (Theorem 2.21) shows the existence of invariant manifolds associated with the fixed point.

We examine in this section some generalizations of these results. In particular, we would like to know

- if there is a diffeomorphism taking the orbits of the nonlinear system to those of its linearization,
- what happens near equilibrium points with purely imaginary eigenvalues.

The first question can be answered using the theory of **normal forms**. One can devise a systematic procedure in order to eliminate nonlinear terms, keeping only a small number of resonant terms. We expose this method in the first subsection. A partial answer to the second question is given by the **center manifold theorem**, which we expose in the following section. This result is an important tool in order to reduce the system to a lower–dimensional one.

Another important phenomenon associated with non–hyperbolic fixed points is the **bifurcation** of equilibria and periodic orbits. Bifurcation theory in itself is a huge subject, and we only outline a few points which are relevant for our future analysis. The most commonly used approach is based on a center manifold reduction, followed by an application of normal form theory. However, the phenomena we are going to study are affected by the changes of variables necessary to obtain the normal form. For this reason, we also present an alternative approach to bifurcation theory based on the implicit function theorem.

For the theory of normal forms, we follow mainly [**GH**] and [**Har**]. The center manifold theorem is discussed at length in [**Ca**] and [**MM**]. The literature on bifurcation theory is very large. A good review article is [Cr], whereas detailed results on computation of bifurcation diagrams can be found in [**GH**]. A complete introduction to bifurcating equilibria is given in [**IJ**], whereas [**IA**] stresses the role of symmetries.

2.4.1 Normal Forms

We have seen that near an equilibrium point, the differential equation $\dot{x} = f(x)$ can be written $\dot{y} = Ay + \mathcal{O}(y)$. After a linear change of variables, which casts the linear part into Jordan form, we obtain the equation

$$\dot{x} = Jx + b(x), \tag{2.75}$$

where $b(x) = \mathcal{O}(x)$. We assume that $b(x) \in \mathcal{C}^k(\mathcal{U}, \mathbb{R}^n)$, where $\mathcal{U}$ is a neighborhood of $x = 0$ and $k \geqslant 2$.

The purpose of normal forms is to simplify as much as possible the nonlinear part $b(x)$, by a change of variables of the form $x = y + h(y)$, which yields

$$[\mathbb{1} + \partial_y h(y)]\dot{y} = Jy + Jh(y) + b(y + h(y)). \tag{2.76}$$

Ideally, we would obtain for y the linearized equation $\dot{y} = Jy$, if $h(y)$ solves the equation

$$\partial_y h(y) Jy - Jh(y) = b(y + h(y)). \tag{2.77}$$

Since this nonlinear functional equation is in general difficult to solve, one usually tries to construct a solution by successive changes of variables. If $h(y) = \mathcal{O}(y^m)$, $2 \leqslant m \leqslant k$, (2.76) gives

$$\dot{y} = Jy + \big[Jh(y) - \partial_y h(y)Jy + b(y)\big] + \mathcal{O}(y^m). \tag{2.78}$$

Hence, if we manage to construct $h(y)$ so that the terms in brackets vanish, we will have removed terms of order m. The equation to be solved,

$$\partial_y h(y)Jy - Jh(y) = b(y), \tag{2.79}$$

is called the **homological equation**. The simplification procedure can be made systematic in the following way.

Notation 2.10. For $m \geqslant 2$, let $\mathcal{H}_m$ be the vector space of homogeneous polynomial maps of degree m from $\mathbb{R}^n$ to $\mathbb{R}^n$. A basis of $\mathcal{H}_m$ is given by the functions $e_{\mathrm{p},i}(y)$, such that, in Notation 2.8, $e_{\mathrm{p},i}(y)_{[j]} = y^{\mathrm{p}}\delta_{ij}$,[12] with $|\mathrm{p}| = m$. We have $\dim \mathcal{H}_m = n\binom{m+n-1}{m}$. For two functions $f, g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we define their **Lie bracket** [**AM**] by

$$[f, g] := \partial_y f\, g - \partial_y g\, f. \tag{2.80}$$

On $\mathcal{H}_m$, we define a linear operator

$$\begin{array}{rccl} \mathrm{ad}_m[J]: & \mathcal{H}_m & \to & \mathcal{H}_m \\ & h(y) & \mapsto & [h(y), Jy] = \partial_y h(y)Jy - Jh(y). \end{array} \tag{2.81}$$

With these notations, (2.79) can be rewritten as $\mathrm{ad}_m[J]h(y) = b(y)$, which is a linear equation in $\mathcal{H}_m$. The basic formal result of normal form theory is the following:

Proposition 2.17. *For each m, $2 \leqslant m \leqslant k$, choose a complement $\mathcal{G}_m$ of $\mathrm{ad}_m[J]$ in $\mathcal{H}_m$, such that $\mathcal{H}_m = \mathrm{ad}_m[J]\mathcal{H}_m \oplus \mathcal{G}_m$. Then there is an analytic change of variables $x = y + h(y)$, transforming* (2.75) *into*

$$\dot{y} = Jy + g_2(y) + \dots + g_k(y) + \mathcal{O}(y^k), \tag{2.82}$$

where $g_m(y) \in \mathcal{G}_m$, $2 \leqslant m \leqslant k$.

Corollary 2.5. *If $\mathrm{ad}_m[J]$ is invertible for each m, $2 \leqslant m \leqslant k$, one can transform* (2.75) *into $\dot{y} = Jy + \mathcal{O}(y^k)$ by an analytic change of variables.*

Remark 2.5.

1. If $\mathcal{G}_m \neq \{0\}$, the term $g_m(y)$ in (2.82) is called **resonant**. The problem of finding the resonant terms reduces to linear algebra in $\mathcal{H}_m$, and depends only on the linear part J. However, it can be technically quite cumbersome to compute exactly the *coefficients* of the resonant terms $g_m(y)$ (with respect to a basis in $\mathcal{H}_m$).
2. In some cases, we have an explicit expression for the projector on the subspace $\mathcal{G}_m$. For instance, if J has purely imaginary eigenvalues, such a projector is given by [**GSS**]

$$\begin{array}{rccl} P_m: & \mathcal{H}_m & \to & \mathcal{H}_m \\ & b(y) & \mapsto & \displaystyle\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathrm{e}^{-sJ}\, b(\mathrm{e}^{sJ}y)\,\mathrm{d}s. \end{array} \tag{2.83}$$

[12]For instance, if $n = m = 2$, $e_{(2,0),1}(\xi,\eta) = (\xi^2, 0)$, $e_{(1,1),2}(\xi,\eta) = (0, \xi\eta)$, and so on.

Similar results hold for iterated maps. If $x_{p+1} = Jx_p + b(x_p)$, the homological equation for maps reads

$$h(Jy) - Jh(y) = b(y). \tag{2.84}$$

In fact there is a close relationship between the linearization of flows and maps. If the flow at unit time ϕ_1 can be linearized, one can construct a transformation linearizing ϕ_t at all times [St2].

Let us return to the ordinary differential equation (2.75), and examine more closely the condition that $\mathrm{ad}_m[J]$ be invertible. If $J = \mathrm{diag}(a_1, \dots, a_n)$, we see that $\mathrm{ad}_m[J]e_{\mathrm{p},i} = \big[\mathrm{p}[1]a_1 + \dots + \mathrm{p}[n]a_n - a_i\big]e_{\mathrm{p},i}$. In other words, the operator $\mathrm{ad}_m[J]$ is diagonal in the canonical basis of $\mathcal{H}_m$, with eigenvalues given by linear combinations of the a_j with integer coefficients. The operator $\mathrm{ad}_m[J]$ is invertible if and only if none or these combinations vanishes. This remains true for non–diagonal matrices (since, by Lemma 2.1, the off–diagonal terms can be made as small as we like, see also [**Har**], Lemma 12.1):

Lemma 2.5. *Denote by $a = (a_1, \dots, a_n)$ the n–tuple made of eigenvalues of J (each eigenvalue being repeated $\mathrm{m_a}(a_i, J)$ times). Let $2 \leqslant m \leqslant k$. If*

$$\langle \mathrm{p}|a\rangle \neq a_i \qquad \textit{for each } i \textit{ and each } \mathrm{p} \in \mathbb{N}^n \textit{ such that } |\mathrm{p}| = m, \tag{2.85}$$

then $\mathrm{ad}_m[J]$ is invertible.

Equation (2.85) is called a **Diophantine condition** since it involves integer coefficients. Proposition 2.17 tells us that if these Diophantine conditions are satisfied up to order k, then all nonlinear terms can be removed up to this order. The analytic problem of removing all nonlinear terms of order *larger* than k is more difficult [**Har**, Ch].

Theorem 2.22 (Sternberg–Chen). *Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of the origin. For a given integer N, $2 \leqslant N \leqslant \infty$, let $b_1, b_2 \in \mathcal{C}^N(\mathcal{U}, \mathbb{R}^n)$ be such that $b_1(x) = \mathcal{O}(x^2)$ and $b_2(x) = b_1(x) + \mathcal{o}(x^N)$. Let $A \in \mathbb{M}_n(\mathbb{R})$ have eigenvalues with nonzero real part. Then, if $N - k$ is large enough, there exists a map $h(y)$ of class $\mathcal{C}^k$ such that the change of variables $x = y + h(y)$ transforms the equation*

$$\dot{x} = Ax + b_1(x) \tag{2.86a}$$

into

$$\dot{y} = Ay + b_2(y). \tag{2.86b}$$

Remark 2.6. From [**Har**], we have the following bound on N. If the eigenvalues of J satisfy $0 < a_- < \mathrm{e}^{-|\mathrm{Re}\, a_i|} < a_+ < 1$, then N is sufficiently large if it satisfies

$$\frac{n^2 a_+^{N-k}}{a_-^{k+2}} \binom{n+k-1}{k} < 1. \tag{2.87}$$

In [St1], it is shown that if all eigenvalues real parts with the same sign, and $k > \max_{ij}|a_i/a_j|$, we can take $N = k$.

Theorem 2.22 implies that for the normal form $\dot{y} = Jy + \sum_{m=2}^{N} g_m(y) + \mathcal{O}(y^N)$, the term $\mathcal{O}(y^N)$ can be removed by a change of variables of class $\mathcal{C}^k$, provided N is sufficiently large. In particular, if the Diophantine conditions (2.85) are satisfied for large enough $|\mathrm{p}|$, we can transform the system $\dot{x} = Jx + b(x)$ into $\dot{y} = Jy$.

Example 2.7. Consider the two–dimensional case where $J = \operatorname{diag}(-1, -a)$ and $y = (\xi, \eta)$. Condition (2.85) implies that the resonant terms are of the form $\xi^p \eta^q$, with $aq = 1-p$ or $a(1-q) = p$. If a is positive, there is at most one resonant term, occurring only if $a = m$ or $\frac{1}{m}$, for an integer $m \geqslant 2$. For instance, when $a = 2$, the normal form is

$$\begin{aligned} \dot{\xi} &= -\xi, & \xi(t) &= \xi_0 \, \mathrm{e}^{-t}, \\ \dot{\eta} &= -2\eta + c\,\xi^2, & \eta(t) &= [c\,\xi_0^2\, t + \eta_0]\, \mathrm{e}^{-2t}. \end{aligned} \tag{2.88}$$

The resonant term $c\xi^2$ introduces a secular term in the solution, which cannot be removed. In the hyperbolic case ($a < 0$), there are more resonances. For instance, when $a = -1$ the normal form is

$$\begin{aligned} \dot{\xi} &= -\xi(1 + b_1\,\xi\eta + b_2\,\xi^2\eta^2 + \cdots), \\ \dot{\eta} &= \eta(1 + c_1\,\xi\eta + c_2\,\xi^2\eta^2 + \cdots). \end{aligned} \tag{2.89}$$

We recover the existence of invariant manifolds $\xi = 0$ and $\eta = 0$.

2.4.2 Center Manifolds

Theorem 2.23 (Center manifold theorem). *Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of 0, and $f \in \mathcal{C}^k(\mathcal{U}, \mathbb{R}^n)$, $k \geqslant 2$, a vector field such that $f(0) = 0$ and $\partial_x f(0) =: A$. Let $E^{\pm,0}$ be the invariant subspaces of Definition 2.19. Then there exist unique invariant stable and unstable manifolds W^{s} and W^{u}, which are of class $\mathcal{C}^k$ and tangent to $E^{\pm}$ at 0. Furthermore, there exists an invariant manifold W^{c}, called the* **center manifold***; it is of class $\mathcal{C}^{k-1}$, tangent to E^0 at 0, and not necessarily unique.*

We illustrate the computation of the center manifold in the case where there is no unstable subspace. After a linear change of variables, the equation becomes

$$\begin{aligned} \dot{u} &= Cu + c(u, v), \\ \dot{v} &= Bv + b(u, v), \end{aligned} \tag{2.90}$$

where C has eigenvalues with zero real part, and B has eigenvalues with negative real part. The center manifold has, in the vicinity of the origin, a parametric equation of the form $v = h(u)$. Thus we have

$$\begin{aligned} Bh(u) + b(u, h(u)) = \dot{v} &= \partial_u h(u) \dot{u} \\ &= \partial_u h(u)[Cu + c(u, h(u))]. \end{aligned} \tag{2.91}$$

In other words, $h(u)$ is a fixed point of the operator[13]

$$Th(u) := B^{-1}\big[\partial_u h(u)[Cu + c(u, h(u))] - b(u, h(u))\big]. \tag{2.92}$$

On the center manifold, the reduced equation will be

$$\dot{u} = Cu + c(u, h(u)). \tag{2.93}$$

[13] This operator allows to compute an expansion of the center manifold in the analytic case. To prove its existence, one uses another, integral operator, admitting $h(u)$ as fixed point.

Theorem 2.24. *If the origin of* (2.93) *is asymptotically stable (resp. stable, unstable), then the origin of* (2.90) *is asymptotically stable (resp. stable, unstable). Moreover, if the origin of* (2.93) *is stable and* $x(t)$ *is a solution of* (2.90) *with sufficiently small* $x(0)$*, then there exists a solution of* (2.93) *such that*

$$x(t) = (u(t), h(u(t))) + \mathcal{O}(\mathrm{e}^{-\gamma t}), \tag{2.94}$$

where γ *is a positive constant depending only on* B*. The center manifold is said to be* **locally attractive***.*

Theorem 2.25. *If* $\phi(u)$ *is a differentiable function in a neighborhood of the origin such that* $\phi(0) = 0$*,* $\partial_u\phi(0) = 0$ *and* $T\phi(u) = \phi(u) + \mathcal{O}(u^k)$*, then* $h(u) = \phi(u) + \mathcal{O}(u^k)$*.*

Comprehensive proofs of these results can be found in [**Ca**]. They remain true when there is also an unstable manifold, the sole difference being that only the solutions belonging to the stable manifold are locally attracted to the center manifold. Note that Theorem 2.25 implies that if there are multiple center manifolds, then they must be exponentially close.

2.4.3 Bifurcations and the Implicit Function Theorem

We consider a dynamical system depending on a parameter,

$$\dot{x} = f(x, \lambda), \tag{2.95}$$

where $f(x,\lambda) \in \mathcal{C}^k(\mathcal{D} \times I, \mathbb{R}^n)$, $k \geqslant 2$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ and I is an interval in $\mathbb{R}$. We assume that $x_0^\star$ is a fixed point of the system at the parameter value λ_0, i.e., $f(x_0^\star, \lambda_0) = 0$.

The implicit function theorem (Theorem 2.7) tells us that there exists, in a neighborhood of λ_0, a unique curve of equilibria $x^\star(\lambda) \in \mathcal{C}^k$, provided the determinant of the stability matrix $A(\lambda_0) := \partial_x f(x_0^\star, \lambda_0)$ is different from 0. We call such a curve an **equilibrium branch** or **bifurcation branch**. In other words, fixed points are isolated and depend smoothly on λ as long as A has no zero eigenvalue.

Assume now that $\det A(0) = 0$. The existence of fixed points in the neighborhood of the **bifurcation point** $(x^\star(0), 0)$ can be analyzed by the **Lyapunov–Schmidt procedure.** After a linear change of variables, the equation $f(x,\lambda) = 0$ becomes

$$\begin{aligned} B(\lambda)v + b(u,v,\lambda) &= 0, \\ C(\lambda)u + c(u,v,\lambda) &= 0, \end{aligned} \tag{2.96}$$

where C has zero eigenvalues and B has nonzero eigenvalues, while b and c are of second order in u and v. By the implicit function theorem, the first equation admits for small u a unique solution $v = h(u,\lambda)$. Inserting this into the second equality, we obtain the **bifurcation equation**

$$C(\lambda)u + c(u, h(u,\lambda), \lambda) =: g(u,\lambda) = 0. \tag{2.97}$$

Generically, $A(0)$ has only one vanishing eigenvalue, so that $u \in \mathbb{R}$. Let us examine this case more closely. The Taylor series of $g(u,\lambda)$ near the origin can be written (see Proposition 2.8)

$$g(u,\lambda) = \sum_{p+q<k} \alpha_{pq} u^p \lambda^q + \sum_{p+q=k} u^p \lambda^q R_{pq}(u,\lambda), \tag{2.98}$$

where the bifurcation at $(0,0)$ implies that $\alpha_{00} = \alpha_{10} = 0$.

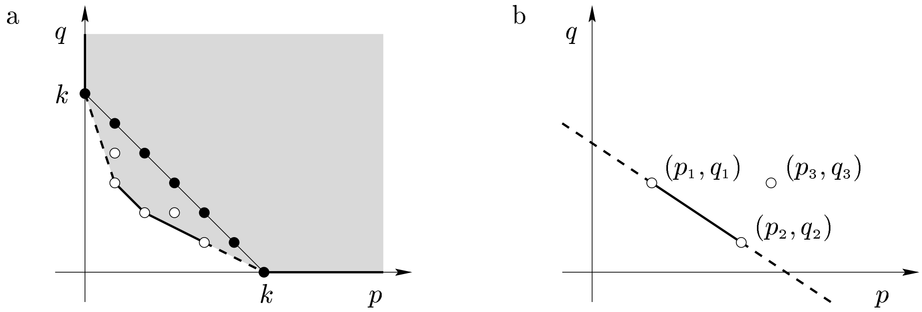

FIGURE 2.8. (a) Definition of the Newton polygon. White circles correspond to points (p, q) such that $\alpha_{pq} \neq 0$, black circles to points (p, q) such that $p + q = k$. The slopes of full lines correspond to possible exponents of branches of fixed points. Dotted lines have a slope which bounds the possible values of the exponents. (b) The setting of the proof of Proposition 2.18.

Definition 2.23. Consider the set of points (p, q) in $\mathbb{N}^2$ such that $\alpha_{pq} \neq 0$ or $p + q = k$. The **Newton polygon** associated with (2.98) is the broken line delimited by the coordinate axes and the convex envelope of these points (see Fig. 2.8).

Proposition 2.18. *If equation* (2.97) *admits a branch of fixed points of the form* $u = C|\lambda|^{\mu}(1 + \mathcal{O}(\lambda))$, *then the Newton polygon must have a segment of slope* $-\mu$, *except in the following cases. If the segment containing the point* $(0, k)$ *has slope* $-\nu$, *then we may have* $\mu \geqslant \nu$. *If the segment containing the point* $(k, 0)$ *has slope* $-\nu$, *then we may have* $0 \leqslant \mu \leqslant \nu$.

PROOF: Consider the case $g(u, \lambda) = \sum_{i=1}^{3} \alpha_i u^{p_i} \lambda^{q_i}$. Then we must have

$$\sum_i \sigma_i \alpha_i C^{p_i} \lambda^{\mu p_i + q_i} (1 + r)^{p_i} = 0, \tag{2.99}$$

where $\sigma_i = \pm 1$ and $r = \mathcal{O}(\lambda)$. We may assume that $p_1\mu + q_1 \leqslant p_2\mu + q_2 \leqslant p_3\mu + q_3$. If $p_1\mu + q_1 < p_2\mu + q_2$, division of (2.99) by $\lambda^{\mu p_1 + q_1}$ gives

$$\sigma_1 \alpha_1 C^{p_1} (1 + r)^{p_1} + \sum_{i=2}^{3} \sigma_i \alpha_i C^{p_i} \lambda^{\mu p_i + q_i - \mu p_1 - q_1} (1 + r)^{p_i} = 0. \tag{2.100}$$

Since λ has a strictly positive exponent, taking $\lambda \to 0$ we obtain $C = 0$, so there is no branch. On the other hand, if $p_1\mu + q_1 = p_2\mu + q_2$, putting $\lambda = 0$ we obtain an equation for C which might admit a solution (depending on the signs). We end up with an equation for r and λ, which may be studied with the implicit function theorem. Graphically, the condition $p_1\mu + q_1 = p_2\mu + q_2 \leqslant p_3\mu + q_3$ means that $\mu = \frac{q_2 - q_1}{p_1 - p_2}$ is minus the slope of the segment (p_1, q_1)–(p_2, q_2), and that (p_3, q_3) is above this segment. Repeating this argument for every triple of points, we obtain the conclusion. The exceptions are due to the fact that $R_{k0}(0, 0)$ or $R_{0k}(0, 0)$ may vanish. □

Example 2.8. Consider the equation

$$\lambda(u - 2\lambda) + (\lambda - u)^3 = 0. \tag{2.101}$$

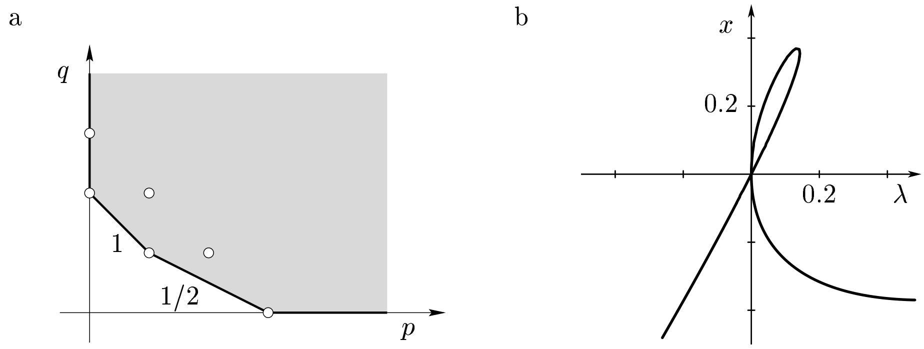

FIGURE 2.9. (a) The Newton polygon associated with Example 2.8 shows that the admissible exponents for the branches of fixed points are 1 and $\frac{1}{2}$. This is confirmed by the actual shape (b) of the curve $\{(\lambda, x) \mid f(x,\lambda) = 0\}$ (note that the global behaviour of the curves can only be guessed by the local analysis).

Newton's polygon (Fig. 2.9a) shows that there may be branches through the origin with exponents $\mu = 1$ and $\frac{1}{2}$. Thus we first look for a solution of the form $u = C\lambda(1 + r)$. Substitution in (2.101) shows that $C = 2$, and solving for r using Newton's method, we obtain a branch of the form

$$u = 2\lambda + \lambda^2 + \mathcal{O}(\lambda^3). \tag{2.102}$$

Next we look for solutions of the form $u = C|\lambda|^{1/2}(1 + r)$. For negative λ, there is no solution, but for positive λ, $C = \pm 1$ works and we get the branches

$$u = \pm\lambda^{1/2} + \frac{1}{2}\lambda + \mathcal{O}(\lambda^{3/2}). \tag{2.103}$$

The equilibrium branches are shown in Fig. 2.9b.

2.4.4 Bifurcations and Normal Forms

We have seen that the branches of equilibria of a parameter–dependent system may bifurcate when the linear part has a zero eigenvalue. More generally, the qualitative behaviour of the orbits can change drastically when eigenvalues of the linear part cross the imaginary axis.

Definition 2.24. The point $(x_0^\star, \lambda_0)$ is said to be a **bifurcation point** of the ordinary differential equation (2.95) if $f(x_0^\star, \lambda_0) = 0$ and $A := \partial_x f(x_0^\star, \lambda_0)$ has at least one eigenvalue with zero real part. If A is real, its eigenvalues are real or occur in complex conjugate pairs. The **codimension** of the bifurcation is the number of zero eigenvalues of A, plus the number of pairs of purely imaginary eigenvalues. The bifurcation is said to be **direct** if the number of equilibrium branches is larger for positive λ than for negative λ. If this number decreases, the bifurcation is called **indirect**.

An alternative to the Lyapunov–Schmidt procedure to reduce the analysis of the bifurcation, which is adequate for this more general setting, is to use the center manifold

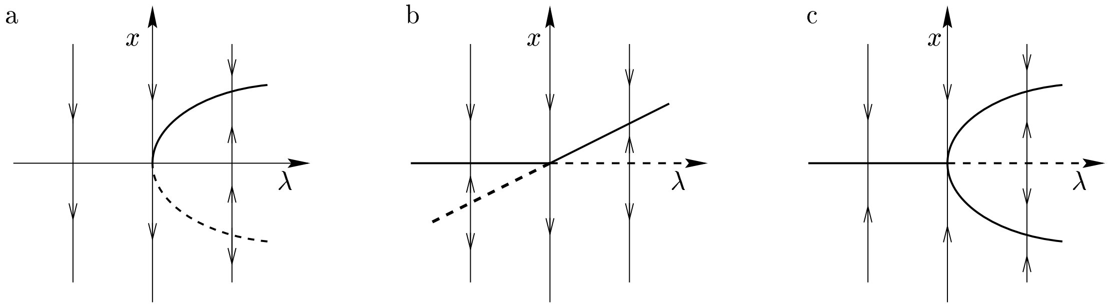

FIGURE 2.10. Bifurcations on a one–dimensional center manifold: (a) saddle–node bifurcation, (b) transcritical bifurcation, (c) pitchfork bifurcation. Full lines indicate stable branches, dotted lines unstable branches. The small arrows show the direction of the flow.

theorem. If $(0,0)$ is a bifurcation point, we may transform (2.95) by a linear change of variables into

$$\begin{aligned}\dot{v} &= Bv + b(v,u,\lambda)\\ \dot{u} &= Cu + c(v,u,\lambda)\\ \dot{\lambda} &= 0,\end{aligned} \tag{2.104}$$

where B has eigenvalues with nonzero real part and C has eigenvalues with zero real part. Considering λ as a dynamic variable in this "suspended" system allows to include terms of order λu and λv into the nonlinear terms. This system has a center manifold of the form $v = h(u,\lambda)$. We thus obtain the reduced system

$$\begin{aligned}\dot{u} &= Cu + c(h(u,\lambda),u,\lambda) =: g(u,\lambda)\\ \dot{\lambda} &= 0.\end{aligned} \tag{2.105}$$

Let us study the most generic bifurcations of codimension 1. The dimension of the center manifold may be 1 or 2, depending on whether there is a zero eigenvalue or two conjugate imaginary eigenvalues. We refer to [**GH**] for precise formulations of existence conditions for equilibria.

Center Manifold of Dimension 1

In this situation, we have $u \in \mathbb{R}$ and $C = 0$. The most generic case is when $\partial_\lambda g(0,0) \neq 0$. Then we can apply the theory of normal forms to (2.105), with the linear part given by $J = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. We obtain that $\mathrm{ad}_m[J](u^p\lambda^q,0) = (pu^{p-1}\lambda^{q+1},0)$, so that all nonlinear terms can be eliminated from (2.105), except the resonant terms u^m, $m \geqslant 2$. The normal form of the equation is thus

$$\dot{u} = \lambda + c_2u^2 + c_3u^3 + \cdots \tag{2.106}$$

If $c_2 \neq 0$, there are two branches of fixed points of opposite stability for one sign of λ, and no fixed points for the other sign. This bifurcation is called the **saddle–node** (Fig. 2.10a).

A less generic case occurs when $\partial_\lambda g(0,0) = 0$. If $\partial_{u\lambda} g(0,0) \neq 0$, the construction of Newton's polygon implies existence of a smooth branch of fixed points $u^\star(\lambda)$ through the

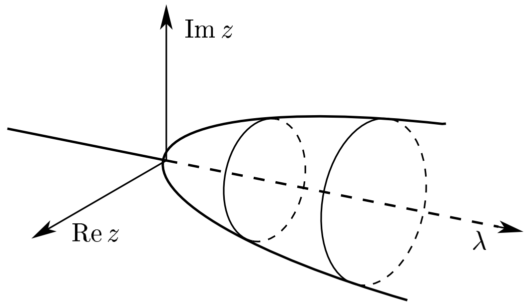

FIGURE 2.11. Hopf bifurcation.

origin. After subtracting this branch, we may thus assume that $f(0,\lambda) = 0$ for all λ near the origin, and thus

$$\dot{u} = a_1(\lambda)u + a_2(\lambda)u^2 + a_3(\lambda)u^3 + \cdots \tag{2.107}$$

Applying again the normal form theory to the one–dimensional equation, we find that changes of variables of the form $u \mapsto u + b(\lambda)u$, with $(m-1)a_1(\lambda)b(\lambda) = [a_m(\lambda) - a_m(0)]$, eliminate all terms except $a_m(0)$. Hence the normal form is

$$\dot{u} = a_1(\lambda)u + c_2u^2 + c_3u^3 + \cdots \tag{2.108}$$

If $c_2 \neq 0$, we obtain the **transcritical bifurcation** (also called **straight–straight**), corresponding to two meeting branches which exchange stability (Fig. 2.10b). If $c_2 = 0$ but $c_3 \neq 0$ (this may happen when there is a symmetry $u \mapsto -u$), we obtain the **pitchfork** or **vertical** bifurcation (Fig. 2.10c).

Center Manifold of Dimension 2

In this case, the equation on the center manifold becomes

$$\begin{aligned} \dot{u} &= Ju + b(u,\lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{2.109}$$

with $J = \left(\begin{smallmatrix} \mathrm{i}\,\omega & 0 \\ 0 & -\mathrm{i}\,\omega \end{smallmatrix}\right)$. The reality condition implies that we can write $u = (z, z^*)$. Here the non–resonance condition (2.85) implies that all terms which are not of the form $z^{q+1}(z^*)^q\lambda^r$ can be cancelled, so that the normal form is

$$\dot{z} = \mathrm{i}\,\omega z + c_0(\lambda)z + c_1(\lambda)|z|^2 z + c_2(\lambda)|z|^3 z + \cdots \tag{2.110}$$

where $c_0(0) = 0$. In polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\,\varphi}$, this equation becomes

$$\begin{aligned} \dot{r} &= \operatorname{Re} c_0(\lambda)r + \operatorname{Re} c_1(\lambda)r^3 + \cdots \\ \dot{\varphi} &= \omega + \operatorname{Im} c_0(\lambda) + \operatorname{Im} c_1(\lambda)r^2 + \cdots \end{aligned} \tag{2.111}$$

If $\operatorname{Re} c_1(0) \neq 0$, there is either creation of a stable periodic orbit or annihilation of an unstable one. This bifurcation is called the **Hopf bifurcation** (Fig. 2.11).

Bifurcations for Maps

In the case of maps, the situation is similar. The point $(x_0^\star, \lambda_0)$ is a bifurcation point if the linearization of the map has an eigenvalue of modulus 1. There are three cases

1. If the eigenvalue is equal to 1, we have, as before, a saddle–node, transcritical or pitchfork bifurcation.
2. The case of an eigenvalue equal to -1 has no counterpart for flows. It corresponds generically to the appearance of a new periodic orbit of period 2, and is thus called **period doubling** of **flip** bifurcation.
3. When two complex conjugate eigenvalues cross the unit circle, we may also have a **Hopf bifurcation**. The difference with the flows is that, if the eigenvalue has a rational argument, there are much more resonant terms. Under generic conditions, however, there is appearance or destruction of an invariant curve.

2.A Some Important Functions

Exponential Decay

The function $f(z) = \mathrm{e}^{-1/z}$ satisfies the following relation for $\operatorname{Re} z > 0$:

$$\mathrm{e}^{-1/z} = \mathcal{O}(z^k)\ \forall\, k \geqslant 0 \quad \Rightarrow \quad \mathrm{e}^{-1/z} \asymp 0. \tag{2.112}$$

Thus it admits the identically zero asymptotic series in the sector $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$. By contrast, it admits no limit in the complementary sector. In fact, we have the relation

$$f(x + \mathrm{i}\,y) = \mathrm{e}^{-x/(x^2+y^2)}\,\mathrm{e}^{\mathrm{i}\,y/(x^2+y^2)}\,. \tag{2.113}$$

The locations of constant module and argument are circles given by (see Fig. 2.12b)

$$\begin{aligned} \big\{z \,\big|\, |f(z)| = a\big\} &= \left\{x + \mathrm{i}\,y \,\middle|\, \Big(x + \frac{1}{2\ln a}\Big)^2 + y^2 = \Big(\frac{1}{2\ln a}\Big)^2\right\} \\ \big\{z \,\big|\, \arg f(z) = b\big\} &= \left\{x + \mathrm{i}\,y \,\middle|\, x^2 + \Big(y - \frac{1}{2\ln b}\Big)^2 = \Big(\frac{1}{2\ln b}\Big)^2\right\} \end{aligned} \tag{2.114}$$

Euler's Gamma Function and Factorials

Definition 2.25. The **Euler Gamma function** is defined by the integral

$$\Gamma(z) := \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t. \tag{2.115}$$

The function $\Gamma(z)$ is analytic in the whole complex plane, except at $z = -n$, $n \in \mathbb{N}$, where it has simple poles. We have in particular $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$. By integration by parts, we obtain $\Gamma(z+1) = z\Gamma(z)$, and thus for $n \in \mathbb{N}$, $\Gamma(n+1) = n!$ The integrand in (2.115) is maximal at $t = z - 1$ and decreases rapidly for large $\operatorname{Re} z$. Using the saddle point method, one obtains the following equality in the half–plane of positive real part, yielding the famous **Stirling formula**

$$\Gamma(z) = \sqrt{2\pi z}\,z^z\,\mathrm{e}^{-z}\,f(z), \tag{2.116}$$

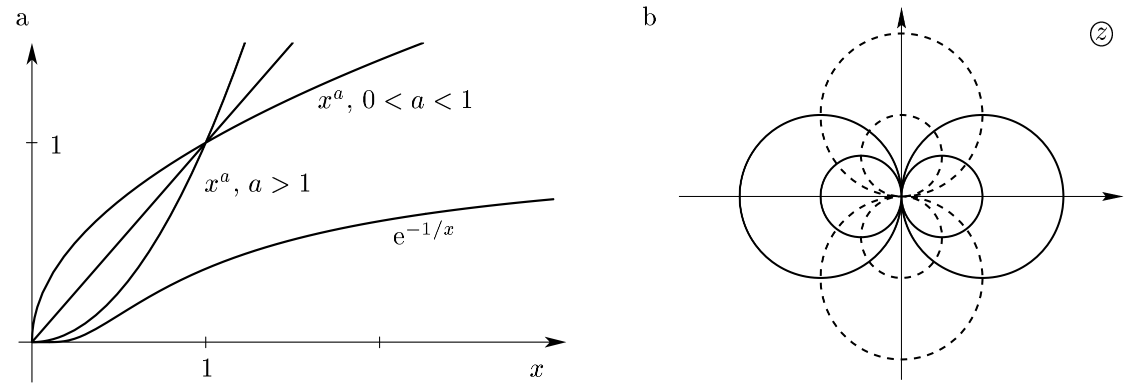

FIGURE 2.12. (a) Comparison of the functions x^a and $\mathrm{e}^{-1/x}$ for small positive x. The function $\mathrm{e}^{-1/x}$ goes to zero faster than any power law. (b) Behaviour of the function $\mathrm{e}^{-1/z}$ in the complex plane: full circles represent loci if constant module, dashed circles loci of constant argument.

where $f(z)$ admits the asymptotic series

$$f(z) \asymp \sum_k c_k z^k, \qquad c_0 = 1, \quad c_1 = \frac{1}{12}, \quad c_2 = \frac{1}{288}, \dots \tag{2.117}$$

The Airy Function

We will sometimes encounter the second order differential equation

$$f''(x) = x f(x) \tag{2.118}$$

which is called **Airy's equation**. In analogy with the p–representation is quantum mechanics, we may look for a solution of the form $\int_{\mathcal{C}} v(z)\,\mathrm{e}^{xz}\,\mathrm{d}z$ on an appropriate path. If $\mathcal{C}$ is such that $v(z)\,\mathrm{e}^{xz}$ vanishes at its extremities, we obtain by integration by parts that $v(z)$ must satisfy $v'(z) = -z^2 v(z)$, which has the solution $v(z) = \mathrm{e}^{-z^3/3}$. This motivates the following definition.

Definition 2.26. Let $\mathcal{C}(t)$ be a path such that $\lim_{t\to\pm\infty}|\mathcal{C}(t)| = \infty$, $\lim_{t\to-\infty} \arg \mathcal{C}(t) \in (\frac{7\pi}{6}, \frac{3\pi}{2})$ and $\lim_{t\to+\infty} \arg \mathcal{C}(t) \in (\frac{\pi}{2}, \frac{5\pi}{6})$ (see Fig. 2.13). **Airy's integral** is defined by

$$\mathrm{Ai}(x) := \frac{1}{2\pi\,\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{xz - z^3/3}\,\mathrm{d}z. \tag{2.119}$$

One also defines

$$\mathrm{Bi}(x) := \mathrm{e}^{\mathrm{i}\,\pi/6}\,\mathrm{Ai}(\mathrm{e}^{\mathrm{i}\,2\pi/3}\,x) + \mathrm{e}^{-\mathrm{i}\,\pi/6}\,\mathrm{Ai}(\mathrm{e}^{-\mathrm{i}\,2\pi/3}\,x). \tag{2.120}$$

Ai(x) and Bi(x) are independent solutions of equation (2.118), which are real for real x. They satisfy

$$W\{\mathrm{Ai}(x), \mathrm{Bi}(x)\} := \mathrm{Ai}(x)\,\mathrm{Bi}'(x) - \mathrm{Bi}(x)\,\mathrm{Ai}'(x) = 1/\pi \tag{2.121}$$

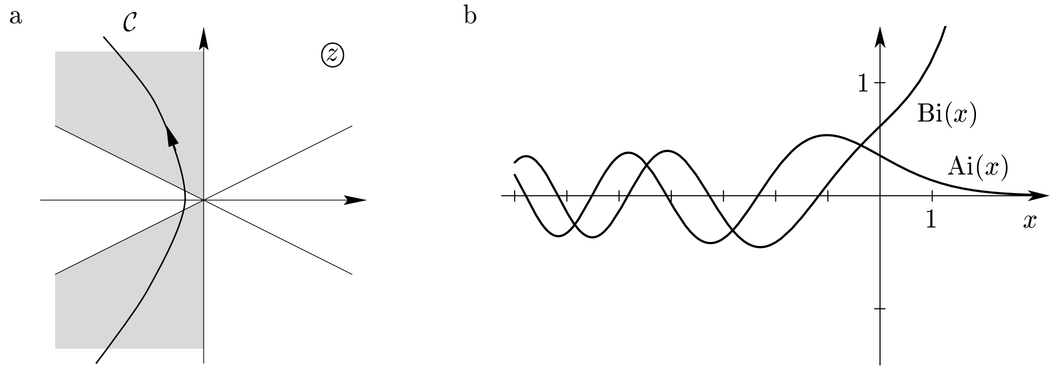

FIGURE 2.13. (a) Path of integration in the definition of Airy's integral. (b) The functions Ai(x) and Bi(x) for real argument.

Near $x = 0$, they behave like

$$\mathrm{Ai}(0) = \frac{1}{\sqrt{3}}\,\mathrm{Bi}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}, \qquad -\mathrm{Ai}'(0) = \frac{1}{\sqrt{3}}\,\mathrm{Bi}'(0) = \frac{1}{3^{1/3}\Gamma(1/3)}, \tag{2.122a}$$

$$\frac{1}{2\,\mathrm{Ai}(0)}\left[\mathrm{Ai}(x) + \frac{1}{\sqrt{3}}\,\mathrm{Bi}(x)\right] = 1 + \frac{1}{3!}x^3 + \frac{1\cdot 4}{6!}x^6 + \frac{1\cdot 4\cdot 7}{9!}x^9 + \cdots, \tag{2.122b}$$

$$\frac{1}{2\,\mathrm{Ai}'(0)}\left[\mathrm{Ai}(x) - \frac{1}{\sqrt{3}}\,\mathrm{Bi}(x)\right] = x + \frac{2}{4!}x^4 + \frac{2\cdot 5}{7!}x^7 + \frac{2\cdot 5\cdot 8}{10!}x^{10} + \cdots. \tag{2.122c}$$

On Fig. 2.13, we see that the Airy functions are growing or decreasing exponentially fast for positive x and are oscillating for negative x. This is confirmed by the following asymptotic expressions [**Ol**]. For large positive x, we define $\xi := \frac{2}{3}x^{2/3}$. Then

$$\mathrm{Ai}(x) = \frac{x^{-1/4}}{2\sqrt{\pi}}\,\mathrm{e}^{-\xi}\,u(-\xi), \qquad \mathrm{Ai}'(x) = -\frac{x^{1/4}}{2\sqrt{\pi}}\,\mathrm{e}^{-\xi}\,v(-\xi), \tag{2.123a}$$

$$\mathrm{Bi}(x) = \frac{x^{-1/4}}{\sqrt{\pi}}\,\mathrm{e}^{\xi}\,u(\xi), \qquad \mathrm{Bi}'(x) = \frac{x^{1/4}}{\sqrt{\pi}}\,\mathrm{e}^{\xi}\,v(\xi), \tag{2.123b}$$

where

$$u(\xi) \asymp \sum_{k=0}^{\infty} u_k \xi^{-k}, \qquad u_0 = 1, \qquad u_k = \frac{(2k+1)(2k+3)\ldots(6k-1)}{6^{3k}k!}, \tag{2.123c}$$

$$v(\xi) \asymp \sum_{k=0}^{\infty} v_k \xi^{-k}, \qquad v_0 = 1, \qquad v_k = -\frac{6k+1}{6k-1}u_k. \tag{2.123d}$$

Similarly,

$$\mathrm{Ai}(-x) = \frac{x^{-1/4}}{\sqrt{\pi}}\,\mathrm{Re}\big[\mathrm{e}^{\mathrm{i}(\xi-\pi/4)}\,u(\mathrm{i}\,\xi)\big], \quad \mathrm{Ai}'(-x) = -\frac{x^{1/4}}{\sqrt{\pi}}\,\mathrm{Re}\big[\mathrm{e}^{\mathrm{i}(\xi+\pi/4)}\,v(\mathrm{i}\,\xi)\big], \tag{2.123e}$$

$$\mathrm{Bi}(-x) = \frac{x^{-1/4}}{\sqrt{\pi}}\,\mathrm{Re}\big[\mathrm{e}^{\mathrm{i}(\xi+\pi/4)}\,u(\mathrm{i}\,\xi)\big], \quad \mathrm{Bi}'(-x) = \frac{x^{1/4}}{\sqrt{\pi}}\,\mathrm{Re}\big[\mathrm{e}^{\mathrm{i}(\xi-\pi/4)}\,v(\mathrm{i}\,\xi)\big]. \tag{2.123f}$$

Chapter 3

Physical Models

"Un physicien éminent me disait un jour à propos de la loi des erreurs: Tout le monde y croit fermement parce que les mathématiciens s'imaginent que c'est un fait d'observation, et les observateurs que c'est un théorème de mathématiques."

Henri Poincaré, "La Science et l'Hypothèse"

"There are no physicists in the hottest parts of hell, because the existence of a 'hottest part' implies a temperature difference, and any marginally competent physicist would immediately use this to run a heat engine and make some other part of hell comfortably cool. This is obviously impossible."

Richard Davisson

We present in this chapter some background information on the physical models used in the applications of Chapters 6 and 7. In this short overview, we would like to introduce a number of basic concepts, definitions and properties of these models, without going too much into detail. As in the previous chapter, we cite some reference books for more detailed information.

- **Section 3.1** The damped motion of a particle in a potential may serve as paradigm for a large class of Dynamical Systems. It presents the advantage of displaying dynamics which are close to the physicist's intuition. We show how to use the geometric theory of Dynamical Systems to draw the phase portrait, and that in the overdamped case (that is, for large enough friction), the motion is governed by an effective one–dimensional equation, called Aristotle's law.
- **Section 3.2** We discuss two approaches to the description of a ferromagnet at equilibrium. Thermostatics is a macroscopic, phenomenological description: the system is characterized by a small set of macroscopic observables and a thermodynamic potential. Equilibrium Statistical Mechanics provides a more precise description, starting from a microscopic model. We introduce a class of lattice spin models, and show how they can be used to derive thermostatic functions.
- **Section 3.3** We present some elements of Non–Equilibrium Statistical Mechanics, which are necessary to describe the dynamics of a magnet out of equilibrium. The evolution of the system's probability distribution is viewed as a stochastic process, and may be described by a master equation. We introduce the most common master equation for a lattice spin model, and show how it can be used to obtain a deterministic

equation for the magnetization in the thermodynamic limit, while fluctuations around this limit obey a Langevin equation.
- **Section 3.4** We briefly describe existing phenomenological models of hysteresis, in particular the Preisach model, in order to outline the differences between such models, and the derivation of hysteresis from explicit deterministic dynamics.

3.1 Damped Particle in a Potential

Let us consider the motion of a point mass in a potential $\Phi(q)$ (of class $\mathcal{C}^2$), subject to a friction force $-2\gamma\dot{q}$, where $\gamma > 0$ and the factor 2 has been introduced for later convenience.

In dimensionless units, its equation of motion can be written as

$$\ddot{q} + 2\gamma\dot{q} + \Phi'(q) = 0, \tag{3.1}$$

where the prime denotes differentiation with respect to q. This equation is equivalent to the first order system

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -2\gamma p - \Phi'(q). \end{aligned} \tag{3.2}$$

Equilibrium points of this system have coordinates $x^\star = (q^\star, 0)$, where $q^\star$ is a stationary point of Φ, i.e., $\Phi'(q^\star) = 0$. The linearization of (3.2) around such a point is given by the matrix

$$A(q^\star) = \begin{pmatrix} 0 & 1 \\ -\Phi''(q^\star) & -2\gamma \end{pmatrix}, \tag{3.3}$$

which has eigenvalues

$$a_\pm(q^\star) = -\gamma \pm \sqrt{\gamma^2 - \Phi''(q^\star)}, \tag{3.4}$$

and eigenvectors $(1, a_\pm(q^\star))$. Thus the stationary point $x^\star$ is
- a **saddle** if $\Phi''(q^\star) < 0$,
- a **stable node** if $0 < \Phi''(q^\star) < \gamma^2$,
- a **stable focus** if $\Phi''(q^\star) > \gamma^2$.

With these informations on equilibrium points, it is relatively easy to draw the phase portrait (see Fig. 3.1).[1]

Definition 3.1. The system (3.1) is said to be **overdamped** if $\Phi''(q) < \gamma^2$ for all q.

Proposition 3.1. *Assume the system* (3.1) *is overdamped. Let q_1 be a hyperbolic stationary point, and q_2 any neighbouring stationary point, i.e., $\Phi'(q)$ does not vanish between q_1 and q_2. Then there exists a unique invariant manifold $p = h(q)$ defined between q_1 and q_2, such that $h(q_1) = h(q_2) = 0$.*

[1] There is, admittedly, a difficulty associated with the global behaviour of the invariant manifolds of saddles. At zero friction, they are level lines of the energy, while at sufficiently high friction, they always connect neighboring equilibria. Between these extremes, the phase portrait may go through several nonlocal bifurcations involving saddle connections.

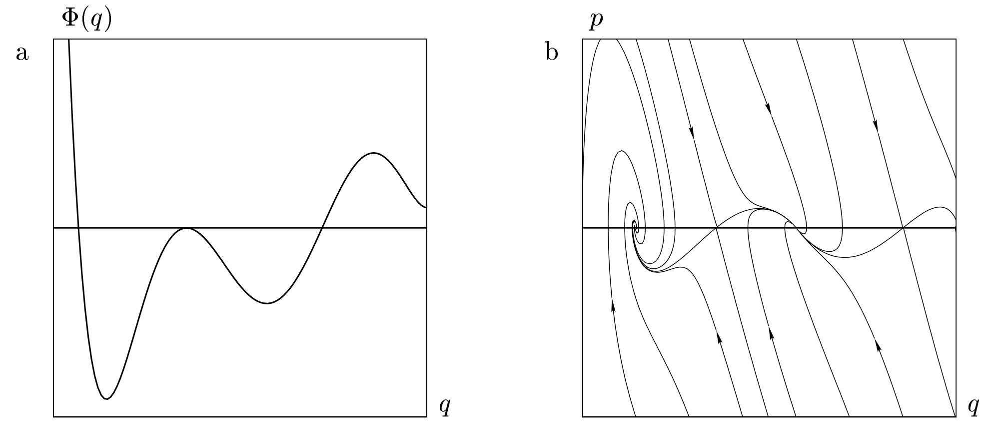

FIGURE 3.1. Example of a potential $\Phi(q)$ (a), and corresponding phase portrait in the (q,p)–plane (b). Maxima of the potential correspond to saddle points, minima to stable nodes or focuses, depending on the relation between friction and curvature of the potential.

PROOF: We take the case $q_2 > q_1$, so that $\Phi'(q) < 0$ for $q_1 < q < q_2$ (Fig. 3.2). We know by the stable manifold theorem that there exists a local unstable manifold $h(q)$ near q_1, with $0 < h'(q_1) = a_+(q_1) \leqslant -\Phi''(q_1)/2\gamma$. We know that this manifold exists globally (see Definition 2.21). We have to prove that $h(q)$ is a graph and $h(q_2) = 0$.

Consider the function $g(q) = -\frac{1}{\gamma}\Phi'(q)$, and the domain

$$\mathcal{D} = \bigl\{(p,q) \in \mathbb{R}^2 \bigm| q_1 \leqslant q \leqslant q_2, 0 \leqslant p \leqslant g(q)\bigr\}. \tag{3.5}$$

We prove that $\mathcal{D}$ is invariant. On the lower boundary, $\dot{p} > 0$. The upper boundary admits $v_\perp(q) = (\frac{1}{\gamma}\Phi''(q), 1)$ as normal vector. If $f(q,p) = (\dot{q},\dot{p})$ is the vector field (3.2), we have

$$\langle v_\perp(q) | f(q, g(q))\rangle = \Phi'(q)\Bigl(1 - \frac{\Phi''(q)}{\gamma^2}\Bigr) \leqslant 0, \tag{3.6}$$

because of the overdamping condition. It follows that the flow cannot leave the domain $\mathcal{D}$, in particular the unstable manifold $p = h(q)$ remains inside $\mathcal{D}$. The fact that $\dot{q} > 0$ inside $\mathcal{D}$ implies that $h(q)$ is a graph, with $\lim_{q\to q_2} h(q) = 0$. $\square$

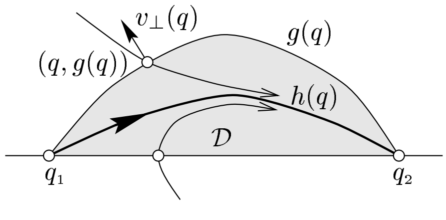

FIGURE 3.2. Geometry of the orbits in the overdamped case. The point $(q_1, 0)$ is a saddle, and $(q_2, 0)$ is a node. The shaded set $\mathcal{D}$ is shown to be invariant under the flow, so that the unstable manifold $p = h(q)$ must remain in $\mathcal{D}$ and reach the point $(q_2, 0)$.

To compute the equation of this invariant manifold $p = h(q)$, we may use the fact that

$$h'(q)h(q) = h'(q)\dot{q} = \dot{p} = -2\gamma h(q) - \Phi'(q), \tag{3.7}$$

which implies that $h(q)$ is a fixed point of the operator

$$(Th)(q) = -\frac{1}{2\gamma}\bigl[\Phi'(q) + h'(q)h(q)\bigr]. \tag{3.8}$$

Iteration of this operator yields (for sufficiently smooth potential) an expansion of the form

$$h(q) = -\frac{1}{2\gamma}\Phi'(q)\Big[1 + \frac{1}{4\gamma^2}\Phi''(q) + \mathcal{O}\Big(\frac{1}{\gamma^4}\Big)\Big]. \tag{3.9}$$

On this invariant manifold, dynamics is governed by the equation

$$\dot{q} = h(q) = -\frac{1}{2\gamma}\Phi'(q) + \mathcal{O}(\gamma^{-3}). \tag{3.10}$$

Using an appropriate scaling of time, we obtain in the high friction limit the equation $\dot{q} = -\Phi'(q)$, which is known as **Aristotle's law**, i.e., velocity is proportional to the force. Thus, any one–dimensional equation of the form $\dot{q} = f(q)$ may be thought of as describing the overdamped motion of a particle in a potential.

Let us briefly discuss the n–dimensional case. If $q \in \mathbb{R}^n$, the equation of motion is still given by (3.1), with $\Phi'(q)$ replaced by the gradient $\partial_q\Phi(q)$. Equilibrium points are $(q^\star, 0)$ where $\partial_q\Phi(q^\star) = 0$. The linearized vector field is described by

$$A(q^\star) = \begin{pmatrix} 0 & 1\!\!1 \\ -\partial_{qq}\Phi(q^\star) & -2\gamma 1\!\!1 \end{pmatrix}. \tag{3.11}$$

The eigenvalues of $A(q^\star)$ must satisfy

$$-\partial_{qq}\Phi(q^\star)q - 2\gamma a q = a^2 q, \tag{3.12}$$

and are thus given by

$$a_{\pm,j}(q^\star) = -\gamma \pm \sqrt{\gamma^2 - \varphi_j(q^\star)}, \tag{3.13}$$

where $\varphi_j(q^\star)$, $j = 1, \dots, n$ are the eigenvalues of the Hessian $\partial_{qq}\Phi(q^\star)$ (they are real since the Hessian is symmetric). Note that this formula allows for a larger variety of saddles and sinks, depending on the number of positive and negative eigenvalues of $\partial_{qq}\Phi$.

The system is **overdamped** if the largest eigenvalue of $\partial_{qq}\Phi$ is smaller than γ^2 for all q. In this case, an invariant manifold of the form $p = h(q)$ satisfies

$$h(q) = -\frac{1}{2\gamma}\big[\partial_q h(q)h(q) + \partial_q\Phi(q)\big] = -\frac{1}{2\gamma}\partial_q\Phi(q)\big[1\!\!1 + \mathcal{O}(\gamma^{-2})\big], \tag{3.14}$$

and the motion on this manifold is governed by the equation $\dot{q} = h(q)$.

3.2 Magnets at Equilibrium

Ferromagnets often display two types of phase transition. When the temperature is low enough, and the magnetic field h is increased from negative to positive values, the magnetization is discontinuous at $h = 0$: there is a first order phase transition. For zero magnetic field, two phases with negative and positive spontaneous magnetization can coexist. As temperature is increased, the spontaneous magnetization becomes zero: this corresponds to a second order phase transition.

There exist two kinds of descriptions of a large physical system at equilibrium. Thermostatics is a powerful phenomenological approach, that postulates existence of a small number of macroscopic variables, describing the system's equilibrium states. All relevant quantities can be derived from a thermodynamic potential, the form of which has to be determined experimentally. Phase transitions are characterized by singularities in this potential.

Equilibrium Statistical Mechanics aims at deriving the system's properties from its microscopic interactions. One usually describes a magnet by a lattice model, where the system is assumed to consist in a regular lattice of atoms, with a magnetic moment, the spin, attached to each site. Existence of phase transitions has been proved rigorously for some of these models.

For more details on the theory presented here, we refer to [**Hu**].

3.2.1 Thermostatics

Let us consider an isolated magnet which has reached a stationary, stable state, called an **equilibrium state**. The basic postulate of Thermostatics is that the equilibrium states of such a system can be labeled by a small set of variables.

Postulate 3.1 (Zeroth principle of thermostatics).
The system's equilibrium states are not modified when further constraints are added. If two systems are in equilibrium with a third one, then they are in equilibrium with each other. There is a one–to–one correspondence between the set of all equilibrium states and the values of a small number of conserved quantities.

For our magnet, these conserved quantities are for instance **magnetization** M and **volume** V of the sample. The first principle states that another such quantity is the energy.

Postulate 3.2 (First principle of thermostatics).
There exists a scalar, extensive, conserved quantity, called the **energy** U.

We will assume that the magnet has a set of homogeneous, isotropic equilibrium states (such a system is called **simple**), which can be labeled by the three variables (U, M, V).

In Thermostatics, one assumes that the system can be completely described by a **thermodynamic potential** $\Phi(U, M, V)$, in particular all physically interesting quantities can be expressed in terms of partial derivatives of the potential. There exist different representations, depending on the choice of the potential and the independent variables.

Entropy representation

Postulate 3.3 (Second principle of thermostatics).
There exists a scalar, extensive, state function $S(U, M, V)$, called the **entropy**, *with the following property: let the system be composed of N identical subsystems,*[2] *then*

$$S(U, M, V) = \max_{\substack{(U_i, M_i, V_i) \\ \sum U_i = U, \sum M_i = M, \sum V_i = V}} \sum_{i=1}^{N} S(U_i, M_i, V_i). \tag{3.15}$$

[2]One can write a similar relation for subsystems which are not identical.

The relation (3.15) implies that S is a homogeneous function of degree 1: for every λ,

$$S(\lambda U, \lambda M, \lambda V) = \lambda S(U, M, V). \tag{3.16}$$

As a consequence, we can write

$$S(U, M, V) = V s(u, m), \tag{3.17}$$

where $u = U/V$ and $m = M/V$ are the energy and magnetization density, and $s(u, m) = S(u, m, 1)$ is the entropy density. Moreover, (3.15) implies that the entropy is a concave function.

The physically important variables are obtained by partial derivation of S. Indeed, **temperature** T, **magnetic field** h and **pressure** p are given by the relations

$$\begin{aligned} \frac{1}{T(U,M,V)} &= \frac{\partial S}{\partial U}(U,M,V) \\ h(U,M,V) &= T\frac{\partial S}{\partial M}(U,M,V) \\ p(U,M,V) &= T\frac{\partial S}{\partial V}(U,M,V). \end{aligned} \tag{3.18}$$

The system is called **paramagnetic** if M and h have the same sign, otherwise it is **diamagnetic**.

Differentiating (3.16) with respect to λ and setting $\lambda = 1$, we obtain the **Euler relation**

$$S(U,M,V) = \frac{\partial S}{\partial U}U + \frac{\partial S}{\partial M}M + \frac{\partial S}{\partial V}V = \frac{1}{T}\big[U + hM + pV\big]. \tag{3.19}$$

Energy representation

It often happens that one prefers to take some conjugate quantities of (3.18) as independent variables. This can be done by introducing new thermodynamic potentials. The simplest reformulation uses the energy as potential. If the temperature is always positive, one can solve the relation $S = S(U, M, V)$ with respect to U, giving $U = U(S, M, V)$. The corresponding Euler relation is

$$U(S, M, V) = TS - hM - pV, \tag{3.20}$$

which means that we can also express h and p as functions of (S, M, V):[3]

$$\begin{aligned} h(S,M,V) &= -\frac{\partial U}{\partial M}(S,M,V) \\ p(S,M,V) &= -\frac{\partial U}{\partial V}(S,M,V). \end{aligned} \tag{3.21}$$

Equation (3.20) is also written in differential form,

$$\mathrm{d}U = T\,\mathrm{d}S - h\,\mathrm{d}M - p\,\mathrm{d}V. \tag{3.22}$$

[3]The physicist's convention is to use the same letter for the functions $h(U, M, V)$ and $h(S, M, V)$.

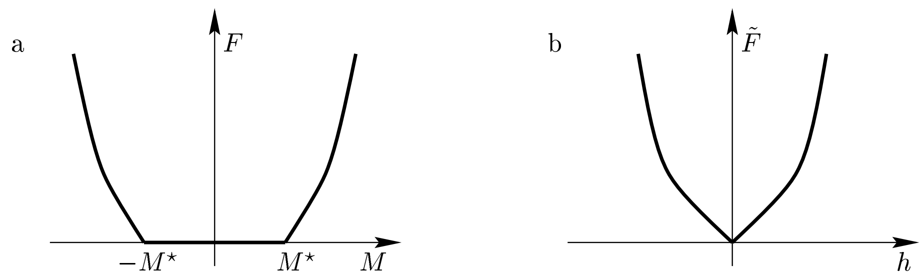

FIGURE 3.3. Qualitative behaviour of the free energy (a) and Helmholtz free energy (b) of a typical ferromagnet at low temperature. The Helmholtz free energy has a discontinuity of $2M^\star$ in the slope at the origin. Thus at zero magnetic field, the magnetization can take any value between $-M^\star$ and $M^\star$. This fact is reflected by a plateau in the free energy. The equilibrium state corresponding to those magnetizations need not be homogeneous. We have a first order phase transition at $h = 0$. By contrast, the free energies are smooth at high temperature.

Other representations

New thermodynamic potentials are obtained by **Legendre transform**.[4] The most important are the **free energy**

$$F(T, M, V) = \inf_S \big[U(S, M, V) - TS\big], \tag{3.23}$$

which satisfies the relation

$$\mathrm{d}F = -S\,\mathrm{d}T - h\,\mathrm{d}M - p\,\mathrm{d}V, \tag{3.24}$$

and the **Helmholtz free energy**

$$\tilde F(T, h, V) = \inf_M \big[F(T, M, V) + Mh\big], \tag{3.25}$$

which satisfies the relation

$$\mathrm{d}\tilde F = -S\,\mathrm{d}T + M\,\mathrm{d}h - p\,\mathrm{d}V. \tag{3.26}$$

Phase transitions

Let us consider a system composed of several subsystems. If the entropy (or some other thermodynamic potential) is strictly concave, the relation (3.15) has a unique solution, given by $(U_i, M_i, V_i) = (U/N, M/N, V/N)$, thus the equilibrium state is homogeneous. By contrast, if the entropy is affine on some set (i.e., it admits a flat part), (3.15) admits several solutions, which are not all homogeneous: we may have phase coexistence. Such a situation is associated with a **first order phase transition**, and the discontinuity of some first derivatives of a thermodynamic potential.

For magnetic systems at low temperature, one often observes a qualitative behaviour of the free energy and Helmholtz free energy as shown in Fig. 3.3. The quantity

$$M^\star = \lim_{h \to 0+} \frac{\partial \tilde F}{\partial h} \tag{3.27}$$

[4]If U is strictly convex, we can write $F = U - TS$, where it is understood that all variables are expressed as functions of (T, M, V).

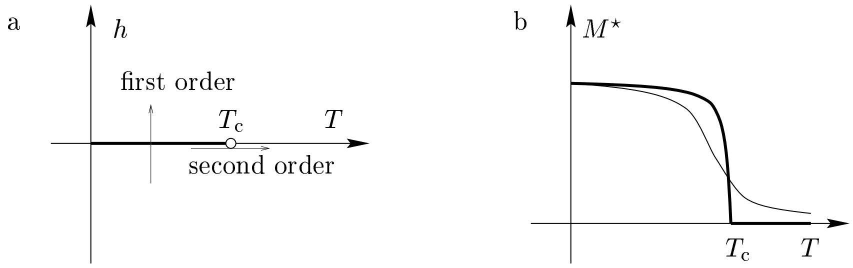

FIGURE 3.4. (a) Phase diagram of a ferromagnet. When the temperature is below T_c, the system undergoes a first order phase transition when h changes sign. When one keeps $h = 0$ and increases the temperature beyond T_c, the system undergoes a second order phase transition. (b) The spontaneous magnetization $M^\star$ is positive below T_c, and then becomes zero, with a discontinous slope at T_c. By contrast, for positive magnetic field, the magnetization is a smooth function of h (thin line).

is called the **spontaneous magnetization**. The system is said to be **ferromagnetic** if $M^\star \neq 0$. In this case, we have a first order phase transition at $h = 0$, where the magnetization is discontinuous.

There exists a **critical temperature** T_c beyond which the spontaneous magnetization is zero (Fig. 3.4). If the magnetic field h is fixed at 0, and the temperature moves through T_c, the magnetization remains continuous, but is not differentiable. This situation is called a **second order phase transition**.

3.2.2 Equilibrium Statistical Mechanics

The aim of Equilibrium Statistical Mechanics is to derive the relations of Thermostatics, and more detailed properties of the system, by starting from microscopic interactions. Magnetic systems are often described by **lattice models**. They assume that the magnet is composed of a crystalline lattice, with a magnetic moment, called the **spin**, attached to each site.

Let us consider the following class of lattice models:

- The cubic lattice is a subset $\Lambda \in \mathbb{Z}^d$, with $|\Lambda| = N$ sites; $d = 1, 2$ or 3 is the dimension of the lattice.
- To each site i of the lattice, we attach a classical **spin**, described by a vector of unit length in $\mathbb{R}^n$, $\sigma_i \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. A **configuration** of the system is described by specifying the spin at each site: $\sigma \in \Omega = (\mathbb{S}^{n-1})^\Lambda$.
- To each configuration, we associate the energy

$$H(\sigma) = -\frac{1}{2} \sum_{i \neq j \in \Lambda} \langle \sigma_i | J_{ij} \sigma_j \rangle - \sum_{i \in \Lambda} \langle h | \sigma_i \rangle, \tag{3.28}$$

  where $h \in \mathbb{R}^n$ is the **magnetic field**, and the matrices $J_{ij} = J_{ji} = J_{ij}^{\mathrm{T}} \in \mathbb{M}_n(\mathbb{R})$ are called **coupling constants**. The system is assumed to be homogeneous, so that J_{ij} depends only on $j - i$.

We will work in the canonical ensemble, which means that the system is coupled to a heat bath at temperature T.[5] The basic postulate of Equilibrium Statistical Mechanics is:

[5]We will not enter here the debate about the motivation of Postulate 3.4. The traditional procedure is

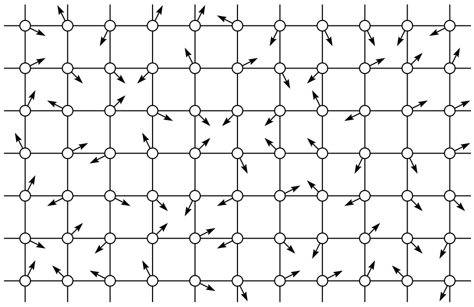

FIGURE 3.5. The lattice models we consider consist in a lattice $\Lambda \subset \mathbb{Z}^d$, with a spin $\sigma_i \in \mathbb{S}^{n-1}$ attached to each site $i \in \Lambda$.

Postulate 3.4. *At equilibrium, the system is in the configuration σ with probability proportional to $\mathrm{e}^{-\beta H(\sigma)}$, where $\beta = 1/k_{\mathrm{B}}T$ is the inverse temperature, and the Boltzmann constant k_{B} defines the temperature scale.*

The normalization constant

$$Z(\beta, h, \Lambda) = \int_\Omega \mathrm{d}\mu(\sigma)\, \mathrm{e}^{-\beta H(\sigma)} \tag{3.29}$$

is called **canonical partition function**, where $\mathrm{d}\mu(\sigma)$ is a product of uniform measures on the sphere (φ, θ denote polar or spherical coordinates):

$$\mathrm{d}\mu(\sigma) = \prod_{i\in\Lambda} \mathrm{d}\mu_i(\sigma_i), \qquad \mathrm{d}\mu_i(\sigma_i) = \begin{cases} [\delta(\sigma_i - 1) + \delta(\sigma_i + 1)]\, \mathrm{d}\sigma_i & \text{if } n = 1, \\ \frac{1}{2\pi}\, \mathrm{d}\varphi_i & \text{if } n = 2, \\ \frac{1}{4\pi^2}\, \mathrm{d}\varphi_i \sin\theta_i\, \mathrm{d}\theta_i & \text{if } n = 3. \end{cases} \tag{3.30}$$

Thus the probability of a configuration $\sigma \in \Omega$ is written

$$p_0(\sigma) = \frac{1}{Z}\, \mathrm{e}^{-\beta H(\sigma)}, \tag{3.31}$$

and the equilibrium value of an observable $\mathcal{C}(\sigma)$ is given by

$$\langle \mathcal{C} \rangle_0 = \int_\Omega \mathrm{d}\mu(\sigma) p_0(\sigma) \mathcal{C}(\sigma) = \frac{1}{Z} \int_\Omega \mathrm{d}\mu(\sigma)\, \mathrm{e}^{-\beta H(\sigma)}\, \mathcal{C}(\sigma). \tag{3.32}$$

Relation to thermostatics

The partition function contains all information necessary for the thermostatic description of the system (but not necessarily for more detailed properties, like correlation functions). The (Helmholtz) **free energy** is defined by

$$F(\beta, h, \Lambda) = \frac{1}{\beta} \ln Z(\beta, h, \Lambda). \tag{3.33}$$

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to start from the ergodic hypothesis, and to couple the system to a large “heat bath”. There are, however, several conceptual (and controversial) difficulties associated with this procedure.

To see that this definition makes sense, let us consider the observable $\mathcal{M} = \frac{1}{N}\sum_{i\in\Lambda}\sigma_i$, i.e., the **magnetization density**. According to (3.32), its average at equilibrium is given by

$$\begin{aligned} M = N\langle\mathcal{M}\rangle_0 &= \frac{1}{Z}\int_\Omega \mathrm{d}\mu(\sigma)\sum_{i\in\Lambda}\sigma_i\,\mathrm{e}^{-\beta H(\sigma)} \\ &= \frac{1}{Z}\int_\Omega \mathrm{d}\mu(\sigma)\frac{1}{\beta}\frac{\partial}{\partial h}\,\mathrm{e}^{-\beta H(\sigma)} = \frac{1}{\beta}\frac{\partial}{\partial h}\ln Z = \frac{\partial F}{\partial h}, \end{aligned} \tag{3.34}$$

which is exactly (3.26). For a *finite* number of sites N, the partition function is analytic and cannot describe phase transitions. But one can define a thermodynamic limit for $f(\beta,h,\Lambda) = F(\beta,h,\Lambda)/N$; this limiting potential may admit singularities, associated with phase transitions.[6]

Ising model

Let us consider the case $n = 1$, i.e., when the spins take values $\sigma_i = \pm 1$. The **Ising model** is obtained when the spins interact only with their nearest neighbours, that is $J_{ij} = J$ if $\|i-j\|_1 = 1$ and 0 otherwise. The configuration space is $\{-1,1\}^\Lambda$, and the partition function becomes

$$Z = \sum_{\sigma\in\Omega}\mathrm{e}^{-\beta H(\sigma)} = \prod_{i\in\Lambda}\sum_{\sigma_i=\pm1}\mathrm{e}^{-\beta H(\sigma)}\,. \tag{3.35}$$

We should pay attention to boundary conditions. In the simple case of a homogeneous system, it is reasonable to take for Λ a hypercubic lattice with periodic boundary conditions, that is corresponding spins on opposite sides of the hypercube are considered as nearest neighbours. Depending on the problem one considers, other boundary conditions may be more appropriate.

We refer to [Ga] for a nice summary of results on the Ising model, and a discussion of issues like boundary conditions and phase transitions.

The partition function (3.35) can be computed exactly for a lattice of dimension $d = 1$ and $d = 2$, when $h = 0$. For $d = 1$, the spontaneous magnetization is always zero, and there is no phase transition. For $d = 2$, there is a phase transition, and the system behaves exactly as in Fig. 3.16. The existence of a phase transition can be shown in a simple way for any $d > 1$, using the Peierls argument [Pe].

Mean field approximation

A physically very different situation arises when the spins have long range interaction. The limiting case occurs when all couplings J_{ij} have the same value J/N, and the Hamiltonian becomes

$$H(\sigma) = -\frac{1}{2N}\sum_{i,j\in\Lambda}\langle\sigma_i|J\sigma_j\rangle - \sum_{i\in\Lambda}\langle h|\sigma_i\rangle. \tag{3.36}$$

[6] An important issue is the behaviour of the probability measure (3.31) in the thermodynamic limit. Under appropriate conditions, sequences of these measures converge to a well–defined limit, called a **Gibbs state**. Phase transitions can be characterized by the existence of several Gibbs states.

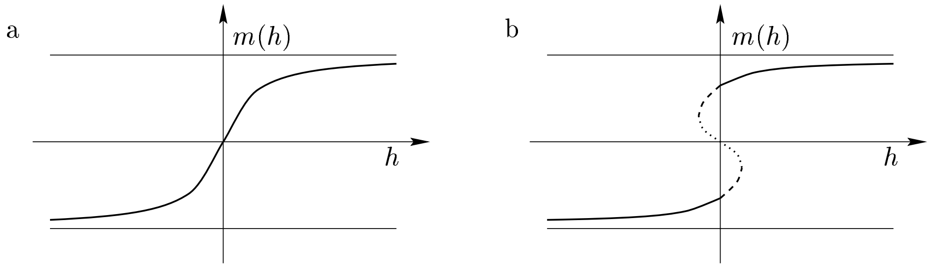

FIGURE 3.6. The curves $m(h)$ which solve the self–consistency equation (3.41), (a) for $\beta < 1$ (high temperature) and (b) for $\beta > 1$ (low temperature). Full lines indicate solutions which are assumed to be thermodynamically stable, dashed lines indicate metastable solutions, and dotted lines unstable ones.

If $n = 1$, this system is known as the **Curie–Weiss** model.

The basic idea of the **mean field approximation** is that each spin will feel a **local field**

$$h_i(\sigma) = h + J \sum_{j\in\Lambda} \frac{\sigma_j}{N}. \tag{3.37}$$

In the thermodynamic limit, the term $\sum_{j\in\Lambda} \sigma_j/N$ should behave, by the central limit theorem, as $m + \mathcal{O}(N^{-1/2})$, where $m = M/N$ is the average magnetization per site. The mean field approximation consists in replacing (3.36) by the effective Hamiltonian

$$H_{\mathrm{mf}}(\sigma) = -\sum_{i\in\Lambda} \langle \sigma_i | h + Jm \rangle. \tag{3.38}$$

The partition function becomes

$$Z_{\mathrm{mf}} = \int_\Omega \mathrm{d}\mu(\sigma)\, \mathrm{e}^{\beta \sum_i \langle \sigma_i | h+Jm\rangle} = \biggl[\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_i(\sigma_i)\, \mathrm{e}^{\beta\langle \sigma_i | h+Jm\rangle} \biggr]^N. \tag{3.39}$$

The magnetization density m has to satisfy the self–consistency equation

$$\begin{aligned} m = \langle \mathcal{M} \rangle_0 &= \frac{1}{Z_{\mathrm{mf}}} \prod_{i\in\Lambda} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_i(\sigma_i) \frac{1}{N} \sum_j \sigma_j\, \mathrm{e}^{\beta\langle \sigma_i | h+Jm\rangle} \\ &= \frac{\displaystyle\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_j(\sigma_j) \sigma_j\, \mathrm{e}^{\beta\langle \sigma_j | h+Jm\rangle}}{\displaystyle\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_j(\sigma_j)\, \mathrm{e}^{\beta\langle \sigma_j | h+Jm\rangle}}. \end{aligned} \tag{3.40}$$

For instance, if $n = 1$, we obtain for the partition function, free energy density and self–consistency condition

$$\begin{aligned} Z_{\mathrm{mf}}(\beta, h, N) &= \bigl(2 \operatorname{ch} \beta(h + Jm(h))\bigr)^N, \\ f_{\mathrm{mf}}(\beta, h) &= \ln\bigl(2 \operatorname{ch} \beta(h + Jm(h))\bigr), \\ m(h) &= \operatorname{th} \beta(h + Jm(h)). \end{aligned} \tag{3.41}$$

This last equation admits a single solution if $\beta < 1$. When $\beta > 1$, it admits one solution for large $|h|$, but three solutions for small $|h|$. It is generally assumed that since the free energy should be convex, only the solution with the same sign than h is thermodynamically stable, the others being metastable or unstable (Fig. 3.6). Note that in the mean field approximation, the dimension d of the lattice has disappeared.

3.3 Magnets out of Equilibrium

While Equilibrium Statistical Mechanics is a relatively well established domain, the understanding of Non–Equilibrium Statistical Mechanics is still much less advanced, and rather relies on a number of more or less phenomenological concepts.

One of these concepts is the stochastic process. In Equilibrium Statistical Mechanics, the system is described by a probability distribution, rather than by specifying its state in detail. Similarly, the evolution of a large system out of equilibrium is described by a probability distribution over all possible evolutions: this is precisely what we call a **stochastic process**.

After stating some basic properties in Subsection 3.3.1, we outline in Subsection 3.3.2 how to construct an evolution equation for the probability of finding the system in a specific state, called a **master equation**. Finally, we show in Subsection 3.3.3 how to derive a deterministic equation for a macroscopic observable like magnetization in the thermodynamic limit. Under appropriate hypotheses, fluctuations around this macroscopic value obey a **Langevin equation**.

This section follows [**Ma1**]. See [**Sp**] for similar results on lattice gases.

3.3.1 Stochastic Processes

A **stochastic process** can be understood as a large number of realizations of a function $x(t)$, each one occurring with some probability. More precisely, let $x(t_1), x(t_2), \dots, x(t_n)$, be n random variables, and $I_1 = [x_1, x_1 + \mathrm{d}x_1], \dots, I_n = [x_n, x_n + \mathrm{d}x_n]$ be n intervals. We denote by

$$p(x_1, t_1; x_2, t_2; \dots; x_n, t_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{3.42}$$

the probability of finding simultaneously $x(t_i)$ in I_i for $i = 1, \dots, n$. It satisfies the relation

$$\begin{aligned}\int \mathrm{d}x_i p(x_1, t_1; \dots; x_{i-1}, t_{i-1}; x_i, t_i; x_{i+1}, t_{i+1}; \dots; x_n, t_n)& \\ = p(x_1, t_1; \dots; x_{i-1}, t_{i-1}; x_{i+1}, t_{i+1}; \dots; x_n, t_n).&\end{aligned} \tag{3.43}$$

The average of a function depending on x at times $t_1, \dots, t_n$ is defined by

$$\langle f(x(t_1), \dots, x(t_n)) \rangle = \int \mathrm{d}x_1 \dots \mathrm{d}x_n f(x_1, \dots, x_n)\, p(x_1, t_1; x_2, t_2; \dots; x_n, t_n). \tag{3.44}$$

Let us denote conditional probabilities by

$$p(x_{k+1}, t_{k+1}; \dots; x_n, t_n | x_1, t_1; \dots; x_k, t_k) = \frac{p(x_1, t_1; \dots; x_n, t_n)}{p(x_1, t_1; \dots; x_k, t_k)}. \tag{3.45}$$

The stochastic process is called **Markovian** if

$$p(x_k, t_k | x_1, t_1; \dots; x_{k-1}, t_{k-1}) = p(x_k, t_k | x_{k-1}, t_{k-1}). \tag{3.46}$$

Such a process has only a local memory in time. For instance, the solution of an ODE has this property. In this case, all probabilities can be calculated if one knows only the

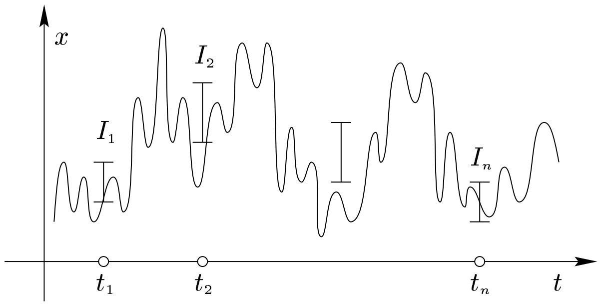

FIGURE 3.7. One realization of a stochastic process. The intervals I_i enter in the definition of the probability measure (3.42).

functions $p(x_1, t_1)$ and $p(x_2, t_2|x_1, t_1)$. It follows from (3.43) that a Markovian stochastic process satisfies the **Chapman–Kolmogorov equation**: for $t_1 \leqslant t_2 \leqslant t_3$,

$$p(x_3, t_3|x_1, t_1) = \int \mathrm{d}x_2\, p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1). \tag{3.47}$$

Let us consider a **stationary** Markovian process, that is we assume that

$$p(x_1, t_1 + \Delta t|x_0, t_0 + \Delta t) = p(x_1, t_1|x_0, t_0) \qquad \forall \Delta t. \tag{3.48}$$

The process is said to be **diffusive** if its moments satisfy the relations

$$\begin{aligned} \langle \Delta x \rangle_{x_0} &:= \int \mathrm{d}x (x - x_0)\, p(x, \Delta t|x_0, 0) = a(x_0)\Delta t + o(\Delta t), \\ \langle (\Delta x)^2 \rangle_{x_0} &:= \int \mathrm{d}x (x - x_0)^2 p(x, \Delta t|x_0, 0) = b(x_0)\Delta t + o(\Delta t), \\ \langle (\Delta x)^k \rangle_{x_0} &:= \int \mathrm{d}x (x - x_0)^k p(x, \Delta t|x_0, 0) = o(\Delta t), \qquad k \geqslant 3. \end{aligned} \tag{3.49}$$

Then the conditional probability satisfies the **Fokker–Planck equation**

$$\frac{\partial}{\partial t} p(x, t|x_0, 0) = -\frac{\partial}{\partial x}\big(a(x)\, p(x, t|x_0, 0)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(b(x)\, p(x, t|x_0, 0)\big). \tag{3.50}$$

3.3.2 Master Equations

Let us assume that the conditional probability of a stationary Markovian process behaves as

$$p(x_2, \Delta t|x_1, 0) = (1 - a_0(x_1)\Delta t)\delta(x_1 - x_2) + w(x_2|x_1)\Delta t + \mathcal{O}(\Delta t^2), \tag{3.51}$$

where $w(x_2|x_1)$ is called **transition probability** by unit time. The function $a_0(x_1)$ is determined by the condition

$$\int \mathrm{d}x_2\, p(x_2, t|x_1, 0) = 1 \qquad \Rightarrow \qquad a_0(x_1) = \int \mathrm{d}x_2\, w(x_2|x_1). \tag{3.52}$$

Using the Chapman–Kolmogorov equation (3.47), we obtain that the probability satisfies the equation

$$\frac{\partial}{\partial t} p(x, t) = \int \mathrm{d}y \big[w(x|y)\, p(y, t) - w(y|x)\, p(x, t)\big]. \tag{3.53}$$

Such a relation is called a **master equation**. It describes the balance between gains and losses of the probability to be in the state x at time t.

We now consider the special case where $p(\sigma, t)$ describes the probability of finding the lattice model in the configuration σ at time t. We would like to determine an expression for the transition probabilities, in such a way that they define a reasonable dynamics for the model. One condition is that the equilibrium distribution (3.31), $p_0(\sigma) = \mathrm{e}^{-\beta H(\sigma)}/Z$, be a stationary solution of (3.53). A sufficient (but not necessary) condition is that $w(\sigma'|\sigma)$ satisfy the **detailed balance**

$$w(\sigma|\sigma')\, p_0(\sigma') = w(\sigma'|\sigma)\, p_0(\sigma). \tag{3.54}$$

In the case of one–dimensional spins $\sigma_i = \pm 1$, one usually chooses **Glauber spin–flip dynamics** [Ka]. Let σ^i be the configuration obtained by turning the i^{th} spin of the configuration σ:

$$\sigma^i_j := \begin{cases} -\sigma_j & \text{if } j = i \\ \sigma_j & \text{otherwise.} \end{cases} \tag{3.55}$$

The transition probabilities of the Glauber prescription are given by

$$w(\sigma'|\sigma) = \begin{cases} \frac{\gamma}{2}\big[1 - \sigma_i \operatorname{th} \beta h_i(\sigma)\big] & \text{if } \sigma' = \sigma^i \\ 0 & \text{otherwise,} \end{cases} \tag{3.56}$$

where $h_i(\sigma) = \sum_j J_{ij}\sigma_j + h$ is the local field at site i, and γ is a constant. The associated master equation is

$$\begin{aligned} \frac{\partial}{\partial t} p(\sigma, t) &= \sum_{\sigma'} w(\sigma|\sigma') p(\sigma', t) - w(\sigma'|\sigma) p(\sigma, t) \\ &= \frac{\gamma}{2} \sum_{i\in\Lambda} \big[1 + \sigma_i \operatorname{th} \beta h_i\big] p(\sigma^i, t) - \big[1 - \sigma_i \operatorname{th} \beta h_i\big] p(\sigma, t), \end{aligned} \tag{3.57}$$

where $h_i = h_i(\sigma) = h_i(\sigma^i)$. A straightforward calculation shows that the detailed balance condition (3.54) is satisfied.

The choice (3.56) is not the only one satisfying the detailed balance. But it is not arbitrary, since it can be derived from Hamiltonian dynamics [**Ma1**, Ma2]. One considers a *quantum* spin model with an Ising–like Hamiltonian. Each spin is coupled to an independent free Fermi field at temperature T. Under appropriate hypotheses, one can derive an integrodifferential equation of motion for the density matrix of the spin system, which is called a **generalized master equation**. The quantum spin system admits a set of particular states, which are in one–to–one correspondance with the configurations of the classical Ising model. In the weak coupling limit, or **Van Hove limit**, their equation of motion reduces to (3.57), with the difference that γ is a function of h_i, related to correlation functions of the Fermi field. We will assume that the function γ is always strictly positive. Then it can be replaced by a constant without modifying qualitative properties of the dynamics.

3.3.3 Langevin Equations

Let us consider the Curie–Weiss model, for which $J_{ij} = \frac{J}{N}$ for all $j \neq i$. We are interested in the observable

$$\mathcal{M}_N(\sigma) := \frac{1}{N} \sum_{j \in \Lambda} \sigma_j, \tag{3.58}$$

which defines the magnetization density of the system with N spins. Its average value at time t is given by

$$m_N(t) = \big\langle \mathcal{M}_N(\sigma) \big\rangle_N := \sum_\sigma p_N(\sigma, t) \frac{1}{N} \sum_j \sigma_j. \tag{3.59}$$

We would like to determine an evolution equation for $m_N(t)$. We have

$$\frac{\mathrm{d}m_N}{\mathrm{d}t} = \sum_\sigma \frac{\partial}{\partial t} p_N(\sigma, t) \frac{1}{N} \sum_j \sigma_j. \tag{3.60}$$

Replacing $\partial_t p_N(\sigma, t)$ by its expression (3.57), we obtain two sums over configurations σ. In the sum involving $p_N(\sigma^i, t)$, we change the summation index σ to σ^i. Using the fact that $\sum_j (\sigma^i_j - \sigma_j) = -2\sigma_i$, we obtain

$$\frac{\mathrm{d}m_N}{\mathrm{d}t} = \gamma \big\langle -\mathcal{M}_N + \operatorname{th} \beta(h + J\mathcal{M}_N + \mathcal{O}(1/N)) \big\rangle_N. \tag{3.61}$$

Now we would like to take the thermodynamic limit $N \to \infty$. To do this, we have to assume that the probability distributions behave well in this limit for a set of "macroscopic" observables. More precisely, we say that a sequence $(p_N(\sigma, t))_N$ is **macroscopic at the point** $m(t)$ if

$$\lim_{N \to \infty} \big\langle \varphi(\mathcal{M}_N) \big\rangle_N = \varphi(m(t)) \tag{3.62}$$

for every sufficiently smooth function $\varphi(m)$. In particular, we have $m = \lim_{N \to \infty} m_N$. In fact, one easily checks that if the sequence $(p_N(\sigma, 0))_N$ is macroscopic at $m(0)$, then $(p_N(\sigma, t))_N$ is macroscopic at some $m(t)$, and equation (3.61) shows that $m(t)$ is the solution of

$$\frac{\mathrm{d}m}{\mathrm{d}t} = f(m) := \gamma \big[-m + \operatorname{th} \beta(h + Jm) \big]. \tag{3.63}$$

Thus we have obtained a *deterministic* evolution equation in the thermodynamic limit.

Let us now examine the behaviour of fluctuations for large, but finite N. By the central limit theorem, we expect these fluctuations to be of order $N^{-1/2}$. Thus we consider the observable

$$\widehat{\mathcal{M}}_N(\sigma, t) := \sqrt{N} \Big(\frac{1}{N} \sum_{j \in \Lambda} \sigma_j - m(t) \Big). \tag{3.64}$$

If $\varphi(m)$ is a twice differentiable function, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\varphi(\widehat{\mathcal{M}}_N)\big\rangle_N &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_\sigma p_N(\sigma,t)\,\varphi\bigg(\sqrt{N}\Big[\frac{1}{N}\sum_{j\in\Lambda}\sigma_j - m(t)\Big]\bigg) \\ &= \sum_\sigma \frac{\partial}{\partial t}p_N(\sigma,t)\varphi(\widehat{\mathcal{M}}_N) - \sqrt{N}f(m)\big\langle\varphi'(\widehat{\mathcal{M}}_N)\big\rangle_N.\end{aligned} \tag{3.65}$$

The first term is evaluated in a similar way than (3.60), using the relations

$$\widehat{\mathcal{M}}_N(\sigma^i,t) - \widehat{\mathcal{M}}_N(\sigma,t) = -\frac{2}{\sqrt{N}}\sigma_i, \tag{3.66}$$

$$\varphi\big(\widehat{\mathcal{M}}_N(\sigma^i,t)\big) - \varphi\big(\widehat{\mathcal{M}}_N(\sigma,t)\big) = -\frac{2}{\sqrt{N}}\sigma_i\varphi'\big(\widehat{\mathcal{M}}_N(\sigma,t)\big) + \frac{2}{N}\varphi''\big(\widehat{\mathcal{M}}_N(\sigma,t)\big) + \mathcal{O}\Big(\frac{1}{N}\Big).$$

We obtain that the terms of order $\sqrt{N}$ cancel, and it remains

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\varphi(\widehat{\mathcal{M}}_N)\big\rangle_N = f'(m)\big\langle\widehat{\mathcal{M}}_N\varphi'\big(\widehat{\mathcal{M}}_N\big)\big\rangle_N + g(m)\big\langle\varphi''\big(\widehat{\mathcal{M}}_N\big)\big\rangle_N + \mathcal{O}(1), \tag{3.67}$$

where

$$g(m) := \gamma\big(1 - m\,\mathrm{th}\,\beta(Jm+h)\big). \tag{3.68}$$

We now assume that the probability distribution has **normal fluctuations** of the form

$$\lim_{N\to\infty}\big\langle\varphi(\widehat{\mathcal{M}}_N)\big\rangle_N = \int_{-\infty}^{\infty}\mu(\hat{m},t)\varphi(\hat{m})\,\mathrm{d}\hat{m}. \tag{3.69}$$

If $\mu(\hat{m},t)$ is sufficiently regular, integrations by part yield

$$\frac{\partial}{\partial t}\mu(\hat{m},t) = -f'(m(t))\frac{\partial}{\partial\hat{m}}\Big[\hat{m}\,\mu(\hat{m},t)\Big] + g(m(t))\frac{\partial^2}{\partial\hat{m}^2}\mu(\hat{m},t), \tag{3.70}$$

which is a **Fokker–Planck diffusion equation** for $\mu(\hat{m},t)$. It is associated with the **Langevin equation**

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{m}(t) = -f'(m(t))\hat{m}(t) + \eta(t), \tag{3.71}$$

where $\eta(t)$ is a Gaussian **white noise**, satisfying

$$\langle\eta(t)\rangle = 0, \qquad \langle\eta(t_1)\eta(t_2)\rangle = 2g(m(t_1))\delta(t_1-t_2). \tag{3.72}$$

Indeed, the solution of (3.71) with initial conditions $m(0)=m_0$ and $\hat{m}(0)=\hat{m}_0$ can be written as[7]

$$\hat{m}(t) - \hat{m}_0 = -f'(m_0)\hat{m}_0 t + \int_0^t \eta(s)\,\mathrm{d}s + \mathcal{O}(t^2), \tag{3.73}$$

which implies that the moments satisfy

$$\begin{aligned}\langle\hat{m}(t)-\hat{m}_0\rangle_{\hat{m}_0} &= -f'(m_0)\hat{m}_0 t + \mathcal{O}(t^2), \\ \langle(\hat{m}(t)-\hat{m}_0)^2\rangle_{\hat{m}_0} &= 2g(m_0)t + \mathcal{O}(t^2),\end{aligned} \tag{3.74}$$

[7] For the sake of simplicity, we ignore here the technicalities of stochastic integration.

while higher moments are $\mathcal{O}(t^2)$. In view of (3.49), this shows that the probability distribution of the solutions of the Langevin equation (3.71) satisfies the Fokker–Planck equation (3.70).

To summarize, we have shown that the average magnetization of the Curie–Weiss model with Glauber dynamics satisfies

$$\langle \mathcal{M}_N(\sigma) \rangle_N = m(t) + \frac{1}{\sqrt{N}}\hat{m}(t) + \mathcal{O}\Big(\frac{1}{N}\Big), \tag{3.75}$$

where $m(t)$ satisfies the deterministic equation (3.63), and $\hat{m}(t)$ satisfies the Langevin equation (3.71).

3.4 Phenomenological Models of Hysteresis

There exist several models which consider hysteresis from a "macroscopic" point of view, as a functional operator transforming an input $u(t)$ into an output $v(t)$. See [MNZ] for a review of such models. Let us briefly describe some of them. One usually distinguishes between **active** and **relay hysteresis**.

In **relay** hysteresis, the output can only take two values $f_\pm(u)$ for a given input u. Given two functions $f_+(u) > f_-(u)$, and two threshold values $\alpha < \beta$, the output is defined by

$$v(t) = F[u](t) = \begin{cases} f_-(u(t)) & \text{if } u(t) \leqslant \alpha, \\ f_+(u(t)) & \text{if } u(t) \geqslant \beta, \\ f_-(u(t)) & \text{if } \alpha < u(t) < \beta \text{ and } u(\tau(t)) = \alpha, \\ f_+(u(t)) & \text{if } \alpha < u(t) < \beta \text{ and } u(\tau(t)) = \beta, \end{cases} \tag{3.76}$$

where $\tau = \sup\{s \leqslant t \,|\, u(s) = \alpha \text{ or } \beta\}$ is the time of the last attained threshold. Thus, the output switches to $f_+(v)$ each time the input becomes larger than β, and switches to $f_-(v)$ each time the input becomes smaller than α (Fig. 3.8a). This is the most simple, purely phenomenological model of hysteresis.

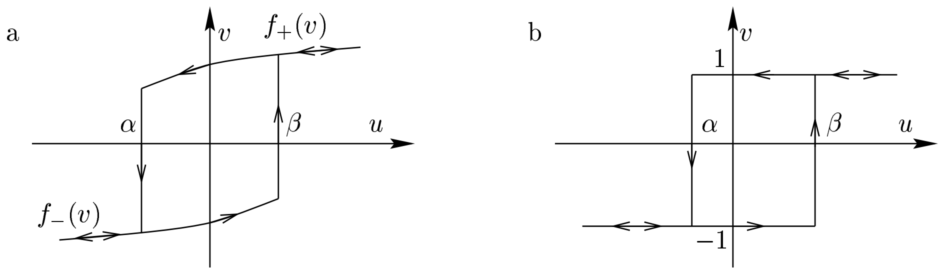

FIGURE 3.8. (a) Definition of relay hysteresis: the output switches between $f_+(v)$ and $f_-(v)$ each time the input becomes larger than β or smaller than α. (b) Every elementary Preisach operator $\hat{\gamma}_{\alpha\beta}$ is a relay operator with $f_\pm = \pm 1$.

In **active** hysteresis, the system may perform minor hysteresis loops, obtained by letting the input oscillate back and forth between two values (Fig. 3.11). One of the

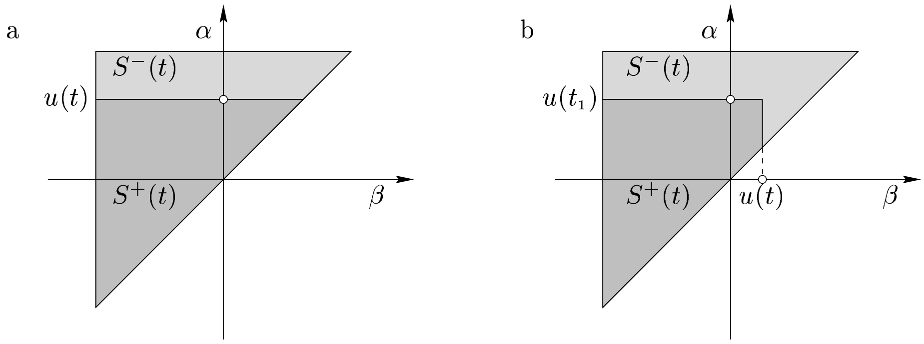

FIGURE 3.9. Geometric interpretation of the Preisach model. The weight function $\mu(\alpha,\beta)$ is assumed to be supported by the triangle T. (a) If the input $u(t)$ is increased monotonically from -1 to $u(t)$, the line $\alpha = u(t)$ separates T into two regions $S^{\pm}(t)$, where the elementary hysteresis operators $\hat{\gamma}_{\alpha,\beta}$ are respectively in the "up" and "down" position. (b) If the input is decreased monotonically, the region $S^+(t)$ shrinks since it is limited by the vertical line $\beta = u(t)$.

models used to describe active hysteresis is the **Duhem model**, defined by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \begin{cases} f_-(v,u)\dot{u} & \text{if } \dot{u} \leqslant 0 \\ f_+(v,u)\dot{u} & \text{if } \dot{u} \geqslant 0. \end{cases} \tag{3.77}$$

The solutions of the differential equations for decreasing and increasing input define two families of curves in the (u,v)–plane, which determine the hysteresis cycles.

Another popular model for active hysteresis is the **Preisach model**, which is discussed in [**May**]. It is defined as follows:

$$v(t) = \hat{\Gamma}[u](t) := \int_{\alpha \geqslant \beta} \mu(\alpha,\beta)\,\hat{\gamma}_{\alpha\beta}\,u(t)\ \mathrm{d}\alpha\,\mathrm{d}\beta. \tag{3.78}$$

Here $\hat{\gamma}_{\alpha\beta}$ is an elementary relay hysteresis operator, defined by (3.76) with $f_+ = 1$ and $f_- = -1$ (Fig. 3.8b).

This model admits the following geometric interpretation. Let us denote by T the triangle $-\alpha_0 \leqslant \beta \leqslant \alpha \leqslant \alpha_0$. We assume that $\mu(\alpha,\beta) = 0$ outside the triangle T. For simplicity, we take $\alpha_0 = 1$.

The elementary hysteresis operators $\hat{\gamma}_{\alpha\beta}$ can be considered as switches, which are either in "up" or in "down" position. Consider the case when $u(t_0) < -1$, and $u(t)$ increases monotonically. Initially, all elementary hysteresis operators are in the "down" position. While $u(t)$ increases, all operators $\hat{\gamma}_{\alpha\beta}$ such that $\alpha < u(t)$ switch to the "up" position. Thus the output is given by the relation

$$v(t) = \int_{S^+(t)} \mu(\alpha,\beta)\ \mathrm{d}\alpha\,\mathrm{d}\beta - \int_{S^-(t)} \mu(\alpha,\beta)\ \mathrm{d}\alpha\,\mathrm{d}\beta, \tag{3.79}$$

where $S^{\pm}(t)$ denote respectively the part of T below and above the line $\alpha = u(t)$. If for $t \geqslant t_1$, we decrease the input $u(t)$ monotonically, then all elementary hysteresis operators with $\beta > u(t)$ are switched back into the "down" position. Thus the output $v(t)$ is determined by (3.79), where $S^+(t)$ is defined by the relations $\alpha \leqslant u(t_1)$ and $\beta \leqslant u(t)$.

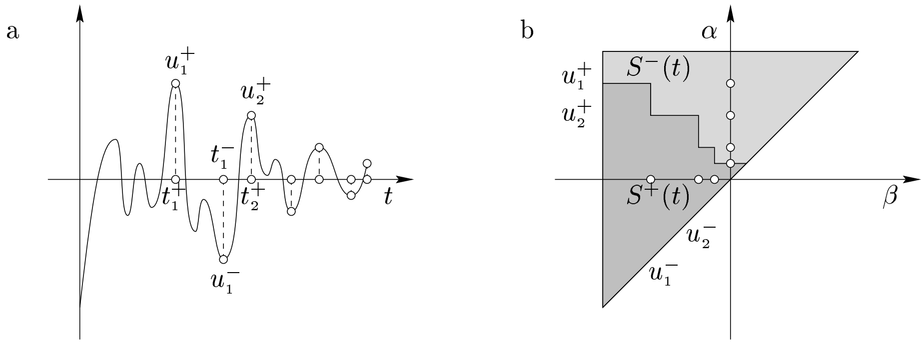

FIGURE 3.10. Wiping–out property: if $u(t_0) = -1$, the output at time t depends on a series of alternating extrema of $u(s)$ (a). The corresponding regions $S^{\pm}(t)$ are shown in (b). All intermediate extrema are wiped out by the model.

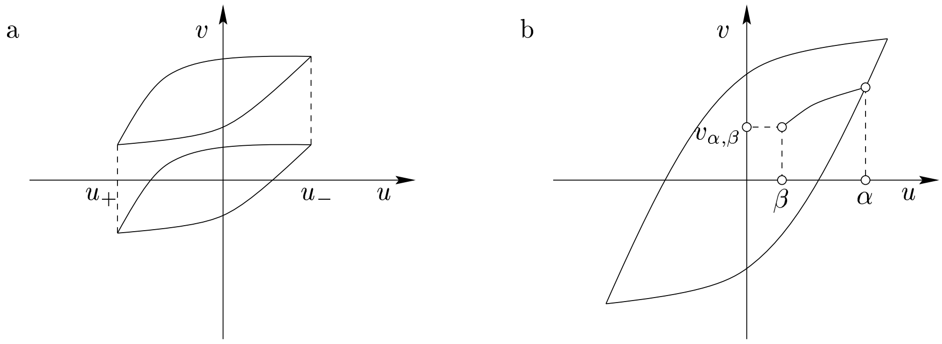

FIGURE 3.11. (a) Two congruent minor hysteresis loops. (b) If $u(t)$ is moved monotonically from -1 to α, and then to β, the output value $v_{\alpha,\beta}$ allows to determine the weights $\mu(\alpha, \beta)$.

Continuing in this way, we find that the output can always be determined by the relation (3.79), where the areas $S^{\pm}(t)$ are separated by a broken line. Each increase of the input adds a horizontal segment to this line, which moves upward, while each decrease of the input creates a vertical segment moving to the left. Moreover, the model has the following two properties:

1. **Wiping-Out Property:** Each local input maximum wipes out the vertices of the separation line whose α–coordinates are below this maximum, and each local minimum wipes out the vertices whose β–coordinates are above this minimum.
   This property states that the output at time t depends only on a set of local input minima and maxima. Assume for instance that $u(t_0) = -1$, and define

$$u_1^+ = \max_{[t_0,t]} u(s), \quad u(t_1^+) = u_1^+, \qquad u_1^- = \min_{[t_1^+,t]} u(s), \quad u(t_1^-) = u_1^-. \tag{3.80}$$

   Subsequent local extrema are defined in a similar way:

$$u_k^+ = \max_{[t_{k-1}^-,t]} u(s), \quad u(t_k^+) = u_k^+, \qquad u_k^- = \min_{[t_k^+,t]} u(s), \quad u(t_k^-) = u_k^-. \tag{3.81}$$

   Then $v(t)$ is determined by the input values $u_k^{\pm}$.

2. **Congruency Property:** All minor hysteresis loops corresponding to back–and–forth variations of input between the same two consecutive extremum values are congruent. Assume that the inputs $u_1(t)$ and $u_2(t)$ have different past histories, but for $t \geqslant t_0$, they vary back and forth between two same consecutive extrema $u_\pm$. Then the corresponding loops in the (u, v)–plane can be transformed into one another by a vertical translation.

It is shown in [**May**] that the wiping-out and congruency property are necessary and sufficient for the hysteresis to be representable by the Preisach model. If this is the case, the coefficients $\mu(\alpha, \beta)$ can be determined in the following way. Let $u(t_0) = -1$, and increase u monotonically until α. Then decrease u monotonically until β. If $v_{\alpha,\beta}$ is the output value at that time, the coefficients are determined by

$$\mu(\alpha, \beta) = \frac{1}{2} \frac{\partial^2 v_{\alpha,\beta}}{\partial\alpha\partial\beta}. \tag{3.82}$$

Chapter 4

One–Dimensional Systems

"Es wird das thermische Verhalten eines linearen, aus Elementarmagneten bestehenden Körpers untersucht (...) Es wird gezeigt, daß ein solches Modell noch keine ferromagnetischen Eigenschaften besitzt (...) Auch bei einem Räumlichen Modell, das den oben aufgestellten Annahmen genügt, gelangt man nicht zu einem anderen Ergebniß"

Ernst Ising

In this chapter, we present a detailed study of one–dimensional (1D) dynamical systems, which depend on a parameter varying adiabatically in time. More precisely, our aim is to provide a method to answer the following questions:

- does this system admit hysteresis cycles?
- how do these cycles scale with the adiabatic parameter ε?

We will develop tools which help to solve these questions in a simple way, partly based on geometrical representations, and avoiding lengthy calculations.

We have two reasons to begin with the study of 1D systems, before considering the n–dimensional (nD) case. The first reason is pedagogical: 1D systems can be analysed in much detail, and their solutions are easy to visualize geometrically. Some properties are easier to understand on 1D systems, but can be generalized to the nD case without difficulty. The other reason is practical: the analysis of nD systems can sometimes be reduced to a 1D problem, for instance using Lyapunov functions or adiabatic manifolds.

The content of this chapter is presented in the following sections:

- **Section 4.1** We give a precise characterization of the 1D systems we want to study, and illustrate on a simple example the mathematical difficulties which may appear in such singular perturbation problems. We also introduce some fundamental notions of the theory of differential inequalities.
- **Section 4.2** We show existence of adiabatic solutions, which remain at a distance of order ε from equilibrium branches of the static equation, as long as these do not bifurcate. We provide an iterative scheme for computing asymptotic series in ε of these solutions, and show that they can be approximated with exponential accuracy.
- **Section 4.3** We develop a practical method to analyse the behaviour of solutions near bifurcations. This method is based on the Newton polygon, and helps to find out where solutions go after the bifurcation, and how they depend on ε at leading order.

- **Section 4.4** We examine a few generalizations of results in the previous section to the complex case. In particular, we state some important results by Neishtadt on the Hopf bifurcation.
- **Section 4.5** We discuss global properties of the flow, in particular the relaxation of orbits starting far from equilibrium branches. We show how to analyse the global aspects of the flow qualitatively.
- **Section 4.6** We particularize the discussion to periodic variation of parameters, and prove existence of periodic orbits. We give a definition of hysteresis and motivate it by a few illustrative examples; we also show how to compute scaling exponents associated with hysteresis cycles.
- Finally, in **Appendix 4.A**, we give the proofs of some results presented in previous sections.

The work presented here has been partially inspired by existing literature. Existence of adiabatic solutions has been known, at least qualitatively, for a long time. Early precise results on relaxation to equilibrium can be found in [PR]. The computation of asymptotic series in ε has been initiated in [Hab], but we follow closely the iterative method introduced by Neishtadt in [Ne1].

Dynamics near bifurcations have been studied in various papers, often involving elaborate mathematical tools [LS]. Techniques using local changes of variables depending on ε have been discussed by Delcroix in [**Ben**]. However, our method based on differential inequalities, which allows to read the scaling exponents directly off Newton's polygon, is new to our knowledge.

The phenomenon of bifurcation delay has been analysed in detail in the case of the Hopf bifurcation, in particular by Neishtadt [Ne1, Ne2] and by Diener and Diener [**Ben**].

Finally, although hysteresis phenomena in dynamical systems have been studied in a few particular cases [JGRM, HL&], we are not aware of any previous work giving a precise definition and a systematic characterization of hysteresis cycles in differential equations.

4.1 Preliminaries

4.1.1 What One–Dimensional Systems?

The simplest one–dimensional equation we are going to consider is

$$\varepsilon \dot{x} = f(x,\tau), \tag{4.1}$$

which we call the **adiabatic system**. We assume that x belongs to an open set $\mathcal{D} \subset \mathbb{R}$, τ varies in an interval $I \subset \mathbb{R}$, and $f \in \mathcal{C}^k(\mathcal{D} \times I, \mathbb{R})$, with $k \geqslant 2$. The adiabatic parameter ε varies in an interval $(0, \varepsilon_0)$, where ε_0 will always be assumed to be sufficiently small. As seen in Section 3.1, this equation can be thought of as describing the overdamped motion of a particle in a slowly varying potential $\Phi(x,\tau)$ with $\partial_x \Phi = f(x,\tau)$.

We would like to compare the solutions of (4.1) with those of the family of **instantaneous systems**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_{\tau_0}(x) := f(x,\tau_0), \qquad \tau_0 \in I \text{ fixed.} \tag{4.2}$$

The solutions of (4.2) are very simple to analyse qualitatively: $x(t)$ is increasing when $f_{\tau_0}(x) > 0$, decreasing when $f_{\tau_0}(x) < 0$. The orbits are attracted exponentially fast by

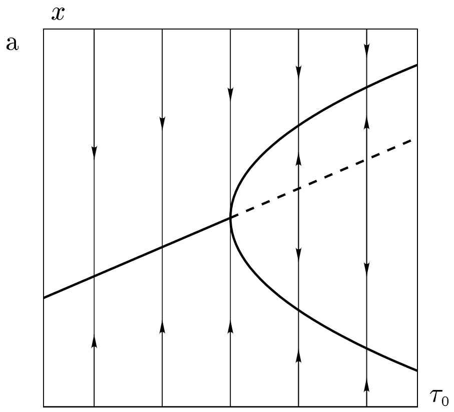

FIGURE 4.1. Orbits in the case $f(x,\tau) = (2x - \tau)(\tau - x^2)$. (a) The solutions of the family of instantaneous equations $\mathrm{d}_t x = f(x, \tau_0)$ (thin lines) are essentially determined by the branches of fixed points (thick lines): they are attracted by stable branches (full lines) and repelled by unstable ones (dotted lines). (b) The orbits of the adiabatic system $\varepsilon\dot{x} = f(x,\tau)$ have a more complex structure, but they obey a simple rule: the branches of fixed points must be crossed horizontally, and separate the plane into regions where $x(\tau)$ is monotonous.

the stationary points $x^\star$, solution of $f_{\tau_0}(x^\star) = 0$ with $\partial_x f_{\tau_0}(x^\star) < 0$, and repelled by the stationary points with $\partial_x f_{\tau_0}(x^\star) > 0$. The case $\partial_x f_{\tau_0}(x^\star) = 0$ corresponds to bifurcation (Fig. 4.1a).

The dynamics of (4.1) can be qualitatively understood by considering the equivalent system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\tau \\ x\end{pmatrix} = \begin{pmatrix}\varepsilon \\ f(x,\tau)\end{pmatrix}. \tag{4.3}$$

This vector field has a small horizontal component, whereas the vertical component is given by (4.2). The orbits of (4.3) must cross the lines of fixed points $x^\star(\tau)$ horizontally, and are increasing or decreasing with a very large slope when $f \neq 0$ (Fig. 4.1b).

We will sometimes consider more general equations than (4.1), because they become important in the n–dimensional case. For instance, the motion near a focus or a Hopf bifurcation is often more adequately described by a 1D complex equation than by a 2D real one. Whenever possible (in particular in the part concerning adiabatic solutions), we will include the complex case in our study.

Another possibility is that f depends also on ε. If this dependence is smooth in the limit $\varepsilon \to 0$ (e.g. the fixed points vary smoothly and there are no new bifurcations generated when ε decreases[1]), it will not affect the qualitative conclusions of our discussion. One way to deal with this situation is to analyse in detail the dependence on δ of solutions of the system

$$\varepsilon\dot{x} = f(x,\tau,\delta) \tag{4.4}$$

for $0 < \delta < \delta_0$, and then to see what happens when we replace δ by ε. We will discuss such a situation in Subsection 7.1.2.

[1]For instance the system $\varepsilon\dot{x} = 1 - \varepsilon x$ is problematic because its fixed point diverges when $\varepsilon \to 0$. Another example is the imperfect bifurcation $\varepsilon\dot{x} = \tau x - x^3 + \delta$ analysed in [EM].

4.1.2 An Important Example

We discuss in this subsection a simple, but very important example, illustrating the problems which may appear because of the singular dependence of (4.1) on ε.

Consider the linear system

$$\varepsilon\dot{x} = -ax + h(\tau), \tag{4.5}$$

where $a \in \mathbb{C}$ and $h(\tau)$ is analytic in a neighborhood of the real axis. If a is a positive real, (4.5) describes an overdamped forced harmonic oscillator. A complex value of a would model a situation with low friction, where the solutions are oscillating.

In principle, ε is supposed to be a small positive real number. But in order to apply methods of complex analysis, it is of interest to consider (4.5) in a complex neighborhood of $\varepsilon = 0$. By a linear change of variables, we may then arrange that $a = 1$.

Equation (4.5) can be explicitly solved (we take $a = 1$):

$$x(\tau) = \mathrm{e}^{-\tau/\varepsilon}\Big[x(0) + \frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{s/\varepsilon}\, h(s)\,\mathrm{d}s\Big], \tag{4.6}$$

but this doesn't tell us much on the nature of the solution, except that there seems to be an essential singularity at $\varepsilon = 0$, because of the factor $\mathrm{e}^{-\tau/\varepsilon}$. However, successive integrations by part transform (4.6) into

$$\begin{aligned} x(\tau) &= x^{(N)}(\tau) + (-\varepsilon)^{N+1}\frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{(s-\tau)/\varepsilon}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^{N+1} h(s)\,\mathrm{d}s + \mathrm{e}^{-\tau/\varepsilon}[x(0) - x^{(N)}(0)] \\ x^{(N)}(\tau) &:= \sum_{j=0}^N (-\varepsilon)^j \Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^j h(\tau) = h(\tau) - \varepsilon\dot{h}(\tau) + \varepsilon^2\ddot{h}(\tau) + \cdots \end{aligned} \tag{4.7}$$

Note that the first term in the expansion (4.7) corresponds to the adiabatic evolution, where the particle follows its static equilibrium $x^\star(\tau) = h(\tau)$.

It seems that there exists a particular solution, $x^{(N)}(\tau)$, which is represented by the asymptotic series (4.7) when $N \to \infty$. However, since, by Cauchy's formula, the N^{th} derivative of h grows like $N!$, the series (4.7) does not converge in general. In fact, this series is Gevrey–1, as in Example 2.4, and the best truncation is located at $N(\varepsilon) \sim [1/\varepsilon]$, which gives a remainder of order $\mathrm{e}^{-1/C|\varepsilon|}$.

The reasons for this lack of convergence can be better understood by considering the special case of a periodic forcing, $h(\tau + T) = h(\tau)$. Then $h(\tau)$ can be decomposed as its Fourier series,

$$h(\tau) = \sum_{p=-\infty}^{\infty} \hat{h}(p)\,\mathrm{e}^{\mathrm{i}\,\Omega p\tau}, \qquad \Omega = \frac{2\pi}{T}, \qquad |\hat{h}(p)| \leqslant \mathrm{e}^{-\kappa|p|}. \tag{4.8}$$

It is natural to look for a periodic solution of (4.5), and indeed we immediately find

$$\bar{x}(\tau) = \sum_{p=-\infty}^{\infty} \frac{\hat{h}(p)}{1 + \mathrm{i}\,\Omega p\varepsilon}\,\mathrm{e}^{\mathrm{i}\,\Omega p\tau}, \tag{4.9}$$

and the general solution of (4.5) is

$$x(\tau) = \mathrm{e}^{-\tau/\varepsilon}(x(0) - \bar{x}(0)) + \bar{x}(\tau). \tag{4.10}$$

The periodic solution (4.9) has simple poles at $\varepsilon = \frac{\mathrm{i}}{\Omega p}$ for $p \in \mathbb{Z}^*$. This is a phenomenon of **resonance** between the harmonics of the forcing and the eigenfrequencies of the oscillator: in the resonant case, the solution is no longer periodic, but there is a secular term growing linearly with time. Unless $h(\tau)$ contains only a finite number of harmonics, these poles accumulate at $\varepsilon = 0$, and thus prevent the existence of a Taylor series in ε of the solution.

Note also that in the non–resonant case, the expansion (4.7) becomes

$$x^{(N)}(\tau) = \sum_{p=-\infty}^{\infty} \hat{h}(p) \sum_{j=0}^{N} (-\,\mathrm{i}\,\Omega p \varepsilon)^j \,\mathrm{e}^{\mathrm{i}\,\Omega p \tau}, \tag{4.11}$$

which is exactly the geometric series which would converge to $\bar{x}(\tau)$ if there were only finitely many harmonics (a similar result is obtained in the resonant case $\varepsilon = \frac{\mathrm{i}}{\Omega q}$, treating separately the mode q).

We conclude from this example that even as simple equations as (4.5) do not, in general, admit solutions with convergent series in ε, although there may exist particular solutions admitting asymptotic series (which can be truncated so as to approach the solution exponentially closely).

4.1.3 A Fundamental Lemma

As explained before, we will be looking for easily obtained, yet precise bounds on the solutions of equations like (4.1), rather than for exact solutions. For this purpose, we introduce some elements of the theory of differential inequalities, taken from [**Hal**].[2]

Lemma 4.1 ("Comparison Lemma"). *Let $g(u,t)$ be continuous on an open connected set $\mathcal{D} \subset \mathbb{R}^2$ and such that the equation*

$$\dot{u} = g(u,t), \qquad u(t_0) = u_0 \tag{4.12}$$

admits a unique solution $u(t)$ on $[t_0, t_1]$. Assume that $v(t)$ is a continuous function on $[t_0, t_1)$, with a right–hand derivative satisfying

$$\lim_{h \to 0+} \frac{v(t+h)}{h} =: \mathrm{d_R} v(t) \leqslant g(v(t), t), \qquad t_0 \leqslant t < t_1, \tag{4.13}$$

and such that $v(t_0) \leqslant u_0$. Then $v(t) \leqslant u(t)$ for $t_0 \leqslant t \leqslant t_1$.

Proof: Let $u_n(t)$ be the solution of $\dot{u}_n = g(u_n, t) + \frac{1}{n}$, $u_n(t_0) = u_0$. Lemma 2.3 implies that $u_n(t)$ exists on $[t_0, t_1]$ for sufficiently large n, and converges uniformly to $u(t)$. Assume by contradiction there exists an interval $(t_-, t_+] \subset (t_0, t_1)$ on which $v(t) > u_n(t)$, and such that $v(t_-) = u_n(t_-)$. Then $v(t) - v(t_-) > u_n(t) - u_n(t_-)$ for $t_- < t \leqslant t_+$ so that

$$\mathrm{d_R} v(t_-) \geqslant \dot{u}_n(t_-) = g(u_n(t_-), t_-) + \frac{1}{n} = g(v(t_-), t_-) + \frac{1}{n} > g(v(t_-), t_-), \tag{4.14}$$

which is a contradiction. Thus, $v(t) \leqslant u_n(t)$ for sufficiently large n, and one concludes using the uniform convergence. □

Assume for instance that we study a vectorial equation $\dot{x} = f(x,t)$. The sup norm of x, $v(t) := |x(t)|$, is in general not differentiable. But is is shown in [**Hal**] that its right–hand derivative exists and satisfies $|\mathrm{d_R}|x(t)|| \leqslant |\dot{x}(t)| = |f(x,t)|$. If $|f(x,t)| \leqslant g(|x|,t)$, Lemma 4.1 implies that $v(t)$ can be bounded by solutions of $\dot{u} = g(u,t)$.

[2] A similar result is called "Fence Theorem" in [**HW**].

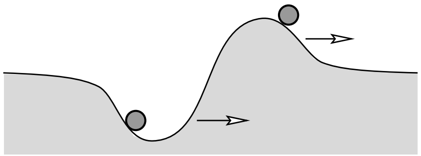

FIGURE 4.2. In the analogy of the particle in a potential, adiabatic solutions are those which remain close to the extrema of the potential. For instance, the particle may settle slightly behind a minimum moving to the right. It may also remain in unstable equilibrium slightly ahead of a potential maximum.

4.2 Adiabatic Solutions

In this section we consider the equation

$$\varepsilon\dot{x} = f(x,\tau), \tag{4.15}$$

where $x \in \mathbb{C}$, $\tau \in I \subset \mathbb{R}$ and f is a complex–valued $\mathcal{C}^{k+1}$ function, $k \geqslant 1$. Equation (4.15) tells us that either $\dot{x} = \mathcal{O}(1/\varepsilon)$, or $f(x,\tau) = \mathcal{O}(\varepsilon)$. In this section, we are interested in the second case, that is in solutions of (4.15) remaining in a neighborhood of order ε of a branch of fixed points $x^\star(\tau)$, solution of $f(x^\star(\tau),\tau) = 0$. More precisely, we introduce the following definition:

Definition 4.1. An **adiabatic solution** of order ε^μ of the equation $\varepsilon\dot{x} = f(x,\tau)$ associated with the equilibrium branch $x^\star(\tau)$ is a solution $\bar{x}(\tau)$ remaining at a distance of order ε^μ from $x^\star(\tau)$.

We will begin by presenting, in Subsection 4.2.1, an iterative scheme which simplifies the system (4.15) near the fixed point. This will allow us, in Subsection 4.2.2, to prove the existence of adiabatic solutions of order ε in the cases without bifurcation. Finally, we examine in Subsection 4.2.3 some properties of the solutions in a neighborhood of an adiabatic solution.

4.2.1 Iterative Scheme

Let us assume the existence of an equilibrium branch $x^\star(\tau)$, such that $f(x^\star(\tau),\tau) = 0$, $\tau \in I$. It is natural to consider the change of variables $x = x^\star(\tau) + y$, giving

$$\varepsilon\dot{y} = a(\tau)y + b(y,\tau) + \varepsilon w(\tau), \tag{4.16}$$

with

$$a(\tau) := \partial_x f(x^\star(\tau),\tau) \qquad \text{linear term,} \tag{4.17a}$$

$$b(y,\tau) := f(x^\star(\tau)+y,\tau) - a(\tau)y = \mathcal{O}(y^2) \qquad \text{nonlinear term,} \tag{4.17b}$$

$$w(\tau) := -\partial_\tau x^\star(\tau) \qquad \text{drift term.} \tag{4.17c}$$

If we assume that $a(\tau) \neq 0$, $\tau \in I$, the implicit function theorem implies that $x^\star(\tau) \in \mathcal{C}^{k+1}(I,\mathbb{C})$, and thus (4.16) is of class $\mathcal{C}^k$.

We will see that the drift term is the most important, since it may prevent the solutions from settling on the fixed point. Since the system (4.16) has a fixed point at $y = -\varepsilon w(\tau)/a(\tau) + \mathcal{O}(\varepsilon^2)$, we can try to decrease the order of this drift term by successive changes of variables. We systematize this procedure in the following way, considering the more general case where the nonlinear term b depends on the conjugate y^* and ε as well.

Iterative Scheme 4.1. *Assume that for some $N \geqslant 1$,*

$$\varepsilon \dot{y}_{N-1} = a(\tau) y_{N-1} + b_{N-1}(y_{N-1}, y_{N-1}^*, \tau, \varepsilon) + \varepsilon^N w_{N-1}(\tau, \varepsilon), \tag{4.18}$$

where all functions are of class $\mathcal{C}^{k-N+1}$, $a(\tau) \neq 0$ and $b_{N-1} = \mathcal{O}(|y_{N-1}|^2) + \mathcal{O}(\varepsilon |y_{N-1}|)$.
The change of variables

$$y_{N-1} = y_N + \varepsilon^N u_N(\tau, \varepsilon), \qquad u_N(\tau, \varepsilon) := -\frac{w_{N-1}(\tau, \varepsilon)}{a(\tau)}, \tag{4.19}$$

transforms (4.18) *into*

$$\varepsilon \dot{y}_N = a(\tau) y_N + b_N(y_N, y_N^*, \tau, \varepsilon) + \varepsilon^{N+1} w_N(\tau, \varepsilon), \tag{4.20}$$

where the functions

$$w_N(\tau, \varepsilon) := \frac{1}{\varepsilon^{N+1}} b_{N-1}(\varepsilon^N u_N, \varepsilon^N u_N^*, \tau, \varepsilon) - \dot{u}_N(\tau, \varepsilon) \tag{4.21a}$$

$$b_N(y_N, y_N^*, \tau, \varepsilon) := b_{N-1}(y_N + \varepsilon^N u_N, y_N^* + \varepsilon^N u_N^*, \tau, \varepsilon) - b_{N-1}(\varepsilon^N u_N, \varepsilon^N u_N^*, \tau, \varepsilon) \tag{4.21b}$$

are of class $\mathcal{C}^{k-N}$ and $b_N = \mathcal{O}(|y_N|^2) + \mathcal{O}(\varepsilon |y_N|)$.

Remark 4.1. It is possible to replace $w_{N-1}(\tau, \varepsilon)$ in (4.19) by $w_{N-1}(\tau, 0)$, if the definition of w_N is changed accordingly. The choice (4.19) simplifies the proof of Lemma 4.2 below, while the other choice simplifies the calculations (but yields equivalent results).

We conclude that if $f \in \mathcal{C}^{k+1}$, it is possible, by k successive changes of variables, to decrease the order of the drift term to ε^{k+1}. We will examine the consequences of this fact in the next subsection. What happens in the analytic case? The example of Subsection 4.1.2 shows that the series $\sum_N \varepsilon^N u_N$ will not converge in general. But it turns out that the drift term can be made exponentially small. We will encounter several generalizations of this important result in subsequent chapters. The proof, which we give in Appendix 4.A.1, is adapted from a paper by Neishtadt [Ne1].

Lemma 4.2. *Consider the equation*

$$\varepsilon \dot{y}_0 = a(\tau) y_0 + b_0(y_0, y_0^*, \tau, \varepsilon) + \varepsilon w_0(\tau, \varepsilon) \tag{4.22}$$

with $b_0 = \mathcal{O}(|y_0|^2) + \mathcal{O}(\varepsilon |y_0|)$ and $a(\tau) \neq 0$.

1. *If a, b and w are of class $\mathcal{C}^k$, there exists a continuous change of variables $y_0 = y_k + \sum_{j=1}^k \varepsilon^j u_j(\tau, \varepsilon)$ transforming* (4.22) *into*

$$\varepsilon \dot{y}_k = a(\tau) y_k + b_k(y_k, y_k^*, \tau, \varepsilon) + \varepsilon^{k+1} w_k(\tau, \varepsilon), \tag{4.23}$$

   with $b_k = \mathcal{O}(|y_k|^2) + \mathcal{O}(\varepsilon |y_k|)$.

2. *If* (4.22) *is analytic for y_0 in a complex neighborhood $\mathcal{D}$ of the origin, and for τ in a complex neighborhood of an interval I, there exists in smaller neighborhoods a change of variables of the form $y_0 = z + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j u_j(\tau, \varepsilon)$ transforming* (4.22) *into*

$$\varepsilon \dot{z} = a(\tau) z + \bar{b}(z, z^*, \tau, \varepsilon) + \mathrm{e}^{-1/C|\varepsilon|}\, \bar{w}(\tau, \varepsilon), \tag{4.24}$$

   with $\bar{b} = \mathcal{O}(|z|^2) + \mathcal{O}(\varepsilon |z|)$. Moreover, $y_0 - z = \mathcal{O}(\varepsilon)$.

Let us introduce some terminology for further reference.

Definition 4.2. If $f(x,\tau)$ in (4.15) is of class $\mathcal{C}^{k+1}$ and admits a smooth branch of fixed points $x^\star(\tau)$, we associate with it the **adiabatic approximation of order** k

$$x^{(k)}(\tau,\varepsilon) := x^\star(\tau) + \sum_{j=1}^{k} \varepsilon^j u_j(\tau,\varepsilon), \tag{4.25}$$

where the functions u_j are defined recursively by Iterative Scheme 4.1. Note that since $u_j \in \mathcal{C}^{k-j+1}$, this expansion may be rewritten as $x^{(k)}(\tau,\varepsilon) = x^\star(\tau) + \sum_j \varepsilon^j x_j(\tau)$ (see also Remark 4.1).

If $f(x,\tau)$ is analytic in an open complex set, we call **adiabatic approximation of exponential order** the function

$$x^{(\omega)}(\tau,\varepsilon) := x^\star(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j u_j(\tau,\varepsilon), \tag{4.26}$$

where $N(\varepsilon)$ is defined by (4.170).

Lemma 4.2 tells us that $x(\tau) = x^{(k)}(\tau,\varepsilon) + y$, where y satisfies an equation of the form $\dot{y} = a(\tau)y + \mathcal{O}(|y|^2) + \mathcal{O}(\varepsilon|y|) + \mathcal{O}(\varepsilon^{k+1})$ (resp. $\mathcal{O}(\mathrm{e}^{-1/C|\varepsilon|})$ in the analytic case). We are now going to show that under appropriate conditions, $y(\tau)$ remains small, so that the adiabatic approximations $x^{(k)}$ are indeed good approximations of the real solutions.

4.2.2 Existence of Adiabatic Solutions

Let us consider the equation

$$\varepsilon\dot{y} = a(\tau)y + b(y,y^*,\tau,\varepsilon) + \varphi(\varepsilon)w(\tau,\varepsilon), \tag{4.27}$$

where $b = \mathcal{O}(|y|^2) + \mathcal{O}(\varepsilon|y|)$ and $\varphi(\varepsilon)$ is a continuous function going to 0 when $\varepsilon \to 0$ (for instance ε^{k+1} or $\mathrm{e}^{-1/C|\varepsilon|}$).

Proposition 4.1 (Stable Case). *Assume that* $\operatorname{Re} a(\tau) \leqslant -a_0 < 0$ *for* $\tau \in I$. *Then there exist* $c_0, \varepsilon_0 > 0$ *such that if* $0 < \varepsilon \leqslant \varepsilon_0$, *any solution of* (4.27) *with* $|y(\tau_1)| \leqslant c_0|\varphi(\varepsilon)|$ *satisfies* $|y(\tau)| \leqslant c_0|\varphi(\varepsilon)|$ *for* $\tau \in I$ *and* $\tau \geqslant \tau_1$.

PROOF: We assume that $|b| \leqslant M(|\varepsilon y| + |y|^2)$ and $|w| \leqslant W$. Let $V(\tau) := |y(\tau)|$. Then

$$\varepsilon\dot{V} = \frac{1}{2V}\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}|y|^2 = \frac{1}{V}\operatorname{Re}(y^*\varepsilon\dot{y}) \leqslant -a_0V + M(\varepsilon V + V^2) + |\varphi(\varepsilon)|W. \tag{4.28}$$

In fact, this equation is defined as long as $V \neq 0$, but if $V = 0$, the inequality remains true when $\dot{V}$ is replaced with its right–hand derivative. Take $M\varepsilon_0 < a_0$, and consider the equation

$$\varepsilon\dot{u} = g(u) := -(a_0 - M\varepsilon_0)u + Mu^2 + |\varphi(\varepsilon)|W. \tag{4.29}$$

The function $g(u)$ has a fixed point at $u^\star = \frac{MW}{a_0 - M\varepsilon_0}|\varphi(\varepsilon)| + \mathcal{O}(\varphi(\varepsilon)^2)$. Since $u(\tau) = u^\star$ is a particular solution of (4.29), it follows from Lemma 4.1 that $|y(\tau_1)| \leqslant u^\star$ implies $|y(\tau)| \leqslant u^\star$ for $\tau \geqslant \tau_1$. □

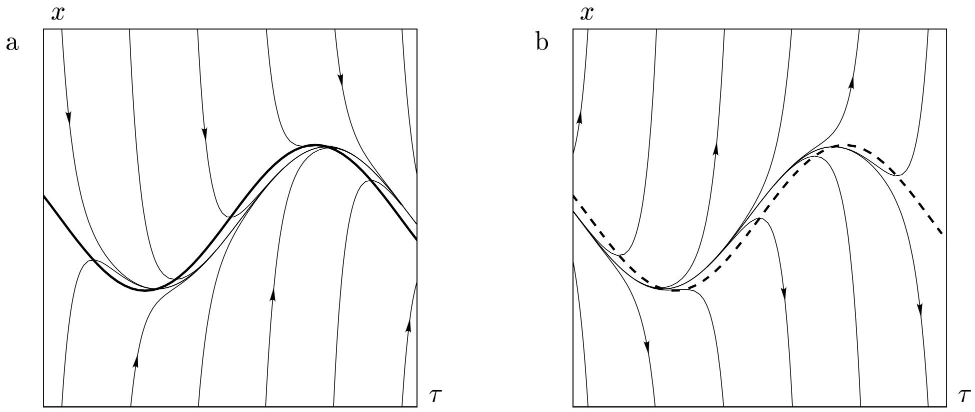

FIGURE 4.3. Orbits near a branch of fixed points. (a) Stable case: the orbits are attracted by the stable branch (thick full line) and converge to an adiabatic solution, at a distance of order ε from the branch. (b) Unstable case: the orbits are repelled by the unstable branch (dotted line). They diverge exponentially fast from an adiabatic solution, which can be constructed by fixing a "final condition" of order ε, and following it backwards in time.

Proposition 4.2 (Unstable Case). *Assume that* $\operatorname{Re} a(\tau) \geqslant a_0 > 0$ *for* $\tau \in I$. *Then, if* $0 < \varepsilon \leqslant \varepsilon_0$, *there exists a solution of* (4.27) *such that* $|y(\tau)| \leqslant c_0|\varphi(\varepsilon)|$ *for* $\tau \in I$.

PROOF: The change of variables $\sigma = -\tau$ transforms (4.27) into

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}\sigma} = -a(-\sigma)y - b(y, y^*, -\sigma, \varepsilon) - \varphi(\varepsilon)w(-\sigma, \varepsilon), \tag{4.30}$$

to which we can apply Proposition 4.1. □

Together with Lemma 4.2, these two propositions imply the existence of adiabatic solutions, and of their asymptotic series. We summarize this central result in the following way.

Theorem 4.1. *Let* $x^\star(\tau)$ *be a hyperbolic equilibrium branch of* $f(x,\tau)$, *that is such that* $\operatorname{Re} a(\tau) := \operatorname{Re} \partial_x f(x^\star(\tau), \tau)$ *is bounded away from zero. Then the equation* $\varepsilon\dot{x} = f(x,\tau)$ *admits an adiabatic solution of order* ε, $\bar{x}(\tau) = x^\star(\tau) + \mathcal{O}(\varepsilon)$. *Moreover,*

1. *If* $f(x,\tau) \in \mathcal{C}^{k+1}$, $\bar{x}(\tau)$ *admits the expansion*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{k} \varepsilon^j x_j(\tau) + \mathcal{O}(\varepsilon^{k+1}), \tag{4.31}$$

where the functions $x_j(\tau)$ *or obtained by rearranging the terms of the adiabatic approximation* (4.25). *Thus they can be calculated recursively by Iterative Scheme 4.1.*

2. *If* $f(x,\tau)$ *is analytic in an open complex domain,* $\bar{x}(\tau)$ *admits the expansion*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j x_j(\tau) + \mathcal{O}(\mathrm{e}^{-1/C|\varepsilon|}), \tag{4.32}$$

where $N(\varepsilon) = \mathcal{O}(1/\varepsilon)$.

In other words, the adiabatic solutions are close to the adiabatic approximations $x^{(k)}$ or $x^{(\omega)}$ of Definition 4.2. The interesting point is that $x^{(k)}$ can be computed using only algebraic manipulations, given in Iterative Scheme 4.1, without having to solve any differential equation. Thus we have established a first link between solutions of the adiabatic system (4.1) and those of the family of instantaneous equations (4.2).

Using the iterative scheme to second order, we obtain for these adiabatic solutions the expansion

$$\bar{x}(\tau) = x^\star(\tau) + \varepsilon \frac{\dot{x}^\star(\tau)}{a(\tau)} + \mathcal{O}(\varepsilon^2). \tag{4.33}$$

In the stable case, the adiabatic solution will follow the equilibrium branch with some delay (Fig. 4.3a), while in the unstable case, the adiabatic solution will stay ahead of the branch (Fig. 4.3b). This result confirms the intuitive picture given in Fig. 4.2.

Corollary 4.1. *If* $\operatorname{Re} a(\tau)$ *is bounded away from* 0, *the adiabatic approximation* (4.25) *satisfies*

$$\varepsilon \dot{x}^{(k)} = f(x^{(k)}, \tau) + \mathcal{O}(\varepsilon^{k+1}). \tag{4.34}$$

PROOF: We know that there exists an adiabatic solution $\bar{x}(\tau) = x^{(k)}(\tau,\varepsilon) + y$, where y satisfies (4.27) with $\varphi(\varepsilon) = \mathcal{O}(\varepsilon^{k+1})$. By the above propositions, $y(\tau) = \mathcal{O}(\varepsilon^{k+1})$ and thus $\varepsilon\dot{y} = \mathcal{O}(\varepsilon^{k+1})$ so that $\varepsilon\dot{x}^{(k)} = \varepsilon\dot{\bar{x}} - \varepsilon\dot{y} = f(\bar{x},\tau) + \mathcal{O}(\varepsilon^{k+1}) = f(x^{(k)},\tau) + \mathcal{O}(\varepsilon^{k+1})$. □

This result has the following important interpretation: let

$$x^{(k)}(\tau,\varepsilon) = x^\star(\tau) + \sum_{j=0}^{k} \varepsilon^j x_j(\tau) \tag{4.35}$$

be any formal series satisfying the equation $\varepsilon\dot{x} = f(x,\tau)$ up to order ε^{k+1}. Then this formal series must coincide with the adiabatic approximation. It is thus possible to compute this approximation by naive substitution of a series in the equation, instead of using the iterative scheme. Of course, even in the analytic case, this series will not converge in general, but has to be truncated after $\mathcal{O}(1/\varepsilon)$ terms in order to give a good approximation of the adiabatic solution.

The situation where $\operatorname{Re} a(\tau)$ changes sign, which corresponds to bifurcation, will be considered in subsequent sections. For further reference, we discuss here the elliptic case, where $\operatorname{Re} a(\tau) = 0$ (but remember that terms of order εy may be included into b). The results are similar but slightly weaker, because of the phenomenon of resonance described in Subsection 4.1.2.

Proposition 4.3 (Elliptic Case). *Assume that* $\operatorname{Re} a(\tau) = 0$ *and* $\varphi(\varepsilon) = o(\varepsilon^2)$. *On any* bounded *interval* $[\tau_1, \tau_2]$, *there exists a solution of* (4.27) *such that* $|y(\tau)| \leqslant c_0|\varphi(\varepsilon)|/\varepsilon$.

PROOF: If $V(\tau) := |y(\tau)|$, then $\varepsilon\dot{V} \leqslant M|\varepsilon|V + MV^2 + |\varphi(\varepsilon)|W$, and the result follows from Lemma 4.7 in Appendix 4.A.2, taking $q = 0$ and $k = 2$. □

4.2.3 Near Adiabatic Solutions

Assume we have found an adiabatic solution of the equation $\varepsilon\dot{x} = f(x,\tau)$ associated with the branch $x^\star(\tau)$, that is $\bar{x}(\tau) = x^\star(\tau) + \mathcal{O}(\varepsilon)$. Then the change of variables $x = \bar{x}(\tau) + y$ transforms the equation into

$$\varepsilon\dot{y} = f(\bar{x}+y,\tau) - f(\bar{x},\tau) = \bar{a}(\tau;\varepsilon)y + b(y,\tau;\varepsilon), \tag{4.36}$$

where $\bar{a}(\tau;\varepsilon) := \partial_x f(\bar{x},\tau) = a(\tau) + \mathcal{O}(\varepsilon)$ and $b = \mathcal{O}(y^2)$. Since $\bar{x}(\tau)$ is in general not analytic in a neighborhood of $\varepsilon = 0$, neither is (4.36), but this equation behaves nevertheless smoothly when $\varepsilon \neq 0$ (e.g. in the analytic case it admits asymptotic series).

We will consider the case where $\operatorname{Re}\bar{a} < 0$ for all times. The results can then easily be extended to the case $\operatorname{Re}\bar{a} > 0$ by inverting the direction of time. Using the same technique as in Proposition 4.1, one can show that the solutions of (4.36) decrease exponentially. But better results are obtained by considering the nonlinear term in (4.36) as an inhomogeneous one. If we define the functional map

$$\begin{aligned}(Ty)(\tau) &:= \mathrm{e}^{\bar{\alpha}(\tau,\tau_0)/\varepsilon}\, y(\tau_0) + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\bar{\alpha}(\tau,s)/\varepsilon}\, b(y(s),s)\,\mathrm{d}s,\\ \bar{\alpha}(\tau,\tau_0) &:= \int_{\tau_0}^{\tau} \bar{a}(s)\,\mathrm{d}s,\end{aligned} \tag{4.37}$$

then the solution of (4.36) is a fixed point of T. We use this fact to prove the following result, which states that the contribution of the nonlinear terms is indeed negligible for orbits starting close enough to the fixed point. Geometrically, it implies that in the vicinity of adiabatic solutions, the orbits look like "rivers", see Fig. 4.3.

Proposition 4.4. *There exists a constant $\delta > 0$ such that the solutions of* (4.36) *with $|y(\tau_0)| = y_0 \leqslant \delta$ are of the form*

$$y(\tau) = \mathrm{e}^{\bar{\alpha}(\tau,\tau_0)/\varepsilon}\big[y_0 + R(y_0,\tau)y_0^2\big], \tag{4.38}$$

where $R(y_0,\tau)$ is a bounded function which can be expanded in powers of y_0.

We give the proof, which is based on the fixed point theorem, in Appendix 4.A.3.

Example 4.1. For the logistic equation $\varepsilon\dot{y} = -y + y^2$, the first iterations of T yield

$$\begin{aligned}y_1(\tau) &= \mathrm{e}^{-\tau/\varepsilon}\, y_0,\\ y_2(\tau) &= \mathrm{e}^{-\tau/\varepsilon}[y_0 + (1-\mathrm{e}^{-\tau/\varepsilon})y_0^2],\\ y_3(\tau) &= \mathrm{e}^{-\tau/\varepsilon}[y_0 + (1-\mathrm{e}^{-\tau/\varepsilon})y_0^2 + (1-\mathrm{e}^{-\tau/\varepsilon})^2 y_0^3].\end{aligned} \tag{4.39}$$

For $y_0 < 1$, the series converges to the Taylor expansion of the solution

$$y(\tau) = \mathrm{e}^{-\tau/\varepsilon}\,\frac{y_0}{1-(1-\mathrm{e}^{-\tau/\varepsilon})y_0}. \tag{4.40}$$

4.3 Bifurcations: Real Case

In this section, we consider the real system

$$\varepsilon\dot{x} = f(x,\tau), \tag{4.41}$$

with $x \in \mathcal{D} \subset \mathbb{R}$, $\tau \in I \subset \mathbb{R}$, and $f \in \mathcal{C}^k$, $k \geqslant 2$. The complex case will be discussed in the next section. If $x^\star(\tau)$ is a branch of fixed points of f, we have seen that the distance to the branch, $y = x - x^\star(\tau)$, satisfies an equation of the form

$$\varepsilon\dot{y} = a(\tau)y + b(y,\tau) + \varepsilon w(\tau). \tag{4.42}$$

As long as $a(\tau) \neq 0$, $x^\star(\tau)$ is a smooth function, with which we can associate adiabatic solutions of order ε.

When $a(\tau)$ vanishes, say, at $\tau = 0$, new phenomena can occur. First of all, the adiabatic approximation breaks down at $\tau = 0$, because the iterative scheme (which involves division by $a(\tau)$) is no more defined. Moreover, several equilibrium branches may meet at the bifurcation point, so that it is not clear which one, if any, the solutions will follow. Finally, some of these branches may have an infinite slope at the bifurcation point, which means that $w(\tau) = -\partial_\tau x^\star(\tau)$ diverges.

Our analysis will show that (4.42) admits in general no adiabatic solution of order ε, but may admit solutions scaling with some other power of ε. In this section, we will present an efficient method to compute these exponents.

In order to do this, we begin by introducing in Subsection 4.3.1 some notations which help to manipulate these exponents. In Subsection 4.3.2, we state some general results on the scaling behaviour of the functions $a(\tau)$, $b(y,\tau)$ and $w(\tau)$ which appear in (4.42).

Subsections 4.3.3 and 4.3.4 discuss in detail the dynamics near a bifurcation, where the phases before and after the bifurcation point are treated separately for convenience. These techniques are illustrated on a few generic examples in Subsection 4.3.5.

Subsection 4.3.6 discusses the particular case of a bifurcation branch which does not depend on τ, and displays the remarkable phenomenon of bifurcation delay. We give some concluding remarks in Subsection 4.3.7.

4.3.1 Orders and Scaling Exponents

Since in this section, we are mainly interested in the scaling laws of the solutions to leading order, we will introduce some notations and definitions which simplify this analysis.

Notation 4.1. Let $x(\tau,\varepsilon)$ and $y(\tau,\varepsilon)$ be two continuous real functions defined for $0 < \varepsilon \leqslant \varepsilon_0$ and $\tau \in I \subset \mathbb{R}$. We write

- $x(\tau,\varepsilon) \approx y(\tau,\varepsilon)$ if there exist positive constants $c_\pm$, independent of τ, ε, such that $c_- y(\tau,\varepsilon) \leqslant x(\tau,\varepsilon) \leqslant c_+ y(\tau,\varepsilon)$.
- $x(\tau,\varepsilon) \sim y(\tau,\varepsilon)$ if $|x(\tau,\varepsilon)| \approx |y(\tau,\varepsilon)|$.
- $x(\tau,\varepsilon) \preccurlyeq y(\tau,\varepsilon)$ if there exists c_+, such that $0 \leqslant x(\tau,\varepsilon) \leqslant c_+ y(\tau,\varepsilon)$.[3]
- $x(\tau,\varepsilon) \succcurlyeq y(\tau,\varepsilon)$ if there exists c_-, such that $0 \leqslant c_- y(\tau,\varepsilon) \leqslant x(\tau,\varepsilon)$.[4]

[3] This definition is similar, but somewhat more precise than $x = \mathcal{O}(y)$.

[4] $x \succcurlyeq y$ is equivalent to $y \preccurlyeq x$. Moreover, $x \approx y$ if and only if $x \preccurlyeq y$ and $x \succcurlyeq y$.

Definition 4.3. Let $x(\tau,\varepsilon)$ be a continuous, strictly positive function for $0<\tau\leqslant\tau_0$ and $0<\varepsilon\leqslant\varepsilon_0<1$. The **scaling index function** associated with $x(\tau,\varepsilon)$ is

$$\tilde{x}(\eta,\varepsilon) := \frac{\ln x(\varepsilon^\eta,\varepsilon)}{\ln\varepsilon}. \tag{4.43}$$

The scaling index function introduces a logarithmic scale, which highlights the power law behaviour of the solution. It follows immediately from the definition that the inverse operation of (4.43) is given by

$$x(\varepsilon^\eta,\varepsilon)=\varepsilon^{\tilde{x}(\eta,\varepsilon)} \quad\Rightarrow\quad x(\tau,\varepsilon)=\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,\varepsilon)} \tag{4.44}$$

Moreover, it has the following properties.

Proposition 4.5. *Let $x(\tau,\varepsilon)$, $x_1(\tau,\varepsilon)$ and $x_2(\tau,\varepsilon)$ be functions with scaling index $\tilde{x}(\eta,\varepsilon)$, $\tilde{x}_1(\eta,\varepsilon)$ and $\tilde{x}_2(\eta,\varepsilon)$ respectively. Then*

$$\tilde{x}(\eta,\varepsilon)=a\eta+b \quad\Rightarrow\quad x(\tau,\varepsilon)=\varepsilon^b\tau^a, \tag{4.45a}$$

$$x(\tau,\varepsilon)=x_1(\tau,\varepsilon)x_2(\tau,\varepsilon) \quad\Rightarrow\quad \tilde{x}(\eta,\varepsilon)=\tilde{x}_1(\eta,\varepsilon)+\tilde{x}_2(\eta,\varepsilon), \tag{4.45b}$$

$$x(\tau,\varepsilon)=x_1(\tau,\varepsilon)^\mu \quad\Rightarrow\quad \tilde{x}(\eta,\varepsilon)=\mu\tilde{x}_1(\eta,\varepsilon), \tag{4.45c}$$

$$\tilde{x}(\eta,\varepsilon)=\tilde{x}(\eta,0)+\mathcal{O}\big(1/\ln\varepsilon\big) \quad\Rightarrow\quad x(\tau,\varepsilon)\approx\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,0)}, \tag{4.45d}$$

$$x(\tau,\varepsilon)=x_1(\tau,\varepsilon)+x_2(\tau,\varepsilon) \quad\Rightarrow\quad \tilde{x}(\eta,\varepsilon)=\min\{\tilde{x}_1(\eta,\varepsilon),\tilde{x}_2(\eta,\varepsilon)\}+\mathcal{O}\big(1/\ln\varepsilon\big). \tag{4.45e}$$

Proof: The properties (4.45a), (4.45b), and (4.45c) are direct consequences of the definition. To prove (4.45d), we note that if $\tilde{x}(\eta,\varepsilon)\geqslant\tilde{x}(\eta,0)+a/\ln\varepsilon$,

$$x(\tau,\varepsilon)=\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,\varepsilon)}\leqslant\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,0)+a/\ln\varepsilon}=\mathrm{e}^a\,\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,0)}. \tag{4.46}$$

Similarly, $\tilde{x}(\eta,\varepsilon)\leqslant\tilde{x}(\eta,0)-b/\ln\varepsilon$ implies $x(\tau,\varepsilon)\geqslant\mathrm{e}^{-b}\,\varepsilon^{\tilde{x}(\ln\tau/\ln\varepsilon,\,0)}$.

To prove (4.45e), assume that $\tilde{x}_1(\eta,\varepsilon)\leqslant\tilde{x}_2(\eta,\varepsilon)$. Then,

$$\begin{aligned}\tilde{x}(\eta,\varepsilon)&=\frac{\ln\big\{x_1(\varepsilon^\eta,\varepsilon)[1+x_2(\varepsilon^\eta,\varepsilon)/x_1(\varepsilon^\eta,\varepsilon)]\big\}}{\ln\varepsilon}=\frac{\ln x_1(\varepsilon^\eta,\varepsilon)}{\ln\varepsilon}+\frac{\ln[1+\varepsilon^{\tilde{x}_2(\eta,\varepsilon)-\tilde{x}_1(\eta,\varepsilon)}]}{\ln\varepsilon}\\&=\tilde{x}_1(\eta,\,\varepsilon)+\mathcal{O}\bigg(\frac{\varepsilon^{\tilde{x}_2(\eta,\varepsilon)-\tilde{x}_1(\eta,\varepsilon)}}{\ln\varepsilon}\bigg),\end{aligned} \tag{4.47}$$

where the last term is of order $1/\ln\varepsilon$ (and even smaller if $\tilde{x}_2>\tilde{x}_1$). □

The main interest of this definition is that, as we are going to show, the scaling index of a solution near a bifurcation is usually a piecewise linear function, plus a correction of order $1/\ln\varepsilon$. Thus it follows from (4.45a) and (4.45d) that to leading order, the solution obeys some simple power law of the form $x\approx\varepsilon^b\tau^a$.

4.3.2 Classification of Bifurcations

Usually, bifurcations are classified by their normal forms, which has the advantage of introducing a concept of genericity. However, the knowledge of the normal form is not sufficient for our purpose. Indeed, the three equations

$$\varepsilon\dot{x}=(x-\tau)(\tau-x^2), \tag{4.48a}$$

$$\varepsilon\dot{x}=(x-\tau^2)(\tau-x^2), \tag{4.48b}$$

$$\varepsilon\dot{x}=x(\tau-x^2), \tag{4.48c}$$

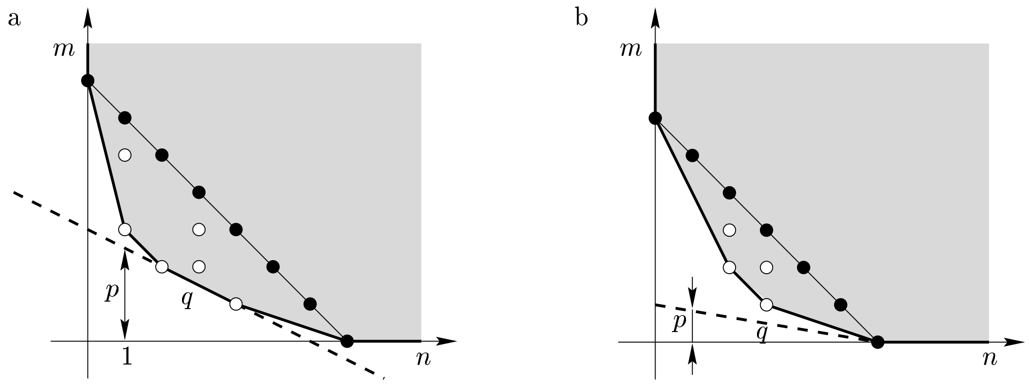

FIGURE 4.4. The exponents p and q, which govern the behaviour of the solution near a bifurcation branch, can be directly read off Newton's polygon. (a) If $x^\star(\tau) \sim |\tau|^q$ is a bifurcation branch, corresponding to a segment of slope $-q$ of the polygon, the ordinate at $n=1$ of the tangent gives p, such that $a(\tau) = \partial_x f(x^\star(\tau), \tau) \sim |\tau|^p$. (b) It may happen, because of the finite differentiability, that q is not the slope of a segment of the polygon (as in the case $f(x,\tau) = \tau^2 - \tau x + x^{5/2}$).

have the same normal form, but we will see that their solutions behave very differently (see Examples 4.9, 4.10 and 4.11).

Let $x^\star(\tau)$ be an equilibrium branch bifurcating at $\tau = 0$. Near the origin, we have $x^\star(\tau) \sim |\tau|^q$, where q may be deduced from Newton's polygon (see Subsection 2.4.3). We say that the branch is **regular** if $q \geqslant 1$, and **singular** if $0 < q < 1$.

Let us examine the behaviour of the terms in (4.42). First of all, the drift term is $w(\tau) = -\partial_\tau x^\star(\tau) \sim |\tau|^{q-1}$ (thus the slope at the bifurcation is infinite for a singular branch and finite for a regular branch).

Next, we look for a $p > 0$ such that the linear term scales as $a(\tau) = \partial_x f(x^\star(\tau), \tau) \sim |\tau|^p$. This can be done by writing the Taylor expansion

$$\partial_x f(x,\tau) = \sum_{\substack{n+m<k \\ n \geqslant 1}} n\alpha_{nm} x^{n-1}\tau^m + \sum_{\substack{n+m=k \\ n \geqslant 1}} x^{n-1}\tau^m R_{n-1,m}(x,\tau), \tag{4.49}$$

where $\alpha_{nm} = \partial_{nm} f(0,0)/n!m!$, and $R_{n-1,m}$ is continuous (see Proposition 2.8). Thus, since $x^\star(\tau) \sim |\tau|^q$,

$$|a(\tau)| \preccurlyeq \sum_{\substack{n+m<k \\ n \geqslant 1,\, \alpha_{nm}\neq 0}} |\tau|^{q(n-1)+m} + \sum_{\substack{n+m=k \\ n \geqslant 1}} |\tau|^{q(n-1)+m} |R_{n-1,m}(x^\star(\tau),\tau)|, \tag{4.50}$$

where the sign $\preccurlyeq$ comes from the fact that there may be cancelations in the sum.

Definition 4.4. Let

$$p^\star := \min_{n \geqslant 1,\, m \geqslant 0} \big\{ q(n-1) + m \,\big|\, (n+m<k \text{ and } \alpha_{nm} \neq 0) \text{ or } n+m=k \big\}. \tag{4.51}$$

Graphically, $p^\star$ is the ordinate at $n=1$ of the tangent to the Newton polygon with slope $-q$, see Fig. 4.4. The bifurcation branch is said to be **tame** if $a(\tau) \sim |\tau|^{p^\star}$, i.e., $p = p^\star$ (this is generically true, but because of the cancelations, we might have $p > p^\star$, see Example 4.3 below).

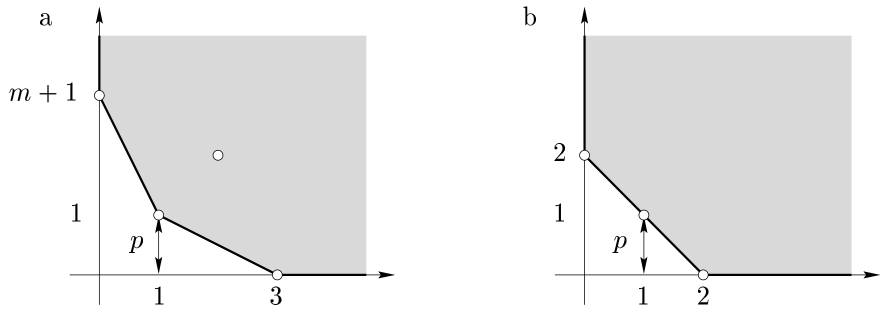

FIGURE 4.5. Newton's polygon for Example 4.2 (a) and for Example 4.3 (b).

Finally, we give a characterization of the nonlinear term $b(y,\tau)$, which is useful in the case $q < p$.

Lemma 4.3. *Let $x^\star(\tau)$ be a tame bifurcation branch.*
1. *If $|y| \preccurlyeq |\tau|^q$, then $y^{-2}|b(y,\tau)| \preccurlyeq |\tau|^{p-q}$.*
2. *If $|\tau| \preccurlyeq \varepsilon^r$ and $|y| \preccurlyeq \varepsilon^{qr}$ for $r \geqslant 0$, then $|b(y,\tau)| \preccurlyeq \varepsilon^{(p+q)r}$.*

PROOF: We know by Taylor's formula that for some $z = x^\star(\tau) + \theta y$, where $\theta \in (0,1)$,

$$\begin{aligned} b(y,\tau) &= y^2 \partial_{2,0} f(z,\tau) \\ &= y^2 \Biggl[\sum_{\substack{n+m<k \\ n \geqslant 2}} n(n-1)\alpha_{nm} z^{n-2}\tau^m + \sum_{\substack{n+m=k \\ n \geqslant 2}} z^{n-2}\tau^m R_{n-2,m}(z,\tau) \Biggr], \end{aligned} \tag{4.52}$$

and thus, if $|y| \preccurlyeq |\tau|^q$,

$$|y^{-2} b(y,\tau)| \preccurlyeq \sum_{\substack{n+m<k \\ n \geqslant 2,\, \alpha_{nm} \neq 0}} |\tau|^{q(n-2)+m} + \sum_{\substack{n+m=k \\ n \geqslant 2}} |\tau|^{q(n-2)+m}. \tag{4.53}$$

If the branch is tame, we have $q(n-1)+m \geqslant p$ whenever $\alpha_{nm} \neq 0$ or $n+m=k$, and thus the last term is of order $|\tau|^{p-q}$ (it may be smaller because of the restriction $n \geqslant 2$). In a similar way, replacing $|\tau| \preccurlyeq \varepsilon^r$ and $|y| \preccurlyeq \varepsilon^{qr}$ in (4.52), we obtain $|b(y,\tau)| \preccurlyeq \varepsilon^{(p+q)r}$. □

Example 4.2 (Pitchfork bifurcation). Consider the family of functions

$$f(x,\tau) = (x - \tau^m)(\tau - x^2), \qquad m \geqslant 1. \tag{4.54}$$

There are two branches of fixed points. The regular branch, $x^\star(\tau) = \tau^m$, corresponds to $q = m$. In this case we have

$$f(x^\star(\tau) + y, \tau) = (\tau - \tau^{2m})y - 2\tau^m y^2 - y^3, \tag{4.55}$$

so that $p = 1$, and if $y \sim \tau^q$, $|y^{-2}b(y,\tau)| = |2\tau^m + y| \sim |\tau|^q \preccurlyeq |\tau|^{1-q}$ since $q = m \geqslant 1$. The singular branches, $x^\star(\tau) = \pm\sqrt{\tau}$, correspond to $q = 1/2$, with

$$f(\sqrt{\tau} + y, \tau) = -2(\tau - \tau^{m+1/2})y - (3\sqrt{\tau} - \tau^m)y^2 - y^3, \tag{4.56}$$

and thus $p = 1$ and if $y \sim \tau^q$, $|y^{-2}b(y,\tau)| = |3\sqrt{\tau} - \tau^m + y| \sim \sqrt{\tau} = \tau^{1-1/2} = \tau^{p-q}$.

Example 4.3 (Counterexample to Lemma 4.3). Consider the function

$$f(x,\tau) = (x-\tau)^2. \tag{4.57}$$

In this case, $x^\star(\tau) = \tau$ so that $q = 1$. The Newton polygon shows that $p^\star = 1$, but due to a cancelation of terms, we have $b(x^\star(\tau) + y, \tau) = y^2$, so that p is infinite and the bifurcation is not tame. Consequently, if $y \sim \tau$, $|y^{-2} b(y,\tau)| \sim 1 = \tau^{p^\star - q} \not\asymp \tau^{p-q}$.

4.3.3 Before the Bifurcation

We know that in absence of bifurcations, we can associate adiabatic solutions with equilibrium branches. For stable branches, these solutions can be constructed in a simple way by choosing any initial condition at a distance of order ε from the branch, and following its future evolution. When the branch undergoes a bifurcation, it is natural to look at what happens to such solutions when the bifurcation point is crossed.

It is convenient to divide this problem into two parts. In this subsection, we will examine what happens to the adiabatic solutions associated with a stable branch, before and up to the bifurcation point. In the next subsection, we will study their behaviour after the bifurcation. By changing the direction of time, which exchanges stable and unstable branches, it is always possible to deduce informations on what happens before the bifurcation, from what happens after it.

We assume that for $-\tau_0 \leqslant \tau < 0$, $x^\star(\tau)$ is a bifurcation branch around which the dynamics is governed by the equation

$$\varepsilon \dot y = a(\tau) y + b(y,\tau) + \varepsilon w(\tau), \tag{4.58}$$

with $a(\tau) \asymp -|\tau|^p$ and $w(\tau) \sim |\tau|^{q-1}$. By changing, if necessary, the sign of y, it is always possible to consider the case of a decreasing branch, that is $w(\tau) \asymp |\tau|^{q-1} > 0$.

Our strategy will be to analyse first the linear equation

$$\varepsilon \dot y = -a_0 |\tau|^p y + \varepsilon w_0 |\tau|^{q-1}, \qquad y(-\tau_0) = y_0, \tag{4.59}$$

where $\tau_0 \asymp 1$ and $y_0 \asymp \varepsilon$. The solution can be written in the form

$$y(-\tau) = y_0 \, \mathrm{e}^{c\tau^{p+1}/\varepsilon} + w_0 \int_\tau^{\tau_0} s^{q-1} \, \mathrm{e}^{c[\tau^{p+1} - s^{p+1}]/\varepsilon} \, \mathrm{d}s, \qquad \text{where } c = \frac{a_0}{p+1}. \tag{4.60}$$

It is easy to see that the scaling index of the function $\mathrm{e}^{c\tau^{p+1}/\varepsilon}$ is of order $1/\ln\varepsilon$ for $\eta \geqslant \frac{1}{p+1}$, and diverges in the limit $\varepsilon \to 0$ for $0 < \eta \leqslant \frac{1}{p+1}$. The scaling properties of the second term are described by the next lemma.

Lemma 4.4. *Let τ_0, c, p and q be strictly positive constants, and consider*

$$z(\tau,\varepsilon) = \int_\tau^{\tau_0} s^{q-1} \, \mathrm{e}^{c[\tau^{p+1} - s^{p+1}]/\varepsilon} \, \mathrm{d}s. \tag{4.61}$$

Then the scaling index of z satisfies $\tilde z(\eta,\varepsilon) = \tilde z(\eta,0) + \mathcal{O}(1/\ln\varepsilon)$, where

$$\tilde z(\eta,0) = \begin{cases} \dfrac{q}{p+1} & \text{for } \eta \geqslant \frac{1}{p+1}, \\ 1 - (p+1-q)\eta & \text{for } 0 \leqslant \eta \leqslant \frac{1}{p+1}. \end{cases} \tag{4.62}$$

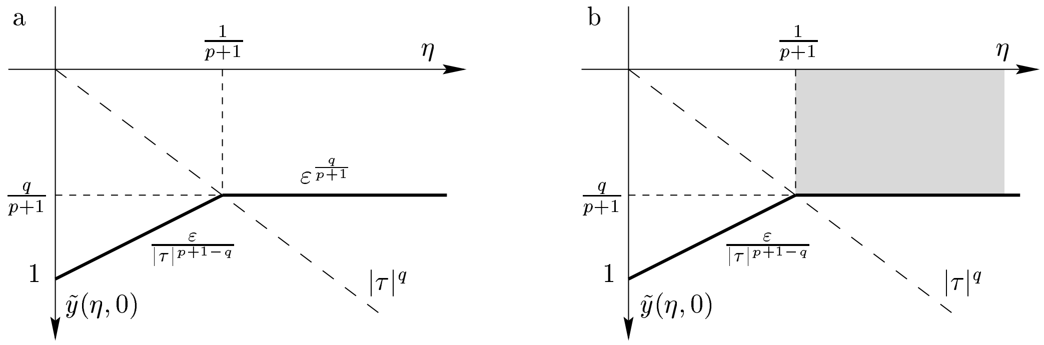

FIGURE 4.6. The graphic representation of the scaling index function allows to understand the scaling behaviour of the solutions at one glance. The direction of the axes is chosen in such a way as to preserve orientation. Thus, $\eta = 0$ corresponds to $\tau \approx -1$ and $\eta \to \infty$ represents $\tau \to 0$. $\tilde{y} = 0$ corresponds to $y \approx 1$ and $\tilde{y} > 0$ means that $y \approx \varepsilon^{\tilde{y}} \preccurlyeq 1$. (a) shows the scaling index for the solution of the linear equation (4.59), as well as equation (4.58) in the case where $f(x^\star(\tau) + y, \tau) < 0$ for positive y. Note that $q/p+1$ need not be smaller than 1. (b) If the hypothesis $f(x^\star(\tau)+y,\tau) < 0$ is not satisfied, the line $y = \varepsilon^{q/p+1}$ represents only a lower bound on the solution for $\tau \succcurlyeq \varepsilon^{1/p+1}$.

In other words, we have

$$z(\tau,\varepsilon) \approx \begin{cases} \varepsilon^{\frac{q}{p+1}} & \text{for } 0 \leqslant \tau \preccurlyeq \varepsilon^{\frac{1}{p+1}}, \\ \dfrac{\varepsilon}{\tau^{p+1-q}} & \text{for } \varepsilon^{\frac{1}{p+1}} \preccurlyeq \tau \leqslant \tau_0. \end{cases} \tag{4.63}$$

We give the proof in Appendix 4.A.4.

If we take y_0 of order ε (since we are looking at an adiabatic solution), it follows from (4.60) that the solution of the linear equation (4.59) obeys two different scaling regimes: for $-\tau_0 \leqslant \tau \preccurlyeq -\varepsilon^{1/p+1}$, it follows the power law $\varepsilon\tau^{q-p-1}$, whereas for $-\varepsilon^{1/p+1} \preccurlyeq \tau \leqslant 0$, it remains of order $\varepsilon^{q/p+1}$. We represent this evolution schematically in Fig. 4.6.

We are now going to prove that under appropriate conditions, the solutions of the general nonlinear equation (4.58) behave in a similar way.

Theorem 4.2. *Assume that for $-\tau_0 \leqslant \tau < 0$, $x^\star(\tau)$ is a tame stable bifurcation branch such that $x^\star(\tau) \approx |\tau|^q$. Then the equation $\varepsilon\dot{x} = f(x,\tau)$ admits a solution of the form $\bar{x}(\tau) = x^\star(\tau) + y$ with the following properties.*

1. *There exists a constant d such that*

$$y(\tau) \approx \frac{\varepsilon}{\tau^{p+1-q}} \qquad \text{for } -\tau_0 \leqslant \tau \leqslant -d\varepsilon^{\frac{1}{p+1}}. \tag{4.64}$$

2. *Either $y(\tau)$ diverges at $\tau = -d\varepsilon^{\frac{1}{p+1}}$, or*

$$y(\tau) \succcurlyeq \varepsilon^{\frac{q}{p+1}} \qquad \text{for } -d\varepsilon^{\frac{1}{p+1}} \leqslant \tau \leqslant 0. \tag{4.65}$$

3. *If $f(x^\star(\tau) + y, \tau) < 0$ for $0 < y < y_0$ and $-\tau_0 < \tau < 0$, then $y(\tau)$ exists and*

$$y(\tau) \approx \varepsilon^{\frac{q}{p+1}} \qquad \text{for } -d\varepsilon^{\frac{1}{p+1}} \leqslant \tau \leqslant 0. \tag{4.66}$$

The proof is given in Appendix 4.A.5.

This result distinguishes between an inner and an outer region around the bifurcation. The outer region, $-\tau_0 \leqslant \tau \leqslant -d\varepsilon^{1/p+1}$, is defined by the fact that the nonlinear term $b(y,\tau)$ is negligible with respect to the linear terms.[5] The solution follows a simple power law interpolating between the initial $\mathcal{O}(\varepsilon)$ and $\varepsilon^{q/p+1}$.

In the inner region, $-d\varepsilon^{1/p+1} \leqslant \tau \leqslant 0$, the solution is governed by the rescaled equation

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \tilde{a}(\sigma) + \tilde{b}(z,\sigma) + \tilde{w}(\sigma), \qquad z = \varepsilon^{-q/p+1}y,\ \sigma = \varepsilon^{-1/p+1}\tau, \tag{4.67}$$

and may depend on nonlinear terms. The condition that $f(x^\star(\tau)+y,\tau) < 0$ for $0 < y < y_0$ means that there is no unstable branch above $x^\star(\tau)$ taking part in the bifurcation. If this condition is violated, it may happen that the solution escapes the neighborhood of the bifurcation before $\tau = 0$ (see Example 4.6 below).

Remark 4.2. If the bifurcation is not tame, one can still show similar properties in a smaller outer region. Since we always have $b(y,\tau) = \mathcal{O}(y^2)$, the outer region includes at least the domain $|\tau| \succcurlyeq \varepsilon^{1/2p+1-q}$, which is defined by the condition $\varepsilon|\tau|^{q-p-1} \preccurlyeq |\tau|^p$.

Example 4.4 (Direct pitchfork bifurcation). Consider the equation

$$\varepsilon\dot{x} = (x - (-\tau)^m)(\tau - x^2). \tag{4.68}$$

Here we have $x^\star(\tau) = (-\tau)^m$, so that $q = m$ and $p = 1$, as we know from Example 4.2. Thus we have $\frac{1}{p+1} = \frac{1}{2}$ and $\frac{q}{p+1} = \frac{m}{2}$. Fig. 4.7 shows the behaviour of the solution in the generic case $m = 1$. Note however that if $m = 2$, the solution remains of order ε all the time, and for $m > 2$, it decreases when approaching the bifurcation.

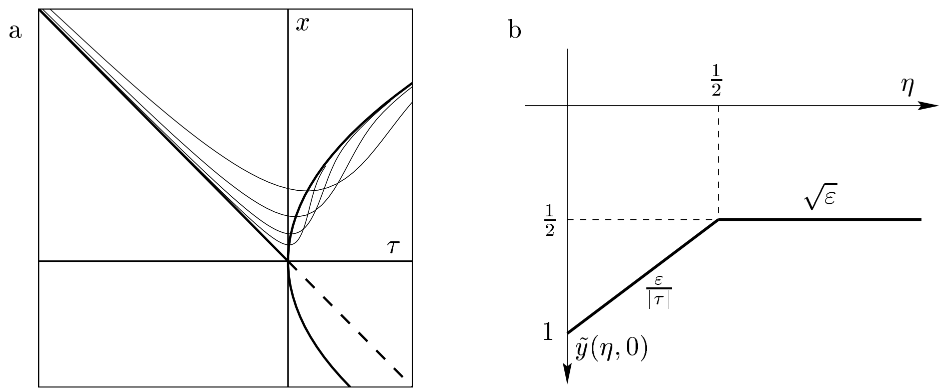

FIGURE 4.7. Adiabatic solutions for the equation $\varepsilon\dot{x} = (x+\tau)(\tau - x^2)$, see Example 4.4. (a) The orbits follow the branch $x^\star(\tau) = -\tau$ until the bifurcation. The four thin lines are the orbits obtained with $\varepsilon = 10^{-1/2}$, 10^{-1}, $10^{-3/2}$ and 10^{-2}. (b) The scaling index shows that $y = x - x^\star(\tau)$ grows as $\frac{\varepsilon}{|\tau|}$ for $\tau \leqslant -\sqrt{\varepsilon}$, and remains then of order $\sqrt{\varepsilon}$.

[5] The basic idea of the proof could be systematized: if the solution of $\varepsilon\dot{x} = f(x,\tau)$ is such that $r(x,\tau)$ is negligible compared to $f(x,\tau)$, then the solution of $\varepsilon\dot{y} = f(y,\tau) + r(y,\tau)$ with the same initial condition is of the same order than $x(\tau)$.

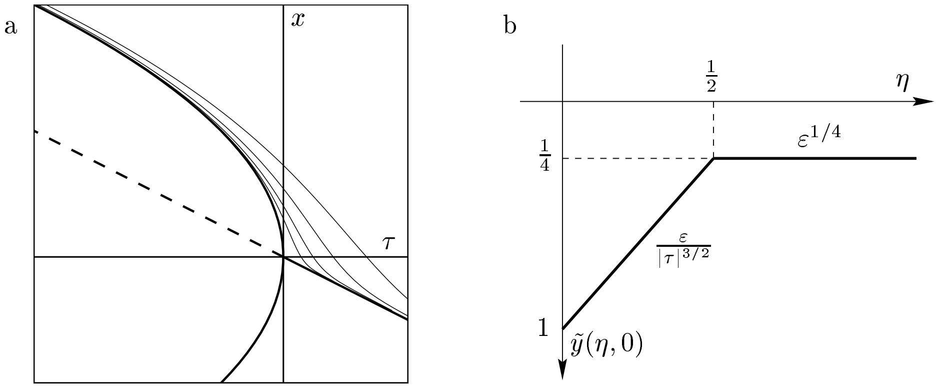

FIGURE 4.8. Same as Fig. 4.7, but for the equation $\varepsilon\dot{x} = -(\tau + 2x)(\tau + x^2)$, see Example 4.5. This time, y grows as $\frac{\varepsilon}{|\tau|^{3/2}}$ for $\tau \leqslant -\sqrt{\varepsilon}$, and remains then of order $\varepsilon^{1/4}$.

Example 4.5 (Indirect pitchfork bifurcation). Consider the equation

$$\varepsilon\dot{x} = -(\tau + 2x)(\tau + x^2). \tag{4.69}$$

There is a singular branch $x^\star(\tau) = \sqrt{|\tau|}$, for which $q = \frac{1}{2}$ and $p = 1$. This implies that $\frac{1}{p+1} = \frac{1}{2}$ and $\frac{q}{p+1} = \frac{1}{4}$, see Fig. 4.8.

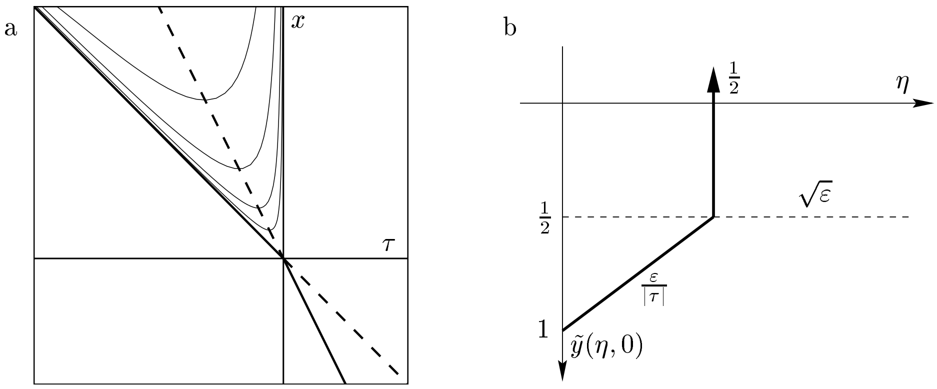

FIGURE 4.9. Same as Fig. 4.7, but for the equation $\varepsilon\dot{x} = (x + \tau)(x + 2\tau)$ of Example 4.6. The solutions follow the stable branch until $\tau \approx -\sqrt{\varepsilon}$, where they cross the unstable branch and escape to infinity.

Example 4.6 (Transcritical bifurcation with escaping trajectories). In the case of the equation

$$\varepsilon\dot{x} = (x + \tau)(x + 2\tau), \tag{4.70}$$

an unstable branch $x^\star(\tau) = -2\tau$ lies above the stable branch $x^\star(\tau) = -\tau$ with $q = p = 1$, so that the condition $f(x^\star(\tau) + y, \tau) > 0$ for positive y is violated. Here $\frac{1}{p+1} = \frac{q}{p+1} = \frac{1}{2}$. Putting $y = \sqrt{\varepsilon}z$ and $\tau = \sqrt{\varepsilon}\sigma$, we obtain the rescaled equation

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \sigma z + z^2 + 1, \tag{4.71}$$

which admits solutions escaping to infinity, see Fig. 4.9.

4.3.4 After the Bifurcation

We assume now that at the instant of the bifurcation $\tau = 0$, we have determined that the solution we are interested in behaves as $x(0) \approx \varepsilon^{\mu}$ for some positive μ. We would like to determine what happens to this solution after the bifurcation. There are two possibilities:

1. There is no bifurcation branch for positive times, as in the case of the indirect saddle–node bifurcation. We will discuss such a case in the next subsection, where we show that the vicinity of the bifurcation can also be separated into an inner region, where the solution does not yet react to the disappearance of the equilibrium, and an outer region where it quickly leaves the neighborhood of the bifurcation point.
2. There exist one or several equilibrium branches for positive times. In this case, we would like to determine whether the solution converges to an adiabatic state associated with one of those branches, or follows some other direction.

We will not present an exhaustive discussion of everything that can happen after a bifurcation, but rather give a general methodology, which we will illustrate on a number of generic examples in the next subsection. Note that merely by considering the sign of $f(x,\tau)$ between the different branches, it is already possible to deduce valuable information. Then we follow the procedure below:

1. For each unstable branch, determine, by inverting the direction of time and applying Theorem 4.2, the behaviour at $\tau = 0$ of the corresponding adiabatic solutions. These solutions act as boundaries between the basins of attraction of the stable branches.
2. Beginning from above, analyse the equation in the neighborhood of each equilibrium branch that the solution might follow. In particular, determine whether the solution remains above this branch, or if it crosses the branch. In the first case, we have found the branch which will be followed adiabatically. In the second case, we may have to repeat the procedure for other branches below the first one.

In the remainder of this subsection, we compute the scaling indices of the solutions of a few simple equations, which can be used as reference equations, in order to analyse the orbits in a vicinity of a branch of fixed points.

Let $\sigma_{1,2} = \pm 1$ and a_0, b_0, w_0 be positive constants. We consider in particular the linear equation

$$\varepsilon \dot{y} = \sigma_1 a_0 \tau^p y + \sigma_2 \varepsilon w_0 \tau^{q-1}, \tag{4.72}$$

which has the solution

$$y(\tau) = y(0)\, \mathrm{e}^{\sigma_1 c \tau^{p+1}/\varepsilon} + \sigma_2 w_0 \int_0^{\tau} s^{q-1}\, \mathrm{e}^{\sigma_1 c(\tau^{p+1} - s^{p+1})/\varepsilon}\, \mathrm{d}s, \tag{4.73}$$

where $c = a_0/(p+1)$. In order to bound the influence of certain nonlinear terms, it is also of interest to consider the Bernouilli equation

$$\varepsilon \dot{y} = \sigma_1 a_0 \tau^p y + \sigma_2 b_0 \tau^r y^m, \tag{4.74}$$

where $m \geqslant 2$, which has the solution

$$y(\tau) = y(0)\, \mathrm{e}^{\sigma_1 c \tau^{p+1}/\varepsilon} \left[1 - \sigma_2 y(0)^{m-1} b_0 (m-1) \frac{1}{\varepsilon} \int_0^{\tau} s^r\, \mathrm{e}^{\sigma_1 c(m-1) s^{p+1}/\varepsilon}\, \mathrm{d}s \right]^{-\frac{1}{m-1}}. \tag{4.75}$$

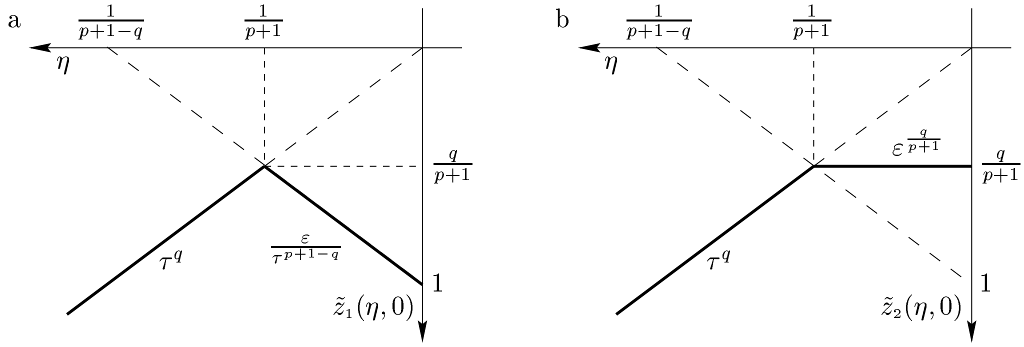

FIGURE 4.10. Scaling indices, in the limit $\varepsilon \to 0$, of the two functions in Lemma 4.5. The direction of the axes is again chosen in such a way as to preserve orientation; (a) scaling index of $z_1(\tau,\varepsilon)$ (see equation (4.76)), (b) scaling index of $z_2(\tau,\varepsilon)$ (see equation (4.78)).

Lemma 4.5. *Let c, p and q be strictly positive constants.*

1. *The scaling index of*

$$z_1(\tau,\varepsilon) = \int_0^\tau s^{q-1}\,\mathrm{e}^{c(s^{p+1}-\tau^{p+1})/\varepsilon}\,\mathrm{d}s \tag{4.76}$$

satisfies $\tilde{z}_1(\eta,\varepsilon) = \tilde{z}_1(\eta,0) + \mathcal{O}(1/\ln\varepsilon)$, *with*

$$\tilde{z}_1(\eta,0) = \begin{cases} q\eta & \text{for } \eta \geqslant \frac{1}{p+1}, \\ 1-(p+1-q)\eta & \text{for } 0 \leqslant \eta \leqslant \frac{1}{p+1}. \end{cases} \tag{4.77}$$

2. *The scaling index of*

$$z_2(\tau,\varepsilon) = \int_0^\tau s^{q-1}\,\mathrm{e}^{-cs^{p+1}/\varepsilon}\,\mathrm{d}s \tag{4.78}$$

satisfies $\tilde{z}_2(\eta,\varepsilon) = \tilde{z}_2(\eta,0) + \mathcal{O}(1/\ln\varepsilon)$, *with*

$$\tilde{z}_2(\eta,0) = \begin{cases} q\eta & \text{for } \eta \geqslant \frac{1}{p+1}, \\ \dfrac{q}{p+1} & \text{for } 0 \leqslant \eta \leqslant \frac{1}{p+1}. \end{cases} \tag{4.79}$$

PROOF: Let us denote $\frac{1}{p+1}$ by ν.

1. If $s \leqslant \tau \leqslant \varepsilon^\nu$, we have, $\mathrm{e}^{c(s^{p+1}-\tau^{p+1})/\varepsilon} \approx 1$, and thus $z_1(\tau,\varepsilon) \approx \int_0^\tau s^{q-1}\,\mathrm{d}s = q^{-1}\tau^q$. For $\tau \geqslant \varepsilon^\nu$, the proof is similar to that of Lemma 4.4, using the bounds $(p+1)\tau^p(s-\tau) \leqslant s^{p+1} - \tau^{p+1} \leqslant \tau^p(s-\tau)$.
2. If $\tau \leqslant \varepsilon^\nu$, we proceed as before. For $\tau \geqslant \varepsilon^\nu$, $z_2(\tau,\varepsilon) = z_2(\varepsilon^\nu,\varepsilon) + \int_{\varepsilon^\nu}^\tau s^{q-1}\,\mathrm{e}^{-cs^{p+1}/\varepsilon}\,\mathrm{d}s$, where the integral is shown to be of order $\varepsilon^{q\nu}$ using $s^{p+1}/\varepsilon \geqslant 1+(p+1)\varepsilon^{-\nu}(s-\varepsilon^\nu)$. □

Corollary 4.2. *The solutions of the linear equation* (4.72), *with any initial condition scaling as* $y(0) \approx \varepsilon^\mu$, *admit a scaling index* $\widetilde{|y|}(\eta,\varepsilon) = \widetilde{|y|}(\eta,0) + \mathcal{O}(1/\ln\varepsilon)$, *where* $\widetilde{|y|}(\eta,0)$ *depends on the signs* $\sigma_{1,2}$ *and is shown in Fig. 4.11.*

PROOF: If $\sigma_1 = -1$, then $y(\tau) = y(0)\,\mathrm{e}^{-c\tau^{p+1}/\varepsilon} + w_0 z_1(\tau,\varepsilon)$, and the result follows from Lemma 4.5 and Proposition 4.5. In particular, when $\sigma_2 = -1$, the solution changes sign as soon as $z_1 \approx \varepsilon^\mu\,\mathrm{e}^{-c\tau^{p+1}/\varepsilon}$. If $\sigma_1 = 1$, we can write $y(\tau) = \varepsilon^{c\tau^{p+1}/\varepsilon}[y(0) + w_0 z_2(\tau,\varepsilon)]$, and conclude in the same way. □

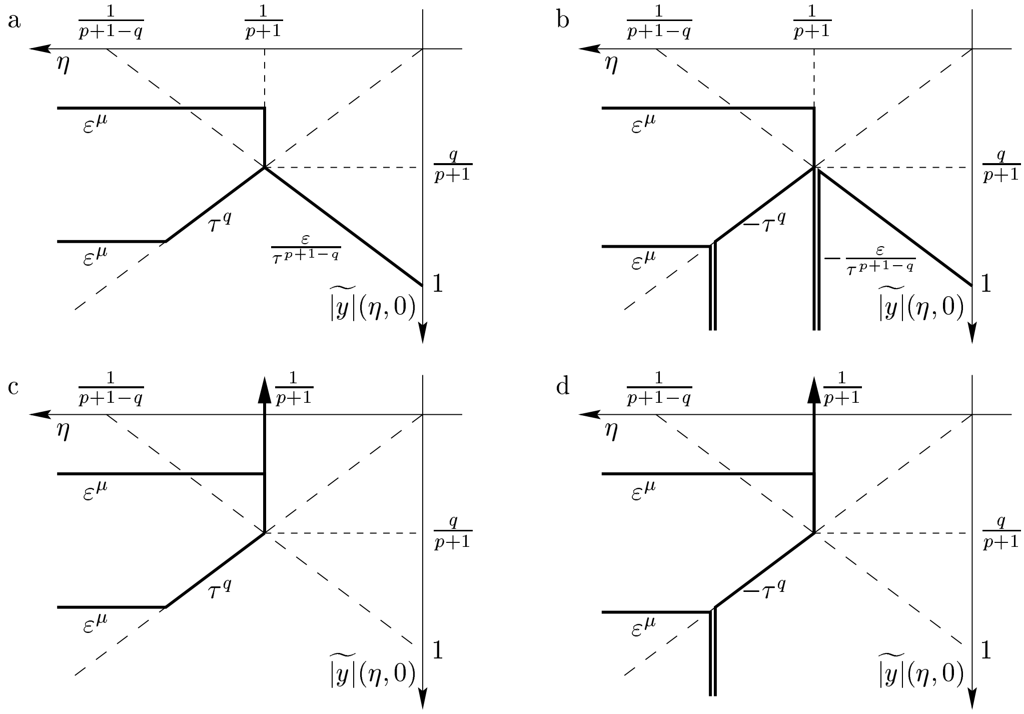

FIGURE 4.11. Scaling indices of the solutions of the linear equations in Corollary 4.2. We show the results obtained for two different values of μ (the order of the initial condition). Parallel vertical bars indicate that the solution changes sign, vertical arrows indicate that the solution grows exponentially. (a) Index associated with $\varepsilon\dot{y} = -a_0\tau^p y + \varepsilon w_0 \tau^{q-1}$, (b) index associated with $\varepsilon\dot{y} = -a_0\tau^p y - \varepsilon w_0\tau^{q-1}$, (c) with $\varepsilon\dot{y} = a_0\tau^p y + \varepsilon w_0\tau^{q-1}$ and (d) with $\varepsilon\dot{y} = a_0\tau^p y - \varepsilon w_0 \tau^{q-1}$.

Corollary 4.3. *The solutions of the Bernouilli equation* (4.74)*, with $y(0) \approx \varepsilon^\mu$, admit a scaling index $\tilde{y}(\eta,\varepsilon) = \tilde{y}(\eta,0) + \mathcal{O}(1/\ln\varepsilon)$, where $\tilde{y}(\eta,0)$ depends on the signs $\sigma_{1,2}$ and is shown in Fig. 4.12.*

PROOF: Let us denote again $\frac{1}{p+1}$ by ν.

1. If $\sigma_1 = \sigma_2 = -1$, we obtain from the solution (4.75) that for $\tau \leqslant \varepsilon^\nu$,
$$\tilde{y}(\eta,0) = \mu - \tfrac{1}{m-1}\min\bigl\{0, \mu(m-1) - 1 + (r+1)\eta\bigr\} = \max\bigl\{\mu, \tfrac{1-(r+1)\eta}{m-1}\bigr\}. \qquad (4.80)$$
For $\tau \geqslant \varepsilon^\nu$, the solution is exponentially small.
2. If $\sigma_1 = -\sigma_2 = -1$, the situation is similar, but the solution diverges if the denominator of (4.75) vanishes, which happens when $\varepsilon^{\mu(m-1)-1}\tau^{r+1} \approx 1$. This is only possible if $\mu \leqslant (p-r)/(p+1)(m-1)$, otherwise the solution decreases exponentially for $\tau \geqslant \varepsilon^\nu$.
3. If $\sigma_1 = 1$, the solution can be written as
$$y(\tau) = y(0)\Biggl[\mathrm{e}^{-(m-1)c\tau^{p+1}/\varepsilon} - \sigma_2 y(0)^{m-1} b_0 \tfrac{m-1}{\varepsilon}\int_0^\tau s^r\,\mathrm{e}^{c(m-1)(s^{p+1}-\tau^{p+1})/\varepsilon}\,\mathrm{d}s\Biggr]^{-\frac{1}{m-1}}.$$
If $\sigma_2 = -1$, we obtain the same result as in 1. for $\tau \leqslant \varepsilon^\nu$, whereas for $\tau \geqslant \varepsilon^\nu$,
$$\tilde{y}(\eta,0) = \mu - \tfrac{1}{m-1}\bigl[\mu(m-1) - 1 + 1 - (p-r)\eta\bigr] = \tfrac{p-r}{m-1}\eta. \qquad (4.81)$$
4. If $\sigma_1 = \sigma_2 = 1$, the solution diverges when $\varepsilon^{\mu(m-1)-1}\tau^{r+1} \approx \mathrm{e}^{-(m-1)c\tau^{p+1}/\varepsilon}$, which happens for $\tau \approx \min\{\varepsilon^{(1-\mu(m-1))/(r+1)}, \varepsilon^\nu\}$. □

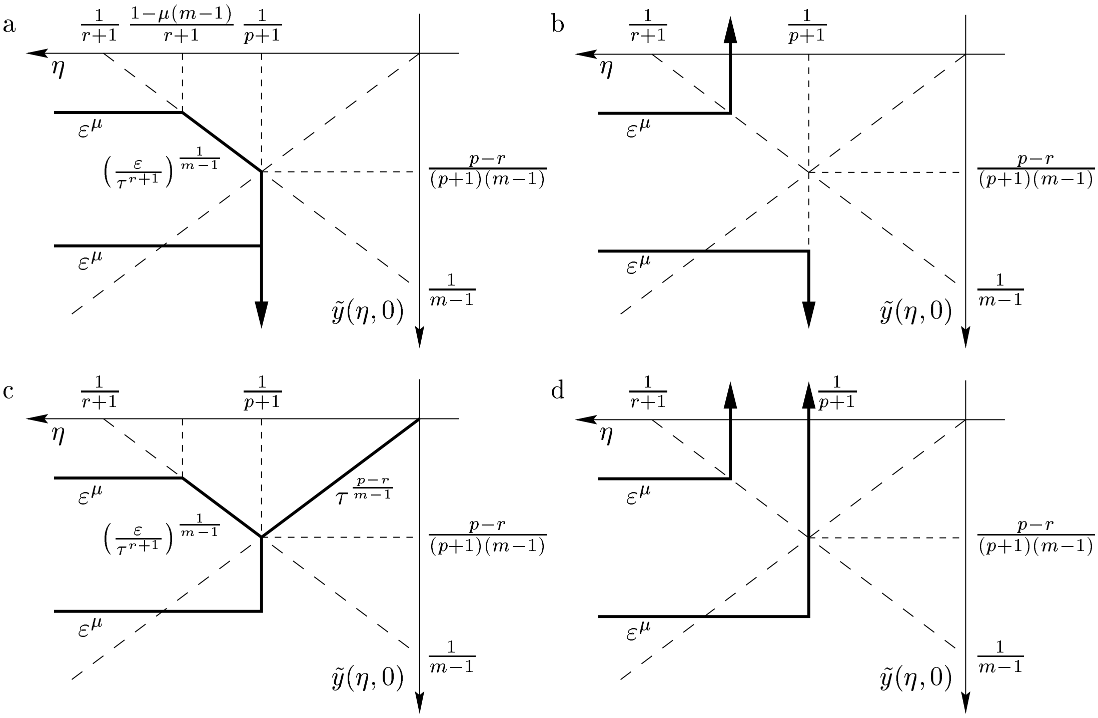

FIGURE 4.12. Scaling indices of the solutions of the Bernouilli equations in Corollary 4.3. We show the results obtained for two different values of μ (the order of the initial condition). Vertical arrows pointing upward indicate that the solution diverges, while vertical arrows pointing downward indicate that the solution decreases exponentially. (a) Index associated with $\varepsilon\dot{y} = -a_0\tau^p y - b_0\tau^r y^m$, (b) index associated with $\varepsilon\dot{y} = -a_0\tau^p y + b_0\tau^r y^m$, (c) with $\varepsilon\dot{y} = a_0\tau^p y - b_0\tau^r y^m$ and (d) with $\varepsilon\dot{y} = a_0\tau^p y + b_0\tau^r y^m$.

4.3.5 Examples

In this subsection, we illustrate the methods developed in the preceding subsections on a few examples. We discuss the most generic cases, namely saddle–node and pitchfork bifurcations, but the same procedure can be used to analyse less generic cases. As we pointed out, the behaviour of the orbits depends not only on the normal form of the bifurcation, but also on the way the equilibrium branches scale with time in the vicinity of the bifurcation point.

Example 4.7 (Indirect saddle–node bifurcation). Consider an equation of the form

$$\varepsilon\dot{x} = -\tau - x^2 + \mathcal{O}(x^3) + \mathcal{O}(\tau x) + \mathcal{O}(\tau^2). \tag{4.82}$$

Using the method of Newton's polygon, one readily shows that there exist two branches of fixed points for $\tau \leqslant 0$, a stable branch $x^\star(\tau) \approx \sqrt{|\tau|}$ and an unstable branch with the opposite sign. For positive times, there is no branch that the solutions could follow.

We begin by examining the behaviour of the adiabatic solution associated with the stable branch for $\tau \leqslant 0$. Following the procedure described in Subsection 4.3.3, we find that Theorem 4.2 can be applied, with $q = p = \frac{1}{2}$, so that $\frac{1}{p+1} = \frac{2}{3}$ and $\frac{q}{p+1} = \frac{1}{3}$. It follows that the $y = x - x^\star(\tau)$ scales as $\varepsilon/|\tau|$ in an outer region $\tau \leqslant -d\varepsilon^{2/3}$, and remains of order $\varepsilon^{1/3}$ in the inner region $-d\varepsilon^{2/3} \leqslant \tau \leqslant 0$, see Fig. 4.13b.

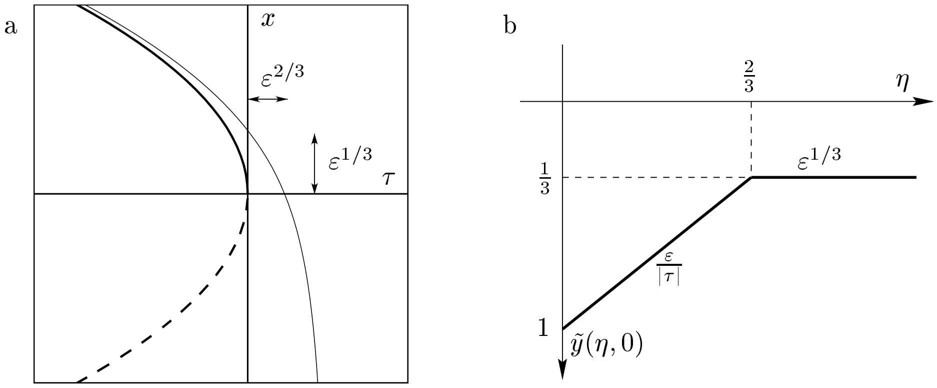

FIGURE 4.13. (a) Adiabatic solution associated with the equation $\varepsilon\dot{x} = -\tau - x^2$ of Example 4.7. (b) The scaling index shows that the distance to the stable branch for $\tau \leqslant 0$ first grows like $\varepsilon/|\tau|$, and then remains of order $\varepsilon^{1/3}$. For positive times, the solution does not react to the disappearance of the equilibria for a period of order $\varepsilon^{2/3}$, before diverging.

It turns out that this inner region can be continued to positive times. To see this, we use the rescaled variables $x = \varepsilon^{1/3} z$ and $\tau = \varepsilon^{2/3}\sigma$. In these variables, (4.82) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = -\sigma - z^2 + \text{higher order terms.} \tag{4.83}$$

Using the comparison Lemma, we can bound the solutions by those of the Ricatti equation $\mathrm{d}_\sigma z = -\sigma - z^2$ (with some constants in front of the terms which can be removed by scaling). This equation can be transformed into Airy's equation by the change of variables $z = \psi'/\psi$. The important fact is that z diverges for some σ of order 1.[6]

Going back to the initial equation, we find that $x \approx \varepsilon^{1/3}$ for $0 \leqslant \tau \leqslant d'\varepsilon^{2/3}$, and then x changes sign and quickly leaves the neighborhood of the bifurcation, as seen on Fig. 4.13a (its future then depends on the global structure of the equation).

Example 4.8 (Indirect pitchfork bifurcation). Consider an equation of the form

$$\varepsilon\dot{x} = -(x - \varphi(\tau))(\tau + x^2), \tag{4.84}$$

where $\varphi(\tau) \sim \tau^m$ for some $m \geqslant 1$. For $\tau \leqslant 0$, we have two stable branches $x^\star(\tau) = \pm\sqrt{|\tau|}$. We already examined the behaviour of the associated adiabatic solutions in Example 4.5, in particular we know that $\pm x(0) \approx \varepsilon^{1/4}$.

For positive times, there is a unique stable branch $x^\star(\tau) = \varphi(\tau)$. Just by sketching the vector field, it is clear that the solutions have to follow this branch. Let us examine in more detail the behaviour of $y = x - \varphi(\tau)$, which obeys the equation

$$\varepsilon\dot{y} = -(\tau + \varphi^2)y - (2\varphi + y)y^2 - \varepsilon\dot{\varphi}. \tag{4.85}$$

We consider the case where $y(0) \approx +\varepsilon^{1/4}$ and $\varphi(\tau)$ is decreasing. As an upper bound, we can compare (4.85) with the linear equation $\varepsilon\dot{y} = -\tau y + \varepsilon\tau^{m-1}$. Using Corollary 4.2 with $p = 1$ and $q = m$, we find that $y(\tau)$ remains of order $\varepsilon^{1/4}$ for a time of order $\sqrt{\varepsilon}$, and

[6]Note that the same result can be found without knowing Airy's equation, by comparing to the simple equation $\varepsilon\dot{x} = -x^2$. As long as $\tau \preccurlyeq \varepsilon^{2/3}$, it is negligible with respect to x^2.

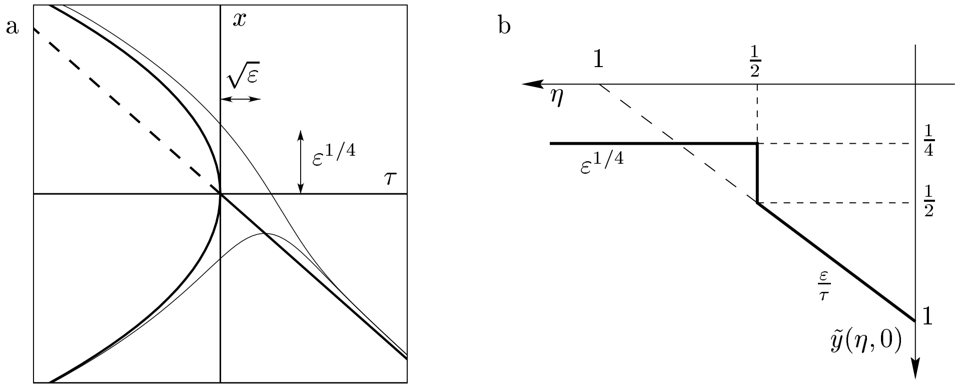

FIGURE 4.14. Solutions of $\varepsilon\dot{x} = -(x+\tau)(\tau+x^2)$, as discussed in Example 4.8. (a) Adiabatic solutions associated with the two stable equilibria are of order $\varepsilon^{1/4}$ at the instant of the bifurcation. After a time of order $\sqrt{\varepsilon}$, they converge to the new stable equilibrium branch. The scaling index of the distance between the upper solution and the equilibrium for $\tau > 0$ is shown in (b).

then scales as $\varepsilon\tau^{m-2}$. Then we can use this upper bound on y to show that the nonlinear terms are of smaller order than the linear terms in (4.85). For instance, in the generic case $m = 1$, we can write, for $\tau \succcurlyeq \sqrt{\varepsilon}$

$$\varepsilon\dot{y} = -(\tau + \tau^2 + \mathcal{O}(\varepsilon^{1/2})\tau)y - \varepsilon(1 + \mathcal{O}(\varepsilon^{1/2})), \tag{4.86}$$

while for $\tau \preccurlyeq \sqrt{\varepsilon}$, we can use the Bernouilli equation $\varepsilon\dot{y} = -\tau y - y^3$ as lower bound. The result is plotted in Fig. 4.14b.

For the adiabatic solution resulting from the lower branch (or if $\varphi(\tau)$ is increasing), we obtain a similar result with the difference that y changes sign at a time of order $\sqrt{\varepsilon}$, see Fig. 4.14a.

Example 4.9 (Direct pitchfork bifurcation). Let us study the system

$$\varepsilon\dot{x} = (x+\tau)(\tau - x^2). \tag{4.87}$$

We know that for negative time, we can associate an adiabatic solution with the unique stable branch, which behaves as $x(0) \approx \sqrt{\varepsilon}$. For positive times, let us first examine the adiabatic solution associated with the unstable equilibrium $x^\star(\tau) = -\tau$. Setting $x = -\tau - y$ and $\tau = -\sigma$, we obtain

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}\sigma} = (\sigma + \sigma^2)y - 2\sigma y^2 + y^3 + \varepsilon, \tag{4.88}$$

to which we can apply Theorem 4.2, with $p = q = 1$. This result tells us that $y(\sigma) \approx \varepsilon/|\sigma|$ for $|\sigma| \geqslant \sqrt{\varepsilon}$, and that $y(\sigma) \succcurlyeq \sqrt{\varepsilon}$ for $|\sigma| \leqslant \sqrt{\varepsilon}$. By rescaling the equation, we find that this solution diverges. Thus it is clear that the adiabatic state coming from the left has to follow the upper branch (this can already be found by sketching the vector field, see Fig. 4.15a).

Let us now write $x = \sqrt{\tau} + y$, giving

$$\varepsilon\dot{y} = -2(\tau + \tau^{3/2})y - (3\sqrt{\tau} + \tau)y^2 - y^3 - \frac{\varepsilon}{2\sqrt{\tau}}. \tag{4.89}$$

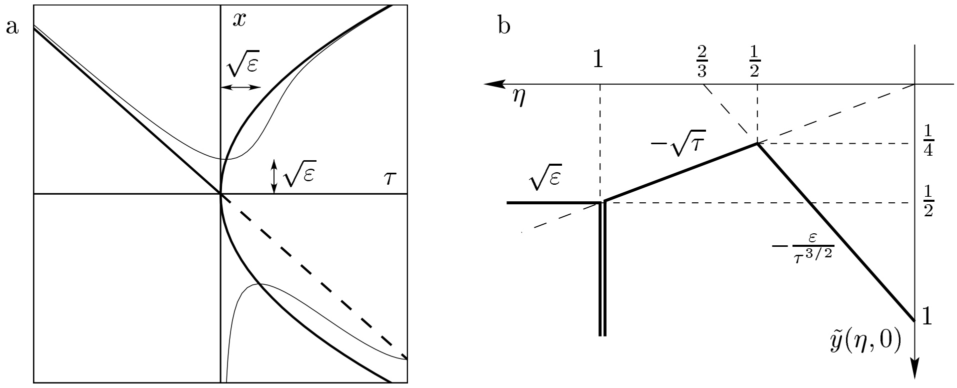

FIGURE 4.15. Solutions of the equation $\varepsilon\dot{x} = (x + \tau)(\tau - x^2)$ of Example 4.9. (a) The adiabatic solution associated with the branch which is stable before the bifurcation follows the upper branch after the bifurcation, while the solution associated with the unstable branch diverges. (b) Scaling index of $y = x - \sqrt{\tau}$ for positive τ.

Applying Corollary 4.2 to the linearized equation, we find that the solution remains of order $\sqrt{\varepsilon}$ for $\tau \preccurlyeq \varepsilon$, then it changes sign (i.e., the solution crosses the branch), and scales as $-\sqrt{\tau}$ until $\tau \approx \sqrt{\varepsilon}$. In fact, this simply means that $x = \sqrt{\tau} + y$ remains of order $\sqrt{\varepsilon}$. For larger times, the solution approaches the upper branch according to the law $y \approx -\varepsilon/|\tau|^{3/2}$ (Fig. 4.15b).

We now show that this behaviour is not modified by the nonlinear terms. There are three different regions:

- $\tau < \varepsilon$: As long as y is positive, the linearization provides an upper bound, yielding $y \preccurlyeq \sqrt{\varepsilon}$. But this implies that $3\sqrt{\tau}y^2 + y^3 \preccurlyeq \varepsilon^{3/2}$, which is smaller than the drift term $\varepsilon/2\sqrt{\tau}$. Thus the linearization also provides a lower bound.
- $\varepsilon < \tau < \sqrt{\varepsilon}$: The variable $z = -y$ satisfies the equation

$$\varepsilon\dot{z} = -2(\tau + \tau^{3/2})z + (3\sqrt{\tau} + \tau)z^2 - z^3 + \frac{\varepsilon}{2\sqrt{\tau}}. \tag{4.90}$$

  Since $\varepsilon\dot{z} \geqslant -3\tau z - z^3$, we obtain from Corollary 4.3 that $z \succcurlyeq \varepsilon^{1/2}$. But then we have $\varepsilon\dot{z} \leqslant -2\tau z + \varepsilon/2\sqrt{\tau} + \mathcal{O}(\sqrt{\tau}\varepsilon)$, which gives us the upper bound $z \leqslant \sqrt{\tau}$. This finally implies that $z^3 = \mathcal{O}(\tau^{3/2}) = \mathcal{O}(\varepsilon^2/\sqrt{\tau})$, and thus the term z^3 is bounded by the drift term.
- $\tau > \sqrt{\varepsilon}$: The argument is the same than in the proof of Theorem 4.2.

Example 4.10 (Direct pitchfork bifurcation). Let us slightly modify the previous example:

$$\varepsilon\dot{x} = (x - \tau^2)(\tau - x^2). \tag{4.91}$$

The only difference is that the regular branch is $x^\star(\tau) = \tau^2$ instead of $-\tau$. Writing $x = \tau^2 + y$, we get the equation

$$\varepsilon\dot{y} = (\tau - \tau^4)y - 2\tau^2y^2 - y^3 - 2\varepsilon\tau. \tag{4.92}$$

For $\tau \leqslant 0$, we obtain by Theorem 4.2 that the adiabatic solution remains above the branch, at a distance of order ε for all times. The difficulty is that the same is true for the adiabatic

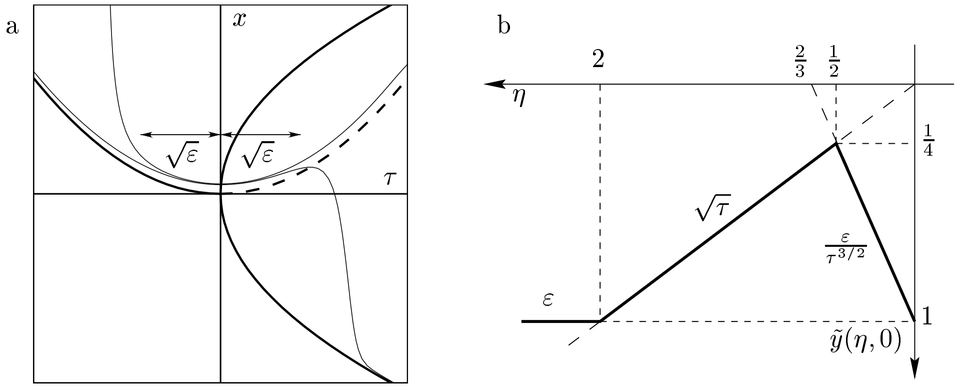

FIGURE 4.16. Solutions of the equation $\varepsilon\dot{x} = (x-\tau^2)(\tau - x^2)$ discussed in Example 4.10. (a) The adiabatic solution associated with the equilibrium $x^\star(\tau) = \tau^2$ remains of order ε until a time of order $\sqrt{\varepsilon}$ after the bifurcation. The adiabatic solution associated with the unstable equilibrium coming from the right behaves in a similar way, but remains above the first solution. (b) Scaling index of the distance between the adiabatic solution and the lower branch.

solution associated with the unstable branch for positive time. Since both solutions are of order ε at $\tau = 0$, it is not obvious which one is above the other.

This problem is due to the fact that there is almost a common adiabatic state for the two solutions, passing through the bifurcation. For times of order 1, we know the existence of an adiabatic solution

$$\bar{y}(\tau) = \frac{2\varepsilon}{1-\tau^3} \begin{cases} < 2\varepsilon & \text{if } \tau < 0 \\ > 2\varepsilon & \text{if } \tau > 0. \end{cases} \tag{4.93}$$

If we carry out the change of variables $y = 2\varepsilon + z$, we obtain that $\varepsilon\dot{z} = -8\varepsilon^2\tau^2 - 2\varepsilon\tau^4 - 8\varepsilon^3$ if $z = 0$. This term being negative, the adiabatic solution coming from the left must lie below the solution coming from the right, and will join the lower stable branch (Fig. 4.16a). By studying the evolution of $z = x + \sqrt{\tau}$, we obtain the scaling behaviour indicated in Fig. 4.16b.

4.3.6 Bifurcation Delay

Up to now, we have discussed in detail the situation of a branch $x^\star(\tau) \sim |\tau|^q$, bifurcating at $\tau = 0$. We found that although solutions may stay close to an unstable branch after a bifurcation, they are usually attracted by a stable branch after a time of order ε^ν (generically $\nu = \frac{1}{2}$).

The situation is very different when a fixed point is independent of time, so that there is no drift term. Such a situation may seem improbable, but the existence of a reflection symmetry, for instance, may cause the origin to be stationary for all time. In such a case, orbits may follow the unstable branch for a macroscopic time, i.e., a time which does not shrink to zero in the limit $\varepsilon \to 0$. This phenomenon of **bifurcation delay** has been extensively studied in the case of an analytic Hopf bifurcation [Ne1, Ne2, ME1], where the stationary solution is not required to be independent of time. We will discuss that case in the next section.

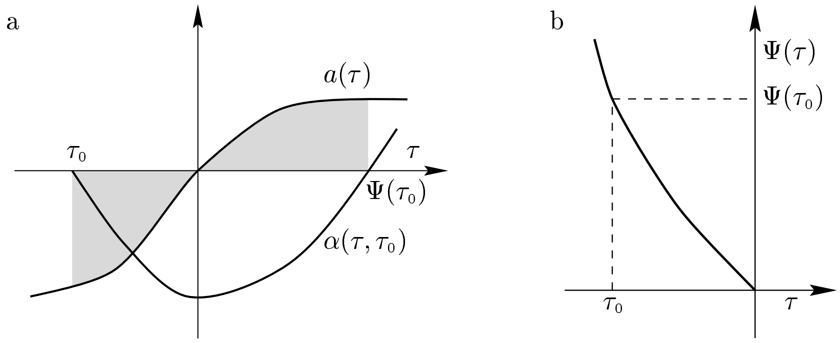

FIGURE 4.17. (a) Plots of a function $a(\tau)$ and its integral $\alpha(\tau,\tau_0)$, associated with a system showing bifurcation delay. The delay time $\Psi(\tau_0)$ is obtained by equating the two shaded areas. (b) The corresponding bifurcation delay function.

In this subsection, we examine the system $\varepsilon\dot{x} = f(x,\tau)$, where f need only be of class $\mathcal{C}^2$, but such that $f(0,\tau) = 0$ for all τ in some interval (τ_1,τ_2) containing 0. Then

$$\varepsilon\dot{x} = a(\tau)x + b(x,\tau), \tag{4.94}$$

where $b(x,\tau) = \mathcal{O}(x^2)$. We further assume that $a(\tau) < 0$ for $\tau_1 < \tau < 0$, and $a(\tau) > 0$ for $0 < \tau < \tau_2$, i.e., there is a bifurcation at $\tau = 0$ (for instance a pitchfork or a transcritical bifurcation).

Let us first analyse the linearized system

$$\varepsilon\dot{x} = a(\tau)x \qquad \Rightarrow \qquad x(\tau) = \mathrm{e}^{\alpha(\tau,\tau_0)/\varepsilon}\,x(\tau_0), \quad \alpha(\tau,\tau_0) := \int_{\tau_0}^{\tau} a(s)\,\mathrm{d}s, \tag{4.95}$$

where we assume that $\tau_1 < \tau_0 < 0$. Since $a(\tau)$ is negative for negative times, $\alpha(\tau,\tau_0)$ is decreasing until it reaches its minimum at the instant of the bifurcation. As long as α remains negative, and this is still true for some time after the bifurcation, $x(\tau)$ is exponentially small.

Definition 4.5. Let $\tau_0 \in (\tau_1, 0)$. The smallest positive solution $\Psi(\tau_0)$ of the equation

$$\alpha(\Psi(\tau_0),\tau_0) := \int_{\tau_0}^{\Psi(\tau_0)} a(s)\,\mathrm{d}s = 0 \tag{4.96}$$

is called **bifurcation delay time**, if it exists. If no such solution exists (i.e., if $\alpha(\tau_2,\tau_0) < 0$), we say by convention that the bifurcation delay time is infinite.

Is is easy to show that $\Psi(\tau)$ exists and is decreasing for negative τ sufficiently close to 0, with $\lim_{\tau\to 0_-}\Psi(\tau) = 0$. Moreover, differentiating (4.96), we get

$$\Psi'(\tau) = \frac{a(\tau)}{a(\Psi(\tau))}, \tag{4.97}$$

and the implicit function theorem implies $\lim_{\tau\to 0_-}\Psi'(\tau) = -1$. Geometrically, $\Psi(\tau)$ is obtained by equating two areas, see Fig. 4.17. If $\tau_2 = \infty$ and $a(\tau)$ decreases sufficiently rapidly, $\Psi(\tau)$ may diverge at a finite τ_0, such that $\alpha(0,\tau_0) = -\alpha(\infty,0)$.

Definition 4.5 implies that the solution of the linear equation (4.95), with initial condition $x(\tau_0) \sim 1$, is exponentially small for $\tau_0 < \tau < \Psi(\tau_0)$, and then increases exponentially fast. Similar properties hold for the nonlinear system (4.94), as we now show.

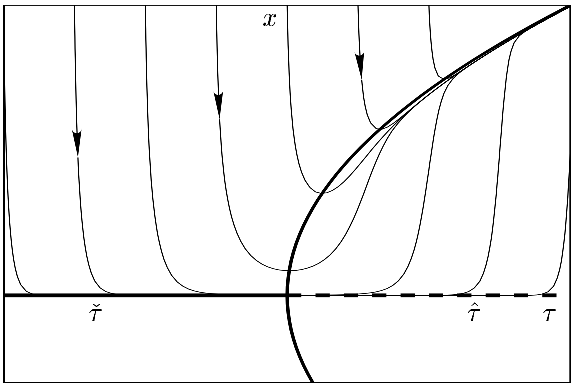

FIGURE 4.18. Solutions of the equation $\varepsilon\dot{x} = \tau x - x^3$ of Example 4.11 remain close to the origin for $\check{\tau} \leqslant \tau \leqslant \hat{\tau} = \Psi(\check{\tau}) = -\check{\tau}$, where $\check{\tau} = \tau_0 + \mathcal{O}(\varepsilon|\ln\varepsilon|)$.

Proposition 4.6. *There exist positive constants c_0 to c_3 such that the solution of* (4.94) *with $|x(\tau_0)| \leqslant c_0$ remains of order ε for $\tau_0 + c_1\varepsilon|\ln\varepsilon| \leqslant \tau \leqslant \Psi(\tau_0) - c_2\varepsilon|\ln\varepsilon|$. Furthermore, if $x(\tau_0) \sim 1$, then $|x(\tau)| > c_0$ for $\tau > \Psi(\tau_0) + c_3\varepsilon|\ln\varepsilon|$.*

PROOF: We may assume that $x(\tau_0) > 0$, otherwise we change x into $-x$. We know by Proposition 4.4 that $x(\tau) = \mathrm{e}^{\alpha(\tau,\tau_0)/\varepsilon}[x(\tau_0) + \mathcal{O}(x(\tau_0)^2)]$ for $\tau < 0$, provided $x(\tau_0)$ is sufficiently small. Thus, $x(\check{\tau}) = c\varepsilon$ and is decreasing at $\check{\tau} = \tau_0 + \mathcal{O}(\varepsilon|\ln\varepsilon|)$. By continuity, there exists a time $\hat{\tau} > \check{\tau}$ such that $x(\tau) \leqslant c\varepsilon$ for $\check{\tau} \leqslant \tau \leqslant \hat{\tau}$. If $|b(x,\tau)| \leqslant Mx^2$ for some $M > 0$, we have in this interval

$$[a(\tau) - Mc\varepsilon]x \leqslant \varepsilon\dot{x} \leqslant [a(\tau) + Mc\varepsilon]x, \tag{4.98}$$

$$c\varepsilon\,\mathrm{e}^{\alpha(\tau,\check{\tau})/\varepsilon - Mc(\tau-\check{\tau})} \leqslant x(\tau) \leqslant c\varepsilon\,\mathrm{e}^{\alpha(\tau,\check{\tau})/\varepsilon + Mc(\tau-\check{\tau})}. \tag{4.99}$$

Let $\bar{\tau}$ be the smallest positive solution of $\alpha(\tau,\check{\tau}) + \varepsilon Mc(\tau-\check{\tau}) = 0$. By the implicit function theorem, $\bar{\tau} = \Psi(\check{\tau}) + \mathcal{O}(\varepsilon) = \Psi(\tau_0) + \mathcal{O}(\varepsilon|\ln\varepsilon|)$. We have thus $c\,\mathrm{e}^{-2Mc(\bar{\tau}-\check{\tau})}\,\varepsilon \leqslant x(\bar{\tau}) \leqslant c\varepsilon$. Inverting the direction of time and using Proposition 4.4, we find that $x(\tau) = x(\tau_0)$ at $\tau = \bar{\tau} + \mathcal{O}(\varepsilon|\ln\varepsilon|)$. □

This result gives only very rough bounds, but can easily be used to compute better approximations. If we write $b(x,\tau) = x^2 r(x,\tau)$, we can use the fact that $x(\tau)$ is a fixed point of the operator

$$(Tx)(\tau) := x(\tau_0)\,\mathrm{e}^{\alpha(\tau,\tau_0)/\varepsilon}\left[1 - \frac{x(\tau_0)}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\alpha(s,\tau_0)/\varepsilon}\, r(x(s),s)\,\mathrm{d}s\right]^{-1}. \tag{4.100}$$

Example 4.11 (Pitchfork bifurcation). The equation

$$\varepsilon\dot{x} = \tau x - x^3 \tag{4.101}$$

satisfies our hypotheses, and the bifurcation delay time is simply $\Psi(\tau) = -\tau$. This is confirmed by the solution

$$x(\tau) = x(\tau_0)\,\mathrm{e}^{(\tau^2-\tau_0^2)/2\varepsilon}\left[1 + \frac{2x(\tau_0)^2}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{(s^2-\tau_0^2)/\varepsilon}\,\mathrm{d}s\right]^{-1/2}, \tag{4.102}$$

which is dominated by the numerator as long as $\tau < -\tau_0$ (see Fig. 4.18).

4.3.7 Concluding Remarks

Let us briefly summarize the method we have introduced to analyse the adiabatic solutions near a bifurcation.

1. Using Newton's polygon, determine the branches of fixed points, their stability and the associated numbers p and q determining the scaling exponents (see Fig. 4.4).
2. For each stable branch, use Theorem 4.2 to determine the behaviour of the associated adiabatic solution for $\tau \leqslant 0$.
3. Inverting time, do the same for the unstable branches for $\tau \geqslant 0$. These particular solutions delimit the basins of attraction of the different stable branches.
4. Using this information and the sign of $f(x,\tau)$, try to single out the branch followed after the bifurcation. Analyse the equation near each possible branch, using the reference equations of Corollaries 4.2 and 4.3.

This method allows to determine the solutions to leading order in ε, which is in general sufficient for our purposes (mainly, determining existence and shape of hysteresis cycles, and their scaling with ε). To obtain more precise information, we have to push further the analytic calculations. This can be done again by dividing the region before or after the bifurcation into two domains, depending on exponents ν and μ which have been determined above.

1. In the outer region $|\tau| \succcurlyeq \varepsilon^\nu$, it is possible to use Iterative Scheme 4.1 to compute an asymptotic series of the solution.
2. In the inner region $|\tau| \preccurlyeq \varepsilon^\nu$, one has to analyse the equation rescaled with $x = \varepsilon^\mu z$, $\tau = \varepsilon^\nu \sigma$.

Example 4.12. Let us examine more closely the adiabatic solution associated with the branch $x = -\tau$ in the equation

$$\varepsilon \dot{x} = (x+\tau)(\tau - x^2) \tag{4.103}$$

of Example 4.4. The change of variable $x = y - \tau$ gives

$$\varepsilon \dot{y} = a(\tau)y + 2\tau y^2 - y^3 + \varepsilon, \tag{4.104}$$

where $a(\tau) = \tau - \tau^2$. In the outer region $\tau \leqslant -d\sqrt{\varepsilon}$, application of Iterative Scheme 4.1 yields an asymptotic series of the form

$$y = -\frac{\varepsilon}{a(\tau)} + \mathcal{O}\left(\frac{\varepsilon^2}{\tau^3}\right) + \mathcal{O}\left(\frac{\varepsilon^3}{\tau^5}\right) + \cdots \tag{4.105}$$

When $\tau = -d\sqrt{\varepsilon}$, the terms of this series are of order $\sqrt{\varepsilon}/d$, $\sqrt{\varepsilon}/d^3$, $\sqrt{\varepsilon}/d^5$ and so on, which gives an asymptotic series in $1/d^2$.

In the inner region $-d\sqrt{\varepsilon} \leqslant \tau \leqslant 0$, the rescaling $x = \sqrt{\varepsilon} z$, $\tau = \sqrt{\varepsilon}\sigma$ gives

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \sigma z + \sigma^2 - \sqrt{\varepsilon}(\sigma z^2 + z^3), \tag{4.106}$$

which can be studied using more traditional methods of perturbation theory.

4.4 Bifurcations: Complex Case

In this section, we examine a few generalizations of the results given in the previous section to complex equations. We are mainly interested in two particular types of complex equations, because of their importance in the n–dimensional case or in physical applications.

The first type of equation is

$$\varepsilon \dot{z} = f(z, \tau), \qquad z \in \mathbb{C}. \tag{4.107}$$

It occurs either in physical situations which are naturally described by complex variables, like in quantum mechanics, or in the study of linear nD equations, as we will see in Section 5.3.

Another situation we will consider is that of a 2D system

$$\varepsilon \dot{x} = f(x, \tau), \qquad x \in \mathbb{R}^2. \tag{4.108}$$

Assume that f admits an equilibrium branch $x^\star(\tau)$, such that $A(\tau) := \partial_x f(x^\star(\tau), \tau)$ has complex conjugate eigenvalues $a(\tau) = \rho(\tau) + \mathrm{i}\,\omega(\tau)$ and $a(\tau)^*$. Then there exists a linear change of variables of the form $x = x^\star(\tau) + S(\tau)(z, z^*)$ transforming (4.108) into

$$\varepsilon \dot{z} = a(\tau) z + b(z, z^*, \tau, \varepsilon) + \varepsilon w(\tau), \qquad z \in \mathbb{C}, \tag{4.109}$$

where $b = \mathcal{O}(|z|^2) + \mathcal{O}(\varepsilon |z|)$.

We know by Theorem 4.1 that as long as $\rho(\tau)$ does not change sign, (4.109) admits adiabatic solutions of order ε. The case where $\rho(\tau)$ changes sign at $\tau = 0$, but with $\omega(0) \neq 0$ corresponds to Hopf bifurcation. It has been analysed in detail by Neishtadt [Ne1, Ne2, Ne3], and we state in Subsection 4.4.1 some of his results which are important for our future developments.

In Subsection 4.4.2, we consider equation (4.108) when the origin is a bifurcation point. We briefly discuss the differences between the complex and the real case, and analyse in particular an equation which will become important in the context of complex eigenvalue crossings in Subsection 5.3.6.

4.4.1 Hopf Bifurcation

We describe a two–dimensional Hopf bifurcation by equation (4.109), where we assume that $\rho(\tau)$ changes sign from negative to positive at $\tau = 0$, and $\omega(0) < 0$. In the analytic case, we know from Lemma 4.2 that the drift term can be made exponentially small. This fact is used in [Ne1] to prove existence of a bifurcation delay similar to the one encountered in Subsection 4.3.6. There are however differences with the pitchfork bifurcation:

1. the delay phenomenon subsists even with a drift term (i.e., it is not required that the equilibrium be independent of time);
2. a macroscopic delay time however requires analyticity of the equation;
3. there is a maximal value for the delay, called **buffer point**.

To prove these properties, [Ne2] uses a technique involving deformation of the integration path into the complex plane. This requires the introduction of a few geometrical definitions.

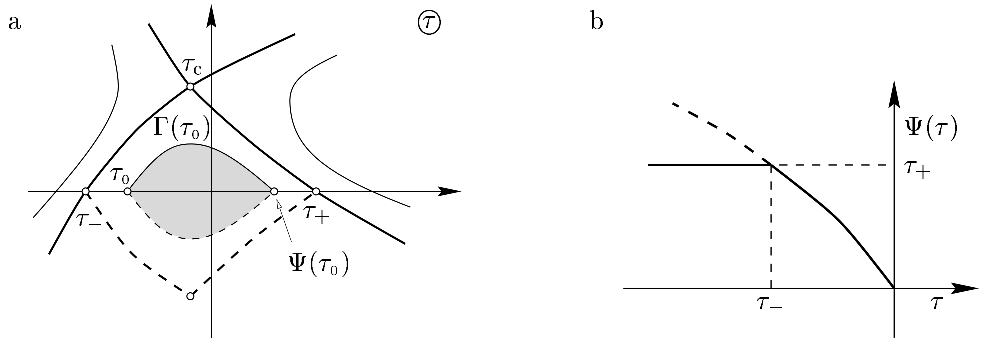

FIGURE 4.19. (a) Level lines of the function $\operatorname{Re}\alpha(\tau)$ in the complex τ–plane. The point τ_c is a zero of $a(\tau)$ and a saddle point of $\operatorname{Re}\alpha(\tau)$. Thick lines are Stokes lines, which define the largest possible path $\Gamma(\tau)$ of constant $\operatorname{Re}\alpha$, and the buffer points (τ_-, τ_+). The shaded area is $\mathcal{D}(\tau_0)$. (b) The bifurcation delay is given by $\min\{\Psi(\tau), \tau_+\}$.

Definition 4.6. Let

$$\alpha(\tau) := \int_0^\tau a(s)\,\mathrm{d}s. \tag{4.110}$$

For $\tau_0 < 0$, we define as is Definition 4.5, the **delay time** $\Psi(\tau_0)$ as the smallest positive solution of $\operatorname{Re}\alpha(\Psi(\tau_0)) = \operatorname{Re}\alpha(\tau_0)$. The function $\Psi(\tau_0)$ is defined for $\tau_0 \in (\tau_1, 0]$.

Let us now continue analytically the solutions of (4.109) and the function $\Psi(\tau)$ to a complex neighborhood of $\tau = 0$. For sufficiently small $|\tau_0|$, the real times τ_0 and $\Psi(\tau_0)$ can be connected by a path $\Gamma(\tau_0)$ in the upper half plane, on which $\operatorname{Re}\alpha(\tau)$ is constant (Fig. 4.19a). Let $\mathcal{D}(\tau_0)$ be the domain bounded by $\Gamma(\tau_0)$ and its conjugate $\Gamma(\tau_0)^*$. Let τ_- be the smallest real time such that

1. $\Gamma(\tau_-)$ exists, and has nowhere a vertical tangent,

and for every $\tau \in \mathcal{D}(\tau_-)$,

2. equation (4.109) as analytic at $z = 0$,
3. $a(\tau) \neq 0$,
4. $a(\tau)^* \neq a(\tau)$.

Then τ_- is called **negative buffer point**, and

$$\tau_+ := \sup_{\tau_- < \tau < 0} \Psi(\tau) \tag{4.111}$$

is called **positive buffer point**.

A major limitation to the existence of the path $\Gamma(\tau_0)$ comes from the fact that $a(\tau)$ may vanish for some complex τ_c. In such a case, τ_c is a saddle point of $\operatorname{Re}\alpha(\tau)$. The lines $\operatorname{Re}\alpha = \text{constant}$ going through τ_c are called **Stokes lines**, and their intersections with the real axis define the buffer points $\tau_\pm$ (Fig. 4.19a). For a pictorial interpretation of this phenomenon, see Diener & Diener in [**Ben**].

Theorem 4.3. *Assume* (4.109) *admits finite buffer points* (τ_-, τ_+). *There exist constants* c_0 *to* c_3 *such that*

1. *If* $\tau_- < \tau_0 < 0$ *and* $|z(\tau_0)| \leqslant c_0$, *then* $|z(\tau)| \leqslant c_1^{-1}\varepsilon$ *in the domain* $\tau_0 + c_2\varepsilon|\ln\varepsilon| \leqslant \tau \leqslant \Psi(\tau_0) - c_3\varepsilon|\ln\varepsilon|$.

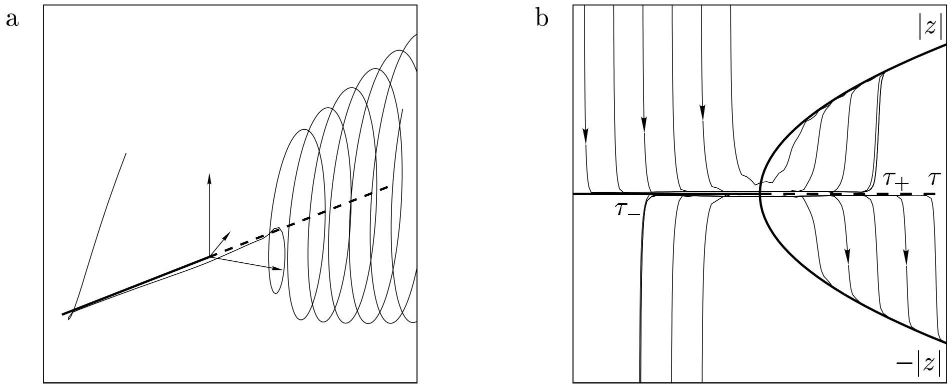

FIGURE 4.20. (a) Trajectory of a dynamical Hopf bifurcation. The unstable equilibrium is followed for some macroscopic time after the bifurcation, before the onset of oscillations. (b) Evolution of the distance to the equilibrium for different solutions. For clarity, we plot $|z|$ for some solutions, and $-|z|$ for some others. Trajectories reaching the equilibrium at a time τ_0 between the buffer point τ_- and the bifurcation time leave it again at $\Psi(\tau_0)$. Most trajectories reaching the equilibrium before τ_- leave it again at the positive buffer point τ_+. One can, however, construct trajectories following the unstable equilibrium for arbitrary time. These solutions reach the branch at the negative buffer point τ_-.

2. *If $\tau_0 < \tau_-$ and $|z(\tau_0)| \leqslant c_0$, then $|z(\tau)| \leqslant c_1^{-1}\varepsilon$ in the domain* $\tau_0 + c_2\varepsilon|\ln\varepsilon| \leqslant \tau \leqslant \tau_+ - c_3\varepsilon|\ln\varepsilon|$.

The proof [Ne2] is mainly based on deformation of the integration path to the curve $\Gamma(\tau_0)$ or $\Gamma(\tau_-)$, on which the linear part of (4.109) is $\varepsilon\dot{z} = \mathrm{i}\,\omega z$. This result shows that the bifurcation delay is bounded below by $\min\{\Psi(\tau_0), \tau_+\}$. [Ne3] and Diener & Diener in [**Ben**] also give upper bounds on the delay time, under more restrictive hypotheses.

Remark 4.3. We have only stated a simplified version of Neishtadt's theorem, which applies in fact to n–dimensional slow–fast systems. If the equation is only $\mathcal{C}^k$, it is shown in [Ne1] that generically, the delay is only of order $\sqrt{(k-3/2)\varepsilon|\ln\varepsilon|}$.

The important new phenomenon is the existence of a **buffer** or **maximal delay** τ_+ (see Fig. 4.19b and Fig. 4.20b). Note however that by inverting time, we can construct solutions which remain close to the origin for all $\tau \geqslant \tau_-$.

Example 4.13. If $a(\tau) = \tau - \mathrm{i}$, then $\alpha(\tau) = \frac{1}{2}\tau^2 - \mathrm{i}\,\tau$, and the level lines of

$$\operatorname{Re}\alpha(\tau) = \frac{1}{2}\big[(\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau - 1)^2 + 1\big] \tag{4.112}$$

are hyperbolas centered at $\tau_c = \mathrm{i}$. The Stokes lines have equations $\operatorname{Im}\tau = \mathrm{i} \pm \operatorname{Re}\tau$, and the buffer points are $\tau_\pm = \pm 1$, so that the delay time is given by $\min\{-\tau_0, 1\}$.

4.4.2 Bifurcations with Zero Eigenvalue

Let us briefly discuss the case of the complex equation

$$\varepsilon\dot{z} = f(z,\tau) \tag{4.113}$$

admitting $(0,0)$ as a bifurcation point. The discussion of Subsection 4.3.2, on the relation between Newton's polygon and the scaling exponents associated with equilibrium branches, still applies here. The only difference is that the number of equilibrium branches may be larger. For instance, the function $f(z,\tau) = -z^2 - \tau^2$ admits the branches $z = \pm\sqrt{-\tau}$ for negative τ, but also the branches $z = \pm\,\mathrm{i}\sqrt{\tau}$ for positive τ.

The motion near a tame bifurcation branch $z^\star(\tau) \sim |\tau|^q$ is described by the equation

$$\varepsilon\dot{y} = a(\tau)y + b(y,\tau) + \varepsilon w(\tau), \tag{4.114}$$

where $a(\tau) \sim |\tau|^p$, p is given by Newton's polygon, $w(\tau) \sim |\tau|^{q-1}$, and $b(y,\tau)$ satisfies Lemma 4.3. In polar coordinates $y = r\,\mathrm{e}^{\mathrm{i}\,\varphi}$, it becomes

$$\begin{aligned} \varepsilon\dot{r} &= \operatorname{Re} a(\tau) r + \cos\varphi \operatorname{Re} b(r\,\mathrm{e}^{\mathrm{i}\,\varphi},\tau) + \varepsilon\cos\varphi\operatorname{Re} w(\tau) \\ \varepsilon\dot{\varphi} &= \operatorname{Im} a(\tau) - \sin\varphi\frac{1}{r}\operatorname{Im} b(r\,\mathrm{e}^{\mathrm{i}\,\varphi},\tau) - \varepsilon\sin\varphi\frac{1}{r}\operatorname{Im} w(\tau). \end{aligned} \tag{4.115}$$

We have thus $\varepsilon\dot{r} \leqslant \operatorname{Re} a(\tau)r + |b(r\,\mathrm{e}^{\mathrm{i}\,\varphi},\tau)| + \varepsilon|w(\tau)|$, and the results of Section 4.3 give *upper* bounds on $r(\tau)$ when $\operatorname{Re} a(\tau) \sim |\tau|^p$. Due to the oscillatory behaviour of the solutions, lower bounds are more difficult to obtain.

Let us now consider the case when $a(\tau) = \mathrm{i}\,\omega(\tau)$ is imaginary. Let us first give a generalization of Lemma 4.4 on the behaviour of solutions of the linear system $\varepsilon\dot{y} = \mathrm{i}\,\omega(\tau)y + \varepsilon w(\tau)$.

Lemma 4.6. *Let $p, q, \tau_0, w_0 > 0$ be constants. Let $\omega(\tau)$ be a continuous real–valued function on $(0,\tau_0]$ such that $\omega(\tau) \sim |\tau|^p$, and let $w(\tau)$ be a differentiable complex–valued function on $(0,\tau_0]$ such that $|w(\tau)| \leqslant w_0|\tau|^{q-1}$ and $|\dot{w}(\tau)| \leqslant w_0|\tau|^{q-2}$. Consider the integral*

$$z(\tau,\varepsilon) = \int_\tau^{\tau_0} \mathrm{e}^{\mathrm{i}[\varphi(\tau)-\varphi(\tau_0)]/\varepsilon}\, w(s)\,\mathrm{d}s, \qquad \varphi(\tau) := \int_0^\tau \omega(s)\,\mathrm{d}s. \tag{4.116}$$

Then there exists a constant $c > 0$, depending only on $\omega(\tau)$, such that

$$\text{If } q < p+1, \qquad |z(\tau,\varepsilon)| \leqslant \begin{cases} cw_0\varepsilon^{q/p+1} & \text{if } \tau \leqslant \varepsilon^{1/p+1}, \\ cw_0\varepsilon\tau^{q-p-1} & \text{if } \tau \geqslant \varepsilon^{1/p+1}. \end{cases} \tag{4.117a}$$

$$\text{If } q = p+1, \qquad |z(\tau,\varepsilon)| \leqslant \begin{cases} cw_0\varepsilon|\ln\varepsilon| & \text{if } \tau \leqslant \varepsilon^{1/p+1}, \\ cw_0\varepsilon|\ln\tau| & \text{if } \tau \geqslant \varepsilon^{1/p+1}. \end{cases} \tag{4.117b}$$

$$\text{If } q > p+1, \qquad |z(\tau,\varepsilon)| \leqslant cw_0\varepsilon. \tag{4.117c}$$

Proof: For $\tau \geqslant \varepsilon^{1/p+1}$, the result is obtained by integration by parts. For $\tau \leqslant \varepsilon^{1/p+1}$, one can use the change of variables $\xi = \varphi(s)/\varepsilon$ to bound $z(\tau,\varepsilon) - z(\varepsilon^{1/p+1},\varepsilon)$. □

It is more difficult to control the influence of nonlinear terms. We will do it in a particular case, that we will need when studying eigenvalue crossings in Subsection 5.3.6.

Example 4.14. Consider the equation

$$\varepsilon\dot{y} = \mathrm{i}\,\omega(\tau)y + \varepsilon\rho(\tau)y + \varepsilon b(\tau)y^2 + \varepsilon w(\tau), \tag{4.118}$$

where $\omega(\tau)$, $\rho(\tau)$ and $b(\tau)$ are differentiable in a neighborhood of $\tau = 0$, $w(\tau)$ is twice differentiable, and $\omega(\tau) \sim |\tau|$ is real–valued, as well as $\rho(\tau)$.

We divide the motion into three steps:

1. $\tau_0 \leqslant \tau \leqslant -d\sqrt{\varepsilon}$:
   We first construct a particular solution, as fixed point of the operator

$$(Ty)(\tau) := \mathrm{i}\,\varepsilon \frac{w(\tau_0)}{\omega(\tau_0)} \mathrm{e}^{\mathrm{i}[\varphi(\tau)-\varphi(\tau_0)]/\varepsilon} + \mathrm{e}^{\mathrm{i}\,\varphi(\tau)/\varepsilon} \int_{\tau_0}^{\tau} \mathrm{e}^{-\mathrm{i}\,\varphi(s)/\varepsilon} \big[\rho(s)y(s) + b(s)y(s)^2 + w(s)\big]\,\mathrm{d}s, \qquad (4.119)$$

   where $\varphi(\tau)$ is given in (4.116). On the space of differentiable functions on $[\tau_0, -d\sqrt{\varepsilon}]$, we introduce the norm

$$\|y\| := \sup_{\tau_0 \leqslant \tau \leqslant -d\sqrt{\varepsilon}} \left| y(\tau) - \mathrm{i}\,\varepsilon \frac{w(\tau)}{\omega(\tau)} \right| \frac{|\tau|^3}{\varepsilon^2} + \sup_{\tau_0 \leqslant \tau \leqslant -d\sqrt{\varepsilon}} |\dot{y}| \frac{|\tau|^2}{\varepsilon}. \qquad (4.120)$$

   Is is easy to show that T is a contraction on the space of differentiable functions with $\|y\| \leqslant c$, for sufficiently small c, ε, τ_0 and d^{-1}. We conclude that T admits a fixed point,

$$\bar{y}(\tau) = \mathrm{i}\,\varepsilon \frac{w(\tau)}{\omega(\tau)} + \mathcal{O}\left(\frac{\varepsilon^2}{\tau^3}\right), \qquad (4.121)$$

   which is a particular solution of (4.118).
   To construct the general solution, we write $y(\tau) = \bar{y}(\tau) + z(\tau)$. Then $|\dot{z}| \leqslant M(|z| + |z|^2)$, which shows that solutions starting sufficiently close to $\bar{y}$ remain close.
2. $-d\sqrt{\varepsilon} \leqslant \tau \leqslant d\sqrt{\varepsilon}$:
   Let us carry out the rescaling $y = \sqrt{\varepsilon} z$, $\tau = \sqrt{\varepsilon}\sigma$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \mathrm{i}\,\tilde{\omega}(\sigma)z + \sqrt{\varepsilon}\tilde{\rho}(\sigma)z + \varepsilon\tilde{b}(\sigma)z^2 + \tilde{w}(\sigma), \qquad (4.122)$$

   where

$$\begin{aligned} \tilde{\omega}(\sigma) &= \varepsilon^{-1/2}\omega(\sqrt{\varepsilon}\sigma) = \omega_1\sigma + \mathcal{O}(\sqrt{\varepsilon}\sigma^2), \\ \tilde{\rho}(\sigma) &= \rho(\sqrt{\varepsilon}\sigma) = \rho_0 + \mathcal{O}(\sqrt{\varepsilon}\sigma), \\ \tilde{b}(\sigma) &= b(\sqrt{\varepsilon}\sigma) = b_0 + \mathcal{O}(\sqrt{\varepsilon}\sigma), \\ \tilde{w}(\sigma) &= w(\sqrt{\varepsilon}\sigma) = w_0 + \mathcal{O}(\sqrt{\varepsilon}\sigma). \end{aligned} \qquad (4.123)$$

   One easily shows that (4.122) admits a solution

$$z(\sigma) = \mathrm{i}\,\frac{w_0}{\omega_1} + \left(z(-d) - \mathrm{i}\,\frac{w_0}{\omega_1}\right) \mathrm{e}^{\mathrm{i}\,\omega_1(\sigma+d)} + \mathcal{O}(\sqrt{\varepsilon}). \qquad (4.124)$$

3. $\tau \geqslant d\sqrt{\varepsilon}$:
   We can construct a particular solution as in point 1.

We conclude from this analysis that any solution with initial condition $y(\tau_0) = \mathcal{O}(\varepsilon)$ will first grow as $\varepsilon/|\tau|$ until $\tau = -d\sqrt{\varepsilon}$. Note that the particular solution (4.121) does not oscillate, while the general solution oscillates more and more slowly. Then it will remain of order $\sqrt{\varepsilon}$, while it rotates by an angle $2d\omega_1$. For $\tau \geqslant d\sqrt{\varepsilon}$, although there exists a particular solution decreasing as $\varepsilon/|\tau|$, the orbit we have constructed has no reason to match this particular solution. Thus it will in general remain of order $\sqrt{\varepsilon}$, while it starts oscillating faster and faster, see Fig. 5.6.

4.5 Global Properties of the Flow

Up to now, we have studied solutions remaining in a neighborhood of equilibrium branches. These are the most important solutions to determine the global structure of the flow. We also need, however, to characterize those solutions which start far away from an equilibrium. We will show that in general, such orbits join some adiabatic solution after a very short time.

In this section, we consider again the real system $\varepsilon\dot{x} = f(x,\tau)$, where f is of class $\mathcal{C}^2$. In Subsection 4.5.1, we examine the relaxation properties of general solutions to stable equilibria. In Subsection 4.5.2, we then analyse the consequences of the previously gathered information on the global structure of the flow.

4.5.1 Approach to Equilibrium

Assume that $x^\star(\tau)$ is a stable equilibrium branch in a neighborhood of $\tau = \tau_0$. This means that $f(x^\star(\tau) + y, \tau)$ must be negative for small positive y, and positive for small negative y. A general method for estimating the relaxation time of a solution to the branch $x^\star(\tau)$ proceeds as follows.

If $v := |x - x^\star(\tau)|$ and $w(\tau) := -\dot{x}^\star(\tau)$, we obtain

$$\varepsilon\dot{v} = \begin{cases} f(x^\star(\tau) + v, \tau) + \varepsilon w(\tau) & \text{if } v > 0, \\ -f(x^\star(\tau) - v, \tau) - \varepsilon w(\tau) & \text{if } v < 0. \end{cases} \tag{4.125}$$

If we know a uniform bound in time $f_-(v) \leqslant \varepsilon\dot{v} \leqslant f_+(v) < 0$, we obtain the estimation

$$\varepsilon \int_{v(\tau)}^{v_0} \frac{\mathrm{d}v}{-f_-(v)} \leqslant \tau - \tau_0 \leqslant \varepsilon \int_{v(\tau)}^{v_0} \frac{\mathrm{d}v}{-f_+(v)} \tag{4.126}$$

on the time necessary for the solution with initial condition $v(\tau_0) = v_0$ to reach $v(\tau)$. This bound is at the base of the following result.

Proposition 4.7. *Let $g(v,\tau)$ be a function of class $\mathcal{C}^2$ in a neighborhood of $(0,\tau_0)$, such that $\partial_v g(0,\tau_0) = -a_0 < 0$, $g(0,\tau) = 0$ and $g(v,\tau) < 0$ for $\tau_0 \leqslant \tau \leqslant \tau_1$ and $0 < v \leqslant v^\star$. Let $w(\tau)$ be of class $\mathcal{C}^1$ in a neighborhood of τ_0. Then any solution of*

$$\varepsilon\dot{v} = g(v,\tau) + \varepsilon w(\tau), \tag{4.127}$$

with initial condition $v(\tau_0) = v_0 \in (0, v^\star)$ (where v_0 is independent of ε), will reach the $\mathcal{O}(\varepsilon)$–neighborhood of the origin at the time

$$\tau^\star(\varepsilon) = \tau_0 + \frac{1}{a_0}\varepsilon|\ln\varepsilon| + \mathcal{O}(\varepsilon). \tag{4.128}$$

Proof: Let $\delta(\varepsilon) := 2\varepsilon|\ln\varepsilon|/a_0$. Using a Taylor series to second order,[7] we find that for $\tau_0 \leqslant \tau \leqslant \tau_0 + \delta(\varepsilon)$ and $0 \leqslant v \leqslant v^\star$,

$$g(v,\tau) \leqslant -a_0 v + M_1 v\delta(\varepsilon) + M_2 v^2, \tag{4.129}$$

[7]If $v^\star$ is not in the neighborhood of the origin where $g \in \mathcal{C}^2$, the solution will reach this neighborhood in a time of order ε, see Remark 4.5 below.

where M_1 and M_2 are positive bounds on second derivatives of g. We further assume that $|w(\tau)| \leqslant w_0$ on the same domain. Then it is straightforward to check that, if ε is so small that $|\ln\varepsilon| \geqslant w_0 M_2/M_1$ and $\varepsilon|\ln\varepsilon| \leqslant a_0^2/8M_1$, (4.127) can be bounded by the parabola

$$\varepsilon\dot{v} \leqslant \Big(v - 2\frac{w_0}{a_0}\varepsilon\Big)\Big(M_2 v - a_0 + 2M_1\delta(\varepsilon)\Big). \tag{4.130}$$

If $c_0 > 2w_0/a_0$, this implies by (4.126) that $v(\tau) = c_0\varepsilon$ at a time $\tau^\star(\varepsilon)$ such that

$$\tau^\star(\varepsilon) - \tau_0 \leqslant \varepsilon \int_{c_0\varepsilon}^{v_0} \frac{\mathrm{d}v}{\big(v - 2\frac{w_0}{a_0}\varepsilon\big)\big(M_2 v - a_0 + 2M_1\delta(\varepsilon)\big)} = \frac{1}{a_0}\varepsilon|\ln\varepsilon| + \mathcal{O}(\varepsilon). \tag{4.131}$$

Using a lower bound similar to (4.129), we find that for any constant $c_1 \geqslant c_0$, the time necessary to reach $c_1\varepsilon$ is bounded below by $\frac{1}{a_0}\varepsilon|\ln\varepsilon| + \mathcal{O}(\varepsilon)$. □

When applied to (4.125), this result simply means the following. Assume that there is no fixed point between $x(\tau_0)$ and the stable equilibrium $x^\star(\tau_0)$. Then $x(\tau)$ will reach the $\mathcal{O}(\varepsilon)$–neighborhood of $x^\star(\tau)$ in the time $\frac{1}{a_0}\varepsilon|\ln\varepsilon| + \mathcal{O}(\varepsilon)$, where $a_0 = \partial_x f(x^\star(\tau_0), \tau_0)$ is the linearization of f around the fixed point *at the initial time* τ_0. Once this neighborhood is reached, we have already developed all the tools necessary to study the future of the solution.

Remark 4.4. The interval $\varepsilon|\ln\varepsilon|$ is small on the scale of the slow time τ, but going back to the original time scale $t = \tau/\varepsilon$, it diverges as $|\ln\varepsilon|$. This is due to the fact that we have used the small parameter ε both as adiabatic parameter, and as a length scale defining a notion of closeness to the equilibrium. Since it is more natural, in the adiabatic limit, to plot the solutions in the (τ, x)–plane rather than in the (t, x)–plane, this fact is not a problem. Proposition 4.7 simply means that the orbit will be close to a vertical line until it reaches an equilibrium.

Remark 4.5. Assume that $f(x,\tau)$ is uniformly bounded by a constant $-c < 0$ in the domain $x_- \leqslant x \leqslant x_+$, $\tau_0 \leqslant \tau \leqslant \tau_1$. Then equation (4.126) implies that the solution with initial condition $x(\tau_0) = x_+$ will reach x_- at $\tau = \tau_0 + \mathcal{O}(\varepsilon)$. This means that the solution reaches any $\mathcal{O}(1)$–neighborhood of an equilibrium branch in a time of order ε.

4.5.2 Flow Sections

Let us denote by $\phi(\tau, \tau_0, x_0)$ the solution at time τ of the real differential equation

$$\varepsilon\dot{x} = f(x,\tau), \qquad x(\tau_0) = x_0. \tag{4.132}$$

The solution is not necessarily defined for all times, but we know (see Theorem 2.11) that if it ceases to exist at time $\tau^\star$, then $(x(\tau^\star), \tau^\star)$ must belong to the boundary of the domain of definition of f. A simple way to avoid such difficulties is to assume that (4.132) is defined for $x \in \mathcal{D} = [x_-, x_+]$, and such that $f(x_-,\tau) \geqslant 0$ and $f(x_+,\tau) \leqslant 0$ for all τ. In the analogy of the particle in a potential, this amounts to requiring that the potential is attractive at long distance. All orbits starting in $\mathcal{D}$ must remain in this interval, so that ϕ is defined for $x_0 \in \mathcal{D}$ and $\tau \geqslant \tau_0$.

We would like to think of the dynamics as of a map

$$\begin{aligned} T[\tau_0,\tau_1] :\ \mathcal{D} &\to \mathcal{D} \\ x &\mapsto \phi(\tau_1,\tau_0,x), \end{aligned} \tag{4.133}$$

where τ_0 and τ_1 are fixed times, separated by an interval of order 1. We call $T = T[\tau_0, \tau_1]$ the **flow map** from τ_0 to τ_1.

It follows from unicity of solutions of (4.132) that T is an increasing function, admitting (where defined) an inverse $T[\tau_0, \tau_1]^{-1} = T[\tau_1, \tau_0]$. It can also be given a probabilistic interpretation. Indeed, the probability that $x(\tau_1)$ belongs to the interval $[a, b]$ is

$$\mathbb{P}\,[a \leqslant x(\tau_1) \leqslant b] = \mathbb{P}\,[T^{-1}(a) \leqslant x(\tau_0) \leqslant T^{-1}(b)]. \tag{4.134}$$

If $\rho(x, \tau)$ represents the probability distribution of $x(\tau)$, this is equivalent to

$$\int_a^b \rho(x, \tau_1)\,\mathrm{d}x = \int_{T^{-1}(a)}^{T^{-1}(b)} \rho(x, \tau_0)\,\mathrm{d}x, \tag{4.135}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}x} T^{-1}(x) = \frac{\rho(x, \tau_1)}{\rho(T^{-1}(x), \tau_0)}. \tag{4.136}$$

Thus the derivative of T^{-1} can be interpreted as a conditional probability density. In fact, it follows from regularity properties of the flow (see Theorem 2.13) that the derivative with respect to x_0 of the flow satisfies the equation

$$\varepsilon \partial_\tau \phi'(\tau, \tau_0, x_0) = \partial_x f(\phi(\tau, \tau_0, x_0), \tau) \phi'(\tau, \tau_0, x_0). \tag{4.137}$$

Thus, the derivative of T is given by

$$T[\tau_0, \tau_1]'(x) = \exp \frac{1}{\varepsilon} \int_{\tau_0}^{\tau} \partial_x f(\phi(s, \tau_0, x_0), s)\,\mathrm{d}s. \tag{4.138}$$

We first examine the structure of T in absence of bifurcations. Consider the case of an unstable branch $x^\star_{\mathrm{u}}(\tau)$ above a stable one $x^\star_{\mathrm{s}}(\tau)$. The orbits (see Fig. 4.21a) pass from the unstable adiabatic solution to the stable one in a "soliton–like" way, in a time of order $\varepsilon|\ln \varepsilon|$. This implies that the flow map is exponentially close either to a horizontal or to a vertical line.

Proposition 4.8. *Let $x^\star_{\mathrm{s}}(\tau) < x^\star_{\mathrm{u}}(\tau)$ be a stable and an unstable equilibrium branch of f, such that $f(x, \tau) < 0$ for $x^\star_{\mathrm{s}}(\tau) < x < x^\star_{\mathrm{u}}(\tau)$. Let $\bar{x}_{\mathrm{s,u}}(\tau)$ be associated adiabatic solutions. Then there exists a positive constant δ such that*

$$\bar{x}_{\mathrm{s}}(\tau_1) < T[\tau_0, \tau_1](x) < \bar{x}_{\mathrm{s}}(\tau_1) + \mathrm{e}^{-(\tau_1 - \tau_0)/\delta\varepsilon} \qquad \text{if} \quad \bar{x}_{\mathrm{s}}(\tau_0) < x < \bar{x}_{\mathrm{u}}(\tau_0) - \mathrm{e}^{-(\tau_1 - \tau_0)/\delta\varepsilon}. \tag{4.139}$$

Proof: Let us fix some point $(\tilde{x}, \tilde{\tau})$ (independent of ε) between the two branches. We know by Proposition 4.7 that $x(\tau) := \phi(\tau, \tilde{\tau}, \tilde{x}) = x^\star_{\mathrm{s}}(\tau) + \mathcal{O}(\varepsilon)$ as soon as $\tau \geqslant \tau_+ = \tilde{\tau} + \mathcal{O}(\varepsilon|\ln \varepsilon|)$. For larger times, Proposition 4.4 shows that $x(\tau) = \bar{x}_{\mathrm{s}}(\tau) + \mathrm{e}^{\alpha_{\mathrm{s}}(\tau, \tau_+)/\varepsilon}\,\mathcal{O}(\varepsilon)$, where

$$\alpha_{\mathrm{s}}(\tau, \tau_+) := \int_{\tau_+}^{\tau} \partial_x f(\bar{x}_{\mathrm{s}}(\tau), \tau)\,\mathrm{d}s < -a_0(\tau - \tau_+), \tag{4.140}$$

where $-a_0 < 0$. In particular, $x(\tau_1) \leqslant \bar{x}_{\mathrm{s}}(\tau_1) + \mathrm{e}^{-a_0(\tau_1 - \tilde{\tau})/\varepsilon}$. Inverting time and following $x(\tau)$ backwards, we find in a similar way $x(\tau_0) \geqslant \bar{x}_{\mathrm{u}}(\tau_0) - \mathrm{e}^{-a_0(\tilde{\tau} - \tau_0)/\varepsilon}$. We have thus constructed a solution starting exponentially close to $\bar{x}_{\mathrm{u}}(\tau_0)$ and arriving exponentially close to $\bar{x}_{\mathrm{s}}(\tau_1)$. The result follows from the monotonicity of T. □

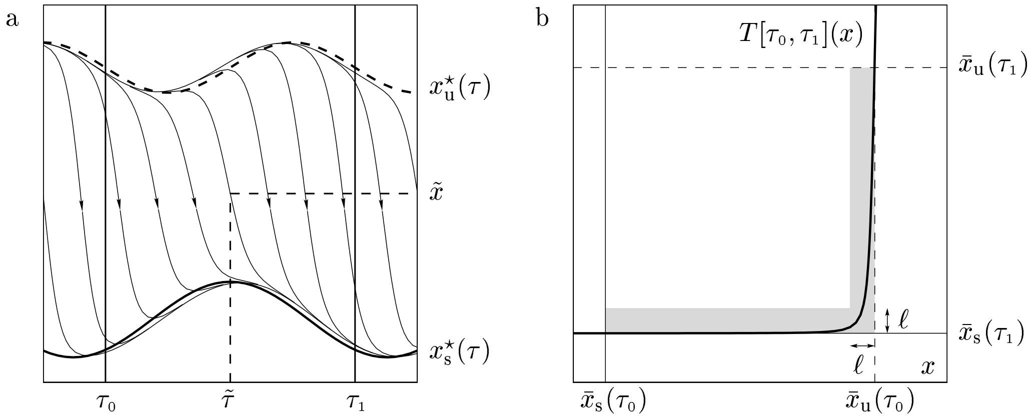

FIGURE 4.21. (a) Orbits move from the $\mathcal{O}(\varepsilon)$–neighborhood of an unstable branch $x_u^\star(\tau)$ to the $\mathcal{O}(\varepsilon)$–neighborhood of a stable branch $x_s^\star(\tau)$ in a soliton–like way, in a time of order $\varepsilon|\ln\varepsilon|$. The point $(\tilde{\tau},\tilde{x})$ appears in the proof of Proposition 4.8. (b) The corresponding flow map is exponentially close either to the horizontal $x(\tau_1)=\bar{x}_s(\tau_1)$ or to the vertical $x(\tau_0)=\bar{x}_u(\tau_0)$, in other words the width of the gray bands is $\ell=\mathrm{e}^{-(\tau_1-\tau_0)/\delta\varepsilon}$ (in fact, in this picture, ε has been considerably exaggerated in order to distinguish the curve from the horizontals and verticals).

Observe that (4.138) implies that $T'(\bar{x}_s(\tau_0))$ is exponentially small, while $T'(\bar{x}_u(\tau_0))$ is exponentially large. Between these adiabatic solutions (but close to the unstable one) is a point x such that $T'(x)=1$, and the corresponding solution spends comparable times near the unstable and the stable branch.

Example 4.15. For the logistic equation

$$\varepsilon\dot{x}=x(x-1), \tag{4.141}$$

the flow map is given by

$$T[\tau_0,\tau_0+\Delta](x)=\frac{x}{x+\mathrm{e}^{\Delta/\varepsilon}(1-x)}. \tag{4.142}$$

In particular, $T(1-\mathrm{e}^{-\Delta/2\varepsilon})<\mathrm{e}^{-\Delta/2\varepsilon}$. Moreover, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}T^{-1}(x)=\frac{1}{(\mathrm{e}^{\Delta/2\varepsilon}x+\mathrm{e}^{-\Delta/2\varepsilon}(1-x))^2}, \tag{4.143}$$

so that the probability density of x is sharply peaked around the stable origin.

In the more general situation where $f(x,\tau)$ admits several equilibrium branches, of alternating stability and which do not bifurcate, repeated application of Proposition 4.8 shows that the flow map is exponentially close to a staircase function, with horizontal segments following stable adiabatic solutions $\bar{x}_s(\tau_1)$ and vertical segments following unstable adiabatic solutions $\bar{x}_u(\tau_0)$.

Let us now illustrate the effect of bifurcations on two examples. Fig. 4.22 shows a case involving a pitchfork and a transcritical bifurcation. We already know from Example 4.9 that the adiabatic solution associated with the regular branch of the pitchfork bifurcation will follow the upper singular branch. The transcritical bifurcation admits one adiabatic

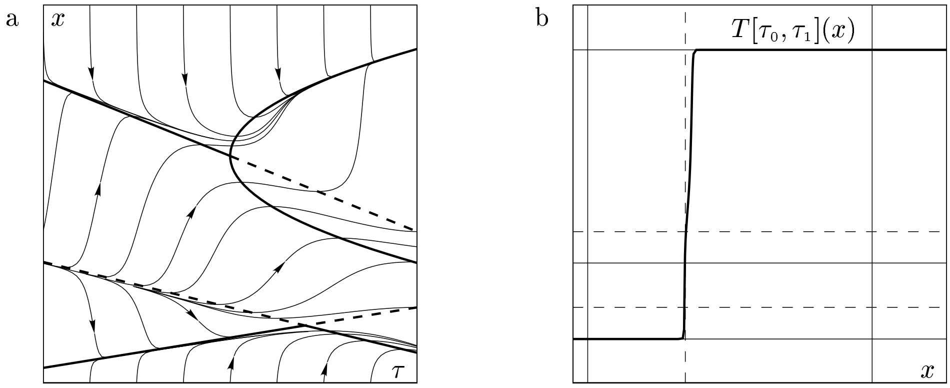

FIGURE 4.22. (a) Example of a system with a pitchfork and a transcritical bifurcation. (b) The corresponding flow map (again with an exaggerated value of ε), where thin lines indicate the position of stable (full lines) or unstable (dashed lines) equilibrium branches. It shows that the probability of ending on one of the three intermediate branches is very small. Most trajectories arrive at the uppermost or lowest stable branch, and the boundary of their basins of attraction is located close to the initially unstable branch.

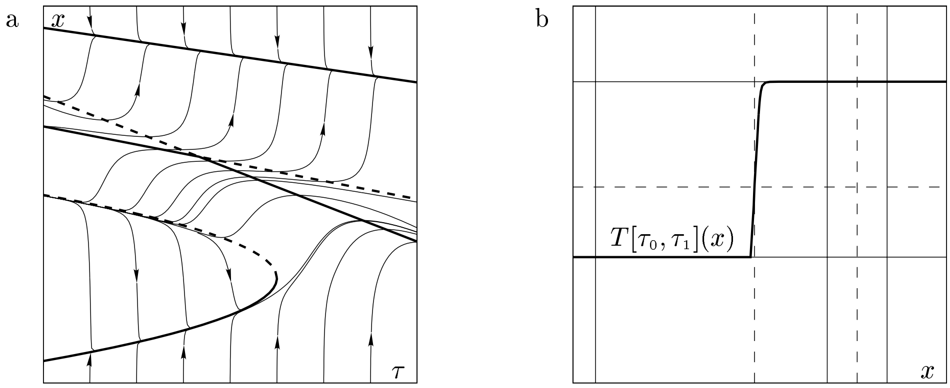

FIGURE 4.23. (a) Example of a system with a transcritical and a saddle–node bifurcation. (b) The corresponding flow map. The adiabatic solution associated with the uppermost branch is a fixed point of the map. The solutions involved in the transcritical bifurcation are shifted: the incoming stable branch is connected to the upper branch, while the outgoing stable branch is connected to lower incoming branch, which undergoes saddle–node bifurcation. The basins of attraction of the two stable branches are separated by a solution connecting the lower incoming unstable branch with the outgoing one.

solution always following the stable branch, and another one following the unstable branch. Orbits of the system will thus end up with a high probability on the uppermost or lowest branch. Following backwards the adiabatic solutions of the two outgoing unstable branches, we find that they all start exponentially close to the incoming unstable branch. These solutions delimit the basins of attraction of the stable branches, and we see that trajectories arrive at the central stable branch only with exponentially small probability.

Fig. 4.23 shows a situation involving a saddle–node and a transcritical bifurcation, which has the particularity of consisting of an unstable branch lying above a decreasing stable one. As in Example 4.4, the adiabatic state following the stable branch escapes before the bifurcation and joins the uppermost branch. The stable branch disappearing in the saddle–node bifurcation is connected with the lower outgoing branch.

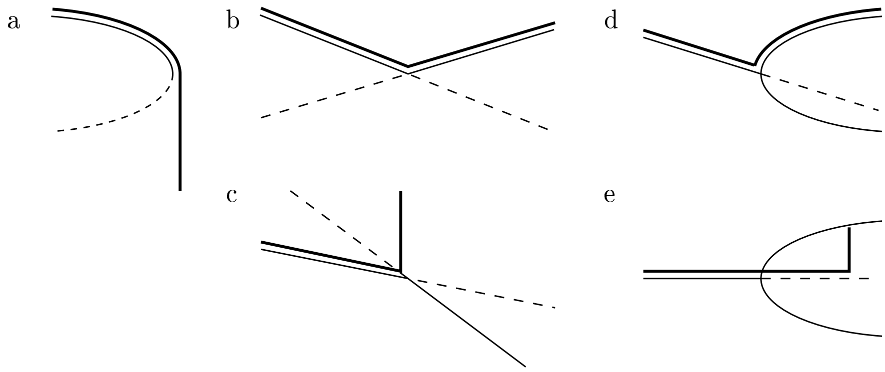

FIGURE 4.24. Behaviour of solutions in the adiabatic limit, for the most generic bifurcations. (a) Saddle–node bifurcation: the limiting trajectory is composed of the upper stable branch and a vertical. (b) Transcritical bifurcation: if the decreasing stable branch lies above the unstable one, the solution simply follows the new stable branch; (c) but if the unstable branch is above, the solution escapes on a vertical. (d) Pitchfork bifurcation: generically, the solution follows the outgoing branch with the same sign as the incoming one; (e) if the origin is always an equilibrium, the phenomenon of bifurcation delay causes the solution to follow the unstable branch for some time.

In general, the phase portraits are governed by the following rules:

- Solutions starting near stable branches remain close to the branch for increasing time, at least as long as this branch is stable.
- Solutions close to an unstable branch have been close to the same branch in previous times since it has become unstable.
- The behaviour near a bifurcation can be determined by a local analysis. In the limit $\varepsilon \to 0$, the solution either follows a stable or an unstable branch, or a vertical line, until meeting another equilibrium branch. We have sketched the most generic cases in Fig. 4.24.

Because of (4.138), solutions spending most of the time near stable branches are exponentially stable, while those which remain close to unstable branches are exponentially unstable.

On the other hand, it is difficult to predict the future of a solution starting close to an unstable branch (or the past of an orbit starting near a stable branch). By continuity, between a stable and an unstable solution, there must be at least one trajectory such that $T' = 1$, spending comparable times near stable and unstable branches.

4.6 Periodic Systems and Hysteresis

We finally come to the problem of periodic variation of parameters, existence and scaling properties of hysteresis cycles. More precisely, let us consider a dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,\lambda), \tag{4.144}$$

where x and λ are real and $F(x,\lambda)$ is of class $\mathcal{C}^2$. We impose a slow variation of the parameter $\lambda(\varepsilon t)$, where $\lambda(\tau)$ is a periodic function, say of period 1 (and also of class $\mathcal{C}^2$). On the scale of the slow time $\tau = \varepsilon t$, the system becomes

$$\varepsilon\dot{x} = f(x,\tau), \tag{4.145}$$

where $\dot{x} := \mathrm{d}_\tau x$ and $f(x,\tau) := F(x,\lambda(\tau))$ is also periodic in τ.

We would like to propose the following definition of hysteresis.

Definition 4.7. Let $x(\tau,\varepsilon)$ be the solution of (4.145) with initial condition $x(\tau_0,\varepsilon) = x_0$. If there exists a single–valued function $X(\lambda,x_0)$ such that

$$\lim_{\varepsilon\to 0} x(\tau,\varepsilon) = X(\lambda(\tau),x_0) \tag{4.146}$$

for almost any τ, then we say that this solution **does not display hysteresis**. If no such function exists, we say that the solution **displays hysteresis**. A periodic solution of (4.145) displaying hysteresis is called a **hysteresis cycle**.

In fact, this definition involves two different problems. One of them is the existence of periodic solutions, and the asymptotic properties of non–periodic solutions. In Subsection 4.6.1, we show that (4.145) admits periodic solutions under very general conditions, and that non–periodic solutions relax to such a cycle. This implies in particular that the function $X(\lambda,x_0)$, if it exists, does almost not depend on x_0 (i.e., $\partial_{x_0}X(\lambda,x_0) = 0$ almost everywhere).

Another problem is the dependence of these cycles on ε, and the question whether they display hysteresis. We examine these issues in Subsection 4.6.2, where we also show how to compute the scaling properties of the cycles with ε. In particular, we examine the dependence of their area on the adiabatic parameter in a few important examples.

4.6.1 Existence of Periodic Solutions

We assume that $f(x,\tau)$ in (4.145) is a periodic function of τ, of period 1. For any τ_0, the flow map $T[\tau_0,\tau_0+1](x)$ is nothing but the Poincaré map associated with the Poincaré section at τ_0. Periodic orbits are fixed points of T, and their stability is determined by the sign of $T(x) - x$ near the fixed point.

Proposition 4.9. *Assume there exist $x_- < x_+$ such that $F(x_-,\lambda) \geqslant 0$ and $F(x_+,\lambda) \leqslant 0$ for all λ. Then* (4.145) *admits at least one stable periodic orbit. Generically, there exist $N+1$ asymptotically stable and N unstable periodic orbits ($N \geqslant 0$). Any solution of* (4.145) *is either a periodic orbit, or attracted by a stable periodic orbit.*

PROOF: As discussed in Subsection 4.5.2, the hypotheses imply that the Poincaré map is defined for $x_- \leqslant x \leqslant x_+$. Since $f(x_-,\tau) \geqslant 0$, the comparison Lemma 4.1 implies that $T(x_-) \geqslant x_-$, and similarly we have $T(x_+) \leqslant x_+$. Consider the function $u(x) = T(x) - x$. There exists at least one point $x^\star$ such that $u(x^\star) = 0$, $u(x) \geqslant 0$ for slightly smaller x and $u(x) \leqslant 0$ for slightly larger x, so that $x^\star$ is a stable fixed point of T. If $u'(x) \neq 0$ every time $u(x) = 0$, these points have alternating stability. The last statement follows from elementary properties of monotonous iterated maps. □

To construct the Poincaré map, it is more convenient (but not necessary) to choose τ_0 such that $f(x,\tau)$ admits no bifurcation point at τ_0. Then we proceed as follows:
- Determine the future evolution, during one period, of each orbit originating in a stable fixed point of $f(x,\tau_0)$.
- Determine the past evolution, during one period, of each orbit originating in an unstable fixed point of $f(x,\tau_0)$.
- Using (4.138), compute the derivative of T at these points, and use this information to draw the Poincaré map.

In the case without bifurcation, the fixed points of T are necessarily given by the adiabatic solutions associated with the different fixed points of $f(x,\tau_0)$. From the iterative scheme (see equation (4.33)), we know that the adiabatic solution associated with the branch $X^\star(\lambda)$ of $F(x,\lambda)$ has an expansion of the form

$$\bar{x}(\tau) = \overline{X}(\lambda(\tau),\tau), \qquad \overline{X}(\lambda,\tau) = X^\star(\lambda) + \varepsilon \frac{\partial_\lambda X^\star(\lambda)}{\partial_x F(X^\star(\lambda),\lambda)} \dot{\lambda}(\tau) + \mathcal{O}(\varepsilon^2). \tag{4.147}$$

If x_0 is not exponentially close to an unstable adiabatic solution, the orbit with initial condition $x(\tau_0) = x_0$ will reach the $\mathcal{O}(\varepsilon)$–neighborhood of a stable branch in a time of order $\varepsilon|\ln\varepsilon|$. In the adiabatic limit, this orbit will consist of a vertical segment and the stable branch $X^\star(\lambda)$, so that by Definition 4.7, this system does not display hysteresis, since we can take $X(\lambda,x_0) = X^\star(\lambda)$. We conclude that the presence of bifurcations is necessary for the existence of hysteresis.

4.6.2 Scaling of Hysteresis Cycles

Once we have established existence of a periodic orbit in the adiabatic limit $\varepsilon \to 0$, we can examine the dependence of its shape (and, in particular, of its area) on ε, using the methods introduced in previous sections.

In the case without bifurcation, it follows from (4.147) that in the (λ,x)–plane, the periodic orbit encloses an area

$$\oint \bar{x}\,\mathrm{d}\lambda = \varepsilon \int_0^1 \frac{\dot{x}^\star(\tau)}{\partial_x f(x^\star(\tau),\tau)} \dot{\lambda}(\tau)\,\mathrm{d}\tau + \mathcal{O}(\varepsilon^2), \qquad x^\star(\tau) := X^\star(\lambda(\tau)). \tag{4.148}$$

In presence of bifurcations, we use the results of Section 4.3 to compute the scaling behaviour to leading order. Consider for instance the behaviour just before a bifurcation. Under the hypotheses of Theorem 4.2, the contribution of this phase of motion to the area is

$$\mathcal{A}(\varepsilon) \approx \int_0^{\varepsilon^{1/p+1}} \varepsilon^{\frac{q}{p+1}}\,\mathrm{d}\tau + \int_{\varepsilon^{1/p+1}}^1 \frac{\varepsilon}{\tau^{p+1-q}}\,\mathrm{d}\tau \approx \begin{cases} \varepsilon^{\frac{q+1}{p+1}} & \text{if } q < p \\ \varepsilon|\ln\varepsilon| & \text{if } q = p \\ \varepsilon & \text{if } q > p. \end{cases} \tag{4.149}$$

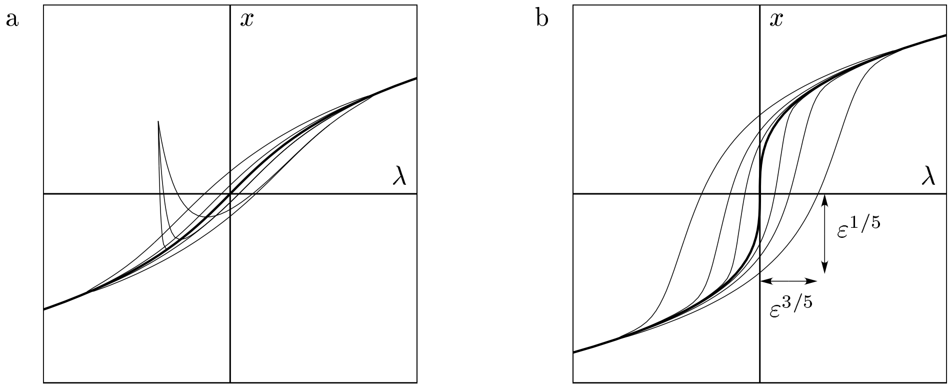

FIGURE 4.25. (a) Orbits, in the (λ, x)–plane, of the system $\varepsilon\dot{x} = -\mu x - x^3 + \lambda(\tau)$, for $\mu = 1$ and $\varepsilon = 10^{-3/2}, 10^{-2}$ and $10^{-5/2}$. The solutions are attracted by a periodic orbit, enclosing an area of order ε. (b) Same as (a), but for $\mu = 0$. At $\lambda = 0$, the periodic orbit is at a distance of order $\varepsilon^{1/5}$ of the equilibrium point, and it encloses an area of order $\varepsilon^{4/5}$.

One can proceed in a similar way for the motion after a bifurcation. The only difficulty is that the solution may cross the branch that it will follow, so that the area can have contributions with positive or negative sign.

We illustrate this method on a few examples. For simplicity, we will assume in these examples that $\lambda(\tau)$ is a "sine–like" function, i.e., it admits exactly one maximum and one minimum, and no other stationary point.

Example 4.16 (Ginzburg–Landau potential in external field).
Consider the function

$$F(x, \lambda) = -\mu x - x^3 + \lambda, \tag{4.150}$$

where $\mu \in \mathbb{R}$ is a static parameter, and $\lambda(\tau)$ is slowly oscillating around 0. This problem can be thought of as describing the overdamped motion of a particle in a Ginzburg–Landau–type potential

$$\Phi(x, \lambda) = \frac{1}{2}\mu x^2 + \frac{1}{4}x^4 - \lambda x \tag{4.151}$$

with temperature $T = T_c + \mu$ and external field λ. It has also been used to model a laser [JGRM, HL&].

The fixed points of F lie on the curve $\lambda = x^3 + \mu x$. There are three different cases, depending on the value of μ.

1. **Case 1:** $\mu > 0$
   There exists a unique stable equilibrium branch $X^\star(\lambda)$, solution of $x^3 + \mu x = \lambda$. Thus, the system admits a single stable periodic orbit, given by (4.147), lying at a distance of order ε from the static equilibrium. This cycle encloses an area of order ε, given to leading order by (4.148), see Fig. 4.25a. According to Definition 4.7, the system does not display hysteresis, since we can choose $X(\lambda, x_0) = X^\star(\lambda)$, independently of x_0. Physically, this means that in the adiabatic limit, the order parameter is a function of the external field only.

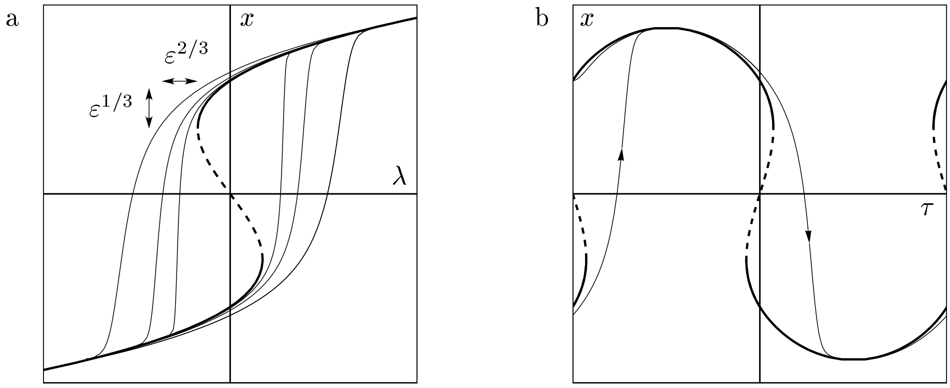

FIGURE 4.26. (a) Same as Fig. 4.25, for $\mu = -0.7$. The periodic orbit lies at a distance at most $\mathcal{O}(\varepsilon^{1/3})$ from a limiting hysteresis cycle, composed of stable equilibrium branches and two vertical lines. The enclosed area is $\mathcal{A}(\varepsilon) = \mathcal{A}(0) + \mathcal{O}(\varepsilon^{2/3})$. (b) Plot of a trajectory in the (τ, x)–plane, which allows to construct the Poincaré map.

2. **Case 2:** $\mu = 0$
The equilibrium curve $X^\star(\lambda) = \lambda^{1/3}$ admits the origin as a bifurcation point. Writing $x = \lambda(\tau)^{1/3} - y$, we get
$$\varepsilon\dot{y} = -3\lambda^{2/3}y + 3\lambda^{1/3}y^2 - y^3 + \frac{1}{3}\varepsilon\lambda^{-2/3}\dot{\lambda}, \tag{4.152}$$
to which we can apply the methods of Subsections 4.3.3 and 4.3.4, with $p = \frac{2}{3}$ and $q = \frac{1}{3}$. We may assume that $\lambda(0) = 0$ and $\dot{\lambda}(0) > 0$. We find that near $\lambda = 0$, $y(\tau)$ scales as $\varepsilon/|\tau|^{4/3}$ for $|\tau| \succcurlyeq \varepsilon^{3/5}$ and as $\varepsilon^{1/5}$ for $|\tau| \preccurlyeq \varepsilon^{3/5}$.
Since $\partial_x F(x, \lambda)$ is always negative, the Poincaré map has a derivative smaller than one, and admits a single fixed point. The corresponding cycle (see Fig. 4.25b) lies at a distance at most $\varepsilon^{1/5}$ from the equilibrium branch. Using (4.149), we obtain that its area scales as
$$\mathcal{A}(\varepsilon) \asymp \varepsilon^{4/5}. \tag{4.153}$$
This confirms a result which has been obtained in [HL&] using exact solutions.
3. **Case 3:** $\mu < 0$
In this case, the equilibrium curves admit two saddle–node bifurcation points (see Fig. 4.26a) at $(\lambda, x) = \pm\sqrt{-\mu/3}(2\mu/3, 1)$. We assume that the amplitude of $\lambda(\tau)$ is so large that these points are crossed with nonzero velocity. Then the equilibria in the (τ, x)–plane look as in Fig. 4.26b. Following during on period the solutions originating on the upper and lower stable branch, we find that they become exponentially close. In fact, the same is true for every solution starting with a finite x. This implies the existence of a unique stable fixed point of the Poincaré map.
The corresponding periodic orbits are plotted in Fig. 4.26a for several values of ε. From the local analysis of the saddle–node bifurcation in Example 4.7, we know that the solution remains at a distance of order $\varepsilon^{1/3}$ from the bifurcation point during a time of order $\varepsilon^{2/3}$. Afterwards, it joins the other branch in a time of order $\varepsilon|\ln\varepsilon|$. In the adiabatic limit, the periodic orbit approaches a cycle consisting of two vertical lines, and the stable parts of the equilibrium branch. According to our definition,

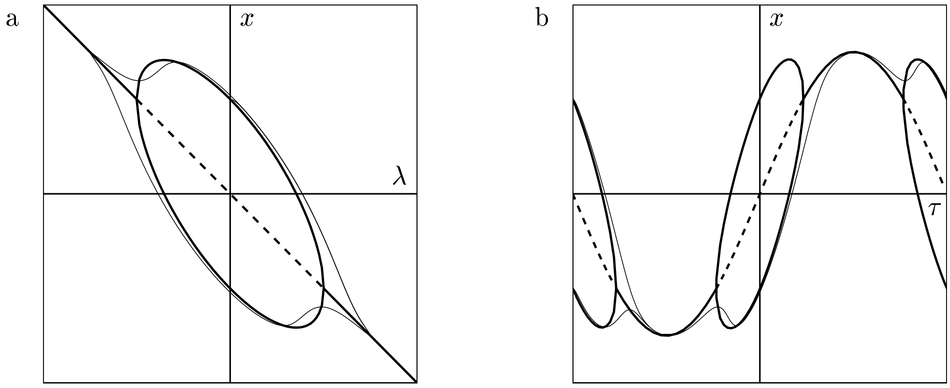

FIGURE 4.27. (a) Hysteresis cycle of the equation (4.155), involving two pitchfork bifurcations, (b) same orbit in the (τ, x)–plane.

this cycle is a hysteresis cycle, since the solution does not follow the same branch for increasing or decreasing λ.
The area of the cycle increases with ε. Using a computation similar to (4.153), we find that the main contribution to the increase in area comes from the part between the vertical line starting at the bifurcation point, and the actual solution, which is delayed by a time of order $\varepsilon^{2/3}$. Hence, the area scales as

$$\mathcal{A}(\varepsilon) - \mathcal{A}(0) \asymp \varepsilon^{2/3}, \tag{4.154}$$

a result which was already obtained in [JGRM].

Example 4.17 (Two pitchfork bifurcations).
Hysteresis may occur each time the equation involves bifurcations with a different number of incoming and outgoing branches. In Fig. 4.27a, we show another example involving a direct and an indirect pitchfork bifurcation. Such a bifurcation diagram is realized, for instance, by the function

$$F(x, \lambda) = (x + \lambda)(1 - \lambda^2 - (x + \lambda)^2). \tag{4.155}$$

As in the previous example, the orbits originating on the two stable branches become exponentially close (Fig. 4.27b), implying existence of a unique periodic orbit. That this orbit displays hysteresis is due to the fact that the solution follows the upper branch when λ increases, and the lower one when λ decreases. The local behaviour near the bifurcation points has been analysed in Examples 4.8 and 4.9. For instance, using (4.149) with $p = 1$ and $q = \frac{1}{2}$, we obtain that the indirect pitchfork bifurcation contributes to the area with a term of order $\varepsilon^{3/4}$. The direct bifurcation gives a contribution of the same order, but with opposite sign. Strictly speaking, we have thus only shown that the area of the hysteresis cycle scales as

$$|\mathcal{A}(\varepsilon) - \mathcal{A}(0)| \precsim \varepsilon^{3/4}. \tag{4.156}$$

To determine the exact behaviour, we have to analyse the local solutions in a more precise way, using the remarks of Subsection 4.3.7. From Fig. 4.27, it seems likely that the positive contributions dominate, so that $\mathcal{A}(\varepsilon) - \mathcal{A}(0) \asymp \varepsilon^{3/4}$.

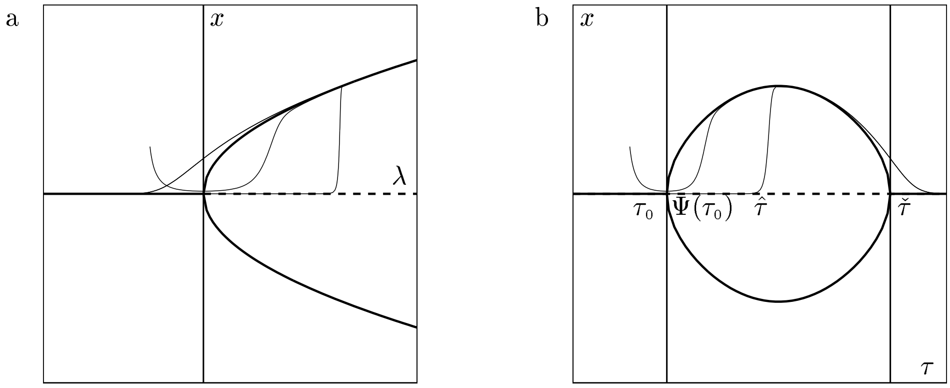

FIGURE 4.28. (a) Hysteresis cycle of the equation $\varepsilon\dot{x} = \lambda x - x^3$. (b) Plot in the (τ, x)–plane shows that the first cycle is a transient motion, where the solution drops on the origin at τ_0, and leaves it again at $\Psi(\tau_0)$. At $\check\tau$, $x \approx \varepsilon^{1/4}$, and the orbit follows the origin until $\hat\tau = \Psi(\check\tau)$, defining the hysteresis cycle.

Example 4.18 (Ginzburg–Landau potential with varying temperature). Let us consider the function

$$F(x, \lambda) = \lambda x - x^3, \tag{4.157}$$

which describes the overdamped motion in a Ginzburg–Landau potential of the form (4.151), but with no external field and a varying temperature. Because of the symmetry $x \mapsto -x$, this system displays the phenomenon of bifurcation delay. The delay function is defined by the equation

$$\int_{\tau_0}^{\Psi(\tau_0)} \lambda(\tau)\,\mathrm{d}\tau = 0, \tag{4.158}$$

where $\lambda(\tau_0) < 0$ and $\lambda(\Psi(\tau_0)) > 0$.

Let us consider a trajectory starting with positive x at τ_0 (Fig. 4.28b). Since f is negative and by Proposition 4.6, it will follow the origin until $\Psi(\tau_0)$, and then jumps on the upper branch in a time of order $\varepsilon|\ln\varepsilon|$. The upper branch is followed adiabatically until the indirect bifurcation at $\check\tau$, where we know that $x \approx \varepsilon^{1/4}$. The trajectory remains close to the origin for $\check\tau \leqslant \tau \leqslant \hat\tau = \Psi(\check\tau)$. If

$$\langle\lambda\rangle := \int_0^1 \lambda(\tau)\,\mathrm{d}\tau, \tag{4.159}$$

denotes the average of λ, the delay time is finite only if $\langle\lambda\rangle > 0$, otherwise the orbit remains indefinitely close to the origin.

We can now construct the Poincaré map T at $\check\tau$. We know by (4.138) that $T'(0) = \mathrm{e}^{\langle\lambda\rangle/\varepsilon}$. If $\langle\lambda\rangle > 0$, the origin is an unstable periodic orbit. Orbits starting at an x which is not exponentially small leave the origin at $\hat\tau < \check\tau + 1$, so that $T(x) \approx \varepsilon^{1/4}$ and

$$T'(x) = \exp\frac{1}{\varepsilon}\biggl[\int_{\hat\tau}^{\check\tau+1} \partial_x f(\sqrt{\lambda(\tau)}, \lambda(\tau))\,\mathrm{d}\tau + \mathcal{O}(\varepsilon^{1/4})\biggr] \tag{4.160}$$

is exponentially small. Thus T admits a positive stable fixed point $x^\star \approx \varepsilon^{1/4}$ (and its symmetric $-x^\star$). The corresponding periodic orbit (Fig. 4.28b) is a hysteresis cycle, of area

$$\mathcal{A}(\varepsilon) - \mathcal{A}(0) \approx \varepsilon^{3/4}. \tag{4.161}$$

This time, the hysteretic behaviour is due to the phenomenon of bifurcation delay. Physically, it can be interpreted as metastability of the homogeneous phase at low temperature.

4.7 Summary and Conclusion

We have introduced a method allowing to determine existence and scaling properties of hysteresis cycles, for one–dimensional adiabatic systems of the form $\varepsilon\dot{x} = F(x,\lambda(\tau))$. To obtain qualitative information, it is not necessary to know the equilibrium branches of the static system exactly; it is sufficient to know the first terms of Taylor series around bifurcation points, and how these points are connected by equilibrium branches.

We can then construct the flow map, knowing that in the adiabatic limit,

- orbits starting far from equilibrium branches follow a vertical line until meeting an equilibrium branch;
- orbits starting close to stable branches follow them until the next bifurcation; the same is true in the past for solutions starting near unstable branches;
- the behaviour at a bifurcation is determined by a local analysis, where the most generic cases are sketched in Fig. 4.24. The solution either follows one of the outgoing equilibrium branches, or a vertical line.

In the adiabatic limit, the flow map consists of horizontal and vertical lines. In the periodic case, its fixed points determine periodic orbits, which may or may not be hysteresis cycles.

Once existence of a cycle is established, it is possible to determine its scaling properties to leading order in ε. For each bifurcation point, two numbers p and q, which can be read off Newton's polygon (see Fig. 4.4), generically determine the scaling behaviour of the solution. It is then sufficient to add up contributions of the different bifurcation points.

We have shown that the presence of bifurcation points is necessary for existence of hysteresis. In such a case, the state of the system in the adiabatic limit is not a function of the parameter only, but depends on the history of $\lambda(\tau)$ as well. Still, the dependence is relatively simple. All solutions are attracted by periodic ones, and satisfy

$$\lim_{\varepsilon\to 0} x(\tau,\varepsilon) = X^\star_{j[\lambda(\cdot)]}(\lambda(\tau)), \tag{4.162}$$

where the $X^\star_j(\lambda)$ are equilibrium branches of $F(x,\lambda)$. The hysteresis behaviour is concentrated into the function $j[\lambda(\cdot)]$, which depends on the way bifurcation points are crossed.

4.A Proofs of Some Results

4.A.1 Proof of Lemma 4.2

We consider the equation

$$\varepsilon\dot{y}_0 = a(\tau)y_0 + b_0(y_0, y_0^*, \tau, \varepsilon) + \varepsilon w_0(\tau, \varepsilon) \tag{4.163}$$

with $b_0 = \mathcal{O}(|y_0|^2) + \mathcal{O}(\varepsilon|y_0|)$ and $a(\tau) \neq 0$.

The first conclusion of the lemma is obtained by applying Iterative Scheme 4.1 k times, which decreases the order of the drift term to ε^{k+1}. In particular, as long as $b_{N-1} \in \mathcal{C}^1$, (4.21b) implies that $b_N = \mathcal{O}(|y_N|^2) + \mathcal{O}(\varepsilon|y_N|)$.

To prove the second part, we assume that (4.163) is analytic for ε in an open complex set $\mathcal{U}$ containing 0 in its closure, and for $(y_0, \tau) \in \Gamma(D_0)$, where

$$\Gamma(D) := \bigl\{(y,\tau) \in \mathbb{C}^2 \bigm| |\mathrm{Re}\, y|, |\mathrm{Im}\, y|, |\mathrm{Im}\, \tau| < D, |\mathrm{Re}\, \tau - s| < D\, \forall s \in I\bigr\}. \tag{4.164}$$

Let d, ε_0 be positive constants and $D_N = D_0 - Nd\varepsilon_0$. This means that we consider the equations on a sequence of domains, $\Gamma(D_{N+1}) \subset \Gamma(D_N)$, with boundaries at a distance of order ε_0. We assume by induction that for $y_N \in \Gamma(D_N)$ and $\varepsilon \in \mathcal{U}$, $|\varepsilon| < \varepsilon_0$, (4.20) is analytic with

$$|w_N(\tau,\varepsilon)| < W_N, \quad |\partial_y b_N(0,0,\tau,\varepsilon)|, |\partial_{y^*} b_N(0,0,\tau,\varepsilon)| < M_N\varepsilon_0, \tag{4.165}$$

and the second derivatives of b_N are bounded by M. We will show that for sufficiently small d^{-1} and ε_0, the bounds W_N and M_N depend on N in the following way:

$$\begin{aligned} W_N &= \frac{W_0}{(2\varepsilon_0)^N} \\ M_N &= M_0 + 4KW_0M\left(1 - \frac{1}{2^N}\right) \leqslant M_0 + 4KW_0M =: \overline{M}. \end{aligned} \tag{4.166}$$

Using a second order Taylor series, we deduce from (4.165) that

$$|b_N(y_N, y_N^*, \tau, \varepsilon)| < 2M_N\varepsilon_0|y_N| + 2M|y_N|^2. \tag{4.167}$$

If $|a(\tau)| > K^{-1}$ for all τ, then (4.19) implies that $|u_{N+1}(\tau,\varepsilon)| < KW_N$. It follows that if $(y_{N+1}, \tau) \in \Gamma(D_{N+1})$, and $d \geqslant \varepsilon_0^N KW_N$, then $(y_N, \tau) \in \Gamma(D_N)$ and w_{N+1}, b_{N+1} are analytic. Moreover, by construction, every point in $\Gamma(D_{N+1})$ is the center of a ball of radius $d\varepsilon_0$, contained in $\Gamma(D_N)$. Thus Cauchy's inequality implies that $|\dot{u}_{N+1}| < KW_N/\varepsilon_0 d$, and it follows from (4.21) that

$$\begin{aligned} |w_{N+1}(\tau,\varepsilon)| &< KW_N[2M_N + 2M\varepsilon_0^N KW_N + 1/\varepsilon_0 d] \\ |\partial_y b_{N+1}(0,0,\tau,\varepsilon)| &= |\partial_y b_N(\varepsilon^{N+1}u_{N+1}, \varepsilon^{N+1}u_{N+1}^*, \tau, \varepsilon)| < M_N\varepsilon_0 + 2\varepsilon_0^{N+1}KW_NM, \end{aligned} \tag{4.168}$$

and the second derivatives of b_{N+1} are bounded by M. Let us take $d = \max\{6K, W_0K\}$ and $\varepsilon_0 \leqslant [d(\overline{M} + KW_0M)]^{-1}$. Then $d \geqslant \varepsilon_0^N KW_N$ and

$$\begin{aligned} |w_{N+1}| &< \frac{KW_N}{\varepsilon_0}\left[\varepsilon_0(2\overline{M} + 2KW_0M) + \frac{1}{d}\right] \leqslant \frac{W_0}{(2\varepsilon_0)^{N+1}} \\ |\partial_y b_{N+1}| &< \varepsilon_0\left[M_N + \frac{1}{2^{N-1}}KW_0M\right] = \varepsilon_0\left[M_0 + 4KW_0M\left(1 - \frac{1}{2^{N+1}}\right)\right] \end{aligned} \tag{4.169}$$

which proves (4.166) by induction. The final step is to fix a number of changes of variables of order $1/\varepsilon$, for instance

$$N(\varepsilon_0) := \left[\frac{D_0}{2d\varepsilon_0}\right] \tag{4.170}$$

where the brackets denote the integer part. Then the final map is analytic in the set $\Gamma(D_{N(\varepsilon_0)}) \supset \Gamma(D_0/2)$, and the drift term is bounded by

$$|\varepsilon^{N(\varepsilon_0)} w_{N(\varepsilon_0)}| \leqslant 2^{-N(\varepsilon_0)} W_0 \leqslant 2W_0\, \mathrm{e}^{-\ln 2D_0/2d|\varepsilon|}\,. \tag{4.171}$$

Finally, the change of variables $y_0 = z + \sum_{j=1}^{N(\varepsilon_0)} \varepsilon^j u_j(\tau,\varepsilon)$ is such that

$$|z - y_0| \leqslant \sum_{j=1}^{N(\varepsilon_0)} |\varepsilon^j u_j| \leqslant |\varepsilon| \sum_{j=0}^{N(\varepsilon_0)-1} KW_0 2^{-j} \leqslant 2KW_0|\varepsilon| = \mathcal{O}(\varepsilon), \tag{4.172}$$

which proves the lemma. □

4.A.2 Proof of Proposition 4.3

Lemma 4.7. *Let T, b_0, w_0 and v_0 be arbitrary positive constants, and $a(\tau)$ a non–negative differentiable function on $[0,T]$. Let $q \geqslant 0$, $k \geqslant \max\{1, 2-q\}$ and let $\varphi(\varepsilon) = o(\varepsilon^{k-1})$. Then there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \leqslant \varepsilon_0$, any solution of*

$$\varepsilon\dot{v} = \varepsilon a(\tau)v + \varepsilon^q b_0 v^2 + \varepsilon\varphi(\varepsilon)w_0 \tag{4.173}$$

with $0 \leqslant v(0) \leqslant v_0\varphi(\varepsilon)$ satisfies $v(\tau) = \mathcal{O}(\varphi(\varepsilon))$ for $0 \leqslant \tau \leqslant T$.

Proof: Let $\alpha(\tau) := \int_0^\tau a(s)\,\mathrm{d}s \geqslant 0$. We set $\gamma := \mathrm{e}^{\alpha(T)}[v_0 + 2w_0T] > v_0$, and assume that ε_0 is so small that $\varphi(\varepsilon) \leqslant \varepsilon^{k-1}w_0/b_0\gamma^2$. By continuity of the solution, we can define $\tau_1 \in (0,T]$ such that $v(\tau) < \gamma\varphi(\varepsilon)$ for $0 \leqslant \tau < \tau_1$, and either $v(\tau_1) = \gamma\varphi(\varepsilon)$ or $\tau_1 = T$. Thus, for $0 \leqslant \tau \leqslant \tau_1$, we have

$$\varepsilon\dot{v} \leqslant \varepsilon a(\tau)v + \varphi(\varepsilon)[w_0\varepsilon + b_0\gamma^2\varepsilon^q\varphi(\varepsilon)] \leqslant \varepsilon a(\tau)v + 2w_0\varepsilon\varphi(\varepsilon). \tag{4.174}$$

By Lemma 4.1, this implies

$$v(\tau) \leqslant \varphi(\varepsilon)\,\mathrm{e}^{\alpha(\tau)}\left[v_0 + 2w_0\int_0^\tau \mathrm{e}^{-\alpha(s)}\,\mathrm{d}s\right] \leqslant \varphi(\varepsilon)\,\mathrm{e}^{\alpha(\tau)}[v_0 + 2w_0\tau]. \tag{4.175}$$

If $\tau_1 < T$, we would have $v(\tau_1) < \gamma\varphi(\varepsilon)$, which is a contradiction. Thus, $\tau_1 = T$ and the lemma is proved. □

Remark 4.6. The proof of Lemma 4.7 shows that $x(\tau)$ can grow exponentially with T, while ε_0 decreases exponentially. The requirement $\varphi(\varepsilon) = o(\varepsilon^{k-1})$ is necessary as shows the following example: if $\varepsilon\dot{v} = v^2 + \varepsilon^p$, the lemma applies when $p > 2$. Indeed, the solution $v(\tau) = \varepsilon^{p/2}\,\mathrm{tg}(\varepsilon^{p/2-1}\tau + c)$ diverges at $\tau = \varepsilon^{1-p/2}(\frac{\pi}{2} - c)$, where $c = \mathrm{Arctg}(\varepsilon^{-p/2}v(0))$.

4.A.3 Proof of Proposition 4.4

We give a constructive proof, based on the fixed point theorem. The solution we are looking for should be a fixed point of the operator

$$\begin{aligned}(Ty)(\tau) &:= \mathrm{e}^{\bar{\alpha}(\tau,\tau_0)/\varepsilon}\, y(\tau_0) + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\bar{\alpha}(\tau,s)/\varepsilon}\, b(y(s),s)\,\mathrm{d}s,\\ \bar{\alpha}(\tau,\tau_0) &:= \int_{\tau_0}^{\tau} \bar{a}(s)\,\mathrm{d}s.\end{aligned} \tag{4.176}$$

Assume that for τ in some interval I and sufficiently small y, (4.36) satisfies the bounds $\operatorname{Re}\bar{a}(\tau) \leqslant -K^{-1}$, $|b(y,\tau)| \leqslant M|y|^2$ and $|\partial_y b(y,\tau)| \leqslant M|y|$. We take δ_1 such that $0 < KM\delta_1 < 1$, and fix a $y_0 \leqslant \delta = (1-KM\delta_1)\delta_1$. We introduce the norm

$$\|y\| := \sup_{\tau\in I,\tau \geqslant \tau_0} \left|\mathrm{e}^{-\bar{\alpha}(\tau,\tau_0)/\varepsilon}\, y(\tau)\right|, \tag{4.177}$$

which satisfies $|y(\tau)| \leqslant \mathrm{e}^{\operatorname{Re}\bar{\alpha}(\tau,\tau_0)/\varepsilon}\|y\|$. Let $\mathcal{E}$ be the space of functions $y(\tau)$ with $y(\tau_0) = y_0$ and $\|y\| \leqslant \delta_1$. If $y \in \mathcal{E}$,

$$\begin{aligned}|(Ty)(\tau)| &\leqslant \mathrm{e}^{\operatorname{Re}\bar{\alpha}(\tau,\tau_0)/\varepsilon}\, y_0 + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\operatorname{Re}\bar{\alpha}(\tau,s)/\varepsilon}\, M\,\mathrm{e}^{2\operatorname{Re}\bar{\alpha}(s,\tau_0)/\varepsilon}\,\delta_1^2\,\mathrm{d}s\\ &\leqslant \mathrm{e}^{\operatorname{Re}\bar{\alpha}(\tau,\tau_0)/\varepsilon}[y_0 + KM\delta_1^2]\end{aligned} \tag{4.178}$$

and thus $\|Ty\| \leqslant \delta + KM\delta_1^2 = \delta_1$, which implies that $Ty \in \mathcal{E}$. If y_1, y_2 are two functions in $\mathcal{E}$ with $\|y_1\|, \|y_2\| \leqslant y_0 \leqslant \delta_1$, we have

$$\begin{aligned}|b(y_2(s),s) - b(y_1(s),s)| &\leqslant M\,\mathrm{e}^{\operatorname{Re}\bar{\alpha}(s,\tau_0)/\varepsilon}\, y_0|y_2(s) - y_1(s)|,\\ |(Ty_1)(\tau) - (Ty_2)(\tau)| &\leqslant \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\operatorname{Re}\bar{\alpha}(\tau,s)/\varepsilon}\, M\,\mathrm{e}^{\operatorname{Re}\bar{\alpha}(s,\tau_0)/\varepsilon}\, y_0\,\mathrm{e}^{\operatorname{Re}\bar{\alpha}(s,\tau_0)/\varepsilon}\|y_2 - y_1\|\\ \|Ty_2 - Ty_1\| &\leqslant KMy_0\|y_2 - y_1\|,\end{aligned} \tag{4.179}$$

which implies that T is a contraction with contraction constant $\lambda = KMy_0 < 1$, and admits a unique fixed point $y^\star(\tau)$, which is the solution of (4.36). In particular, if we choose the sequence of functions defined by $y_1(\tau) = y_0\,\mathrm{e}^{\bar{\alpha}(\tau,\tau_0)/\varepsilon}$ and $y_{N+1} = Ty_N$, we have

$$y_2(\tau) = \mathrm{e}^{\bar{\alpha}(\tau,\tau_0)/\varepsilon}\left[y_0 + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{-\bar{\alpha}(s,\tau_0)/\varepsilon}\, b(y_0\,\mathrm{e}^{\bar{\alpha}(s,\tau_0)/\varepsilon}, s)\,\mathrm{d}s\right] \tag{4.180}$$

which means that $\|y_2 - y_1\| \leqslant KMy_0^2$. We easily see that the structure (4.38) is preserved by further iterations, and it follows from the properties of contractions (see Theorem 2.1) that $\|y_N - y^\star\| \leqslant \lambda^{N-1}\|y_2 - y_1\|/(1-\lambda) = \mathcal{O}(y_0^N)$, so that each iteration gives one additional order of $R(y_0,\tau)$. □

4.A.4 Proof of Lemma 4.4

We wish to examine the scaling behaviour of the integral

$$z(\tau,\varepsilon) = \int_{\tau}^{\tau_0} s^{q-1}\,\mathrm{e}^{c[\tau^{p+1} - s^{p+1}]/\varepsilon}\,\mathrm{d}s. \tag{4.181}$$

Let us note $\mu = \frac{q}{p+1}$, $\tau = \varepsilon^{\frac{1+\delta}{p+1}}$, and $c\tau_0^{p+1} = d$.

1. Case $0 \leqslant \tau \leqslant \varepsilon^{\frac{1}{p+1}}$ **(**$\delta \geqslant 0$**):**
The change of variables $s = [\varepsilon\xi/c]^{1/p+1}$ transforms the integral into

$$z(\tau,\varepsilon) = \frac{1}{p+1}\left(\frac{\varepsilon}{c}\right)^{\mu} \mathrm{e}^{c\tau^{p+1}/\varepsilon} \int_{c\tau^{p+1}/\varepsilon}^{d/\varepsilon} \xi^{\mu-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi. \tag{4.182}$$

Since $\frac{\tau^{p+1}}{\varepsilon} = \varepsilon^{\delta}$, $\mathrm{e}^{c\tau^{p+1}/\varepsilon} \approx 1$. The integral can be written as $\Gamma(\mu) - R_1 - R_2$, where

$$\begin{aligned} 0 \leqslant R_1 &= \int_0^{c\varepsilon^{\delta}} \xi^{\mu-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi \leqslant \mu^{-1}[c\varepsilon^{\delta}]^{\mu} \approx \varepsilon^{\delta\mu} \\ 0 \leqslant R_2 &= \int_{d/\varepsilon}^{\infty} \xi^{\mu-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi = \varepsilon^{1-\mu}\,\mathrm{e}^{-d/\varepsilon} \int_0^{\infty} (d+\varepsilon\zeta)^{\mu-1}\,\mathrm{e}^{-\zeta}\,\mathrm{d}\zeta. \end{aligned} \tag{4.183}$$

Thus, R_1 and R_2 are small compared to $\Gamma(\mu)$, and $z(\tau,\varepsilon) \approx \varepsilon^{\mu}$.

2. Case $\varepsilon^{\frac{1}{p+1}} \leqslant \tau \leqslant \tau_0$ **(**$0 < \delta < 1$**):**
We first use the upper bound $\tau^{p+1} - s^{p+1} \leqslant (p+1)\tau^p(\tau - s)$, which follows from convexity of s^{p+1}. Then the change of variables $s = \tau + \varepsilon u/a\tau^p$, $a = c(p+1)$, yields

$$z(\tau,\varepsilon) \leqslant \frac{1}{a}\frac{\varepsilon}{\tau^p}\tau^{q-1} \int_0^{\infty} \left[1 + \frac{\varepsilon}{a\tau^{p+1}}u\right]^{q-1} \mathrm{e}^{-u}\,\mathrm{d}u \approx \frac{\varepsilon}{\tau^{p+1-q}}. \tag{4.184}$$

Finally, if $\bar\tau := \min\{2\tau, \tau_0\}$, we can use for $\tau \leqslant s \leqslant \bar\tau$ the lower bound $\tau^{p+1} - s^{p+1} \geqslant \kappa(\tau - s)$, where $\kappa = (\bar\tau^{p+1} - \tau^{p+1})/(\bar\tau - \tau) \approx \tau^p$. Taking $s = \tau + \varepsilon u/\kappa c$, we find

$$z(\tau,\varepsilon) \geqslant \frac{\varepsilon}{\kappa c}\tau^{q-1} \int_0^{c(\bar\tau^{p+1} - \tau^{p+1})/\varepsilon} \left[1 + \frac{\varepsilon u}{\kappa c \tau}\right]^{q-1} \mathrm{e}^{-u}\,\mathrm{d}u \approx \frac{\varepsilon}{\tau^{p+1-q}}. \tag{4.185}$$

From (4.184) and (4.185), we conclude that $z(\tau,\varepsilon) \approx \dfrac{\varepsilon}{\tau^{p+1-q}}$. □

4.A.5 Proof of Theorem 4.2

We know that $\varepsilon\dot y = a(\tau)y + b(y,\tau) + \varepsilon w(\tau)$, where $a(\tau) \approx -|\tau|^p$, $w(\tau) \approx |\tau|^{q-1}$, and, because of Lemma 4.3, $|y^{-2}b(y,\tau)| \preccurlyeq |\tau|^{p-q}$ whenever $|y| \preccurlyeq |\tau|^q$. We choose a solution with initial condition $y(-\tau_0) \approx \varepsilon$. Let us note $\mu = \frac{q}{p+1}$ and $\nu = \frac{1}{p+1}$. The proof is divided into two steps.

Step 1, outer region: We know that $a(\tau) \leqslant -a_+|\tau|^p$ for some $a_+ \geqslant 0$. By continuity, there exists a $\tau^\star \in (-\tau_0, 0]$ such that, for $-\tau_0 \leqslant \tau \leqslant \tau^\star$, $y(\tau) \leqslant a_+|\tau|^q/2b_0$ and $b(y,\tau) \leqslant b_0 y^2|\tau|^{p-q}$, for some positive b_0.[8] Moreover, we may assume that either $\tau^\star = -\varepsilon^{\nu}$, or $\tau^\star < -\varepsilon^{\nu}$ and $y(\tau^\star) = a_+|\tau|^q/2b_0$. Then, for $-\tau_0 \leqslant \tau \leqslant \tau^\star$,

$$\varepsilon\dot y \leqslant -\frac{a_+}{2b_0}|\tau|^p y + \varepsilon w_0|\tau|^{q-1}. \tag{4.186}$$

Let us write $\tau^\star = -\varepsilon^{\nu-\delta}$, $\delta \geqslant 0$. Using Lemmas 4.4 and 4.1, we obtain

$$y(\tau) \leqslant c_+\frac{\varepsilon}{|\tau|^{p+1-q}}, \tag{4.187}$$

[8] This is always possible if we take b_0 sufficiently large, and ε sufficiently small.

and $y(\tau^\star) \leqslant c_+\varepsilon^{\delta(p+1)}|\tau^\star|^q$. By definition, we have either $\tau^\star = -\varepsilon^\nu$, or $y(\tau^\star) = a_+|\tau^\star|^q/2b_0$, which implies $\varepsilon^{\delta(p+1)} \geqslant a_+/2b_0c_+$, so that $\tau^\star \geqslant -(2b_0c_+/a_+)^\nu\varepsilon^\nu$. In exactly the same way, we obtain a lower bound similar to (4.187), which proves (4.64).

Step 2, inner region: For $\tau^\star \leqslant \tau \leqslant 0$, we rescale the equation according to the relations $y = \varepsilon^\mu z$ and $\tau = \varepsilon^\nu\sigma$, giving

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \tilde{a}(\sigma) + \tilde{b}(z,\sigma) + \tilde{w}(\sigma), \tag{4.188}$$

where $\tilde{a}(\sigma) = \varepsilon^{\nu-1}a(\varepsilon^\nu\sigma) \approx -|\sigma|^p$, $\tilde{w}(\sigma) = \varepsilon^{\nu-\mu}w(\varepsilon^\nu\sigma) \approx |\sigma|^{q-1}$, and, because of Lemma 4.3, $\tilde{b}(\sigma,z) = \varepsilon^{\nu-\mu-1}b(\varepsilon^\mu z, \varepsilon^\nu\sigma) \preccurlyeq 1$. Using the fact that $\mathrm{d}_\sigma z \geqslant -a_-|\sigma|^p z - b_0 z^2$, we obtain by solving this Bernouilli equation that $z(\sigma) \succcurlyeq 1$, and thus $y(\tau) \succcurlyeq \varepsilon^\mu$. If the solution diverges, it does so at a σ of order 1. This completes the proof of (4.65).

If, moreover, the hypothesis $f(x^\star(\tau) + y, \tau) < 0$ is satisfied for positive y, we obtain that $a(\tau)y + b(y,\tau) < 0$, which yields the upper bound $\mathrm{d}_\sigma z \leqslant w_0|\sigma|^{q-1}$. Using Lemma 4.1 again, we conclude that we also have $z(\sigma) \preccurlyeq 1$, and thus $y(\tau) \preccurlyeq \varepsilon^\mu$, which proves (4.66). □

Chapter 5

n–Dimensional Systems

"The problem of Ising's model in more than one dimension has led to a good deal of controversy and in particular since the opinion has often been expressed that the solution of the three–dimensional problem could be reduced to that of the linear model and would lead to similar results, it may be worthwhile to give its solution. (...) It follows rigorously that for sufficiently low temperatures the Ising model in two dimensions shows ferromagnetism and the same holds a fortiori *also for the three–dimensional model."*

R. Peierls

In this chapter, we generalize results from the previous chapter to the n–dimensional (nD) case. In particular, we will study local properties like existence of adiabatic solutions, and the linear and nonlinear behaviour of nearby solutions. We do this analysis in the following steps:

- **Section 5.1** We define precisely what kind of nD systems we are going to study, and make a few comments on differences between the 1D and nD cases.
- **Section 5.2** We discuss the existence of adiabatic solutions, remaining close to non–bifurcating equilibrium branches. The discussion is similar to the one of Section 4.2, with the difference that we have to deal with general hyperbolic equilibria, which may admit contracting and expanding directions.
- **Section 5.3** We analyse linear systems of the form $\varepsilon\dot{y} = A(\tau,\varepsilon)y$, which describe in particular the linearized motion around an adiabatic solution. We present a method allowing to diagonalize such an equation dynamically, when the eigenvalues of A do not cross. We also examine the most generic eigenvalue crossings, which in some cases turn out to behave in a similar way than adiabatic solutions near bifurcations.
- **Section 5.4** We examine the influence of nonlinear terms in the equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y + \mathcal{O}(|y|^2)$. We present two main approaches: the first one deals with adiabatic invariant manifolds associated with hyperbolic equilibria; the second one uses dynamic normal forms and sometimes allows to diagonalize the equation dynamically. We also present a generalization of the center manifold theorem, which allows to reduce the dimension of the motion near a bifurcation point.
- **Section 5.5** We particularize the discussion to the case of a periodic parameter variation. We consider in particular the problem of existence of periodic solutions, and the influence of the slowly changing parameter on the behaviour of nearby solutions.

- **Appendix 5.A** We discuss some properties of matrices, in particular the matricial equation $AX - XB = C$, which is often encountered when dealing with dynamic bloc–diagonalization and Lyapunov functions.
- To make the text more readable, we have postponed some of the longer proofs to **Appendix 5.B**.

This work is inspired by the following literature. The iterative scheme in Section 5.2 is due to Neishtadt [Ne1]. Lyapunov function have already been used to prove stability of attracting equilibrium branches in [PR]. The proof of existence of adiabatic solutions for hyperbolic branches, however, is new.

Our study of linear singularly perturbed systems relies partly on [**Wa**]. Wasow, however, mainly uses formal series and proves existence of asymptotic series. We use here a new method aiming at separating a diagonal part of evolution, including the singular dependence on ε, from a smooth nondiagonal part. This approach considerably simplifies the proofs, and makes them more general.

Eigenvalue crossings have been mainly analysed in quantum mechanics, see for instance [Hag]. The generic case has been studied in [**Wa**], but we obtain better estimates than [**Ol**]. Our new approach yields a different understanding of the other crossings, where the matrix remains diagonal.

The results on adiabatic manifolds and normal forms in Section 5.4 are new to our knowledge. Hale gives some results on existence of invariant manifolds for non–autonomous equations [**Hal**], but they are restricted to the situation when the linear part of the equation can be reduced to a constant matrix, as in the case of periodically time–dependent systems. Moreover, no method is known to compute these manifolds in the general case. We will show that for adiabatically time–dependent systems, it is possible to represent these manifold by an asymptotic series in the adiabatic parameter.

5.1 Preliminaries

5.1.1 What n–Dimensional Systems?

In this chapter, we consider equations of the form

$$\varepsilon \dot{x} = f(x, \tau), \tag{5.1}$$

where x belongs to an open set $\mathcal{D} \subset \mathbb{R}^n$ or $\mathbb{C}^n$, $n \geqslant 2$, and τ varies in an interval $I \subset \mathbb{R}$. The function $f(x,\tau)$ belongs either to $\mathcal{C}^k(\mathcal{D} \times I, \mathbb{R}^n)$ or to $\mathcal{C}^k(\mathcal{D} \times I, \mathbb{C}^n)$, where $k \geqslant 2$.

Most properties we are going to discuss do not depend on the fact that x is real or complex, so that we will in general consider the complex situation. Only in some circumstances, for instance when studying eigenvalue crossings, will the discussion be limited to the real case, when this restriction brings substantial simplifications.

The remarks of Subsection 4.1.1 on a possible dependence of f on ε also apply to the present situation.

The following terminology will be useful to simplify the discussion.

Definition 5.1. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be
- **contracting** or **asymptotically stable** if all its eigenvalues have negative real part;
- **expanding** if $-A$ is contracting;
- **hyperbolic** if it has no eigenvalue with zero real part;
- **elliptic** if all its eigenvalues have zero real part.

5.1.2 What's New in the n–Dimensional Case?

The discussion of adiabatic solutions, in particular the iterative scheme allowing to compute them, easily carries over from the 1D to the nD case. Differences occur when one wants to show existence of adiabatic solutions, since there is no analogue of the Comparison Lemma in the nD case. For asymptotically stable equilibria, the problem can be transformed into a 1D one using Lyapunov functions. The case of general hyperbolic equilibria has to be studied more carefully, by separating the stable and unstable directions.

The problem becomes more difficult when considering the linearized equation around an adiabatic solution, which has the form $\varepsilon\dot{y} = A(\tau,\varepsilon)y$. Unlike in the 1D case, this equation cannot be solved exactly in general. In fact, the problem of solving time–dependent linear equations is a very difficult one, which may involve phenomena like parametric resonance. In the adiabatic case, however, one can do more, since solutions tend to stay close to the instantaneous eigenspaces of the matrix A; we are thus going to construct changes of variables which diagonalize the system dynamically.

The case of eigenvalue crossings has to be discussed separately. It is interesting to note that in some cases, the equations describing the eigenspaces' motion near eigenvalue crossings resemble those of adiabatic solutions near bifurcation points, so that these phenomena can be described in a unified way.

The effect of nonlinear terms is also more difficult to control. There is no simple analogue of Proposition 4.4, and even if the linear part has been diagonalized, nonlinear terms may induce annoying transitions between components. In fact, these problems already exist in the autonomous case, so that we provide generalizations of the methods used in that case to deal with such problems, namely invariant manifolds and dynamical normal forms.

The difficulty of analyzing bifurcation problems grows with the dimensionality of the system, just as in the autonomous case. However, by generalizing the center manifold theorem, it becomes possible to reduce the dimension of the problem to the number of zero eigenvalues at the bifurcation, which is usually small (in finite dimension). In the generic case, the problem can thus be reduced to a bifurcation problem discussed in the previous chapter.

Global properties of the system, even in the periodic case, are of course even more difficult to analyse, since it is well known that systems with 3 or more degrees of freedom may be chaotic. We will thus only give some general remarks, postponing more detailed discussions to the examples in subsequent chapters. In particular, it will be shown in an example in the next chapter that such systems may admit chaotic solutions even in the adiabatic limit.

5.2 Adiabatic Solutions

We consider in this section the equation

$$\varepsilon\dot{x} = f(x,\tau), \tag{5.2}$$

where $f \in \mathcal{C}^{k+1}(\mathcal{D}\times I,\mathbb{C}^n)$, I is an interval in $\mathbb{R}$, $\mathcal{D}\subset\mathbb{C}^n$ and $k \geqslant 1$. Let $x^\star(\tau)$ be an equilibrium branch of (5.2), i.e., $f(x^\star(\tau),\tau) = 0$ for $\tau \in I$. We define adiabatic solutions as in the one–dimensional case.

Definition 5.2. An **adiabatic solution** of order ε^μ of the equation (5.2) associated with the equilibrium branch $x^\star(\tau)$ is a solution $\bar{x}(\tau)$ remaining at a distance of order ε^μ from $x^\star(\tau)$.

The main result of this section will be the following analogue of Theorem 4.1.

Theorem 5.1. *Let $x^\star(\tau)$ be a hyperbolic equilibrium branch of (5.2), i.e., assume that the matrix $A(\tau) := \partial_x f(x^\star(\tau),\tau)$ has no purely imaginary eigenvalue. Then*

1. *Equation (5.2) admits an adiabatic solution $\bar{x}(\tau)$ of order ε associated with $x^\star(\tau)$.*
2. *If $f(x,\tau) \in \mathcal{C}^{k+1}$, this solution admits an expansion of the form*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{k-1} \varepsilon^j x_j(\tau) + \mathcal{O}(\varepsilon^k). \tag{5.3}$$

3. *If $f(x,\tau)$ is analytic in an open complex set, the adiabatic solution admits an expansion*

$$\bar{x}(\tau) = x^\star(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j x_j(\tau) + \mathcal{O}(\mathrm{e}^{-1/C|\varepsilon|}), \tag{5.4}$$

where $N(\varepsilon) = \mathcal{O}(1/\varepsilon)$.

As in the 1D case, the proof will be divided into two steps. The first one is an iterative scheme, very similar to Iterative Scheme 4.1, which decreases the order of a drift term. We present it in Subsection 5.2.1, together with a generalization of Lemma 4.2 on exponential bounds.

In Subsection 5.2.2, we treat the problem of existence of adiabatic solutions for hyperbolic equilibria. The analysis is more difficult than in the 1D case, and uses Lyapunov functions in an essential way. The elliptic case, which requires a more careful control of the linearized equation, will be discussed later in Subsection 5.3.3.

5.2.1 Iterative Scheme

As in the 1D case, the change of variables $x = x^\star(\tau) + y$ yields

$$\varepsilon\dot{y} = A(\tau)y + b(y,\tau) + \varepsilon w(y,\tau), \tag{5.5}$$

where $A(\tau) := \partial_x f(x^\star(\tau),\tau)$, $b(y,\tau) = \mathcal{O}(|y|^2)$, and $w(\tau) := -\dot{x}^\star(\tau)$ is the drift term. If $A(\tau)$ has no zero eigenvalue, the implicit function theorem implies that (5.5) is of class $\mathcal{C}^k$. The order of the drift term can thus be decreased using the following iterative procedure.

Iterative Scheme 5.1. *Assume that for some $N \geqslant 1$,*

$$\varepsilon\dot{y}_{N-1} = A(\tau)y_{N-1} + b_{N-1}(y_{N-1},\tau,\varepsilon) + \varepsilon^N w_{N-1}(\tau,\varepsilon), \tag{5.6}$$

where all functions are of class $\mathcal{C}^{k-N+1}$, $A(\tau)$ is invertible and $b_{N-1} = \mathcal{O}(|y_{N-1}|^2) + \mathcal{O}(\varepsilon|y_{N-1}|)$.

The change of variables

$$y_{N-1} = y_N + \varepsilon^N u_N(\tau,\varepsilon), \qquad u_N(\tau,\varepsilon) := -A(\tau)^{-1} w_{N-1}(\tau,\varepsilon), \tag{5.7}$$

transforms (4.18) *into*

$$\varepsilon \dot{y}_N = A(\tau) y_N + b_N(y_N, \tau, \varepsilon) + \varepsilon^{N+1} w_N(\tau, \varepsilon), \tag{5.8}$$

where the functions

$$w_N(\tau,\varepsilon) := \frac{1}{\varepsilon^{N+1}} b_{N-1}(\varepsilon^N u_N, \tau, \varepsilon) - \dot{u}_N(\tau,\varepsilon) \tag{5.9a}$$

$$b_N(y_N,\tau,\varepsilon) := b_{N-1}(y_N + \varepsilon^N u_N, \tau, \varepsilon) - b_{N-1}(\varepsilon^N u_N, \tau, \varepsilon) \tag{5.9b}$$

are of class $\mathcal{C}^{k-N}$ *and* $b_N = \mathcal{O}(|y_N|^2) + \mathcal{O}(\varepsilon |y_N|)$.

Then the result below essentially follows from [Ne1].

Lemma 5.1. *Consider the equation*

$$\varepsilon \dot{y}_0 = A(\tau) y_0 + b_0(y_0, \tau, \varepsilon) + \varepsilon w_0(\tau, \varepsilon) \tag{5.10}$$

where $b_0 = \mathcal{O}(|y_0|^2) + \mathcal{O}(\varepsilon |y_0|)$ *and* $A(\tau)$ *is invertible.*

1. *If* A, b *and* w *are of class* $\mathcal{C}^k$, *there exists a differentiable change of variables of the form* $y_0 = y_{k-1} + \sum_{j=1}^{k-1} \varepsilon^j u_j(\tau, \varepsilon)$ *transforming* (5.10) *into*

$$\varepsilon \dot{y}_{k-1} = A(\tau) y_{k-1} + b_{k-1}(y_{k-1}, \tau, \varepsilon) + \varepsilon^k w_{k-1}(\tau, \varepsilon), \tag{5.11}$$

   with $b_{k-1} = \mathcal{O}(|y_{k-1}|^2) + \mathcal{O}(\varepsilon |y_{k-1}|)$.

2. *If* (5.10) *is analytic for* y_0 *in a complex neighborhood* $\mathcal{D}$ *of the origin, and for* τ *in a complex neighborhood of an interval* I, *there exists in smaller neighborhoods a change of variables of the form* $y_0 = z + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j u_j(\tau, \varepsilon)$ *transforming* (5.10) *into*

$$\varepsilon \dot{z} = A(\tau) z + \bar{b}(z, \tau, \varepsilon) + \mathrm{e}^{-1/C|\varepsilon|}\, \bar{w}(\tau, \varepsilon), \tag{5.12}$$

   with $\bar{b} = \mathcal{O}(|z|^2) + \mathcal{O}(\varepsilon |z|)$. *Moreover,* $y_0 - z = \mathcal{O}(\varepsilon)$.

PROOF: The proof essentially follows the one of Lemma 4.2, with the following modifications. The absolute value is replaced by the sup norm $|\cdot|$ (see Notation 2.2). The set $\Gamma(D)$ is defined by the conditions $|y| < D$, $|\operatorname{Im} \tau| < D$ and $|\operatorname{Re} \tau - s| < D$ $\forall s \in I$. The constant K is defined by $|A(\tau)^{-1}| < K$ for all τ. □

Definition 5.3. If $f(x,\tau)$ in (5.2) is of class $\mathcal{C}^{k+1}$ and admits a smooth branch of fixed points $x^\star(\tau)$, we associate with it the **adiabatic approximation of order** $k-1$

$$x^{(k-1)}(\tau,\varepsilon) := x^\star(\tau) + \sum_{j=1}^{k-1} \varepsilon^j u_j(\tau,\varepsilon), \tag{5.13}$$

where the functions u_j are defined recursively by Iterative Scheme 5.1. We may, of course, rearrange the terms as a power series in ε, neglecting terms of order ε^k.

If $f(x,\tau)$ is analytic in an open complex set, we call **adiabatic approximation of exponential order** the function

$$x^{(\omega)}(\tau,\varepsilon) := x^\star(\tau) + \sum_{j=1}^{N(\varepsilon)} \varepsilon^j u_j(\tau,\varepsilon), \tag{5.14}$$

where $N(\varepsilon)$ is of order $1/|\varepsilon|$.

5.2.2 Lyapunov Functions

Definition 5.4. Let $f(y) \in \mathcal{C}^2(\mathbb{C}^n, \mathbb{C}^n)$ be such that $f(0) = 0$. An **ideal Lyapunov function** associated with $f(y)$ is a function $V(y, y^*) \in \mathcal{C}^1(\mathcal{U} \setminus \{0\} \times \mathcal{U}^* \setminus \{0\}, \mathbb{R})$, where $\mathcal{U}$ is a neighborhood of the origin in $\mathbb{C}^n$, such that for all $y \in \mathcal{U} \setminus \{0\}$,

1. there exist strictly positive constants $V_\pm$ such that $V_-|y| \leqslant V(y, y^*) \leqslant V_+|y|$;
2. there is a constant $a_0 > 0$ such that $\partial_y V f(y) + \partial_{y^*} V f(y)^* \leqslant -a_0 V$.

Lemma 5.2. *If $A := \partial_y f(0)$ is contracting, then*

$$V(y, y^*) := \langle y | Y y \rangle^{1/2}, \qquad Y := \int_0^\infty \mathrm{e}^{A^* s} \mathrm{e}^{As} \,\mathrm{d}s \tag{5.15}$$

is an ideal Lyapunov function associated with $f(y)$.

Proof: It is shown in Lemma 5.11 of Appendix 5.A that the matrix Y exists and satisfies $A^* Y + Y A = -\mathbb{1}$. Moreover, since $\langle y | Y y \rangle = \int_0^\infty \|\mathrm{e}^{As} y\|_2^2 \,\mathrm{d}s > 0$, the matrix Y is positive definite. Thus, if $y_\pm$ are the extremal eigenvalues of Y, the properties of norms imply that V satisfies the first condition with $V_+ = \sqrt{n y_+}$ and $V_- = \sqrt{y_-}$. To check the second condition, we introduce the solution $z(t)$ of the initial value problem $\dot{z} = f(z)$, $z(0) = y$. Then, writing $b(y) := f(y) - Ay$,

$$\begin{aligned} 2V(\partial_y V f(y) + \partial_{y^*} V f(y)^*) &= \partial_y (V^2) f(y) + \partial_{y^*}(V^2) f(y)^* \\ &= \mathrm{d}_t V(z(t))^2\big|_{t=0} = \langle y | Y f(y) \rangle + \langle f(y) | Y y \rangle \\ &= \langle y | (A^* Y + Y A) y \rangle + \langle y | Y b(y) \rangle + \langle b(y) | Y y \rangle \\ &= -\|y\|_2^2 + 2\,\mathrm{Re}\langle b(y) | Y y \rangle, \end{aligned} \tag{5.16}$$

so that

$$\partial_y V f(y) + \partial_{y^*} V f(y)^* \leqslant -\frac{\|y\|_2}{2V_+} + \frac{\sqrt{n}}{V_-}\Big|\mathrm{Re}\langle b(y) | Y \frac{y}{\|y\|_2} \rangle\Big| \leqslant -\frac{V}{2V_+^2} + \mathcal{O}(V^2), \tag{5.17}$$

and the second condition is satisfied in a neighborhood of the origin. □

Remark 5.1. The Lemma shows that if the origin is asymptotically stable, it is always possible to find an ideal Lyapunov function, which depends only on the linearization of f. In specific cases, it may however be possible to find nonlinear Lyapunov functions which are valid in a larger neighborhood. This would allow a better control of the basin of attraction.

We can now return to the analysis of the equation

$$\varepsilon \dot{y} = A(\tau) y + b(y, \tau, \varepsilon) + \varphi(\varepsilon) w(\tau, \varepsilon), \tag{5.18}$$

which governs the evolution near an adiabatic approximation. Here $b(y, \tau, \varepsilon) = \mathcal{O}(|y|^2) + \mathcal{O}(\varepsilon|y|)$, $\varphi(\varepsilon) = \varepsilon^k$, and all function are differentiable if the initial function is of class $\mathcal{C}^k$, and $\mathrm{e}^{-1/C|\varepsilon|}$ in the analytic case.

Proposition 5.1. *If $A(\tau)$ is contracting, then* (5.18) *admits a solution of order $\varphi(\varepsilon)$.*

Proof: For sufficiently small ε, the function $g(y,\tau,\varepsilon) := A(\tau)y + b(y,\tau,\varepsilon)$ admits a family of ideal Lyapunov functions $V(y,y^*,\tau)$, depending smoothly on τ (given for instance by Lemma 5.2). Thus we have

$$\begin{aligned}\varepsilon\, \mathrm{d}_\tau V(y(\tau),y(\tau)^*,\tau) &= \partial_y V \varepsilon\dot{y} + \partial_{y^*} V \varepsilon \dot{y}^* + \varepsilon\partial_\tau V \\ &= \partial_y V g(y,\tau,\varepsilon) + \partial_{y^*} V g(y,\tau,\varepsilon)^* + \mathcal{O}(\varphi(\varepsilon)) + \mathcal{O}(\varepsilon V) \\ &\leqslant -a_0 V + c_1\varepsilon V + c_2\varphi(\varepsilon).\end{aligned} \tag{5.19}$$

We conclude using the Comparison Lemma 4.1. □

Together with Lemma 5.1, this result proves Theorem 5.1 in the case of an attracting equilibrium branch. By inverting direction of time, we also have the proof in the repelling case.

We now turn to the general hyperbolic case, which is more difficult to analyse, since trajectories are attracted exponentially fast in some directions, and repelled in others. We first simplify system (5.18). We know by Lemma 5.14 of Appendix 5.A that there exists an invertible matrix function $S(\tau)$, as smooth as $A(\tau)$, such that $S(\tau)^{-1}A(\tau)S(\tau) = D(\tau)$ is bloc–diagonal. It follows that the change of variables $y = S(\tau)z$ transforms (5.18) into

$$\varepsilon\dot{z} = D(\tau)z - \varepsilon S^{-1}\dot{S}z + S^{-1}b(Sz,\tau,\varepsilon) + \varphi(\varepsilon)S^{-1}w(\tau,\varepsilon), \tag{5.20}$$

which we rewrite, using the notation $z = (u,v)$, as

$$\begin{aligned}\varepsilon\dot{u} &= A_+(\tau)u + b_+(u,v,\tau,\varepsilon) + \varphi(\varepsilon)w_+(\tau,\varepsilon) \\ \varepsilon\dot{v} &= A_-(\tau)v + b_-(u,v,\tau,\varepsilon) + \varphi(\varepsilon)w_-(\tau,\varepsilon),\end{aligned} \tag{5.21}$$

where $A_+(\tau)$ is an expanding matrix, $A_-(\tau)$ is contracting, and the terms of order $\varepsilon|z|$ have been included in the term $b_\pm(u,v,\tau,\varepsilon) = \mathcal{O}(|u|^2+|v|^2) + \mathcal{O}(\varepsilon(|u|+|v|))$. For this equation, we can prove existence of an adiabatic solution.

Proposition 5.2. *System* (5.21) *admits a solution of order* $\varphi(\varepsilon)$.

Since the proof is relatively long, we give it in Appendix 5.B.1. With this result, we have completed the proof of Theorem 5.1. Note that as in the 1D case, we also have the

Corollary 5.1. *If* $A(\tau)$ *is hyperbolic, the adiabatic approximation* (5.13) *satisfies*

$$\varepsilon\dot{x}^{(k-1)} = f(x^{(k-1)},\tau) + \mathcal{O}(\varepsilon^k). \tag{5.22}$$

The proof is the same as for Corollary 4.1.

Let us briefly examine the behaviour of trajectories close to an adiabatic solution $\bar{x}(\tau)$, associated with the equilibrium branch $x^\star(\tau)$. Since by definition, $f(x^\star,\tau) = 0$ and $\bar{x}(\tau) = x^\star(\tau) + \mathcal{O}(\varepsilon)$, we have

$$\begin{aligned}f(\bar{x}+y,\tau) - f(x^\star+y,\tau) - f(\bar{x},\tau) &= \int_0^1 \partial_x\bigl[f(\bar{x}+sy,\tau) - f(x^\star+sy,\tau)\bigr]\,\mathrm{d}s\; y \\ &= \mathcal{O}(\varepsilon|y|).\end{aligned} \tag{5.23}$$

Thus the change of variables $x = \bar{x}(\tau) + y$ gives

$$\varepsilon\dot{y} = f(\bar{x}+y,\tau) - f(\bar{x},\tau) = f(x^\star+y,\tau) + \mathcal{O}(\varepsilon|y|). \tag{5.24}$$

In the stable case, this yields the following rough result.

Proposition 5.3. *If $A(\tau) := \partial_x f(x^\star(\tau), \tau)$ is contracting, there exists a neighborhood $\mathcal{U}$ of $x^\star(\tau_0)$ such that all orbits starting in this neighborhood at τ_0 satisfy*

$$|x(\tau) - \bar{x}(\tau)| \leqslant K|x(\tau_0) - \bar{x}(\tau_0)|\, \mathrm{e}^{-a(\tau-\tau_0)/\varepsilon} \tag{5.25}$$

for sufficiently small ε, for all $\tau \geqslant \tau_0$ in I, and some strictly positive constants K, a.

PROOF: If $V(y, y^*, \tau)$ is a smooth family of ideal Lyapunov functions associated with $f(x^\star(\tau) + y, \tau)$, we have $\varepsilon\dot{V} \leqslant -a_0 V + c_1\varepsilon V$. □

To obtain a more precise characterization of orbits in a neighborhood of adiabatic solutions, we have to analyse in more detail the solutions of equations of the form $\varepsilon\dot{x} = A(\tau)x + \mathcal{O}(|x|^2)$. This is a rather lengthy task, which will be undertaken in the next two sections.

5.3 Linear Systems

When linearized around an adiabatic solution, the system $\varepsilon\dot{x} = f(x, \tau)$ becomes

$$\varepsilon\dot{y} = A(\tau, \varepsilon)y, \tag{5.26}$$

where $A(\tau, \varepsilon)$ is a matrix–valued function, defined for τ in an interval $I \subset \mathbb{R}$, and for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. We will consider both the differentiable case, where $A(\tau, \varepsilon)$ is of class $\mathcal{C}^k$, $k \geqslant 1$, and the analytic case.

Unlike in one dimension, equation (5.26) is not solved easily. In general, we only know that the solution can be written in the form

$$y(\tau) = U(\tau, \tau_0, \varepsilon)y(\tau_0), \tag{5.27}$$

where the principal solution $U(\tau, \tau_0, \varepsilon)$ is a $\mathcal{C}^k$ matrix–valued function, of which we know a few elementary properties (see Proposition 2.15). In particular, linearity implies that if $a(\tau, \varepsilon)$ is some complex–valued $\mathcal{C}^k$ function (for instance, $a(\tau, \varepsilon) = \operatorname{Tr} A(\tau, \varepsilon)$), we have

$$y = \Big[\exp\frac{1}{\varepsilon}\int_{\tau_0}^{\tau} a(s, \varepsilon)\,\mathrm{d}s\Big]z, \qquad \varepsilon\dot{z} = [A(\tau, \varepsilon) - a(\tau, \varepsilon)]z. \tag{5.28}$$

Thus, only the relative value of the eigenvalues really counts.

The philosophy of adiabatic theory, which has been developed mainly for quantum mechanics, tells us that solutions of (5.26) should follow adiabatically the instantaneous eigenspaces of A. The natural way to study (5.26) is thus to decrease the order of off-diagonal terms in A using an iterative scheme. We present this approach in Subsection 5.3.1.

The iterative method, however, does not allow to remove off–diagonal terms completely. This can be source of serious problems for some systems we are interested in. We are thus led to a new approach, which consists in showing existence of a transformation diagonalizing (5.26) completely. In fact, this problem can be transformed into the problem of finding adiabatic solutions of an auxiliary equation, so that we can use tools developed in the previous section. We apply this method to the hyperbolic case in Subsection 5.3.2, and to the elliptic case in Subsection 5.3.3.

Diagonalization requires in principle the eigenvalues of $A(\tau,\varepsilon)$ to be distinct. Thus the problem of eigenvalue crossings has to be analysed separately. In Subsection 5.3.4, we briefly discuss how to classify the possible crossings, before studying the most generic cases in subsequent subsections. The generic case, discussed in Subsection 5.3.5, occurs when A cannot be diagonalized at the crossing point. This is a well-known turning point problem, which can be reduced to Airy's equation. The second case, discussed in Subsection 5.3.6, occurs when (due, for instance, to a symmetry), the matrix remains diagonalizable at the crossing point. It turns out that in this case, the problem can be transformed into a previously studied bifurcation problem, which allows us to treat bifurcations and eigenvalue crossings in a unified way. This remains true for the "eigenvalue cruising" discussed in Subsection 5.3.7, which is equivalent to a Hopf bifurcation.

When dealing with bloc–diagonalization, we will use the following terminology.

Definition 5.5. Let $\{a_1(\tau),\dots,a_p(\tau)\}$ and $\{a_{p+1}(\tau),\dots,a_n(\tau)\}$ be two groups of continuous complex–valued functions on the real interval I. We call **gap** between these sets the quantity

$$\gamma := \inf_{\substack{\tau\in I\\ 1\leqslant i\leqslant p\\ p+1\leqslant j\leqslant n}} \bigl|a_i(\tau)-a_j(\tau)\bigr|. \tag{5.29}$$

We call **real gap** between these sets the quantity

$$\gamma_{\mathrm{r}} := \inf_{\substack{\tau\in I\\ 1\leqslant i\leqslant p\\ p+1\leqslant j\leqslant n}} \bigl|\mathrm{Re}\bigl(a_i(\tau)-a_j(\tau)\bigr)\bigr|. \tag{5.30}$$

We will use, moreover, some known properties of matrices, which we gathered in Appendix 5.A.

5.3.1 Pseudo–Diagonalization

Consider the equation $\varepsilon\dot y = A(\tau,\varepsilon)y$, where we assume for simplicity[1] that $A(\tau,\varepsilon) = A(\tau,0)+\mathcal{O}(\varepsilon)$. Let us further assume that the eigenvalues of $A(\tau,0)$ can be split into two groups $\{a_1(\tau),\dots,a_p(\tau)\}$ and $\{a_{p+1}(\tau),\dots,a_n(\tau)\}$ with gap $\gamma>0$. If $A(\tau,0)$ is of class $\mathcal{C}^k$, we know by Lemma 5.14 existence of a smooth invertible matrix function, $S_0(\tau)\in\mathcal{C}^k(I,\mathrm{GL}(n,\mathbb{C}))$, such that $S_0(\tau)^{-1}A(\tau,0)S_0(\tau)$ is bloc–diagonal. Thus the change of variables $y=S_0(\tau)y_1$ transforms (5.26) into

$$\varepsilon\dot y_1 = A_1(\tau,\varepsilon)y_1, \qquad A_1 = S_0^{-1}AS_0-\varepsilon S_0^{-1}\dot S_0 = \begin{pmatrix} A^0_{11}+\varepsilon A^1_{11} & \varepsilon A^1_{12}\\ \varepsilon A^1_{21} & A^0_{22}+\varepsilon A^1_{22}\end{pmatrix}, \tag{5.31}$$

where $A^0_{11}(\tau)$ and $A^0_{22}(\tau)$ have respectively the sets of eigenvalues $\{a_1(\tau),\dots,a_p(\tau)\}$ and $\{a_{p+1}(\tau),\dots,a_n(\tau)\}$. Even if we manage to diagonalize $A(\tau,\varepsilon)$ statically, the drift term $-\varepsilon S_0^{-1}\dot S_0$ will produce off–diagonal terms of order ε. The natural idea is to decrease the order of these terms by further instantaneous diagonalizations.

Iterative Scheme 5.2. *Consider the equation*

$$\varepsilon\dot y_N = A_N(\tau,\varepsilon)y_N, \qquad A_N(\tau,\varepsilon) = \begin{pmatrix} A^0_{11}(\tau)+\varepsilon A^N_{11}(\tau,\varepsilon) & \varepsilon^N A^N_{12}(\tau,\varepsilon)\\ \varepsilon^N A^N_{21}(\tau,\varepsilon) & A^0_{22}(\tau)+\varepsilon A^N_{22}(\tau,\varepsilon)\end{pmatrix}, \tag{5.32}$$

[1] This assumption is not necessary if the eigenvalues and eigenspaces of A are known for positive ε.

where $A_N(\tau,\varepsilon)$ is of class $\mathcal{C}^{k-N}$. Consider the change of variables

$$y_N = S_N(\tau,\varepsilon) y_{N+1}, \qquad S_N(\tau,\varepsilon) = \begin{pmatrix} \mathbb{1} & \varepsilon^N S_{12}^N(\tau,\varepsilon) \\ \varepsilon^N S_{21}^N(\tau,\varepsilon) & \mathbb{1} \end{pmatrix}, \tag{5.33}$$

where $S_{12}^N(\tau,\varepsilon)$ and $S_{21}^N(\tau,\varepsilon)$ are solutions of

$$\begin{aligned} A_{11}^0(\tau) S_{12}^N - S_{12}^N A_{22}^0(\tau) &= -A_{12}^N(\tau,\varepsilon) \\ A_{22}^0(\tau) S_{21}^N - S_{21}^N A_{11}^0(\tau) &= -A_{21}^N(\tau,\varepsilon). \end{aligned} \tag{5.34}$$

Then y_{N+1} obeys the equation

$$\varepsilon \dot{y}_{N+1} = A_{N+1}(\tau,\varepsilon) y_{N+1}, \qquad A_{N+1}(\tau,\varepsilon) = \begin{pmatrix} A_{11}^0 + \varepsilon A_{11}^{N+1} & \varepsilon^{N+1} A_{12}^{N+1} \\ \varepsilon^{N+1} A_{21}^{N+1} & A_{22}^0 + \varepsilon A_{22}^{N+1} \end{pmatrix}, \tag{5.35}$$

where $A_{N+1}(\tau,\varepsilon) \in \mathcal{C}^{k-N-1}$ and $\lim_{\varepsilon\to 0} |A_{ij}^{N+1}(\tau,\varepsilon)| < \infty$.

Remark 5.2. Lemma 5.10 of Appendix 5.A shows that equation (5.34) admits a solution, since A_{11}^0 and A_{22}^0 have no common eigenvalue. Lemma 5.13 shows that this solution can be written as

$$S_{12}^N = \frac{1}{2\pi\,\mathrm{i}} \int_\Gamma (A_{11}^0 - z)^{-1} A_{12}^N (A_{22}^0 - z)^{-1} \,\mathrm{d}z, \tag{5.36}$$

where Γ is a closed complex path encircling only the eigenvalues of A_{11}^0. In the hyperbolic case, when A_{11}^0 is contracting and A_{22}^0 is expanding, the solution can also be written as

$$S_{12}^N = \int_0^\infty \mathrm{e}^{A_{11}^0 s} A_{12}^N \,\mathrm{e}^{-A_{22}^0 s} \,\mathrm{d}s, \tag{5.37}$$

by Lemma 5.11.

Theorem 5.2. *Let $A(\tau,\varepsilon)$ have eigenvalues split into two groups with gap $\gamma > 0$.*

1. *If $A(\tau,\varepsilon) \in \mathcal{C}^k$, there exists a change of variables*

$$y = S^{(k)}(\tau,\varepsilon) y_k, \qquad S^{(k)}(\tau,\varepsilon) = S_0(\tau) + \sum_{j=1}^{N} S_j^{(k)}(\tau)\varepsilon^j, \tag{5.38}$$

transforming the equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ into

$$\varepsilon \dot{y}_k = A_k(\tau,\varepsilon) y_k, \qquad A_k(\tau,\varepsilon) = \begin{pmatrix} A_{11}^0 + \varepsilon A_{11}^k & \varepsilon^k A_{12}^k \\ \varepsilon^k A_{21}^k & A_{22}^0 + \varepsilon A_{22}^k \end{pmatrix}. \tag{5.39}$$

2. *If $A(\tau,\varepsilon)$ is analytic, there exists a change of variables*

$$y = \bar{S}(\tau,\varepsilon) z, \qquad \bar{S}(\tau,\varepsilon) = S_0(\tau) + \sum_{j=1}^{N(\varepsilon)} \bar{S}_j(\tau)\varepsilon^j, \tag{5.40}$$

transforming the equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ into

$$\varepsilon \dot{z} = \bar{A}(\tau,\varepsilon) z, \qquad \bar{A}(\tau,\varepsilon) = \begin{pmatrix} A_{11}^0 + \varepsilon \bar{A}_{11} & \mathrm{e}^{-1/C|\varepsilon|} \bar{A}_{12} \\ \mathrm{e}^{-1/C|\varepsilon|} \bar{A}_{21} & A_{22}^0 + \varepsilon \bar{A}_{22} \end{pmatrix}. \tag{5.41}$$

The proof, which is very similar to that of Theorem 5.1, is given in Appendix 5.B.2. This result is however not very satisfactory (for classical Dynamical Systems), since even exponentially small off–diagonal terms may induce appreciable transition amplitudes when the eigenvalues have different real parts.[2]

Example 5.1. Let $\varphi(\varepsilon) \to 0$ continuously as $\varepsilon \to 0$, and consider the equation

$$\begin{aligned} \varepsilon\dot{\xi} &= -\xi \\ \varepsilon\dot{\eta} &= -2\eta + \varphi(\varepsilon)\xi \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \xi(\tau) &= \mathrm{e}^{-\tau/\varepsilon}\,\xi(0) \\ \eta(\tau) &= \mathrm{e}^{-2\tau/\varepsilon}\,\eta(0) + \varphi(\varepsilon)[\mathrm{e}^{-\tau/\varepsilon} - \mathrm{e}^{-2\tau/\varepsilon}]\xi(0). \end{aligned} \tag{5.42}$$

Even if $\varphi(\varepsilon)$ is exponentially small, the off-diagonal term $\varphi(\varepsilon)\,\mathrm{e}^{-\tau/\varepsilon}\,\xi(0)$ may dominate the term $\mathrm{e}^{-2\tau/\varepsilon}\,\eta(0)$.

To avoid this kind of problem, we will develop an alternative approach, aiming at complete diagonalization of the equation.

5.3.2 Complete Diagonalization - Hyperbolic Case

Example 5.2. Consider the two–dimensional system $\varepsilon\dot{y} = A(\tau,\varepsilon)y$. After a first static diagonalization $y = S_0(\tau)y_1$, this equation becomes

$$\varepsilon\dot{y}_1 = A_1(\tau,\varepsilon)y_1, \qquad A_1 = S_0^{-1}AS_0 - \varepsilon S_0^{-1}\dot{S}_0 = \begin{pmatrix} a_{11}(\tau,\varepsilon) & \varepsilon a_{12}(\tau,\varepsilon) \\ \varepsilon a_{21}(\tau,\varepsilon) & a_{22}(\tau,\varepsilon) \end{pmatrix}, \tag{5.43}$$

where $a_{ii}(\tau,\varepsilon) = a_i(\tau) + \mathcal{O}(\varepsilon)$, $a_i(\tau)$ being the eigenvalues of $A(\tau,0)$. Instead of applying the iterative scheme again, let us look for a transformation $y_1 = S_1(\tau,\varepsilon)z$, yielding a diagonal equation $\varepsilon\dot{z} = D(\tau,\varepsilon)z$, $D = \mathrm{diag}(d_1,d_2)$. The relation between these matrices is given by

$$D = S_1^{-1}A_1S_1 - \varepsilon S_1^{-1}\dot{S}_1 \qquad \Rightarrow \qquad \varepsilon\dot{S}_1 = A_1S_1 - S_1D. \tag{5.44}$$

We would like to consider (5.44) as an initial value problem for the matrix function $S(\tau,\varepsilon)$. If we manage to find an initial condition $S_1(\tau_0,\varepsilon)$ such that $S_1(\tau,\varepsilon)$ remains close to identity, we can use it to diagonalize (5.43) dynamically. Specifically, let us look for a matrix of the form

$$S(\tau,\varepsilon) = \begin{pmatrix} 1 & \varepsilon s_{12}(\tau,\varepsilon) \\ \varepsilon s_{21}(\tau,\varepsilon) & 1 \end{pmatrix}. \tag{5.45}$$

Then (5.44) is equivalent to the system of equations

$$\begin{aligned} 0 &= a_{11} + \varepsilon^2 a_{12}s_{21} - d_1 \\ 0 &= a_{22} + \varepsilon^2 a_{21}s_{12} - d_2 \\ \varepsilon\dot{s}_{12} &= (a_{11} - d_2)s_{12} + a_{12} \\ \varepsilon\dot{s}_{21} &= (a_{22} - d_1)s_{21} + a_{21} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \varepsilon\dot{s}_{12} &= a_{12} + (a_{11} - a_{22})s_{12} - \varepsilon^2 a_{21}s_{12}^2 \\ \varepsilon\dot{s}_{21} &= a_{21} + (a_{22} - a_{11})s_{21} - \varepsilon^2 a_{12}s_{21}^2. \end{aligned} \tag{5.46}$$

The first two equations determine d_1 and d_2, which will be close to the eigenvalues $a_1(\tau)$, $a_2(\tau)$, provided $s_{21}(\tau,\varepsilon)$ and $s_{12}(\tau,\varepsilon)$ remain of order 1. The last two equations are independent differential equations for s_{21} and s_{12}, of the form $\varepsilon\dot{s} = f(s,\tau,\varepsilon)$, which we have already extensively studied in Chapter 4. Let us consider two particular cases.

[2] In the case of quantum mechanics, where the eigenvalues are imaginary, this is not a serious problem since the transition amplitudes are effectively small, and the off–diagonal terms only affect the phase.

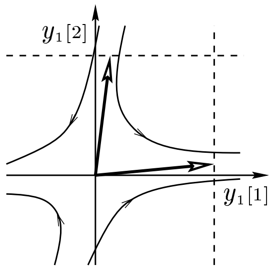

FIGURE 5.1. Dynamic eigenvectors for a 2D hyperbolic system. Thin lines show orbits of (5.43) for a given time. The dynamic eigenvectors (thick lines) follow these orbits, but are constrained to remain on the dashed lines.

1. $\operatorname{Re} a_{11}(\tau,\varepsilon) > \operatorname{Re} a_{22}(\tau,\varepsilon)$:
The equation for s_{21} has a stable fixed point at $s_{21}^\star = -a_{21}/(a_{22}-a_{11}) + \mathcal{O}(\varepsilon^2)$ (see Fig. 5.1 for a geometric interpretation). By Theorem 4.1, we know about existence of an adiabatic solution $\bar{s}_{21}(\tau,\varepsilon) = s_{21}^\star(\tau,\varepsilon) + \mathcal{O}(\varepsilon)$. Similarly, the equation for a_{12} admits an unstable fixed point $s_{12}^\star = -a_{12}/(a_{11}-a_{22}) + \mathcal{O}(\varepsilon^2)$, and an associated adiabatic solution. These adiabatic solutions have the following geometric interpretation: the vectors

$$\begin{pmatrix} 1 \\ \varepsilon s_{21}(\tau,\varepsilon) \end{pmatrix}, \qquad \begin{pmatrix} \varepsilon s_{12}(\tau,\varepsilon) \\ 1 \end{pmatrix} \tag{5.47}$$

can be considered as **dynamic eigenvectors** of (5.43). Each of them is collinear to an actual solution of the equation, but we impose one of their coordinates to be equal to unity. The first vector points in the unstable direction, and thus naturally remains close to the coordinate axis. The second vector is in unstable equilibrium. If its initial condition is not carefully chosen, it will tend to align with the first coordinate axis, and its first component will explode. But we know that a particular solution exists, which follows the stable direction adiabatically.
2. $\operatorname{Re} a_{11}(\tau,\varepsilon) = \operatorname{Re} a_{22}(\tau,\varepsilon)$:
In this case, each equation has the form $\varepsilon\dot{s} = a(\tau) + \mathrm{i}\,\omega(\tau)s + \mathcal{O}(\varepsilon^2 s^2)$, and admits an elliptic fixed point. This case has also been studied in Subsection 4.2.2, and we know that adiabatic solutions exist, at least for finite time.

It is not difficult to extend the technique introduced in this example to the general n–dimensional case. After a first static bloc–diagonalization, the system is cast into the form

$$\varepsilon\dot{y}_1 = A_1(\tau,\varepsilon)y_1, \qquad A_1(\tau,\varepsilon) = \begin{pmatrix} A_{11}(\tau,\varepsilon) & \varepsilon A_{12}(\tau,\varepsilon) \\ \varepsilon A_{21}(\tau,\varepsilon) & A_{22}(\tau,\varepsilon) \end{pmatrix}, \tag{5.48}$$

where the matrix $A_{11}(\tau,0)$ has eigenvalues $a_1(\tau),\dots,a_p(\tau)$, while $A_{22}(\tau,0)$ has eigenvalues $a_{p+1}(\tau),\dots,a_n(\tau)$. We assume that these two groups have a real gap $\gamma_{\mathrm{r}} > 0$. We look again at the equation

$$\varepsilon\dot{S}_1 = AS_1 - S_1D, \qquad S_1(\tau,\varepsilon) = \begin{pmatrix} \mathbb{1}_p & \varepsilon S_{12} \\ \varepsilon S_{21} & \mathbb{1}_{n-p} \end{pmatrix}, \qquad D(\tau,\varepsilon) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{5.49}$$

which is equivalent to

$$D_1 = A_{11} + \varepsilon^2 A_{12} S_{21}, \qquad \varepsilon \dot{S}_{21} = A_{21} + A_{22} S_{21} - S_{21} A_{11} - \varepsilon^2 S_{21} A_{12} S_{21}, \tag{5.50a}$$

$$D_2 = A_{22} + \varepsilon^2 A_{21} S_{12}, \qquad \varepsilon \dot{S}_{12} = A_{12} + A_{11} S_{12} - S_{12} A_{22} - \varepsilon^2 S_{12} A_{21} S_{12}. \tag{5.50b}$$

Each of these equations is of the form $\varepsilon \dot{S} = f(S, \tau, \varepsilon)$, and can be analysed with the method of Section 5.2.

Theorem 5.3. *Let I be a real interval, and $A(\tau,\varepsilon) \in \mathcal{C}^k(I \times [0,\varepsilon_0], \mathbb{M}_n(\mathbb{C}))$, with $k \geqslant 3$. Assume that the eigenvalues of $A(\tau,0)$ can be split into two groups, $\{a_1(\tau),\dots,a_p(\tau)\}$ and $\{a_{p+1}(\tau),\dots,a_n(\tau)\}$, with real gap $\gamma_{\mathrm{r}} > 0$.*

1. *There exists, for sufficiently small ε and for $\tau \in I$, an invertible matrix $S(\tau,\varepsilon) \in \mathcal{C}^{k-1}$ such that the solutions of $\varepsilon \dot{y} = A(\tau,\varepsilon) y$ satisfy*

$$y = S(\tau,\varepsilon) z, \qquad \varepsilon \dot{z} = D(\tau,\varepsilon) z, \tag{5.51}$$

   where $D(\tau,\varepsilon)$ is bloc–diagonal, with blocs of size p and $n-p$.
2. *The matrices $S(\tau,\varepsilon)$ and $D(\tau,\varepsilon)$ may be expanded in powers of ε up to order $k-1$, and for $\varepsilon = 0$ they satisfy $D(\tau,0) = S(\tau,0)^{-1} A(\tau,0) S(\tau,0)$.*
3. *If $A(\tau,\varepsilon)$ is analytic for τ in a complex neighborhood of the interval I, the matrices $S(\tau,\varepsilon)$ and $D(\tau,\varepsilon)$ admit asymptotic series in ε which can be truncated in such a way as to approach their value exponentially closely.*

Proof: If $S_1(\tau,\varepsilon)$ satisfies (5.49), then the matrix $S(\tau,\varepsilon) = S_0(\tau) S_1(\tau,\varepsilon)$ satisfies the theorem. Consider for instance equation (5.50b), which can be written in the form $\varepsilon \dot{S}_{12} = f(S_{12},\tau,\varepsilon)$, where $S_{12} \in \mathbb{M}_{(n-p)\times p}(\mathbb{C})$, which is isomorphic to $\mathbb{C}^{p(n-p)}$. The equation $f(S_{12},\tau,0) = 0$ has the solution $S^\star_{12}(\tau,0) = -L[A_{11}(\tau,0), A_{22}(\tau,0)]^{-1} A_{12}(\tau,0)$, where the linear operator $L[A_{11},A_{22}] : X \mapsto A_{11} X - X A_{22}$ is invertible since A_{11} and A_{22} have no common eigenvalue (see Appendix 5.A.1). Since the Fréchet derivative $\partial_S f(S^\star_{12},\tau,0)$ is identical with the linear operator $L[A_{11},A_{22}]$, the implicit function theorem ensures the existence of a fixed point $S^\star_{12}(\tau,\varepsilon) = -L[A_{11},A_{22}]^{-1} A_{12} + \mathcal{O}(\varepsilon)$ for small ε.

The linearization of f around $S^\star_{12}$ is the linear operator $\partial_S f(S^\star_{12}(\tau,\varepsilon),\tau,\varepsilon)$:

$$\begin{aligned} X \mapsto & \frac{\mathrm{d}}{\mathrm{d}\lambda} \Big[A_{12} + A_{11}(S^\star_{12} + \lambda X) - (S^\star_{12} + \lambda X) A_{22} - \varepsilon^2 (S^\star_{12} + \lambda X) A_{21} (S^\star_{12} + \lambda X) \Big] \Big|_{\lambda=0} \\ &= (A_{11} - \varepsilon^2 S^\star_{12} A_{21}) X - X (A_{22} + \varepsilon^2 A_{21} S^\star_{12}). \end{aligned} \tag{5.52}$$

If we denote the eigenvalues of this operator by $\lambda_{ij}(\tau,\varepsilon)$, $i = 1,\dots,p$, $j = 1,\dots,(n-p)$, we know by Corollary 5.5 in Appendix 5.A.1 that $\lambda_{ij}(\tau,0) = a_i(\tau) - a_{p+j}(\tau)$. The real gap hypothesis implies that $\operatorname{Re}\lambda_{ij}(\tau,0) \neq 0$, and since the eigenvalues depend continuously on ε, the fixed point $S^\star_{12}(\tau,\varepsilon)$ is hyperbolic for small ε. Theorem 5.1 implies the existence of an adiabatic solution $\bar{S}_{12}(\tau,\varepsilon) = S^\star_{12}(\tau,\varepsilon) + \mathcal{O}(\varepsilon)$, admitting asymptotic series in ε. □

Corollary 5.2. *If all eigenvalues $a_j(\tau)$ of $A(\tau,0)$ have a different real part, then the solution of $\varepsilon \dot{y} = A(\tau,\varepsilon) y$ can be written*

$$\begin{aligned} y(\tau) &= S(\tau,\varepsilon) U_0(\tau,\tau_0,\varepsilon) S(\tau_0,\varepsilon)^{-1} y(\tau_0), \\ U_0(\tau,\tau_0,\varepsilon) &= \operatorname{diag}(\mathrm{e}^{\alpha_1(\tau,\tau_0,\varepsilon)/\varepsilon}, \dots, \mathrm{e}^{\alpha_n(\tau,\tau_0,\varepsilon)/\varepsilon}), \\ \alpha_j(\tau,\tau_0,\varepsilon) &= \int_{\tau_0}^{\tau} a_j(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon). \end{aligned} \tag{5.53}$$

Equations (5.51) and (5.53) show that we have achieved to isolate the part of evolution depending in a singular way on ε in a (bloc–)diagonal matrix. We avoid in this way the problems of exponentially large transition amplitudes. The matrix $S(\tau,\varepsilon)$, which describes the motion of the "dynamic eigenspaces", depends smoothly on ε. It can be approximated using Iterative Scheme 5.2, i.e., we have $S(\tau,\varepsilon) = S^{(k)}(\tau,\varepsilon) + \mathcal{O}(\varepsilon^{k+1})$. The matrix $D(\tau,\varepsilon)$ is then computed using (5.50). Both matrices contain remainders which we are not able to compute in general, but unlike off–diagonal terms in A, their influence on the dynamics can be controlled.

Remark 5.3.

1. The choice of the matrix S, satisfying $\varepsilon\dot{S} = AS - SD$, is not unique. In particular, if C is an invertible matrix commuting with D, then the matrix $\tilde{S} := SC$ satisfies the equation
$$\varepsilon\dot{\tilde{S}} = A\tilde{S} - \tilde{S}\tilde{D}, \qquad \tilde{D} = D - \varepsilon C^{-1}\dot{C}. \tag{5.54}$$
2. In the case of a positive real gap, Theorem 5.3, together with Corollary 5.1, provides an alternative, much simpler proof of Theorem 5.2. Indeed, the Corollary states that the adiabatic approximation $S^{(k)}$, obtained by the Iterative Scheme, satisfies (5.49) up to an error of order ε^{k+1}.
3. The real gap condition is satisfied in particular when the eigenvalues a_1 to a_p have real parts smaller than some constant c_1, while the other eigenvalues have real parts larger than $c_2 > c_1$. But the condition may also be satisfied in more general situations, where the eigenvalue's real parts are intermingled.

Example 5.3. Let us slightly modify Example 5.1.

$$\varepsilon\dot{y} = \begin{pmatrix} -1 & 0 \\ \varepsilon h(\tau) & -2 \end{pmatrix} y. \tag{5.55}$$

Equations (5.46) show that we may take $s_{12}(\tau) = 0$, while s_{21} satisfies the equation $\varepsilon\dot{s}_{21} = h(\tau) - s_{21}$. This equation is of course solvable (because we have taken a triangular matrix), but we also know (see the example of Subsection 4.1.2) that its solution admits the asymptotic series $s_{21}(\tau,\varepsilon) = h(\tau) - \varepsilon\dot{h}(\tau) + \cdots$. The solution can be written

$$y(\tau) = \begin{pmatrix} 1 & 0 \\ \varepsilon s_{21}(\tau,\varepsilon) & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\tau/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-2\tau/\varepsilon} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon s_{21}(0,\varepsilon) & 1 \end{pmatrix} y(0). \tag{5.56}$$

5.3.3 Complete Diagonalization - Elliptic Case

We would now like to extend the previous results to the case where the eigenvalues of the matrix $A(\tau,\varepsilon)$ all have the same real part. By integrating out the trace (see equation (5.28)), we can always transform the problem into a similar one, where this real part is zero.

At first sight, the problem of diagonalizing such a matrix dynamically looks like a boot–strap problem. Indeed, we have to control the stability of solutions of equation (5.50), which admits an elliptic fixed point. But to study the equation

$$\varepsilon\dot{y} = A(\tau)y + b(x,\tau,\varepsilon) + \varepsilon^N w(\tau,\varepsilon), \tag{5.57}$$

$A(\tau)$ being elliptic, we need bounds on the solution of the linear equation $\varepsilon\dot{y} = A(\tau)y$.

It turns out that this problem can be solved by induction on the size of the matrix A. We thus obtain the following generalization of Theorem 5.3. The results are slightly weaker, because the phenomenon of resonance may cause solutions of an elliptic equation to grow with time.

Theorem 5.4. *Let $I = [0,T]$ be a* bounded *interval. Let $A(\tau,\varepsilon)$ be a matrix–valued function in $\mathcal{C}^k(I \times [0,\varepsilon_0], \mathbb{M}_n(\mathbb{C}))$ ($k \geqslant 3$), with eigenvalues $a_j(\tau,\varepsilon) = \mathrm{i}\,\omega_j(\tau) + \mathcal{O}(\varepsilon)$, where the real functions $\omega_j(\tau)$ are all distinct.*

1. *There exists an invertible matrix $S(\tau,\varepsilon)$ such that the solutions of $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ satisfy*

$$\begin{aligned} y(\tau) &= S(\tau,\varepsilon)U_0(\tau,\tau_0,\varepsilon)S(\tau_0,\varepsilon)^{-1}y(\tau_0), \\ U_0(\tau,\tau_0,\varepsilon) &= \mathrm{diag}(\mathrm{e}^{\mathrm{i}\,\phi_1(\tau,\tau_0,\varepsilon)/\varepsilon},\dots,\mathrm{e}^{\mathrm{i}\,\phi_n(\tau,\tau_0,\varepsilon)/\varepsilon}), \\ \phi_j(\tau,\tau_0,\varepsilon) &= \int_{\tau_0}^{\tau} \omega_j(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon). \end{aligned} \tag{5.58}$$

2. *As a consequence, the principal solution of $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ satisfies*

$$|U(\tau,0)|, |U(\tau,0)^{-1}| \leqslant c\,\mathrm{e}^{a\tau} \tag{5.59}$$

   for $\tau \in I$, where c and a are positive constants.

3. *The matrix $S(\tau,\varepsilon)$ and the functions $\phi(\tau,\tau_0,\varepsilon)$ admit asymptotic series in ε as in Theorem 5.3.*

We give the proof in Appendix 5.B.3.

Corollary 5.3. *Consider, for $\tau \in I = [0,T]$, the equation*

$$\varepsilon\dot{y} = A(\tau)y + b(y,\tau,\varepsilon) + \varepsilon^N w(\tau,\varepsilon), \tag{5.60}$$

where $A(\tau)$ satisfies the hypotheses of Theorem 5.4, $b(y,\tau,\varepsilon) = \mathcal{O}(|y|^2) + \mathcal{O}(\varepsilon|y|)$, and all functions are of class $\mathcal{C}^3$. Then, if $N \geqslant 2$, any solution of (5.60) with initial condition $y(0) = \mathcal{O}(\varepsilon^{N-1})$ satisfies $|y(\tau)| \leqslant c\varepsilon^{N-1}\,\mathrm{e}^{a\tau}$, for $\tau \in I$, where a and c are positive constants.

Proof: Let $y = U(\tau)z$, where $\varepsilon\dot{U} = A(\tau)U$, $U(0) = \mathbb{1}$. By Theorem 5.4, we know that $|U(\tau)|, |U(\tau)^{-1}| \leqslant c_1\,\mathrm{e}^{a_1\tau} \leqslant c_1\,\mathrm{e}^{a_1 T}$. We have $\varepsilon\dot{z} = U^{-1}b(Uz,\tau,\varepsilon) + \varepsilon^N U^{-1}w(\tau,\varepsilon)$. If $V = \langle z|z\rangle^{1/2}$, we obtain $\varepsilon\dot{V} \leqslant a_0\varepsilon V + b_0V^2 + \varepsilon^N w_0$. By Lemma 4.7, we have $V(\tau) \leqslant c_2\varepsilon^{N-1}$, and thus $|y(\tau)| \leqslant c\varepsilon^{N-1}\,\mathrm{e}^{a\tau}$. □

Remark 5.4. In Theorem 5.4, we have made no assumption on symmetry properties of A. If A is anti-hermitian, for instance, we know that the propagator is unitary, and thus remains bounded in norm.

5.3.4 Eigenvalue Crossings - Classification

Up to now, we have considered either the situation of eigenvalues admitting a gap, in which case the equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ can be pseudo–diagonalized, or the case of a real gap, when the system can be completely diagonalized. We now have to consider cases where these hypotheses are not satisfied, i.e., where some eigenvalues' real parts cross.

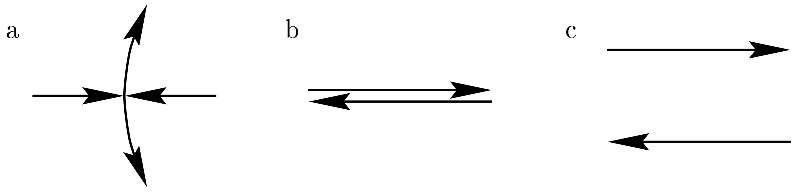

FIGURE 5.2. Eigenvalue crossings: (a) generic case, (b) diagonal case, (c) eigenvalue cruising.

For finite–dimensional equations, these crossings will generically involve pairs of eigenvalues. The other modes may then be decoupled using a dynamical bloc–diagonalization. Crossings involving two eigenvalues can thus be studied through two–dimensional systems $\varepsilon\dot{y} = A(\tau,\varepsilon)y$, $A \in \mathbb{M}_2(\mathbb{C})$. Moreover, we may introduce the following simplifications:

- we assume that the nature of the crossing does not change as $\varepsilon \to 0$, and thus we do not always indicate the ε–dependence of the functions;
- by translating the origin of time, we may assume that the crossing occurs at $\tau = 0$ (regardless of ε);
- using the transformation $y \mapsto \exp\frac{1}{\varepsilon}\int_{\tau_0}^{\tau}\operatorname{Tr}A(s)\,\mathrm{d}s\,y$, it is possible to replace A by a traceless matrix, so that we henceforth assume that $\operatorname{Tr}A = 0$. This means in particular that the propagator has unit determinant.

Crossings involving the real parts of two eigenvalues may then be divided into two classes:

1. The eigenvalues' real parts cross, but the imaginary parts are different. In such a case, Theorem 5.2 on pseudo–diagonalization applies, but not Theorem 5.3 on complete diagonalization, since the equations for S_{12} and S_{21} undergo Hopf bifurcation. We call this situation **eigenvalue cruising**, since the eigenvalues drift past one another at some imaginary distance. We discuss it in Subsection 5.3.7.
2. The eigenvalues are identical at the crossing time (which we may take equal to $\tau = 0$). In this case, we have a doubly degenerate eigenvalue at $\tau = 0$, which we may take equal to zero because of the preceding remarks. By Jordan's decomposition theorem, we know that there exists an invertible matrix S_0 such that

$$S_0^{-1}A(0)S_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.61}$$

We may thus carry out the change of variables $y = S_0 y_0$, giving

$$\varepsilon\dot{y}_0 = A_0(\tau)y_0, \qquad A_0(\tau) = \begin{pmatrix} b(\tau) & d(\tau) \\ c(\tau) & -b(\tau) \end{pmatrix}, \tag{5.62}$$

with the two subcases

(a) $b(0) = c(0) = 0$, but $d(0) = 1$: this is the most generic case, a turning point problem reducible to Airy's equation, which we will consider in Subsection 5.3.5;

(b) $b(0) = c(0) = d(0) = 0$: this case is less generic, but may happen because of the presence of a symmetry (e.g. the matrix is hermitian or anti–hermitian); we consider this situation in Subsection 5.3.6. Generically, the matrix A_0 can be diagonalized smoothly around the origin, and the equation can be transformed into a bifurcation problem.

Crossings involving more than two eigenvalues can be classified in a similar way. They can be reduced either to bifurcation problems of higher codimension, or to higher order turning point problems.

5.3.5 Eigenvalue Crossings - Generic Case

We consider in this Subsection the equation

$$\varepsilon\dot{y} = A(\tau)y, \qquad A(\tau) = \begin{pmatrix} b(\tau) & d(\tau) \\ c(\tau) & -b(\tau) \end{pmatrix} \in \mathcal{C}^2, \tag{5.63}$$

where $b(0) = c(0) = 0$ and $d(0) = 1$ (these functions may depend on ε).

Note that $A(\tau)$ has eigenvalues $\pm\sqrt{b^2+cd}$, and is similar to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ b^2+cd & 0 \end{smallmatrix}\right)$. This fact is used to further simplify (5.63). In a neighborhood of $\tau = 0$ where $d \neq 0$, we may carry out the change of variables

$$y = S(\tau)y_0, \qquad S(\tau) = \frac{1}{\sqrt{d}} \begin{pmatrix} d & 0 \\ -b + \frac{1}{2}\varepsilon\frac{\dot{d}}{d} & 1 \end{pmatrix}, \tag{5.64}$$

which transforms (5.63) into the equation

$$\varepsilon\dot{y}_0 = A_0(\tau)y_0, \qquad A_0(\tau) = \begin{pmatrix} 0 & 1 \\ h(\tau) & 0 \end{pmatrix}, \tag{5.65}$$

with

$$h(\tau) = b^2 + cd + \varepsilon\dot{b} - \varepsilon b\frac{\dot{d}}{d} - \frac{1}{2}\varepsilon^2\frac{\ddot{d}}{d} + \frac{3}{4}\varepsilon^2\frac{\dot{d}^2}{d}. \tag{5.66}$$

Writing $y_0 = (\xi, \eta)$, equation (5.65) is equivalent to

$$\varepsilon^2\ddot{\xi} = h(\tau)\xi, \qquad \eta = \varepsilon\dot{\xi} \tag{5.67}$$

which is well known, since it is equivalent to the stationary Schrödinger equation for a particle in a potential $h(\tau)+E$. The adiabatic limit $\varepsilon \to 0$ is equivalent to the semiclassical limit, which is usually treated with the WKB approximation.

In applications, we will be mainly interested in real matrices $A(\tau)$, so that we henceforth assume $h(\tau)$ to be real. We will consider the generic case where $\mathrm{d}_\tau \det A(\tau)|_{\tau=0} \neq 0$. In this situation, $h(\tau) = \det A(\tau) + \mathcal{O}(\varepsilon)$ vanishes at a time $\tau^\star = \mathcal{O}(\varepsilon)$, which we may set equal to zero by a time translation. Hence, we are going to study (5.65) in the case where $h(0) = 0$ and $\dot{h}(0) \neq 0$, say $\dot{h}(0) > 0$.

Example 5.4. Let us first consider the special case $h(\tau) = \tau$. Then the solution of (5.65) can be expressed in terms of Airy functions,

$$y_0(\tau) = \pi\begin{pmatrix} \mathrm{Bi}(\varepsilon^{-2/3}\tau) & \mathrm{Ai}(\varepsilon^{-2/3}\tau) \\ \varepsilon^{1/3}\,\mathrm{Bi}'(\varepsilon^{-2/3}\tau) & \varepsilon^{1/3}\,\mathrm{Ai}'(\varepsilon^{-2/3}\tau) \end{pmatrix}\begin{pmatrix} -\mathrm{Ai}'(\varepsilon^{-2/3}\tau_0) & \varepsilon^{-1/3}\,\mathrm{Ai}(\varepsilon^{-2/3}\tau_0) \\ \mathrm{Bi}'(\varepsilon^{-2/3}\tau_0) & -\varepsilon^{-1/3}\,\mathrm{Bi}(\varepsilon^{-2/3}\tau_0) \end{pmatrix} y_0(\tau_0). \tag{5.68}$$

Remark that the matrix $A_0(\tau)$ admits eigenvalues $\pm\sqrt{\tau}$, and can be diagonalized everywhere except at $\tau = 0$. Assuming $\tau_0 < 0 < \tau_1$, we would thus like to rewrite (5.68) in a way similar to (5.53), which distinguishes between a singular diagonal part depending on the eigenvalues, and a smooth transformation describing the dynamic eigenspaces' motion. This can be realized by using the asymptotic expressions of Airy functions (2.123), which may be reorganized in the form

$$y_0(\tau_1) = S^+(\tau_1)U^+(\tau_1)TU^-(\tau_0)S^-(\tau_0)^{-1}y(\tau_0), \tag{5.69}$$

where

$$U^+(\tau_1) = \begin{pmatrix} \mathrm{e}^{\frac{2}{3}\tau_1^{3/2}/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\frac{2}{3}\tau_1^{3/2}/\varepsilon} \end{pmatrix}, \qquad U^-(\tau_0) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\frac{2}{3}|\tau_0|^{3/2}/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\frac{2}{3}|\tau_0|^{3/2}/\varepsilon} \end{pmatrix} \tag{5.70}$$

represent the evolution of the diagonalized system, and

$$\begin{aligned} S^+(\tau_1) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \tau^{-1/4}u(\xi_1) & -\tau^{-1/4}u(-\xi_1) \\ \tau^{1/4}v(\xi_1) & \tau^{1/4}v(-\xi_1) \end{pmatrix}, \\ S^-(\tau_0)^{-1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\pi/4}|\tau_0|^{1/4}v(\mathrm{i}\,\xi_0) & \mathrm{e}^{-\mathrm{i}\,\pi/4}|\tau_0|^{-1/4}u(\mathrm{i}\,\xi_0) \\ -\,\mathrm{e}^{\mathrm{i}\,\pi/4}|\tau_0|^{1/4}v(-\,\mathrm{i}\,\xi_0) & \mathrm{e}^{-\mathrm{i}\,\pi/4}|\tau_0|^{-1/4}u(-\,\mathrm{i}\,\xi_0) \end{pmatrix} \end{aligned} \tag{5.71}$$

where $\xi_1 = \frac{2}{3\varepsilon}\tau_1^{3/2}$, $\xi_0 = \frac{2}{3\varepsilon}|\tau_0|^{3/2}$ and the functions u and v admit asymptotic series of the form $u(\xi) \asymp 1 + \sum_{k\geqslant 1} u_k\xi^{-k}$, $v(\xi) \asymp 1 + \sum_{k\geqslant 1} v_k\xi^{-k}$. But this simply means that $S^\pm$ admit asymptotic series in ε, and reduce, when $\varepsilon = 0$, to matrices which diagonalize $A_0(\tau)$.

Finally, the matrix T describes the actual transition, and can be written in polar decomposition as

$$T = \begin{pmatrix} 1 & \mathrm{i} \\ \frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix} = HU, \qquad H = H^* = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \in \mathrm{SU}(2). \tag{5.72}$$

These matrices can be interpreted as small corrections to the contraction/expansion amplitude of $U^+(\tau_1)$ and to the rotation phase of $U^-(\tau_0)$.

We are now going to show that similar results hold in the general case of equation (5.65).

Theorem 5.5. *Let $\tau_0 < 0 < \tau_1$, and let $h(\tau)$ be a real–valued $\mathcal{C}^2$ function on $[\tau_0, \tau_1]$, such that $\dot{h}(\tau) > 0$ and $h(0) = 0$. Then the solution of*

$$\varepsilon\dot{y} = \begin{pmatrix} 0 & 1 \\ h(\tau) & 0 \end{pmatrix} y \tag{5.73}$$

can be written in the form

$$y(\tau_1) = S^+(\tau_1)U^+(\tau_1)TU^-(\tau_0)S^-(\tau_0)^{-1}y(\tau_0), \tag{5.74}$$

where

$$S^+(\tau_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} h(\tau_1)^{-1/4} & -h(\tau_1)^{-1/4} \\ h(\tau_1)^{1/4} & h(\tau_1)^{1/4} \end{pmatrix} + \mathcal{O}(\varepsilon), \tag{5.75a}$$

$$S^-(\tau_0) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\pi/4}|h(\tau_0)|^{1/4} & \mathrm{e}^{-\mathrm{i}\,\pi/4}|h(\tau_0)|^{-1/4} \\ -\,\mathrm{e}^{\mathrm{i}\,\pi/4}|h(\tau_0)|^{1/4} & \mathrm{e}^{-\mathrm{i}\,\pi/4}|h(\tau_0)|^{-1/4} \end{pmatrix} + \mathcal{O}(\varepsilon) \tag{5.75b}$$

are smooth matrices describing the dynamic eigenspaces,

$$U^+(\tau_1) = \begin{pmatrix} \mathrm{e}^{\alpha(\tau_1)/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\alpha(\tau_1)/\varepsilon} \end{pmatrix}, \qquad \alpha(\tau_1) := \int_0^{\tau_1} h(s)^{1/2}\,\mathrm{d}s, \tag{5.75c}$$

$$U^-(\tau_0) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\phi(\tau_0)/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\phi(\tau_0)/\varepsilon} \end{pmatrix}, \qquad \phi(\tau_0) := \int_{\tau_0}^{0} |h(s)|^{1/2}\,\mathrm{d}s \tag{5.75d}$$

contain the singular part of evolution, and the transition is described by the matrix

$$T = \begin{pmatrix} 1 & \mathrm{i} \\ \frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix} + \mathcal{O}(\varepsilon^{2/3}). \tag{5.75e}$$

In the remainder of this subsection, we will prove this theorem and indicate how the corrections in ε can be calculated. To do this, we will divide the interval $[\tau_0, \tau_1]$ into two regions. If $c > 0$ is a constant, we will consider (5.73) separately in the **inner region** $|\tau| \leqslant c\varepsilon^{2/3}$, where it can be reduced to Airy's equation, and in the **outer region** $|\tau| \geqslant c\varepsilon^{2/3}$, where it can be diagonalized dynamically. To simplify the notations, we will assume that $\dot{h}(0) = 1$, but it is straightforward to check that the theorem remains true for arbitrary strictly positive[3] $\dot{h}(0)$.

1. Inner region $|\tau| \leqslant c\varepsilon^{2/3}$

Let us assume that $h(\tau) \in \mathcal{C}^k$, $k \geqslant 2$, and introduce the function

$$\chi_0(\tau) := \left[\frac{3}{2}\int_0^{\tau} \sqrt{h(s)}\,\mathrm{d}s\right]^{2/3}, \tag{5.76}$$

which satisfies the differential equation $\dot{\chi}_0 = \sqrt{h/\chi_0}$. The determination of $z^{2/3}$ can be chosen in such a way that $\chi_0(\tau)$ is $\mathcal{C}^k$ around the origin [**Wa**]. Then the coordinate changes[4]

$$y = \begin{pmatrix} \dot{\chi}_0^{-1/2} & 0 \\ -\frac{1}{2}\varepsilon\ddot{\chi}_0\dot{\chi}_0^{-3/2} & \dot{\chi}_0^{1/2} \end{pmatrix} y_1, \qquad \tau_1 = \chi_0(\tau) + \varepsilon^2 c_1 \tag{5.77}$$

transform the original equation into

$$\varepsilon\frac{\mathrm{d}y_1}{\mathrm{d}\tau_1} = \begin{pmatrix} 0 & 1 \\ \tau_1 + \varepsilon^2 h_1(\tau_1) & 0 \end{pmatrix}, \qquad h_1\bigl(\chi_0(\tau) + \varepsilon^2 c_1\bigr) = \frac{1}{2\dot{\chi}_0^{3/2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\ddot{\chi}_0}{\dot{\chi}_0^{3/2}}\right) - c_1, \tag{5.78}$$

and c_1 is chosen in such a way that $h_1(0) = 0$. Note that we have lost two derivatives in the procedure, while the order of nonlinear terms in h has been decreased to ε^2.

Lemma 5.3. *Let $c > 0$, $I = [-c\varepsilon^{2/3}, c\varepsilon^{2/3}]$, and $h(\tau) \in \mathcal{C}^k(I, \mathbb{C})$, $k \geqslant 2$, such that $h(0) = 0$, $\dot{h}(0) = 1$. Then there exists a matrix $S(\tau) \in \mathcal{C}^k$ and a function $\chi(\tau) \in \mathcal{C}^k$, $\chi(\tau) = \chi_0(\tau) + \mathcal{O}(\varepsilon)$, such that the coordinate changes $y = Sz$ and $\sigma = \chi(\tau)$ transform*

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}\tau} = \begin{pmatrix} 0 & 1 \\ h(\tau) & 0 \end{pmatrix} y \qquad \textit{into} \qquad \varepsilon\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} z. \tag{5.79}$$

[3] Of course, the case $\dot{h}(0) < 0$ can be treated by inverting (5.74).

[4] Transformation 5.77 is obtained by applying a shearing transformation described in [**Wa**], which attempts to eliminate nonlinear terms of h, followed by a change of coordinates similar to (5.64) eliminating diagonal terms.

The proof essentially follows from Theorem 3.1 p. 399 in [**Ol**]. It is shown there that (5.78) admits solutions of the form

$$y_1(\tau) = c_1\,\mathrm{Ai}(\varepsilon^{-2/3}\tau_1) + c_2\,\mathrm{Bi}(\varepsilon^{-2/3}\tau_1) + r(\tau_1,\varepsilon), \tag{5.80}$$

where the error $r(\tau_1,\varepsilon)$ is of order ε in the inner region I. Note, however, that for $\tau \geqslant c\varepsilon^{2/3}$, this remainder diverges as

$$r(\tau_1,\varepsilon) \sim \varepsilon^{1/6}\tau_1^{-1/4}\exp\bigl[\tfrac{2}{3}\tfrac{1}{\varepsilon}\tau_1^{3/2}\bigr]. \tag{5.81}$$

We provide an alternative proof in Appendix 5.B.4.

Let us now indicate how to compute $S(\tau)$ and $\chi(\tau)$. As long as the equation remains sufficiently differentiable, the order of $h(\tau)-\tau$ may be further decreased using the following iterative scheme.

Iterative Scheme 5.3. *Consider, for some $N \geqslant 1$, the equation*

$$\varepsilon\frac{\mathrm{d}y_N}{\mathrm{d}\tau_N} = \begin{pmatrix} 0 & 1 \\ \tau_N + \varepsilon^{2N}h_N(\tau_N) & 0 \end{pmatrix} y_N, \tag{5.82}$$

where h_N is of class $\mathcal{C}^k$, $k \geqslant 2$. Define the function $\chi_N(\tau_N)$ by the relation

$$\tau_N + \varepsilon^{2N}\chi_N(\tau_N) = \left[\frac{3}{2}\int_0^{\tau_N}\sqrt{s+\varepsilon^{2N}h_N(s)}\,\mathrm{d}s\right]^{2/3}. \tag{5.83}$$

Then the coordinate change

$$y_N = \begin{pmatrix} \dfrac{1}{\sqrt{1+\varepsilon^{2N}\dot\chi_N}} & 0 \\ -\dfrac{\varepsilon^{2N+1}\ddot\chi_N}{(1+\varepsilon^{2N}\dot\chi_N)^{3/2}} & \sqrt{1+\varepsilon^{2N}\dot\chi_N} \end{pmatrix} y_{N+1}, \qquad \tau_{N+1} = \tau_N + \varepsilon^{2N}\chi_N(\tau_N) + \varepsilon^{2N+2}c_{N+1} \tag{5.84}$$

transforms (5.82) *into*

$$\varepsilon\frac{\mathrm{d}y_{N+1}}{\mathrm{d}\tau_{N+1}} = \begin{pmatrix} 0 & 1 \\ \tau_{N+1} + \varepsilon^{2N+2}h_{N+1}(\tau_{N+1}) & 0 \end{pmatrix} y_{N+1}, \tag{5.85}$$

where

$$h_{N+1}(\tau_{N+1}) = \frac{1}{2}\frac{1}{(1+\varepsilon^{2N}\dot\chi_N)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}\tau_N}\left[\frac{\ddot\chi_N}{(1+\varepsilon^{2N}\dot\chi_N)^{3/2}}\right] - c_{N+1}, \tag{5.86}$$

and c_{N+1} can be chosen in such a way that $h_{N+1}(0) = 0$.

It follows from these considerations that the solution of our equation in the inner region can be written in the form

$$y(c\varepsilon^{2/3}) = S(c\varepsilon^{2/3})U_0\bigl(\chi(c\varepsilon^{2/3}),\chi(-c\varepsilon^{2/3})\bigr)S(-c\varepsilon^{2/3})^{-1}y(-c\varepsilon^{2/3}), \tag{5.87}$$

where U_0 is the propagator (5.68) associated with Airy's equation. Moreover, since $\chi_0(\tau) = \tau+\mathcal{O}(\tau^2)$ and $\chi(\tau) = \chi_0(\tau)+\mathcal{O}(\varepsilon)$, we conclude that in the inner region, $\dot\chi(\tau) = 1+\mathcal{O}(\varepsilon^{2/3})$ and $S(\tau) = \mathbb{1}+\mathcal{O}(\varepsilon^{2/3})$. Thus the solution (5.87) in the inner region differs only by $\mathcal{O}(\varepsilon^{2/3})$ from the solution of Airy's equation.

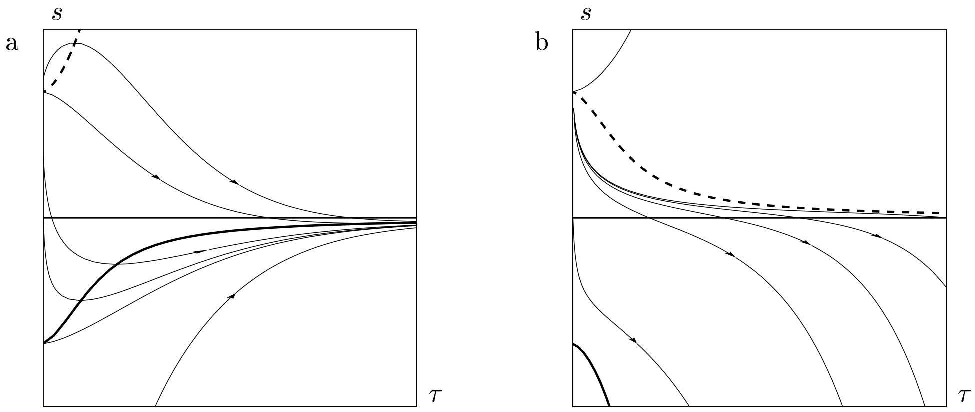

FIGURE 5.3. Orbits of equation 5.92. Thick lines indicate stable equilibrium branches, dashed lines indicate unstable ones. (a) When $\mu = -1$, any solution starting between -1 and 1 remains in this interval. (b) If $\mu = 1$, there exists a solution remaining in the interval $[0, 1]$. For $\tau \geqslant c\varepsilon^{2/3}$, these solutions are of order $\varepsilon/\tau^{3/2}$.

2. Outer region $|\tau| \geqslant c\varepsilon^{2/3}$

In this domain, the bounds given in [**Ol**] are not appropriate, because they diverge as $\exp(\varepsilon^{-2/3}\tau)$ (see (5.81)). We thus use a different approach based on dynamic diagonalization of the equation.

For $\tau \geqslant c\varepsilon^{2/3}$, let us carry out the change of variables

$$y = S_0^+(\tau)y_1, \qquad S_0^+(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{-1/4} & -h^{-1/4} \\ h^{1/4} & h^{1/4} \end{pmatrix}, \tag{5.88}$$

which yields the equation

$$\varepsilon\dot{y}_1 = A_1(\tau)y_1, \qquad A_1(\tau) = \begin{pmatrix} h^{1/2} & -\varepsilon\frac{\dot{h}}{4h} \\ -\varepsilon\frac{\dot{h}}{4h} & -h^{1/2} \end{pmatrix}. \tag{5.89}$$

It is now possible to diagonalize the equation using the transformation

$$y_1 = S_1^+(\tau)z, \qquad S_1^+(\tau) = \begin{pmatrix} 1 & s_2 \\ s_1 & 1 \end{pmatrix}, \qquad \begin{cases} \varepsilon\dot{s}_1 = 2\sqrt{h}\,s_1 + \varepsilon\frac{\dot{h}}{4h}(s_1^2 - 1) \\ \varepsilon\dot{s}_2 = -2\sqrt{h}\,s_2 + \varepsilon\frac{\dot{h}}{4h}(s_2^2 - 1), \end{cases} \tag{5.90}$$

which gives

$$\varepsilon\dot{z} = D(\tau)z, \qquad D(\tau) = \begin{pmatrix} \sqrt{h} + \varepsilon\frac{\dot{h}}{4h}s_1 & 0 \\ 0 & -\sqrt{h} + \varepsilon\frac{\dot{h}}{4h}s_2 \end{pmatrix}. \tag{5.91}$$

For $\tau \leqslant -c\varepsilon^{2/3}$, we obtain similar expressions, with $\sqrt{h}$ replaced by $\mathrm{i}\sqrt{-h}$. We now have to show that (5.90) admits bounded solutions.

Lemma 5.4. *Assume $h(\tau) = \tau + \mathcal{O}(\tau^2)$ is of class $\mathcal{C}^2$ and increasing on $[0, \tau_0]$. Let $\mu = \pm 1$ or $\pm\mathrm{i}$, and consider the equations*

$$\varepsilon\dot{s}^0 = 2\mu\sqrt{\tau}\,s^0 + \varepsilon\frac{1}{4\tau}((s^0)^2 - 1) \tag{5.92a}$$

$$\varepsilon\dot{s} = 2\mu\sqrt{h(\tau)}\,s + \varepsilon\frac{\dot{h}(\tau)}{4h(\tau)}(s^2 - 1). \tag{5.92b}$$

Then there exists a constant $c > 0$ such that (5.92a) *admits a solution $s^0(\tau) = \mathcal{O}(\varepsilon/\tau^{3/2})$ on $[c\varepsilon^{2/3}, \tau_0]$, and* (5.92b) *admits a solution $s(\tau) = s^0(\tau) + \mathcal{O}(\varepsilon/\sqrt{\tau})$.*

PROOF: We first show that both equations admit solutions such that $|s(\tau)| \leqslant 1$. Let $f_\mu(s,\tau)$ denote the right–hand–side of (5.92b). We distinguish 3 cases:

- If $\mu = -1$, the fact that $f_{-1}(1,\tau) \leqslant 0$ and $f_{-1}(-1,\tau) \geqslant 0$ implies that any solution starting in the interval $[-1,1]$ remains in that interval.
- If $\mu = 1$, the fact that $f_1(1,\tau) \geqslant 0$ and $f_1(0,\tau) \leqslant 0$ implies that there exists a solution remaining in the interval $[0,1]$.
- If $\mu = \pm\,\mathrm{i}$, we have $\varepsilon\,\mathrm{d}_\tau|s|^2 = \varepsilon\frac{\dot h}{4h}(s+s^*)(|s|^2-1)$, so that the circle $|s| = 1$ is invariant, and solutions starting within this circle remain inside.

Let $\xi := \frac{2}{3\varepsilon}\tau^{3/2}$. If $h(\tau) = \tau$, the product of the matrices $S_0^+(\tau)$ of (5.88) and $S_1^+(\tau)$ of (5.90) must be equal to the matrix $S^+(\tau)$ of (5.71). Thus $s^0(\tau)$ admits an asymptotic series $s^0 = \sum_{k\geqslant 1} a_k\xi^{-k} = \mathcal{O}(\varepsilon/\tau^{3/2})$.[5]

Finally, writing $s = s^0+s^1$, we obtain for s^1 the equation $\varepsilon\dot s^1 = a(\tau)s^1 + \varepsilon\frac{\dot h}{4h}(s^1)^2 + w(\tau)$, with $a(\tau) = 2\mu\sqrt{h} + \mathcal{O}(\varepsilon^2/\tau^{5/2})$ and $w(\tau) = \mathcal{O}(\varepsilon)$. By Lemma 4.5, with $p = \frac{1}{2}$ and $q = 1$, we know that this equation admits a solution of order $\varepsilon/\sqrt{\tau}$. □

We can now complete the proof of Theorem 5.5. Let us denote by $\tilde U^+$ the principal solution of (5.91), and by the superscript ^{0} all quantities relating to the special case $h(\tau) = \tau$. Using the bounds of Lemma 5.4 when integrating (5.91), we easily find that

$$\tilde U^+(\tau, c\varepsilon^{2/3}) = U^+(\tau)U^{+0}(\tau)^{-1}\tilde U^{+0}\big(\tau, c\varepsilon^{2/3}\big)\big(\mathbb{1} + \mathcal{O}(\varepsilon^{2/3})\big), \tag{5.93}$$

where $U^+(\tau) = \operatorname{diag}(\mathrm{e}^{\alpha/\varepsilon}, \mathrm{e}^{-\alpha/\varepsilon})$ is the matrix in (5.75), and $U^{+0}(\tau)$ is the corresponding one in Example 5.4. If $S^+(\tau) = S_0^+(\tau)S_1^+(\tau)$, we obtain that the propagators of the general case and of the special case $h(\tau) = \tau$ are related by

$$U\big(\tau, c\varepsilon^{2/3}\big) = S^+(\tau)U^+(\tau)U^{+0}(\tau)^{-1}S^{+0}(\tau)^{-1}U^0\big(\tau, c\varepsilon^{2/3}\big)\big(\mathbb{1} + \mathcal{O}(\varepsilon^{2/3})\big). \tag{5.94}$$

Combining this with a similar analysis in the case $\tau \leqslant -c\varepsilon^{2/3}$ and the result (5.87) for the inner region, and using the explicit expression (5.69) of the case $h(\tau) = \tau$, we obtain the conclusion of Theorem 5.5. □

5.3.6 Eigenvalue Crossings - Diagonal Case

We consider in this Subsection the equation

$$\varepsilon\dot y = A(\tau)y, \qquad A(\tau) = \begin{pmatrix} b(\tau) & d(\tau) \\ c(\tau) & -b(\tau) \end{pmatrix} \in \mathcal{C}^2, \tag{5.95}$$

where $b(0) = c(0) = d(0) = 0$ (these functions may depend on ε). The eigenvalues of $A(\tau)$ are $\pm\sqrt{-\det A} = \mathcal{O}(\tau)$. We will assume that $\mathrm{d}_{\tau^2}\det A(\tau)|_{\tau=0} \neq 0$. Then the function

$$a(\tau) := \begin{cases} \sqrt{-\det A(\tau)} & \text{if } \tau \geqslant 0 \\ -\sqrt{-\det A(\tau)} & \text{if } \tau < 0 \end{cases} \tag{5.96}$$

[5] One can also apply Theorem 12.1 of [**Wa**] to the equation $\mathrm{d}_\xi s^0 = 2\mu s^0 + ((s^0)^2-1)/6\xi$.

is a smooth function at $\tau = 0$ (where we have chosen the same determination of the square root in both cases). Generically, $A(\tau)$ can be statically diagonalized in a neighborhood of the origin, for instance by the matrix

$$S_0(\tau) := \frac{1}{\sqrt{ad}} \begin{pmatrix} d & -d \\ a-b & a+b \end{pmatrix} \tag{5.97}$$

We will assume that such a matrix exists and is smooth (of class $\mathcal{C}^2$) in the vicinity of $\tau = 0$. The change of variables $y = S_0(\tau)y_1$ transforms (5.95) into

$$\varepsilon \dot{y}_1 = A_1(\tau) y_1, \qquad A_1(\tau) = S_0^{-1}(A_0 S_0 - \varepsilon \dot{S}_0) = \begin{pmatrix} a(\tau) + \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & -a(\tau) + \varepsilon a_{22} \end{pmatrix}. \tag{5.98}$$

We now carry out a change of variables of the form

$$y_1 = S_1(\tau) z, \qquad S_1(\tau) = \begin{pmatrix} 1 & s_{12}(\tau) \\ s_{21}(\tau) & 1 \end{pmatrix}, \tag{5.99}$$

where the functions $s_{12}(\tau)$ and $s_{21}(\tau)$ are solutions of the equations

$$\varepsilon \dot{s}_{12} = (2a + \varepsilon a_{11} - \varepsilon a_{22}) s_{12} + \varepsilon a_{12} - \varepsilon a_{21} s_{12}^2 \tag{5.100a}$$
$$\varepsilon \dot{s}_{21} = -(2a + \varepsilon a_{11} - \varepsilon a_{22}) s_{21} + \varepsilon a_{21} - \varepsilon a_{12} s_{21}^2. \tag{5.100b}$$

After this transformation, the equation becomes

$$\varepsilon \dot{z} = D(\tau) z, \qquad D(\tau) = \begin{pmatrix} a + \varepsilon a_{11} + \varepsilon a_{12} s_{21} & 0 \\ 0 & -a + \varepsilon a_{22} + \varepsilon a_{21} s_{12} \end{pmatrix}, \tag{5.101}$$

which can be easily solved, but which is meaningful only if the equations (5.100) admit bounded solutions. These equations have the same form as those describing the motion near a 1D bifurcation point, which have been extensively studied in Chapter 4.

We are particularly interested in two cases:

1. $\det A(\tau) < 0$ **for** $\tau \neq 0 \Rightarrow a(\tau)$ **is real;**
   This situation occurs for instance if A is real and $b^2 > -cd$, or if A is hermitian. In particular, linearizations of gradient systems $\varepsilon \dot{x} = \partial_x F(x, \tau)$ have this property.
2. $\det A(\tau) > 0$ **for** $\tau \neq 0 \Rightarrow a(\tau)$ **is imaginary;**
   This situation occurs for instance if A is real and $b^2 < -cd$, or if A is anti–hermitian, as in the case of the time–dependent Schrödinger equation.

1. Real Case

Using the methods of Section 4.3, it is easy to show that

- equation (5.100a) admits bounded solutions of order $\sqrt{\varepsilon}$, and which decrease as $\varepsilon/|\tau|$ for $|\tau| \geqslant \sqrt{\varepsilon}$ (Fig. 5.4a);
- equation (5.100b) generically admits no bounded solution for all times, but it admits a solution $s_{12}^-(\tau)$ which is bounded and of order $\sqrt{\varepsilon}$ for $\tau < 0$, and another solution $s_{12}^+(\tau)$ which is of order $\sqrt{\varepsilon}$ for positive τ; $s_{12}^\pm$ decreases as $\varepsilon/|\tau|$ for $\pm\tau \geqslant \sqrt{\varepsilon}$ (see Fig. 5.4b).

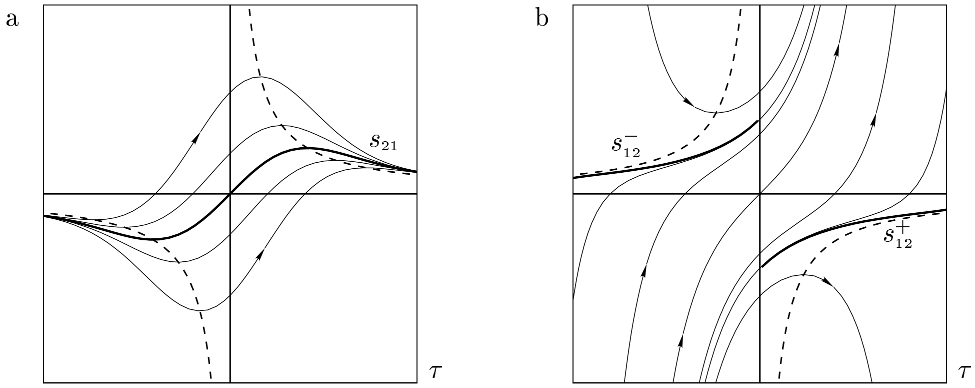

FIGURE 5.4. Orbits of the equations (5.100). Dashed lines indicate branches of fixed points. (a) The equation for s_{21} admits solutions of order $\sqrt{\varepsilon}$. In particular, the orbit going through the origin behaves as τ for $\tau \leqslant \sqrt{\varepsilon}$, and as ε/τ for $\tau \geqslant \sqrt{\varepsilon}$. (b) The equation for s_{12} does not, in general, admit bounded solutions on both sides of the crossing point, but one can choose two solutions $s_{12}^{\pm}$, each of which remains bounded on one side of the crossing.

For this reason, we can construct a bounded matrix $S_1(\tau)$ only if we distinguish between a solution $S^+(\tau)$ for positive τ, and a solution $S^-(\tau)$ for negative τ. If $\tau_0 < 0 < \tau_1$, the solution of (5.98) can thus be written in the form

$$y_1(\tau_1) = S_1^+(\tau_1)U_0^+(\tau_1,0)S_1^+(0)^{-1}S_1^-(0)U_0^-(0,\tau_0)S_1^-(\tau_0)^{-1}y_1(\tau_0), \tag{5.102}$$

where $U_0^{\pm}$ are diagonal propagators obtained by integrating (5.101), with the selected functions s_{12} and s_{21}. Since

$$\begin{aligned} S_1^+(0)^{-1}S_1^-(0) &= \frac{1}{1-s_{12}^+(0)s_{21}(0)}\begin{pmatrix} 1 & -s_{12}^+(0) \\ -s_{21}(0) & 1 \end{pmatrix}\begin{pmatrix} 1 & s_{12}^-(0) \\ s_{21}(0) & 1 \end{pmatrix} \\ &= \frac{1}{1-s_{12}^+(0)s_{21}(0)}\begin{pmatrix} 1-s_{12}^+(0)s_{21}(0) & s_{12}^-(0)-s_{12}^+(0) \\ 0 & 1-s_{12}^-(0)s_{21}(0) \end{pmatrix}, \end{aligned} \tag{5.103}$$

the mismatch between the solutions s_{12}^+ and s_{12}^- at $\tau = 0$ produces an off–diagonal term of order $\sqrt{\varepsilon}$. We have thus obtained

Proposition 5.4. *Assume* $\mathrm{d}_{\tau^2} \det A(\tau)|_{\tau=0} < 0$. *Then, if* $\tau_0 < 0 < \tau_1$, *the solution of* (5.95) *can be written as*

$$y(\tau_1) = S^+(\tau_1)U_0^+(\tau_1,0)TU_0^-(0,\tau_0)S^-(\tau_0)^{-1}y(\tau_0), \tag{5.104}$$

where the singular part of evolution is described by the diagonal propagators

$$U_0^{\pm}(\tau_2,\tau_1) = \begin{pmatrix} \mathrm{e}^{\alpha^{\pm}(\tau_2,\tau_1)/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\alpha^{\pm}(\tau_2,\tau_1)/\varepsilon} \end{pmatrix}, \qquad \alpha^{\pm}(\tau_2,\tau_1) := \int_{\tau_1}^{\tau_2} a(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon), \tag{5.105}$$

(with $a(\tau)$ defined in (5.96)*), and the transition is described by the matrix*

$$T = \begin{pmatrix} 1 & \mathcal{O}(\sqrt{\varepsilon}) \\ 0 & 1 \end{pmatrix}. \tag{5.106}$$

Moreover, if τ_0 and τ_1 are independent of ε, the matrices $S^{\pm}(\tau)$ differ by $\mathcal{O}(\varepsilon)$ from the matrix $S_0(\tau)$ diagonalizing $A(\tau)$ statically.

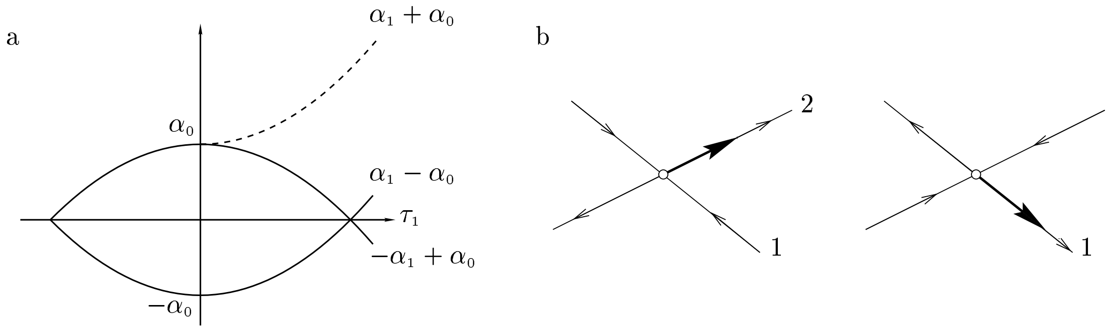

FIGURE 5.5. (a) The functions $\pm\alpha_1 \pm \alpha_0$ occurring in the propagator (5.107). Note that the off–diagonal term $\alpha_1 + \alpha_0$ is the largest one. (b) Two dynamic subspaces can be associated with the equation: the first one is invariant, and the motion on it changes from contracting to expanding; the second one is not invariant, a solution starting on this initially expanding space will tend to align with the invariant subspace after the crossing.

This result can be interpreted as follows. We can define a transformation $y = S^+(\tau)z$ if $\tau > 0$, $y = S^-(\tau)z$ in $\tau < 0$, which admits a discontinuity of order $\sqrt{\varepsilon}$ at $\tau = 0$. The coordinate axes $z_{[2]} = 0$ and $z_{[1]} = 0$ define dynamic eigenspaces of the equation, which are invariant as long as we do not cross the time $\tau = 0$. If $\tau_0 < \tau < \tau_1$, we can write

$$z(\tau_1) = \begin{pmatrix} \mathrm{e}^{(\alpha_1-\alpha_0)/\varepsilon} & \mathcal{O}(\sqrt{\varepsilon})\,\mathrm{e}^{(\alpha_1+\alpha_0)/\varepsilon} \\ 0 & \mathrm{e}^{-(\alpha_1-\alpha_0)/\varepsilon} \end{pmatrix} z(\tau_0), \tag{5.107}$$

$$\alpha_1 = \int_0^{\tau_1} a(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon), \qquad \alpha_0 = \int_{\tau_0}^{0} a(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon).$$

The subspace $z_{[2]} = 0$ is invariant, and the motion on this subspace is first contracting, then expanding. The subspace $z_{[1]} = 0$, however, is not invariant, and solutions starting on it will quickly align with the subspace $z_{[2]} = 0$ for positive times (see Fig. 5.5). We discuss a physical illustration of this phenomenon in Subsection 6.2.1.

Remark 5.5.

1. Proposition 5.4 can be generalized to a complex $a(\tau)$, provided $\operatorname{Re} a(\tau)$ changes sign at $\tau = 0$.
2. If $A(\tau)$ is real symmetric, it may be more appropriate to choose

$$S_0(\tau) = \begin{pmatrix} \cos\theta(\tau) & -\sin\theta(\tau) \\ \sin\theta(\tau) & \cos\theta(\tau) \end{pmatrix} \qquad \Rightarrow \qquad A_1(\tau) = \begin{pmatrix} a(\tau) & \varepsilon\dot{\theta} \\ -\varepsilon\dot{\theta} & -a(\tau) \end{pmatrix}. \tag{5.108}$$

In this case, the choice

$$S_1(\tau) = \begin{pmatrix} \cos\theta_1(\tau) & \cos\theta_2(\tau) \\ \sin\theta_1(\tau) & \sin\theta_2(\tau) \end{pmatrix} \tag{5.109}$$

yields, instead of (5.100), the equations

$$d_1 = a\cos(2\theta_1), \qquad \varepsilon\dot{\theta}_1 = -a\sin(2\theta_1) - \varepsilon\dot{\theta}, \tag{5.110a}$$

$$d_2 = -a\cos(2\theta_2), \qquad \varepsilon\dot{\theta}_2 = a\sin(2\theta_2) - \varepsilon\dot{\theta}, \tag{5.110b}$$

which have a similar behaviour as (5.100).

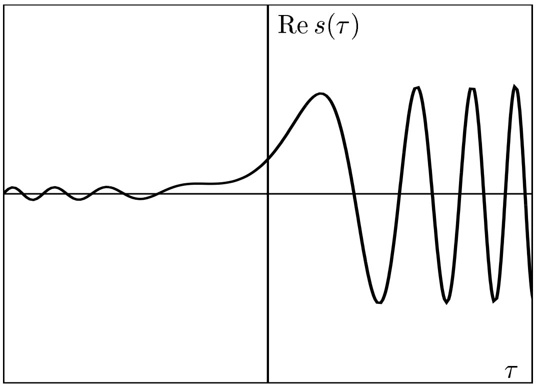

FIGURE 5.6. Solution of (5.100) in the case where $a(\tau) = \mathrm{i}\,\omega(\tau)$ is imaginary. The orbit starts at a distance of order ε from the origin, increases as $\varepsilon/|\tau|$ for $\tau \leqslant -\sqrt{\varepsilon}$, and then remains of order $\sqrt{\varepsilon}$.

2. Imaginary Case

By the discussion of Subsection 4.4.2 (Example 4.14), we know that the system (5.100) admits solutions of order $\sqrt{\varepsilon}$ throughout the bifurcation. More precisely, there exists a solution scaling as $\varepsilon/|\tau|$ for $\tau \leqslant -\sqrt{\varepsilon}$, and as $\sqrt{\varepsilon}$ for larger times, but it does not necessarily decrease to $\mathcal{O}(\varepsilon)$ after the bifurcation (see Fig. 5.6). We thus obtain the following result.

Proposition 5.5. *Assume* $\mathrm{d}_{\tau^2} \det A(\tau)|_{\tau=0} > 0$. *Then, if* $\tau_0 < 0 < \tau_1$, *the solution of* (5.95) *can be written as*

$$y(\tau_1) = S(\tau_1) U_0(\tau_1, \tau_0) S(\tau_0)^{-1} y(\tau_0), \tag{5.111}$$

where the singular part of evolution is described by the diagonal propagators

$$U_0(\tau_1,\tau_0) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\varphi(\tau_1,\tau_0)/\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\varphi(\tau_1,\tau_0)/\varepsilon} \end{pmatrix}, \qquad \varphi(\tau_1,\tau_0) := \int_{\tau_0}^{\tau_1} \omega(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon), \tag{5.112}$$

where $a(\tau) =: \mathrm{i}\,\omega(\tau)$ *(see* (5.96)*), and the matrix* $S(\tau)$ *differs by* $\mathcal{O}(\sqrt{\varepsilon})$ *from the matrix* $S_0(\tau)$ *which diagonalizes* $A(\tau)$ *statically.*

5.3.7 Complex Eigenvalue Cruising

In this subsection, we consider the system

$$\varepsilon \dot{y} = A(\tau) y, \tag{5.113}$$

where $A(\tau)$ is an analytic matrix–valued function in $\mathbb{M}_2(\mathbb{C})$. We assume that the eigenvalues' real parts cross at $\tau = 0$, but with different imaginary parts. Since we may suppose that $\operatorname{Tr} A(\tau) = 0$, these eigenvalues may be written as $\pm a(\tau) = \pm\rho(\tau) \pm \mathrm{i}\,\omega(\tau)$, where $\rho(0) = 0$ and $\omega(0) \neq 0$. Moreover, we assume that $\rho(\tau)$ changes sign from negative to positive at $\tau = 0$. Thus there exists a change of variables $y = S_0(\tau) y_1$, transforming (5.113) into

$$\varepsilon \dot{y}_1 = A_1(\tau) y_1, \qquad A_1(\tau) = \begin{pmatrix} a_{11}(\tau) & \varepsilon a_{12}(\tau) \\ \varepsilon a_{21}(\tau) & a_{22}(\tau) \end{pmatrix}, \tag{5.114}$$

where $a_{11}(\tau) = a(\tau) + \mathcal{O}(\varepsilon)$ and $a_{22}(\tau) = -a(\tau) + \mathcal{O}(\varepsilon)$. To diagonalize this equation dynamically, we look for a transformation $y_1 = S_1(\tau)z$, where

$$S_1(\tau) = \begin{pmatrix} 1 & \varepsilon s_{12} \\ \varepsilon s_{21} & 1 \end{pmatrix}, \qquad \begin{cases} \varepsilon \dot{s}_{12} = a_{12} + (a_{11} - a_{22})s_{12} - \varepsilon^2 a_{21} s_{12}^2 \\ \varepsilon \dot{s}_{21} = a_{21} + (a_{22} - a_{11})s_{21} - \varepsilon^2 a_{12} s_{21}^2, \end{cases} \tag{5.115}$$

which transforms (5.114) into

$$\varepsilon \dot{z} = D(\tau) z, \qquad D(\tau) = \begin{pmatrix} a_{11} + \varepsilon^2 a_{12} s_{21} & 0 \\ 0 & a_{22} + \varepsilon^2 a_{21} s_{12}^2 \end{pmatrix}. \tag{5.116}$$

Let $d(\tau) := a_{11}(\tau) - a_{22}(\tau) = 2a(\tau) + \mathcal{O}(\varepsilon)$. Introducing new variables $s_{12} = -a_{12}/d + u$ and $s_{21} = a_{21}/d + v$, (5.115) becomes

$$\varepsilon \dot{u} = d_1(\tau) u - \varepsilon^2 a_{21} u^2 + \varepsilon w_1(\tau) \qquad d_1(\tau) = d(\tau) + \mathcal{O}(\varepsilon^2) \tag{5.117a}$$
$$\varepsilon \dot{v} = -d_2(\tau) v - \varepsilon^2 a_{12} v^2 + \varepsilon w_2(\tau) \qquad d_2(\tau) = d(\tau) + \mathcal{O}(\varepsilon^2). \tag{5.117b}$$

These equations both describe a Hopf bifurcation, but with different consequences on the solutions.

- In equation (5.117b) the equilibrium changes from unstable to stable. Thus there always exists a bounded solution throughout the bifurcation point.
- In equation (5.117a) the equilibrium changes from stable to unstable. Thus it displays the phenomenon of **bifurcation delay**: there exist solutions which are stable in a neighborhood of the bifurcation point, but they are limited by **buffer points**. Let (τ_-, τ_+) be these buffer points (see Definition 4.6). Then we know by Theorem 4.3 that bounded solutions exist either for $\tau \leqslant \tau_+$ or for $\tau \geqslant \tau_-$.

We have thus the following result.

Proposition 5.6. *There exist buffer points (τ_-, τ_+) such that*

1. *If $\tau_- < \tau_0 < \tau_1$, the solution of (5.113) can be written as*
$$y(\tau_1) = S^+(\tau_1) U_0(\tau_1, \tau_0) S^+(\tau_0)^{-1} y(\tau_0). \tag{5.118}$$
2. *If $\tau_0 < \tau_1 < \tau_+$, the solution of (5.113) can be written as*
$$y(\tau_1) = S^-(\tau_1) U_0(\tau_1, \tau_0) S^-(\tau_0)^{-1} y(\tau_0). \tag{5.119}$$
3. *If $\tau_0 < \tau_- < \tau_+ < \tau_1$, the solution of (5.113) can be written as*
$$y(\tau_1) = S^+(\tau_1) U_0(\tau_1, 0) T U_0(0, \tau_0) S^-(\tau_0)^{-1} y(\tau_0), \tag{5.120}$$
where the transition is described by the matrix
$$T = \begin{pmatrix} 1 & \mathcal{O}(\varepsilon) \\ 0 & 1 \end{pmatrix}. \tag{5.121}$$

In all three cases, U_0 is the diagonal principal solution of (5.116), and $S^\pm(\tau) = S_0(\tau) + \mathcal{O}(\varepsilon)$.

The behaviour of the solutions at a complex cruising is thus similar to the case of diagonal crossing, when the initial and final point are sufficiently far apart. But due to the phenomenon of bifurcation delay, invariant dynamic eigenspaces exist in some neighborhood of the bifurcation point. We discuss a physical illustration of this phenomenon in Subsection 6.2.2.

Remark 5.6. If $A(\tau)$ is only of class $\mathcal{C}^k$, we know by Neishtadt's result that the bifurcation delay is in general only of order $\sqrt{(k-3/2)\varepsilon|\ln\varepsilon|}$ (see remark 4.3). Thus the differentiable case is similar to the case of diagonal eigenvalue crossing.

5.3.8 Concluding Remarks

Let us briefly summarize the method we have constructed to compute solutions of the linear system

$$\varepsilon\dot{y} = A(\tau,\varepsilon)y. \tag{5.122}$$

1. Find the eigenvalues $a_i(\tau,\varepsilon)$ of $A(\tau,\varepsilon)$ and plot their real parts against τ.
2. As long as the real parts are different, use a dynamic diagonalization as described in Subsection 5.3.2. The solution can be written

$$y(\tau) = S(\tau,\varepsilon)U_0(\tau,\tau_0,\varepsilon)S(\tau_0,\varepsilon)^{-1}y(\tau_0), \tag{5.123}$$

   where $U_0(\tau,\tau_0,\varepsilon)$ is diagonal, with diagonal elements of the form $\exp\frac{1}{\varepsilon}\bigl[\int_{\tau_0}^{\tau} a_i(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon)\bigr]$, and $S(\tau,\varepsilon)$ can be expanded in powers of ε.
3. If some eigenvalues' real parts cross at time $\tau^\star$, isolate them using a dynamic bloc–diagonalization. The actual crossing can then be studied by a local analysis in a neighborhood (τ_-,τ_+) of $\tau^\star$. The solution in the dynamic eigenspace of the crossing eigenvalues can generically be written in the form

$$y(\tau_+) = S(\tau_+,\varepsilon)U_0(\tau_+,\tau^\star)TU_0(\tau^\star,\tau_-)S(\tau_-,\varepsilon)^{-1}y(\tau_-), \tag{5.124}$$

   where T is a matrix describing the transition.

Remark 5.7.

1. There is some freedom in the choice of the matrix $S(\tau,\varepsilon)$ (this choice will of course influence the next–to–leading order terms of U_0). In particular, one can choose the diagonalizing transformation in such a way that the matrices $S(\tau,\varepsilon)$ in (5.123) and (5.124) are compatible.
2. Assume that $a_-(\tau) \leqslant \operatorname{Re} a_j(\tau,0) \leqslant a_+(\tau)$ for all eigenvalues of $A(\tau,0)$, and let

$$\alpha_\pm(\tau,\tau_0) := \int_{\tau_0}^{\tau} a_\pm(s)\,\mathrm{d}s. \tag{5.125}$$

   If these eigenvalues have different real part, it follows from (5.123) that for $\tau_0 \leqslant \tau$, the norm of the principal solution $U(\tau,\tau_0)$ of (5.122) satisfies the bounds

$$\begin{aligned}\exp\frac{1}{\varepsilon}\bigl[\alpha_- + \mathcal{O}(\varepsilon)\bigr]\|y\| &\leqslant \|U(\tau,\tau_0)y\| \leqslant \exp\frac{1}{\varepsilon}\bigl[\alpha_+ + \mathcal{O}(\varepsilon)\bigr]\|y\|,\\ \exp\frac{1}{\varepsilon}\bigl[-\alpha_+ + \mathcal{O}(\varepsilon)\bigr]\|y\| &\leqslant \|U(\tau_0,\tau)y\| \leqslant \exp\frac{1}{\varepsilon}\bigl[-\alpha_- + \mathcal{O}(\varepsilon)\bigr]\|y\|.\end{aligned} \tag{5.126}$$

   If some eigenvalues' real parts cross pairwise, and with nonzero velocity, (5.124) implies that these bounds remain true.

5.4 Effect of Nonlinear Terms

We consider in this section the equation

$$\varepsilon \dot{y} = A(\tau, \varepsilon) y + b(y, \tau, \varepsilon), \qquad b(y, \tau, \varepsilon) = \mathcal{O}(|y|^2), \tag{5.127}$$

which describes in particular the motion near an adiabatic solution. In the 1D case, we were able to show that close to the origin, the nonlinear term has only a negligible influence on the dynamics (see Proposition 4.4). In the nD case, the situation is not so simple, as shows the following example.

Example 5.5. Let $m \geq 2$, and consider the autonomous equation

$$\begin{aligned} \varepsilon \dot{\xi} &= -\xi & \xi(0) &= \xi_0 \\ \varepsilon \dot{\eta} &= -3\eta + \xi^m, & \eta(0) &= \eta_0, \end{aligned} \tag{5.128}$$

which has the solution

$$\begin{aligned} \xi(\tau) &= \mathrm{e}^{-\tau/\varepsilon} \xi_0 \\ \eta(\tau) &= \begin{cases} \mathrm{e}^{-3\tau/\varepsilon} \big[\eta_0 + \frac{\tau}{\varepsilon} \xi_0^3\big] & \text{if } m = 3 \\ \mathrm{e}^{-3\tau/\varepsilon} \big[\eta_0 - \frac{\xi_0^m}{3-m}\big] + \mathrm{e}^{-m\tau/\varepsilon} \frac{\xi_0^m}{3-m} & \text{if } m \neq 3. \end{cases} \end{aligned} \tag{5.129}$$

In particular, we observe that

- if $m = 2$, $\eta(\tau)$ decreases only as $\mathrm{e}^{-2\tau/\varepsilon}$, which is much slower than the $\mathrm{e}^{-3\tau/\varepsilon}$ predicted by the linearization;
- if $m = 3$, a resonance produces a secular term $\frac{\tau}{\varepsilon}$;
- if $m > 3$, there exists a particular solution $\eta(\tau) = \eta_0 \, \mathrm{e}^{-m\tau/\varepsilon}$, with initial condition $\eta_0 = \frac{\xi_0^m}{3-m}$, which decreases much faster than the solution of the linearized system.

In the autonomous case, these difficulties are solved either by using normal form theory, or by introducing invariant manifolds. In this section, we present generalizations of these theories to the adiabatic nonautonomous case.

We begin, in Subsection 5.4.1, by introducing some useful bounds on the solutions of (5.127), showing in particular existence of a delay for the dynamic pitchfork bifurcation.

In Subsection 5.4.2, we analyse in detail existence and computation of stable and unstable adiabatic manifolds, generalizing the Stable Manifold Theorem 2.21 of the autonomous case. When restricted to one of these manifolds, the dynamics reduce to a lower–dimensional contracting or expanding motion.

In Subsection 5.4.3, we study the behaviour of dynamic normal forms. In the case of a contracting or expanding motion, we prove the existence of a time–dependent change of variables which transforms (5.127) into its linearization.

Finally, in Subsection 5.4.4, we discuss a different nonlinear problem arising when the origin is a bifurcation point of the system $\varepsilon \dot{x} = f(x, \tau)$. Then there is not necessarily an equilibrium branch through the origin, but we prove the existence of an adiabatic center manifold, on which the low–dimensional motion is easier to study.

5.4.1 Basic Estimates

We first give a bound on the principal solution $U(\tau, \tau_0)$ of the linear system $\varepsilon \dot{y} = A(\tau) y$, which generalizes (5.126).

Lemma 5.5. *Let $A(\tau,\varepsilon) \in \mathbb{M}_n(\mathbb{C})$ be differentiable in τ for $\tau \in I$, where I is an interval in $\mathbb{R}$, and continuous in ε for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. We assume that the eigenvalues $a_j(\tau)$ of $A(\tau,0)$ satisfy $a_-(\tau) \leqslant \operatorname{Re} a_j(\tau) \leqslant a_+(\tau)$ for $j = 1,\dots,n$ and $\tau \in I$. Let $\tau \geqslant \tau_0 \in I$ and $\alpha_\pm(\tau,\tau_0) := \int_{\tau_0}^{\tau} a_\pm(s)\,\mathrm{d}s$. Then the principal solution $U(\tau,\tau_0)$ of the system $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ satisfies the bound*

$$\exp\frac{1}{\varepsilon}\bigl[\alpha_-(\tau,\tau_0) - r(\varepsilon)(1+\tau-\tau_0)\bigr]\|y\| \leqslant \|U(\tau,\tau_0)y\| \leqslant \exp\frac{1}{\varepsilon}\bigl[\alpha_+(\tau,\tau_0) + r(\varepsilon)(1+\tau-\tau_0)\bigr]\|y\|, \tag{5.130}$$

where $r(\varepsilon)$ is a continuous function such that $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$.

We give the proof in Appendix 5.B.5. Note that if the eigenvalues of A have different real parts, or if these real parts cross pairwise with nonzero velocity, the function $r(\varepsilon)$ may be replaced by $\mathcal{O}(\varepsilon)$ (see Remark 5.7).

This result allows us to give a first bound on the solution of the nonlinear system (5.127).

Proposition 5.7. *Assume the propagator of the linear system $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ satisfies the bound $\|U(\tau,\tau_0)\| \leqslant \exp\frac{1}{\varepsilon}\hat{\alpha}(\tau,\tau_0)$, where $\hat{\alpha}(\tau,\tau_0) = \int_{\tau_0}^{\tau}\hat{a}(s)\,\mathrm{d}s$. Assume furthermore that $\|b(y,\tau,\varepsilon)\| \leqslant M\|y\|^2$ and*

$$y_0\frac{M}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\hat{\alpha}(s,\tau_0)/\varepsilon}\,\mathrm{d}s \leqslant 1 - K^{-1}. \tag{5.131}$$

Then the solution of $\varepsilon\dot{y} = A(\tau,\varepsilon)y + b(y,\tau,\varepsilon)$ with initial condition $\|y(\tau_0)\| \leqslant y_0$ satisfies

$$\|y(\tau)\| \leqslant K\,\mathrm{e}^{\hat{\alpha}(\tau,\tau_0)/\varepsilon}\|y(\tau_0)\|. \tag{5.132}$$

Proof: Since

$$y(\tau) = U(\tau,\tau_0)y(\tau_0) + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} U(\tau,s)b(y(s),\tau,\varepsilon)\,\mathrm{d}s, \tag{5.133}$$

we have

$$\|y(\tau)\| \leqslant \mathrm{e}^{\hat{\alpha}(\tau,\tau_0)/\varepsilon}R(\tau), \qquad R(\tau) := \|y(\tau_0)\| + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\hat{\alpha}(\tau_0,s)/\varepsilon}M\|y(s)\|^2\,\mathrm{d}s. \tag{5.134}$$

But $R(\tau)$ satisfies the differential equation

$$\varepsilon\dot{R} = \mathrm{e}^{\hat{\alpha}(\tau_0,\tau)/\varepsilon}M\|y(\tau)\|^2 \leqslant \mathrm{e}^{\hat{\alpha}(\tau,\tau_0)/\varepsilon}MR(\tau)^2, \tag{5.135}$$

so that using the Comparison Lemma 4.1,

$$R(\tau) \leqslant \|y(\tau_0)\|\left[1 - \|y(\tau_0)\|\frac{M}{\varepsilon}\int_{\tau_0}^{\tau}\mathrm{e}^{\hat{\alpha}(s,\tau_0)/\varepsilon}\,\mathrm{d}s\right]^{-1} \leqslant K\|y(\tau_0)\|, \tag{5.136}$$

which proves the proposition, using (5.134). □

Condition (5.131) is easily satisfied if $\hat{a}(\tau)$ is negative. More precisely, if $\hat{a}(\tau) \leqslant -c < 0$, it reduces to $y_0 \leqslant c(1-K^{-1})$. But there are also situations where $\hat{a}(\tau)$ becomes positive, which allow for a y_0 of order 1 in (5.131). This phenomenon results in a bifurcation delay, similar to the situation discussed in Subsection 4.3.6.

Corollary 5.4 (Bifurcation Delay). *Let $\hat{a}(\tau)$ be the largest real part of the eigenvalues of $A(\tau,\varepsilon)$. Assume $\hat{a}(\tau_0) < 0$ and let $\Psi(\tau_0)$ be the smallest time $\tau > \tau_0$ such that*

$$\int_{\tau_0}^{\Psi(\tau_0)} \hat{a}(s)\,\mathrm{d}s = 0, \tag{5.137}$$

if such a time exists (see Definition 4.5). Then there exists a constant $y_0 > 0$ and a continuous function $r(\varepsilon)$, $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$, such that any solution of (5.127) with initial condition $\|y(\tau_0)\| \leqslant y_0$ satisfies $\|y(\tau)\| \leqslant y_0$ for $\tau_0 \leqslant \tau \leqslant \Psi(\tau_0) - r(\varepsilon)$.

PROOF: Since $\hat{a}(\tau_0) < 0$, we may assume that

$$\hat{\alpha}(\tau,\tau_0) := \int_{\tau_0}^{\tau} \hat{a}(s)\,\mathrm{d}s \leqslant -c\max\{1, (\Psi(\tau_0)-\tau)^q\} \tag{5.138}$$

for some positive c, q. Thus we obtain

$$\int_{\tau_0}^{\Psi(\tau_0)-r} \mathrm{e}^{\hat{\alpha}(s,\tau_0)/\varepsilon} \leqslant \frac{\varepsilon}{c} + \int_{\tau_0-\Psi(\tau_0)}^{-r} \mathrm{e}^{-c|s|^q/\varepsilon}\,\mathrm{d}s, \tag{5.139}$$

where, by Lemma 4.4, this last integral behaves as $\varepsilon^{1/q}$ if $|r| \leqslant \varepsilon^{1/q}$, and decreases exponentially fast for $r \geqslant \varepsilon^{1/q}$. Thus there exists an $r(\varepsilon) \approx \varepsilon^{1/q}$ such that the integral is of order ε, and then y_0 is given by (5.131). □

5.4.2 Adiabatic Manifolds

If $x^\star(\tau)$ is a hyperbolic equilibrium branch of the system $\varepsilon\dot{x} = f(x,\tau)$, we know by Theorem 5.1 about existence of an adiabatic solution $\bar{x}(\tau) = x^\star(\tau) + \mathcal{O}(\varepsilon)$. The linearization of the equation around this solution being hyperbolic (for small ε), Theorem 5.3 implies that it may be dynamically bloc–diagonalized. We may thus assume that the system already has the form

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + b_+(x,y,\tau,\varepsilon) \\ \varepsilon\dot{y} &= A_-(\tau,\varepsilon)y + b_-(x,y,\tau,\varepsilon), \end{aligned} \tag{5.140}$$

where $x \in \mathbb{C}^{n_+}$, $y \in \mathbb{C}^{n_-}$, $A_+(\tau,\varepsilon)$ is an expanding matrix in $\mathbb{M}_{n_+}(\mathbb{C})$, $A_-(\tau,\varepsilon)$ is a contracting matrix in $\mathbb{M}_{n_-}(\mathbb{C})$, and $b_\pm(x,y,\tau,\varepsilon) = \mathcal{O}(|x|^2+|y|^2)$. Furthermore, we assume that all functions in (5.140) are of class $\mathcal{C}^k$ in (x,y,τ), $k \geqslant 2$, and continuous in ε.

By the Stable Manifold Theorem (Theorem 2.21), (5.140) admits an instantaneous unstable manifold $y = u_0(x,\tau,\varepsilon) = \mathcal{O}(|x|^2)$, which is a solution of the equation

$$A_-(\tau,\varepsilon)u_0 + b_-(x,u_0,\tau,\varepsilon) - \partial_x u_0\big[A_+(\tau,\varepsilon)x + b_+(x,u_0,\tau,\varepsilon)\big] = 0. \tag{5.141}$$

Remark 5.8. For a given $\lambda^\star$, consider the system

$$\begin{aligned} \dot{x} &= A_+(\lambda^\star,\varepsilon)x + [A_+(\lambda,\varepsilon) - A_+(\lambda^\star,\varepsilon)]x + b_+(x,y,\lambda,\varepsilon) \\ \dot{y} &= A_-(\lambda^\star,\varepsilon)y + [A_-(\lambda,\varepsilon) - A_-(\lambda^\star,\varepsilon)]y + b_-(x,y,\lambda,\varepsilon) \\ \dot{\lambda} &= 0. \end{aligned} \tag{5.142}$$

If the functions are of class $\mathcal{C}^k$ (resp. analytic), then the Center Manifold Theorem shows that (5.142) admits an unstable manifold $y = u_0(x,\lambda,\varepsilon)$, which is also of class $\mathcal{C}^k$ (resp. analytic). Since this manifold coincides with the solution of (5.141), this solution depends smoothly on the variables.

If we carry out the change of variables $y = u_0(x,\tau,\varepsilon) + y_1$ in (5.140), we get

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + b_+(x,u_0+y_1,\tau,\varepsilon) \\ \varepsilon\dot{y}_1 &= \bigl[A_-(\tau,\varepsilon) + B_1(x,y_1,\tau,\varepsilon)\bigr]y_1 + \varepsilon w_1(x,\tau,\varepsilon), \end{aligned} \tag{5.143}$$

where

$$\begin{aligned} B_1(x,y_1,\tau,\varepsilon) &= \int_0^1 \partial_y b_-(x,u_0+sy_1,\tau,\varepsilon)\,\mathrm{d}s - \partial_x u_0 \int_0^1 \partial_y b_+(x,u_0+sy_1,\tau,\varepsilon)\,\mathrm{d}s \\ w_1(x,\tau,\varepsilon) &= -\partial_\tau u_0(x,\tau,\varepsilon). \end{aligned} \tag{5.144}$$

If the drift term $w_1(x,\tau,\varepsilon)$ were absent in (5.143), the manifold $y_1 = 0$ would be invariant. In fact, an invariant manifold of (5.140) with parametric equation $y = \eta_0(x,\tau,\varepsilon)$ should satisfy, instead of (5.141), the partial differential equation

$$\varepsilon\partial_\tau\eta_0(x,\tau,\varepsilon) = A_-(\tau,\varepsilon)\eta_0 + b_-(x,\eta_0,\tau,\varepsilon) - \partial_x\eta_0[A_+(\tau,\varepsilon)x + b_+(x,\eta_0,\tau,\varepsilon)]. \tag{5.145}$$

Indeed, the change of variables $y = \eta_0(x,\tau,\varepsilon) + z$ transforms (5.140) into

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + b_+(x,\eta_0+z,\tau,\varepsilon) \\ \varepsilon\dot{z} &= \bigl[A_-(\tau,\varepsilon) + B_-(x,z,\tau,\varepsilon)\bigr]z, \end{aligned} \tag{5.146}$$

which admits $z = 0$ as an invariant manifold.

We will thus proceed as follows. First we show that if the system is sufficiently differentiable, the order of the drift term in (5.143) can be decreased by an iterative scheme. Then we will prove that equation (5.145) (and similar equations with a smaller drift term) do admit a solution which is close to the instantaneous manifold $u_0(x,\tau,\varepsilon)$. Thus we will have proved at the same time the existence of an adiabatic manifold, and of its expansion in ε. The same procedure can of course be repeated for the stable manifold.

Iterative Scheme 5.4. *Consider the system*

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + b_+(x,u^{(N-1)}+y_N,\tau,\varepsilon) \\ \varepsilon\dot{y}_N &= \bigl[A_-(\tau,\varepsilon) + B_N(x,y_N,\tau,\varepsilon)\bigr]y_N + \varepsilon^N w_N(x,\tau,\varepsilon), \end{aligned} \tag{5.147}$$

where $u^{(N-1)}(x,\tau,\varepsilon) = \sum_{j=0}^{N-1}\varepsilon^j u_j(x,\tau,\varepsilon) = \mathcal{O}(|x|^2)$, $B_N = \mathcal{O}(|x| + |y_N|)$, *and* $w_N = \mathcal{O}(|x|^2)$. *We assume that all functions are of class* $\mathcal{C}^{k-N}$. *Let* $y_N = \varepsilon^N u_N(x,\tau,\varepsilon)$ *be the instantaneous unstable manifold, where* $u_N = \mathcal{O}(|x|^2)$ *is solution of*

$$\bigl[A_- + B_N(x,\varepsilon^N u_N,\tau,\varepsilon)\bigr]u_N + w_N(x,\tau,\varepsilon) - \partial_x u_N\bigl[A_+x + b_+(x,u^{(N-1)}+\varepsilon^N u_N,\tau,\varepsilon)\bigr] = 0. \tag{5.148}$$

Then the change of variables $y_N = \varepsilon^N u_N(x,\tau,\varepsilon) + y_{N+1}$ *transforms* (5.147) *into*

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + b_+(x,u^{(N)}+y_{N+1},\tau,\varepsilon) \\ \varepsilon\dot{y}_{N+1} &= \bigl[A_-(\tau,\varepsilon) + B_{N+1}(x,y_{N+1},\tau,\varepsilon)\bigr]y_{N+1} + \varepsilon^{N+1}w_{N+1}(x,\tau,\varepsilon), \end{aligned} \tag{5.149}$$

with

$$\begin{aligned} B_{N+1}(x,y_{N+1},\tau,\varepsilon) &= B_N(x,\varepsilon^N u_N + y_{N+1},\tau,\varepsilon) - \varepsilon^N\partial_x u_N \int_0^1 \partial_y b_+(x,u^{(N)}+sy_{N+1},\tau,\varepsilon)\,\mathrm{d}s \\ &\quad + \varepsilon^N \int_0^1 \partial_y[B_N(x,\varepsilon^N u_N + sy_{N+1},\tau,\varepsilon)u_N]\,\mathrm{d}s \\ w_{N+1}(x,\tau,\varepsilon) &= -\partial_\tau u_N(x,\tau,\varepsilon). \end{aligned} \tag{5.150}$$

Remark 5.9. A similar iterative scheme can be applied directly to equation (5.145), writing $\eta_N = \varepsilon^N u_N + \eta_{N+1}$, where u_N is the instantaneous unstable manifold of (5.148).

With this iterative scheme, the order of the drift term can be decreased to ε^k if the original system is $\mathcal{C}^k$. If it is analytic, we suspect that this order may be decreased to an exponentially small one. This is confirmed by the following Lemma.

Lemma 5.6. *Assume that* (5.140) *is analytic in* x, y, τ, *for* (x, y) *in a complex neighborhood of the origin, and for* τ *in a complex neighborhood of the real interval* I. *Assume furthermore that it is continuous in* ε *for* $0 \leqslant \varepsilon \leqslant \varepsilon_0$. *Then there exists, in smaller neighborhoods and for sufficiently small* ε, *an analytic function* $u(x, \tau; \varepsilon) = \mathcal{O}(|x|^2)$, *such that the change of variables* $y = u(x, \tau) + z$ *transforms* (5.140) *into the system*

$$\begin{aligned} \varepsilon \dot{x} &= A_+(\tau, \varepsilon) x + b_+(x, u + z, \tau, \varepsilon) \\ \varepsilon \dot{z} &= \big[A_-(\tau, \varepsilon) + B_-(x, z, \tau, \varepsilon)\big] z + \mathrm{e}^{-1/C|\varepsilon|}\, w(x, \tau, \varepsilon), \end{aligned} \tag{5.151}$$

with $w = \mathcal{O}(|x|^2)$, $B_- = \mathcal{O}(|x| + |z|)$ *and* $b_+ = \mathcal{O}(|x|^2 + |z|^2)$.

We prove this lemma in Appendix 5.B.6. The proof is similar to that of Lemma 5.1. The major difficulty is to control norm and domain of analyticity of the solution of (5.148). This is done by writing u as fixed point of an integral operator, as in [**Ca**].

It now remains to prove that systems of the form (5.147) (or (5.151)) admit adiabatic manifolds, close to the instantaneous manifolds. To do this, we should solve the partial differential equation

$$\begin{aligned} \varepsilon \partial_\tau \eta(x, \tau, \varepsilon) = \big[A_-(\tau, \varepsilon) + B_N(x, \eta, \tau, \varepsilon)\big] \eta + \varepsilon^N w_N(x, \tau, \varepsilon) \\ - \partial_x \eta \big[A_+(\tau, \varepsilon) x + b_+(x, u^{(N-1)} + \eta, \tau, \varepsilon)\big]. \end{aligned} \tag{5.152}$$

Since this is not easy, we prefer to transform this equation into an integral equation. A solution of (5.147)satisfies

$$\begin{aligned} y_N(\tau) &= U_-(\tau, \tau_0) y_N(\tau_0) \\ &\quad + \frac{1}{\varepsilon} \int_{\tau_0}^{\tau} U_-(\tau, s) \big[B_N(x(s), y_N(s), s, \varepsilon) y_N(s) + \varepsilon^N w_N(x(s), s, \varepsilon)\big] \,\mathrm{d}s, \end{aligned} \tag{5.153}$$

where $U_-(\tau, \tau_0)$ is the principal solution of $\varepsilon \dot{y} = A_-(\tau, \varepsilon) y$. For a given function $\eta(x, \tau, \varepsilon) = \mathcal{O}(|x|^2)$, we denote by $\xi(\tau, \tau_1, x_1, \eta)$ the solution of the initial value problem

$$\varepsilon \dot{x} = A_+(\tau, \varepsilon) x + b_+(x, u^{(N-1)} + \eta, \tau, \varepsilon), \qquad x(\tau_1) = x_1. \tag{5.154}$$

Then the adiabatic manifold $\eta(x, \tau, \varepsilon)$ we are looking for is a fixed point of the operator

$$\begin{aligned} T\eta &= U_-(\tau, \tau_0) \eta^0(\xi_{\tau_0}, \varepsilon) \\ &\quad + \frac{1}{\varepsilon} \int_{\tau_0}^{\tau} U_-(\tau, s) \big[B_N(\xi_s, \eta(\xi_s, s, \varepsilon), s, \varepsilon) \eta(\xi_s, s, \varepsilon) + \varepsilon^N w_N(\xi_s, s, \varepsilon)\big] \,\mathrm{d}s, \end{aligned} \tag{5.155}$$

where $\xi_s := \xi(s, \tau, x, \eta)$, and $\eta^0(x, \varepsilon) = \eta(x, \tau_0, \varepsilon) = \mathcal{O}(|x^2|)$ is an initial condition for the system (5.152).

In Appendix 5.B.7, we show that T is a contraction in an appropriate function space, and use this fact to prove the following result.

Lemma 5.7. *Assume the functions in* (5.152) *are differentiable in* (x, η, τ). *There exist constants* $\delta_0 > 0$ *and* $\gamma_0 > 0$ *and a neighborhood* $\mathcal{U}$ *of the origin in* $\mathbb{C}^{n+}$ *such that for all differentiable* $\eta^0(x, \varepsilon) = \mathcal{O}(|x|^2)$ *with* $|\eta^0| \leqslant \delta_0$ *and* $|\partial_x \eta^0| \leqslant \gamma_0$ *in* $\mathcal{U}$, (5.152) *admits a Lipschitz continuous solution* $\eta(x, \tau, \varepsilon)$ *such that* $\eta(x, \tau_0, \varepsilon) = \eta^0(x, \varepsilon)$. *If* $|\eta^0| \leqslant H|x|^2$, *then*

$$|\eta(x, \tau, \varepsilon)| \leqslant \left[c_1 H \,\mathrm{e}^{-\kappa(\tau-\tau_0)/\varepsilon} + c_2 \varepsilon^N\right] |x|^2, \tag{5.156}$$

for positive constants c_1, c_2, κ.

In particular, if we take an initial condition $\eta^0(x, \varepsilon)$ of order ε^N, we obtain that (5.152) admits a solution of order ε^N. The lemma remains true if ε^N is replaced by $\mathrm{e}^{-1/C|\varepsilon|}$. Combining the iterative scheme and the lemma, we obtain existence of an adiabatic manifold with expansion $\eta(x, \tau, \varepsilon) = \sum_{j=0}^{N-1} \varepsilon^j u_j(x, \tau, \varepsilon) + \mathcal{O}(\varepsilon^N |x|^2)$. Since the same procedure can be applied to the stable manifold (inverting direction of time), we finally have the following result.

Theorem 5.6. *Assume* (5.140) *is of class* $\mathcal{C}^2$ *in* (x, y, τ), *and* ε *is small enough.*

1. *In a neighborhood of* $x = 0$ *and* $y_1 = 0$, *there exist continuous functions* $\eta(x, \tau, \varepsilon) = \mathcal{O}(|x|^2)$ *and* $\xi(y_1, \tau, \varepsilon) = \mathcal{O}(|y_1|^2)$, *such that the successive changes of variables* $y = \eta(x, \tau, \varepsilon) + y_1$ *and* $x = \xi(y_1, \tau, \varepsilon) + x_1$ *transform* (5.140) *into*

$$\begin{aligned} \varepsilon \dot{x}_1 &= \left[A_+(\tau, \varepsilon) + B_+(x_1, y_1, \tau, \varepsilon)\right] x_1 \\ \varepsilon \dot{y}_1 &= \left[A_-(\tau, \varepsilon) + B_-(x_1, y_1, \tau, \varepsilon)\right] y_1, \end{aligned} \tag{5.157}$$

   where $B_\pm = \mathcal{O}(|x_1| + |y_1|)$.
2. *If* (5.140) *is* $\mathcal{C}^k$, $k \geqslant 2$, η *admits an expansion*

$$\eta(x, \tau, \varepsilon) = \sum_{j=0}^{k-2} \varepsilon^j u_j(x, \tau, \varepsilon) + \mathcal{O}(\varepsilon^{k-1} |x|^2), \tag{5.158}$$

   where the u_j *are constructed successively by Iterative Scheme 5.4, and* $\xi(y_1, \tau, \varepsilon)$ *admits a similar expansion.*
3. *If* (5.140) *is analytic for* (x, y, τ) *in an open complex set,* η *admits an expansion*

$$\eta(x, \tau, \varepsilon) = \sum_{j=0}^{N(\varepsilon)} \varepsilon^j u_j(x, \tau, \varepsilon) + \mathcal{O}(\mathrm{e}^{-1/C|\varepsilon|} |x|^2), \tag{5.159}$$

   and similarly for ξ.

Definition 5.6. The manifold with parametric equation $y = \eta(x, \tau, \varepsilon)$ is called **local adiabatic unstable manifold** associated with the equilibrium branch. The manifold $x = \xi(y_1, \tau, \varepsilon)$ is called **local stable adiabatic manifold**.

Remark 5.10. In Lemma 5.7, we have only proved that the manifolds are Lipschitz continuous, if the equation obtained after applying the iterative scheme has this property. If the system is still Lipschitz differentiable, one can probably show that the manifolds share this property.

5.4.3 Normal Forms

Another way to simplify the equation

$$\varepsilon\dot{y} = A(\tau,\varepsilon)y + b(y,\tau,\varepsilon), \qquad b(y,\tau,\varepsilon) = \mathcal{O}(|y|^2), \tag{5.160}$$

is to look for a nonlinear change of variables $y = z + h(z,\tau;\varepsilon)$, with $h = \mathcal{O}(|z|^2)$, transforming (5.160) into its linearization $\varepsilon\dot{z} = A(\tau,\varepsilon)z$.

Let us briefly recall the results for the autonomous case. The equation $\dot{y} = Ay + b(y)$ is diagonalized by the transformation $y = z + h(z)$, where $h(z)$ satisfies the functional equation

$$\partial_z h(z)Az - Ah(z) = b(z + h(z)). \tag{5.161}$$

Since this equation cannot be solved in general, one usually proceeds in two steps:

1. Let $\mathcal{H}_k$ be the space of homogeneous polynomial functions of degree k. A basis is given by the monomials $e_{\mathrm{p},j}(z) = (0,\dots,0,z^{\mathrm{p}},0,\dots,0)$, with $\mathrm{p} \in \mathbb{N}^n$, $|\mathrm{p}| = \sum_i \mathrm{p}_{[i]} = k$ and $z^{\mathrm{p}} = z_{[1]}{}^{\mathrm{p}[1]} \dots z_{[n]}{}^{\mathrm{p}[n]}$. If $b_k(z) \in \mathcal{H}_k$, the equation
$$[h_k(z), Az] := \partial_z h_k(z)Az - Ah_k(z) = b_k(z) \tag{5.162}$$
is linear in $\mathcal{H}_k$. If $A = \operatorname{diag}(a_1,\dots,a_n)$, we have
$$[e_{\mathrm{p},j}(z), Az] = [\langle \mathrm{p}|a\rangle - a_j]e_{\mathrm{p},j}(z), \tag{5.163}$$
where $\langle \mathrm{p}|a\rangle := \sum_i p_{[i]}a_i$. Thus, (5.162) is solvable in the subspace of non–resonant terms, such that $\langle \mathrm{p}|a\rangle \neq a_j$. The change of variables $y = z + h_k(z)$ yields the equation
$$\begin{aligned}\dot{z} &= Az + [\mathbb{1} + \partial_z h_k(z)]^{-1}[b(z + h_k(z)) - b_k(z)] \\ &= Az + b(z) - b_k(z) + \mathcal{O}(|z|^{k+1}).\end{aligned} \tag{5.164}$$
Thus successive application of this procedure eliminates all non–resonant terms of order $k \leqslant N$, if $b(z) \in \mathcal{C}^N$.
2. If N is sufficiently large (and if A is hyperbolic), the Sternberg–Chen theorem (see Theorem 2.22) states that the remaining terms of order $|z|^N$ can be removed by a change of variables close to identity. In other words, if $b = \mathcal{O}(|z|^N)$ and N is large enough, equation (5.161) admits a solution $h(z) = \mathcal{O}(|z|^N)$.

Let us return to the non–autonomous equation (5.160). It is easy to see that in order to diagonalize the equation with the change of variables $y = z + h(z,\tau,\varepsilon)$, the function h should satisfy

$$\varepsilon\partial_\tau h(z,\tau,\varepsilon) = A(\tau,\varepsilon)h(z,\tau,\varepsilon) - \partial_z h(z,\tau,\varepsilon)A(\tau,\varepsilon)z + b(z + h(z,\tau,\varepsilon),\tau,\varepsilon). \tag{5.165}$$

We would like, once again, to consider (5.165) as an evolution equation for $h(z,\tau,\varepsilon)$, and to prove that it admits adiabatic solutions remaining close to the instantaneous transformations. As usual, we will solve this problem in two steps:

- Devise an iterative scheme, which simplifies the nonlinear terms of the system. In fact, we will apply two successive iterative schemes, the first one decreasing the order in $|z|$ of the remainder, the second one decreasing its order in ε.
- Prove that for the simplified equation, (5.165) admits a solution which is small in some sense.

We will begin with the simplest case, when $A(\tau,\varepsilon)$ has been diagonalized dynamically, and its eigenvalues are not resonant. Then we will examine some generalizations of this case.

1. Diagonal, Non–Resonant Case

In this part of the discussion, we assume

Hypothesis 5.1. *The matrix $A(\tau,\varepsilon)$ is a $\mathcal{C}^N$ function of τ, $N \geqslant 2$, and continuous in ε. It is diagonal, $A(\tau,\varepsilon) = \operatorname{diag}(a_1(\tau,\varepsilon),\dots,a_n(\tau,\varepsilon))$, where the eigenvalues satisfy the non–resonance conditions*

$$\langle \mathrm{p}|a\rangle \neq a_j, \qquad 2 \leqslant |\mathrm{p}| \leqslant N, \quad j = 1,\dots,n \tag{5.166}$$

uniformly in τ,ε.

Iterative Scheme 5.5. *Consider the equation*

$$\varepsilon\dot{y}_k = A(\tau,\varepsilon)y_k + b_k(y_k,\tau,\varepsilon) + r_k(y_k,\tau,\varepsilon), \tag{5.167}$$

where $b_k(\cdot,\tau,\varepsilon) \in \mathcal{H}_k$ and $r_k = \mathcal{O}(|y|^{k+1})$. Let $h_k(y,\tau,\varepsilon)$ be a solution of

$$\varepsilon\partial_\tau h_k = A(\tau,\varepsilon)h_k - \partial_y h_k A(\tau,\varepsilon)y + b_k(y,\tau,\varepsilon). \tag{5.168}$$

Then the change of variables $y_k = y_{k+1} + h_k(y_{k+1},\tau,\varepsilon)$ transforms (5.167) *into*

$$\varepsilon\dot{y}_{k+1} = A(\tau,\varepsilon)y_{k+1} + \tilde{b}_{k+1}(y_{k+1},\tau,\varepsilon), \tag{5.169}$$

where

$$\begin{aligned}\tilde{b}_{k+1}(y,\tau,\varepsilon) &= \big[\mathbb{1} + \partial_y h_k(y)\big]^{-1}\big[r_k(y+h_k,\tau,\varepsilon) + b_k(y+h_k,\tau,\varepsilon) - b_k(y,\tau,\varepsilon)\big]\\ &= \mathcal{O}(|y|^{k+1}),\end{aligned} \tag{5.170}$$

which can be decomposed as $\tilde{b}_{k+1} = b_{k+1} + r_{k+1}$, with $b_{k+1} \in \mathcal{H}_{k+1}$ and $r_{k+1} = \mathcal{O}(|y|^{k+2})$.

We have to show that equation (5.168) admits a bounded solution. To do this, we use the decompositions

$$h_k(y,\tau,\varepsilon) = \sum_{|\mathrm{p}|=k,j} h_{\mathrm{p},j}(\tau,\varepsilon)e_{\mathrm{p},j}(y), \qquad b_k(y,\tau,\varepsilon) = \sum_{|\mathrm{p}|=k,j} b_{\mathrm{p},j}(\tau,\varepsilon)e_{\mathrm{p},j}(y). \tag{5.171}$$

Replacing this in (5.168), and using (5.163), we obtain

$$\varepsilon\dot{h}_{\mathrm{p},j} = -\big[\langle\mathrm{p}|a\rangle - a_j\big]h_{\mathrm{p},j} + b_{\mathrm{p},j} \qquad \forall \mathrm{p}, j, \tag{5.172}$$

which is a *linear* 1D equation, that we can solve explicitly. In particular (see Subsection 4.1.2), we know that it admits a solution

$$\bar{h}_{\mathrm{p},j}(\tau,\varepsilon) = \frac{b_{\mathrm{p},j}(\tau,\varepsilon)}{\langle\mathrm{p}|a\rangle - a_j} + \mathcal{O}(\varepsilon), \tag{5.173}$$

which follows adiabatically the instantaneous solution of (5.161).

By the iterative scheme, we may thus construct a function $h^{(N)}(z,\tau,\varepsilon) = \mathcal{O}(|z|^2)$, such that the change of variables $y = z + h^{(N)}(z)$ transforms (5.160) into

$$\varepsilon\dot{z} = A(\tau,\varepsilon)z + b_N(z,\tau,\varepsilon), \qquad b_N(z,\tau,\varepsilon) = \mathcal{O}(|z|^N). \tag{5.174}$$

Moreover, h admits asymptotic series in ε.

If N is sufficiently large, the Sternberg–Chen theorem asserts existence of a function $\chi(z;\tau,\varepsilon) = \mathcal{O}(|z|^N)$, satisfying

$$\partial_z\chi(z)Az - A\chi(z) = b_N(z+\chi(z),\tau,\varepsilon), \tag{5.175}$$

so that the change of variables $z = z_1 + \chi(z_1)$ transforms (5.175) into

$$\varepsilon\dot{z}_1 = A(\tau,\varepsilon)z_1 + \varepsilon w_1(z_1;\tau,\varepsilon), \tag{5.176}$$

where

$$w_1(z_1;\tau,\varepsilon) = -\big[\mathbb{1} + \partial_z\chi\big]^{-1}\partial_\tau\chi(z_1;\tau,\varepsilon). \tag{5.177}$$

This expression is however meaningful only if we are able to show that χ is differentiable. This is done with the following lemma, that is proved in Appendix 5.B.8.

Lemma 5.8. *Assume $b_N(z,\tau,\varepsilon) = \mathcal{O}(|z|^N)$ is of class $\mathcal{C}^N$ in (z,τ) and continuous in ε, and $A(\tau,\varepsilon)$ satisfies Hypothesis 5.1. If $N-k$ is sufficiently large,* (5.175) *admits a solution $\chi(z,\tau,\varepsilon) = \mathcal{O}(|z|^N)$ of class $\mathcal{C}^k$ in (z,τ) and continuous in ε.*

If the system is sufficiently smooth, it is thus possible to decrease further the order in ε of the nonlinear term by applying the following iterative scheme.

Iterative Scheme 5.6. *Consider the equation*

$$\varepsilon\dot{z}_k = A(\tau,\varepsilon)z_k + \varepsilon^k w_k(z_k;\tau,\varepsilon), \tag{5.178}$$

where $w_k(z_k;\tau,\varepsilon) = \mathcal{O}(|z_k|^N)$. Let $\chi_k(z;\tau,\varepsilon)$ be the solution of

$$\partial_z\chi_k(z)Az - A\chi_k(z) = w_k(z+\chi_k(z),\tau,\varepsilon). \tag{5.179}$$

Then the change of variables $z_k = z_{k+1} + \varepsilon^k\chi_k(z_{k+1},\tau,\varepsilon)$ transforms (5.178) *into*

$$\varepsilon\dot{z}_{k+1} = A(\tau,\varepsilon)z_{k+1} + \varepsilon^{k+1}w_{k+1}(z_{k+1};\tau,\varepsilon), \tag{5.180}$$

where

$$w_{k+1}(z;\tau,\varepsilon) = -\big[\mathbb{1} + \varepsilon^k\partial_z\chi_k\big]^{-1}\partial_\tau\chi_k(z;\tau,\varepsilon). \tag{5.181}$$

We thus know that if the original system is sufficiently differentiable, it can be cast into the form

$$\varepsilon\dot{z}_k = A(\tau,\varepsilon)z_k + \varepsilon^k w_k(z_k;\tau,\varepsilon), \qquad w_k = \mathcal{O}(|z_k|^N). \tag{5.182}$$

To diagonalize this system completely, we have to carry out a transformation of the form $z_k = \zeta + \eta(\zeta;\tau,\varepsilon)$, where η satisfies the equation

$$\varepsilon\partial_\tau\eta = A(\tau,\varepsilon)\eta - \partial_z\eta A(\tau,\varepsilon)z + \varepsilon^k w_k(z+\eta,\tau,\varepsilon). \tag{5.183}$$

Then ζ obeys the linear equation $\varepsilon\dot{\zeta} = A(\tau,\varepsilon)\zeta$.

We would thus like to show that (5.183) admits a solution $\eta(z,\tau,\varepsilon)$ of order $\varepsilon^k|z|^N$. Since this appears to be as difficult as proving the Sternberg–Chen theorem, we will do it under more restrictive hypotheses on the matrix A, which correspond to the contracting case studied in [St1].

Hypothesis 5.2. *The matrix $A(\tau,\varepsilon)$ has eigenvalues $a_1(\tau,\varepsilon),\dots,a_n(\tau,\varepsilon)$, with real parts of the same sign, and such that*

$$\max_{1\leqslant j\leqslant n}|\operatorname{Re} a_j(\tau,\varepsilon)| < N \min_{1\leqslant j\leqslant n}|\operatorname{Re} a_j(\tau,\varepsilon)| \tag{5.184}$$

uniformly in τ,ε.

Lemma 5.9. *Assume $w_k(z,\tau,\varepsilon)=\mathcal{O}(|z|^N)$ is of class $\mathcal{C}^{N+1}$ in z, differentiable in τ and continuous in ε. If $A(\tau,\varepsilon)$ satisfies Hypotheses 5.1 and 5.2, then equation* (5.183) *admits a Lipschitz continuous solution $\eta(z,\tau,\varepsilon)=\mathcal{O}(\varepsilon^k|z|^N)$.*

We give the proof in Appendix 5.B.9. Observe that Hypothesis 5.2 restricts the discussion to equilibrium branches which are sources or sinks. This is however not a too serious restriction, since the contracting and expanding motions can be partially separated near a hyperbolic point, using adiabatic manifolds.

Using both iterative schemes and Lemma 5.9, we arrive to the following conclusion.

Theorem 5.7. *Assume $A(\tau,\varepsilon)$ is a matrix–valued function satisfying Hypotheses 5.1 and 5.2. This amounts to requiring that $A\in\mathcal{C}^N$ has distinct eigenvalues $a_1(\tau,\varepsilon),\dots,a_n(\tau,\varepsilon)$ such that $0<\max_j \operatorname{Re} a_j < N\min_j \operatorname{Re} a_j > 0$ or $0>\min_j \operatorname{Re} a_j > N\max_j \operatorname{Re} a_j < 0$. Let $b(y,\tau,\varepsilon)=\mathcal{O}(|y|^2)$ be a $\mathcal{C}^N$ function of y and τ, which is continuous in ε.*

1. *If N is sufficiently large, there exists, in a neighborhood of the origin, a function $h(z,\tau,\varepsilon)=\mathcal{O}(|y|^2)$, such that the change of variables $y=z+h(z,\tau,\varepsilon)$ transforms*
$$\varepsilon\dot y = A(\tau,\varepsilon)y+b(y,\tau,\varepsilon) \qquad \textit{into} \qquad \varepsilon\dot z = A(\tau,\varepsilon)z. \tag{5.185}$$
2. *The function $h(z,\tau,\varepsilon)$ admits expansions in z and ε, which can be constructed using Iterative Schemes 5.5 and 5.6. In particular, $h(z,\tau,0)$ coincides with the instantaneous transformation putting the equation $\varepsilon\dot y = A(\tau,0)y+b(y,\tau,0)$ into normal form.*

This result implies that the solution of the nonlinear equation $\varepsilon\dot y = A(\tau,\varepsilon)y+b(y,\tau,\varepsilon)$ is determined by the equations

$$\begin{aligned} y(\tau) &= U(\tau,\tau_0)z_0 + h(U(\tau,\tau_0)z_0,\tau,\varepsilon)\\ y(\tau_0) &= z_0 + h(z_0,\tau_0,\varepsilon), \end{aligned} \tag{5.186}$$

where $U(\tau,\tau_0)$ is the propagator of the linearized system $\varepsilon\dot z = A(\tau,\varepsilon)z$. This means in particular that if the solution of the linearized system remains small, then it is a good approximation of the nonlinear system's solution.

2. Diagonal, Resonant Case

Let us consider the equation

$$\varepsilon\dot y = A(\tau,\varepsilon)y + \sum_{\substack{2\leqslant|\mathrm{p}|<N\\ j=1,\dots,n}} b_{\mathrm{p},j}(\tau,\varepsilon)e_{\mathrm{p},j}(y) + b_N(y,\tau,\varepsilon), \tag{5.187}$$

where $b_N(y,\tau,\varepsilon)=\mathcal{O}(|y|^N)$. The following simplifications can be made:

1. Using Iterative Scheme 5.5, it is possible to eliminate all non-resonant terms in (5.187). Thus we may assume that (5.187) contains only those terms such that $\langle \mathrm{p}|a\rangle = a_j$ for some values of (τ,ε).

2. If $A(\tau,\varepsilon)$ satisfies Hypothesis 5.2, it is possible to eliminate the term b_N. Indeed, let $h(z,\tau,\varepsilon) = \mathcal{O}(|z|^N)$ be differentiable, and let

$$\beta(z,h,\tau,\varepsilon) := b^0(z+h,\tau,\varepsilon) - b^0(z,\tau,\varepsilon) - \partial_z h b^0(z+h,\tau,\varepsilon) + b_N(z+h,\tau,\varepsilon), \quad (5.188)$$

where $b^0(y,\tau,\varepsilon)$ is the second term in (5.187). Then it is easy to extend the proof of Lemma 5.9 in order to show that the equation

$$\varepsilon\partial_\tau h = A(\tau,\varepsilon)h - \partial_z h A(\tau,\varepsilon)z + \beta(z,h,\tau,\varepsilon) \quad (5.189)$$

admits a solution. The change of variables $y = z + h(z,\tau,\varepsilon)$ transforms (5.187) into

$$\varepsilon\dot{z} = A(\tau,\varepsilon)z + b^0(z,\tau,\varepsilon), \qquad b^0(z,\tau,\varepsilon) := \sum_{\substack{2\leqslant|\mathrm{p}|<N\\ j=1,\dots,n\\ \text{resonant}}} b_{\mathrm{p},j}(\tau,\varepsilon)e_{\mathrm{p},j}(z), \quad (5.190)$$

where resonant means that condition 5.166 is violated for some values of (τ,ε).

Equation (5.190) can be considered as a **dynamic normal form**, which should be simpler to study than the original equation. Consider for instance the 2D case, where $A = \operatorname{diag}(a_1,a_2)$ and $Na_1 < a_2 < a_1 < 0$ uniformly in (τ,ε). The resonant terms are those for which $j=2$ and $\mathrm{p} = (k,0)$ for $2 \leqslant k \leqslant N$. Thus the dynamic normal form is

$$\begin{aligned} \varepsilon\dot{\xi} &= a_1(\tau,\varepsilon)\xi \\ \varepsilon\dot{\eta} &= a_2(\tau,\varepsilon)\eta + \sum_{k=2}^{N} b_k(\tau,\varepsilon)\xi^k, \end{aligned} \quad (5.191)$$

and admits the explicit solution

$$\begin{aligned} \xi(\tau) &= \mathrm{e}^{\alpha_1(\tau,\tau_0)/\varepsilon}\,\xi(\tau_0) \\ \eta(\tau) &= \mathrm{e}^{\alpha_2(\tau,\tau_0)/\varepsilon}\,\eta(\tau_0) + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{\alpha_2(\tau,s)/\varepsilon} \sum_{k=2}^{N} b_k(s)\,\mathrm{e}^{k\alpha_1(s,\tau_0)/\varepsilon}\,\xi(\tau_0)^k\,\mathrm{d}s, \end{aligned} \quad (5.192)$$

where $\alpha_i(\tau,\tau_0) := \int_{\tau_0}^{\tau} a_i(s;\varepsilon)\,\mathrm{d}s$. It is, however, not obvious to compare the influence of linear and nonlinear terms in (5.192). If resonances occur only for particular values of τ, it is more adequate to carry out a local study of the resonance, and to linearize the equation completely outside the resonances. In fact, it is even possible to continue the linearizing transformation up to the crossing point, as shows the following example.

Example 5.6. Consider the equation

$$\begin{aligned} \varepsilon\dot{\xi} &= -\xi \\ \varepsilon\dot{\eta} &= -2(1+\tau)\eta + \xi^2. \end{aligned} \quad (5.193)$$

The term ξ^2 is resonant at $\tau = 0$. Instead of solving this equation directly, let us look for a transformation $\eta = \zeta + c(\tau)\xi^2$, taking the system into its linearization $\varepsilon\dot{\zeta} = -2(1+\tau)\zeta$. Substitution shows that c should satisfy the equation

$$\varepsilon\dot{c} = 1 - 2\tau c, \quad (5.194)$$

which we know very well. It admits the solution

$$c(\tau) = \frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{(s^2-\tau^2)/\varepsilon}\,\mathrm{d}s \approx \begin{cases} \frac{\tau}{\varepsilon} & \text{for } |\tau| \leqslant \sqrt{\varepsilon} \\ \frac{1}{\tau} & \text{for } |\tau| \geqslant \sqrt{\varepsilon}. \end{cases} \tag{5.195}$$

Similar solutions have been encountered in the case of diagonal eigenvalue crossings, see Fig. 5.5a, but scaled by a factor ε. The solution can be written as

$$\begin{aligned} \xi(\tau) &= \mathrm{e}^{\alpha_1(\tau,\tau_0)/\varepsilon}\,\xi(\tau_0) \\ \eta(\tau) &= \mathrm{e}^{\alpha_2(\tau,\tau_0)/\varepsilon}[\eta(\tau_0) - c(\tau_0)\xi(\tau_0)^2] + c(\tau)\,\mathrm{e}^{2\alpha_1(\tau,\tau_0)}\,\xi(\tau_0)^2, \end{aligned} \tag{5.196}$$

where $\alpha_1(\tau,\tau_0) = -(\tau-\tau_0)$ and $\alpha_2(\tau,\tau_0) = -2(\tau-\tau_0) - (\tau^2-\tau_0^2)$. This formulation is of course equivalent to (5.192), but it has the advantage to outline the influence of nonlinear terms, and the fact that the combination $\eta - c\xi^2$ evolves according to the linear system. The factor $c(\tau)$ being of order $1/\sqrt{\varepsilon}$, the nonlinear terms have a negligible influence at least when $\xi = \mathcal{O}(\varepsilon^{1/4})$.

Let us finally point out that if we replace τ be $-\tau$ in (5.193), the function $c(\tau)$ should satisfy the equation $\varepsilon\dot{c} = 1 + 2\tau c$. As in Fig. 5.5b, this equation does not admit a bounded solution throughout the crossing point, but there exist two solutions $c_\pm(\tau)$, each of which is of order $1/\sqrt{\varepsilon}$ on one side of the crossing point.

3. Eigenvalue Crossing

Let us finally consider the equation

$$\varepsilon\dot{y} = A(\tau,\varepsilon)y + b(y,\tau,\varepsilon), \qquad A(0,0) = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}, \qquad b(y,\tau,\varepsilon) = \mathcal{O}(|y|^2), \tag{5.197}$$

which describes the most generic eigenvalue crossing, discussed in Subsection 5.3.5. The preceding results can be extended to this situation in virtue of the following remarks.

1. Consider in $\mathcal{H}_k$ the linear equation

$$\varepsilon\partial_\tau h_k = Lh_k + b_k(y,\tau,\varepsilon), \qquad Lh_k = A(\tau,\varepsilon)h_k - \partial_y h_k. \tag{5.198}$$

   We know that L is no more diagonal in the basis of the $e_{\mathrm{p},j}(y)$, but it still admits eigenvalues of the form $a_j - \langle \mathrm{p}|a\rangle = (1-|\mathrm{p}|)a_0 + r_{\mathrm{p},j}(\tau)$, where $r_{\mathrm{p},j}(\tau)$ goes continuously to zero when $\tau \to 0$. Thus L is invertible in a neighborhood of $\tau = 0$ and (5.198) admits a solution

$$h_k(y,\tau,\varepsilon) = -L^{-1}b_k(y,\tau,\varepsilon) + \mathcal{O}(\varepsilon). \tag{5.199}$$

   Thus Iterative Scheme 5.5 applies and transforms (5.197) into

$$\varepsilon\dot{y}_N = A(\tau,\varepsilon)y_N + b_N(y_N,\tau,\varepsilon), \qquad b_N = \mathcal{O}(|y|^N). \tag{5.200}$$

2. The Sternberg–Chen Theorem does not require $A(\tau,\varepsilon)$ to be diagonal, so that Lemma 5.8 can be generalized to the present situation. Thus Iterative Scheme 5.6 can be applied, and transforms (5.200) into

$$\varepsilon\dot{z}_k = A(\tau,\varepsilon)z_k + \varepsilon^k w_k(z_k,\tau,\varepsilon), \qquad w_k = \mathcal{O}(|z_k|^N). \tag{5.201}$$

3. To eliminate the non–linear term in (5.201), we have to solve the equation

$$\varepsilon\partial_\tau\eta = A(\tau,\varepsilon)\eta - \partial_z\eta A(\tau,\varepsilon)z + \varepsilon^k w_k(z+\eta,\tau,\varepsilon). \tag{5.202}$$

But since Hypothesis 5.2 is satisfied in the vicinity of the origin, we can extend Lemma 5.9 to show that this equation admits a solution of order $\varepsilon^k|z|^N$. The only observation to be made is that the operator used in the proof of that lemma, which admits the solution as fixed point, has a linear part which is no more diagonal, but still contracting (see Remark 5.13 in Appendix 5.B.9). Thus the operator itself is still contracting and admits a unique fixed point.

With these remarks, we have proved the following result.

Proposition 5.8. *Assume the functions $A(\tau,\varepsilon)$ and $b(y,\tau,\varepsilon)$ in* (5.197) *are $\mathcal{C}^N$ in y and τ and continuous in ε. If N is sufficiently large and $a_0 \neq 0$, there exists in the neighborhood of the origin a transformation $y = z + h(z,\tau,\varepsilon)$ such that $\varepsilon\dot{z} = A(\tau,\varepsilon)z$. The function $h(z,\tau,\varepsilon) = \mathcal{O}(|z|^2)$ differs by $\mathcal{O}(\varepsilon)$ from the static linearizing transformation. The solution of* (5.197) *is thus given by*

$$y(\tau) = U(\tau,\tau_0)z_0 + h(U(\tau,\tau_0)z_0,\tau,\varepsilon), \qquad y(\tau_0) = z_0 + h(z_0,\tau_0,\varepsilon), \tag{5.203}$$

where $U(\tau,\tau_0)$ is the principal solution described in Theorem 5.5.

5.4.4 Bifurcations and Center Manifolds

Let us assume that the system

$$\varepsilon\dot{x} = f(x,\tau), \qquad f \in \mathcal{C}^2, \tag{5.204}$$

admits the origin as a bifurcation point, i.e., $f(0,0) = 0$ and the matrix $A := \partial_x f(0,0)$ has m eigenvalues with zero real part. Then f does not necessarily admit a branch of fixed points through the origin. To analyse the dynamics near the origin, we begin by carrying out a linear change of variables (independent of τ), transforming (5.204) into

$$\begin{aligned} \varepsilon\dot{y} &= By + f_1(y,z,\tau) \\ \varepsilon\dot{z} &= Cz + f_2(y,z,\tau), \end{aligned} \tag{5.205}$$

where B is hyperbolic, C is elliptic, and $f_{1,2} = \mathcal{O}(|y|^2 + |z|^2) + \mathcal{O}\big(\tau(|y| + |z|)\big)$. We are particularly interested in the situation where B is contracting, so we restrict the discussion to that case.

Let us consider the suspended system

$$\begin{aligned} \dot{y} &= By + f_1(y,z,\lambda) \\ \dot{z} &= Cz + f_2(y,z,\lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{5.206}$$

The center manifold theorem shows the existence of a center manifold $y = u(z,\lambda)$, satisfying the equation

$$\partial_z u(z,\lambda)\big[Cz + f_2(u,z,\lambda)\big] = Bu(z,\lambda) + f_1(u,z,\lambda). \tag{5.207}$$

The change of variables $y = u(z,\tau) + y_1$ transforms (5.205) into

$$\begin{aligned} \varepsilon\dot{y}_1 &= \big[B + B_1(y_1,z,\tau)\big]y_1 + \varepsilon w_1(z,\tau) \\ \varepsilon\dot{z} &= Cz + f_2(y_1 + u, z, \tau), \end{aligned} \tag{5.208}$$

where

$$\begin{aligned} B_1(y_1,z,\tau) &= \int_0^1 \partial_y f_1(u+\theta y_1,z,\tau)\,\mathrm{d}\theta - \partial_z u(z,\tau)\int_0^1 \partial_y f_2(u+\theta y_1,z,\tau)\,\mathrm{d}\theta \\ w_1(z,\tau) &= -\partial_\tau u(z,\tau) \end{aligned} \tag{5.209}$$

and $B_1 = \mathcal{O}(|y|+|z|) + \mathcal{O}(\tau)$.

Clearly, the order of the drift term w_1 can be further decreased by an iterative scheme analogous to those of the preceding subsections. Our goal, however, is to eliminate this drift term completely. To do this, we should solve, instead of (5.207), the system

$$\varepsilon\partial_\tau\eta = B\eta + f_1(\eta,z,\tau) - \partial_z\eta\big[Cz + f_2(\eta,z,\tau)\big], \tag{5.210}$$

so that the change of variables $y = y_0 + \eta(z,\tau;\varepsilon)$ transforms (5.205) into the system

$$\begin{aligned} \varepsilon\dot{y}_0 &= \big[B + B_0(y_0,z,\tau)\big]y_0 \\ \varepsilon\dot{z} &= Cz + f_2(y_0 + \eta, z, \tau). \end{aligned} \tag{5.211}$$

Then, the manifold $y = \eta(z,\tau;\varepsilon)$ is invariant, and the dynamics on this manifold is governed by the equation

$$\varepsilon\dot{z} = Cz + g(z,\tau;\varepsilon), \qquad g(z,\tau;\varepsilon) = f_2(\eta(z,\tau;\varepsilon),z,\tau). \tag{5.212}$$

Since in finite dimension, generic bifurcations only involve a small number of eigenvalues, equation (5.212) is in general a low–dimensional one, which describes the motion of slow modes near the bifurcation point. In particular, if z is a real number, we can use the methods developed in Chapter 4.

In Appendix 5.B.10, we prove that (5.210) admits, in the neighborhood of the origin, a solution close to $u(z,\tau)$. Thus we obtain the following result.

Theorem 5.8. *Assume $f(x,\tau) \in \mathcal{C}^2$, $f(0,0) = 0$ and the matrix $A = \partial_x f(0,0)$ is similar to a matrix $\left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$, where B is contracting and C is elliptic. There exist $\delta > 0$ and a function $\eta(z,\tau;\varepsilon) = \mathcal{O}(|z|^2+\tau^2)$, which is defined for $|z|,|\tau| \leqslant \delta$ and Lipschitz continuous, such that the change of variables $y = y_0 + \eta(z,\tau)$ transforms (5.205) into*

$$\begin{aligned} \varepsilon\dot{y}_0 &= \big[B + B_0(y_0,z,\tau)\big]y_0, \qquad B_0 = \mathcal{O}(|y|+|z|+|\tau|), \\ \varepsilon\dot{z} &= Cz + f_2(y_0+\eta,z,\tau). \end{aligned} \tag{5.213}$$

Moreover, if $f \in \mathcal{C}^k$, the function $\eta(z,\tau;\varepsilon)$ can be expanded in powers of ε up to order k.

Remark 5.11. Theorem 5.8 could be extended to show that

1. the adiabatic center manifold still exists if the matrix B also admits an expanding part;
2. there also exist local stable and unstable manifolds;
3. the center manifold is locally attracting.

5.5 Periodic Systems and Hysteresis

In this section, we particularize the discussion to the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,\lambda), \qquad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^p, \quad F \in \mathcal{C}^2, \tag{5.214}$$

where we impose a slow periodic variation $\lambda(\varepsilon t)$ of the parameter, where $\lambda(\tau) = \lambda(\tau+1)$ is a periodic function of class $\mathcal{C}^2$. On the scale of the slow time $\tau = \varepsilon t$, the equation becomes

$$\varepsilon \dot{x} = F(x, \lambda(\tau)) =: f(x,\tau). \tag{5.215}$$

We will denote by $\phi_\varepsilon(\tau,\tau_0,x_0)$ the solution of (5.215) with initial condition $x(\tau_0) = x_0$.

5.5.1 Periodic Orbits

Let us fix $\tau_0 = 0$ as initial time. Then the dynamics during one period is characterized by the **Poincaré map**

$$T_\varepsilon(x) := \phi_\varepsilon(1,0,x). \tag{5.216}$$

For T to be well defined, we may consider the following setting.

Proposition 5.9. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded, connected domain, with differentiable boundary $\partial\mathcal{D}$ admitting an exterior normal vector $v_\perp(x)$ such that*

$$\langle F(x,\lambda) | v_\perp(x) \rangle \leqslant 0 \quad \forall x \in \partial\mathcal{D},\ \forall\lambda. \tag{5.217}$$

If $F(x,\lambda)$ is differentiable for all $x \in \mathcal{D}$, then the Poincaré map is defined on $\mathcal{D}$ and maps $\mathcal{D}$ into itself.

The proof is a trivial consequence of the standard existence theorems.

Let $x(\tau) = \phi_\varepsilon(\tau,0,x_0)$ be a solution of (5.215). Setting $x = x(\tau) + y$, we obtain the equation

$$\varepsilon \dot{y} = A[x(\tau)]y + \mathcal{O}(|y|^2), \qquad A[x(\tau)] := \partial_x f(x(\tau),\tau). \tag{5.218}$$

This implies that the derivative of the Poincaré map is given by

$$\partial_x T_\varepsilon(x_0) = U[x(\tau)](1,0), \tag{5.219}$$

where $U[x(\tau)]$ denotes the principal solution of the system $\varepsilon\dot{y} = A[x(\tau)]y$.

Periodic orbits (with the same period than λ), are fixed points of T_ε. For hyperbolic branches, we have the following existence result.

Proposition 5.10. *Assume that for all τ, $F(x,\lambda(\tau))$ admits a hyperbolic*[6] *fixed point $X^\star(\lambda(\tau))$. Then there exists a periodic orbit $\bar{x}(\tau) = X^\star(\lambda(\tau)) + \mathcal{O}(\varepsilon)$, i.e., we have $T_\varepsilon(\bar{x}(0)) = \bar{x}(0)$.*

[6] By hyperbolic, we mean that the linearization of F around $X^\star$ has no purely imaginary eigenvalues. Sources and sinks are allowed.

Proof: By Theorem 5.1 there exists an adiabatic solution $\bar{x}(\tau) = X^\star(\lambda(\tau)) + \mathcal{O}(\varepsilon)$, where the first term is periodic by construction, but not necessarily the remainder. We assume that $\bar{x}(1) = \bar{x}(0) + \varepsilon z$. Writing $x = \bar{x}(\tau) + y$, we look for a solution $y(\tau)$ such that $y(1) = y(0) - \varepsilon z$. Let us consider the function $\Phi(y(0), z) := y(1) - y(0) + \varepsilon z$. It satisfies $\Phi(0,0) = 0$, while $\partial_y \Phi(0,0) = U[\bar{x}(\tau)](1,0) - \mathbb{1}$. But we know by Theorem 5.3 that U is similar to a bloc–diagonal matrix $\left(\begin{smallmatrix} U_+ & 0 \\ 0 & U_- \end{smallmatrix}\right)$, where U_+ has eigenvalues with exponentially large modulus, and U_- has eigenvalues with exponentially small modulus (see Remark 5.7). Thus $\partial_y \Phi(0,0)$ is invertible (with bounded inverse), and the existence of $y(\tau)$ follows from the implicit function theorem. □

When the equilibrium branch undergoes bifurcation, the situation becomes much more complicated. As in the 1D case, one should use a local analysis around the bifurcation point (using center manifolds described in Subsection 5.4.4) to determine past and future of adiabatic solutions beyond the bifurcation point. Examples in the next chapters will show that even in the adiabatic limit, the solutions are not necessarily periodic.

The linearized Poincaré map around a periodic solution is given by the propagator (5.219) associated with that solution. Let us examine the first terms of the series in ε of this linearization.

5.5.2 Dynamic and Geometric Terms

If $X^\star(\lambda)$ is a hyperbolic equilibrium branch, the associated adiabatic solution admits the expansion

$$\begin{aligned} \bar{x}(\tau) &= x_0(\tau) + \varepsilon x_1(\tau) + \mathcal{O}(\varepsilon^2) \\ x_0(\tau) &= X^\star(\lambda(\tau)) \\ x_1(\tau) &= A(\lambda(\tau))^{-1} \partial_\lambda X^\star(\lambda(\tau)) \dot{\lambda}(\tau), \end{aligned} \tag{5.220}$$

where $A(\lambda) := \partial_x F(X^\star(\lambda), \lambda)$. Setting $x = \bar{x}(\tau) + y$, we obtain the equation $\varepsilon \dot{y} = B(\tau) y + \mathcal{O}(|y|^2)$, where

$$\begin{aligned} B(\tau) &= \partial_x F(\bar{x}(\tau), \lambda(\tau)) \\ &= A(\lambda(\tau)) + \varepsilon L_0(\lambda(\tau)) \dot{\lambda}(\tau) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.221}$$

where $L_0(\lambda)$ is the linear operator

$$\begin{array}{rccl} L_0(\lambda): & \mathbb{R}^p & \to & \mathbb{M}_n(\mathbb{R}) \\ & z & \mapsto & \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \partial_x F(X^\star(\lambda) + \varepsilon A(\lambda)^{-1} \partial_\lambda X^\star(\lambda) z, \lambda)\big|_{\varepsilon=0}. \end{array} \tag{5.222}$$

Let us study the linearized system $\varepsilon \dot{y} = B(\tau) y$. Assume there exists a matrix $S(\lambda)$ such that $S(\lambda)^{-1} A(\lambda) S(\lambda) = D(\lambda)$ is bloc–diagonal. If $y = S(\lambda(\tau)) z$, we have

$$\begin{aligned} \varepsilon \dot{z} = B_1(\tau) z, \qquad B_1(\tau) &= S^{-1} B S - \varepsilon S^{-1} \partial_\lambda S \dot{\lambda} \\ &= D(\lambda(\tau)) + \varepsilon L(\lambda(\tau)) \dot{\lambda} + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.223}$$

where $L = S^{-1} L_0 S - S^{-1} \partial_\lambda S$. We know by Theorem 5.3 that there exists a change of variables $z = S_1(\tau) z_1$ (where $S_1(\tau; \varepsilon)$ is periodic by Proposition 5.10), transforming (5.223) into

$$\varepsilon \dot{z}_1 = B_2(\tau) z_1, \qquad B_2(\tau) = D(\lambda(\tau)) + \varepsilon L^0(\lambda(\tau)) \dot{\lambda} + \mathcal{O}(\varepsilon^2), \tag{5.224}$$

where $B_2(\tau)$ is diagonal, and L^0 denotes the diagonal part of the matrix.[7]

In particular, when $D = \operatorname{diag}(a_1, \dots, a_n)$ is diagonal, we have

$$B_2(\tau) = \operatorname{diag}(a_1(\lambda(\tau)) + \varepsilon L_{11}(\lambda(\tau))\dot{\lambda} + \mathcal{O}(\varepsilon^2), \dots, a_n(\lambda(\tau)) + \varepsilon L_{nn}(\lambda(\tau))\dot{\lambda} + \mathcal{O}(\varepsilon^2)), \tag{5.225}$$

so that

$$z_1(1) = \operatorname{diag}(\mathrm{e}^{\alpha_1/\varepsilon}, \dots, \mathrm{e}^{\alpha_n/\varepsilon}) z_1(0), \tag{5.226}$$

where α_j admits an expansion of the form

$$\begin{aligned} \alpha_j &= \alpha_j^{(0)} + \alpha_j^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2), \\ \alpha_j^{(0)} &= \int_0^1 a_j(\lambda(\tau))\,\mathrm{d}\tau, \\ \alpha_j^{(1)} &= \int_0^1 L_{jj}(\lambda(\tau))\dot{\lambda}(\tau)\,\mathrm{d}\tau = \oint L_{jj}(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{5.227}$$

Definition 5.7.
- The term $\alpha_j^{(0)}$ is called **dynamic term** associated with the j^{th} eigenspace. We call $\rho_j^{(0)} = \operatorname{Re}\alpha_j^{(0)}$ the **dynamic amplitude** and $\phi_j^{(0)} = \operatorname{Im}\alpha_j^{(0)}$ the **dynamic phase**.
- The term $\alpha_j^{(1)}$ is called **geometric term** associated with the j^{th} eigenspace. We call $\rho_j^{(1)} = \operatorname{Re}\alpha_j^{(1)}$ the **geometric amplitude** and $\phi_j^{(1)} = \operatorname{Im}\alpha_j^{(1)}$ the **geometric phase**.

The dynamic term is simply given by the integral of the matrices eigenvalue. The geometric term depends only on the geometric shape of the loop $\lambda(\tau)$ in parameter space. It contains two contributions, one due to the drift term of the adiabatic solution, the other one due to the eigenspaces' motion. In particular, this term is zero when $\lambda(\tau)$ encloses a zero area in parameter space.

If the matrix displays eigenvalue crossings, geometric terms may appear even when $\lambda(\tau)$ encloses a zero area.

5.5.3 Hysteresis

Let us briefly examine the hysteresis phenomenon. As in the 1D case, we propose the following definition:

Definition 5.8. Let $x(\tau, \varepsilon)$ be the solution of (5.215) with initial condition $x(\tau_0, \varepsilon) = x_0$. If there exists a single–valued function $X(\lambda, x_0)$ such that

$$\lim_{\varepsilon \to 0} x(\tau, \varepsilon) = X(\lambda(\tau), x_0) \tag{5.228}$$

for almost any τ, then we say that this solution **does not display hysteresis**. If no such function exists, we say that the solution **displays hysteresis**. A periodic solution of (5.215) displaying hysteresis is called a **hysteresis cycle**.

[7]The transformation affects the diagonal terms of L only at order ε^2, see equation (5.50).

Proposition 5.10 shows that adiabatic solutions associated with hyperbolic equilibria do not display hysteresis, since we may take $X(\lambda, x_0) = X^\star(\lambda)$. Thus, bifurcations are necessary for solutions to display hysteresis (at least when these solutions are most of the time close to adiabatic ones).

Note that Definition 5.8 is mainly interesting in the case where $\lambda(\tau)$ takes the same value for different values of τ. One can wonder about what happens when the problem is considered for *all* functions $\lambda(\tau)$ in some function space. In the 1D case, we have seen that the relation

$$\lim_{\varepsilon\to 0} x(\tau,\varepsilon) = X^\star_{j[\lambda(\cdot)]}(\lambda(\tau)), \tag{5.229}$$

holds, where the $X^\star_j$ are equilibrium branches of the system. This property certainly not always holds in the nD case, since solutions may follow limit cycles or other attractors instead of equilibrium branches. We will, however, see in examples that even when solutions follow equilibrium branches almost all the time, this does not imply that the function $j[\lambda(\cdot)]$ exists.

5.6 Summary and Conclusion

In this chapter, we have presented a number of methods to analyse equations of the form $\varepsilon\dot{x} = f(x,\tau)$. The main steps are

- Find the equilibrium branches of $f(x,\tau)$. Between bifurcation points, compute the adiabatic solutions associated with these branches.
- In the neighborhood of bifurcation points, apply the center manifold theory of Subsection 5.4.4 to analyse the behaviour of adiabatic solutions. In the generic case, one can use methods of Chapter 4 to determine the future evolution of these solutions, and how they scale with ε.
- The behaviour of solutions close to an adiabatic one is governed by an equation of the form $\varepsilon\dot{y} = A(\tau,\varepsilon)y + \mathcal{O}(|y|^2)$. This equation can be simplified by using adiabatic manifolds or dynamic normal forms described in Subsection 5.4.2 and 5.4.3.
- The linearized equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ can be analysed by combining a dynamic diagonalization with a local study of eigenvalue crossings, when they occur (see Subsection 5.3.8).

In general, these methods are not sufficient to solve the problem completely. In particular (as in the autonomous case), it may be hard to control the motion of solutions far from equilibrium branches. Note however that adiabatic manifolds and normal forms may be defined in a fairly large neighborhood of an equilibrium branch, so that they may allow to control solutions which jump from one adiabatic solution to another one. These remarks will become clearer in the particular cases discussed in the next chapters. The basic idea is that if the dynamics of the static dynamical system can be controlled, then it is possible to extend this knowledge to the adiabatic system.

5.A Some Properties of Matrices

We present in this section a few useful properties of matrices and matrix–valued functions.

If I is some interval in $\mathbb{R}$ and $A(\tau) \in \mathcal{C}^k(I, \mathbb{M}_n(\mathbb{C}))$ is a differentiable matrix–valued function, its eigenvalues are continuous, but not necessarily differentiable functions of τ. We examine under which conditions the matrix can be diagonalized by a transformation depending smoothly on τ. To do this, we need to understand the matrix equation $AX - XB = C$. More details are found in [**Wa**, **Bel**, **Kr**].

5.A.1 The equation $AX - XB = C$.

Notation 5.1. Let $A \in \mathbb{M}_n(\mathbb{C})$, $B \in \mathbb{M}_m(\mathbb{C})$. We denote by $L[A, B]$ the linear operator

$$\begin{aligned} L[A,B]: \quad \mathbb{M}_{n\times m}(\mathbb{C}) &\to \mathbb{M}_{n\times m}(\mathbb{C}) \\ X &\mapsto AX - XB. \end{aligned} \tag{5.230}$$

Lemma 5.10. *The equation $AX - XB = 0$ has a solution different from $X = 0$ if and only if A and B have at least one common eigenvalue.*

PROOF:

$\Leftarrow$ If λ is an eigenvalue of A and B, there exist $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$ such that $Av = \lambda v$ and $B^{\mathrm{T}} w = \lambda w$, i.e., $w^{\mathrm{T}} B = \lambda w^{\mathrm{T}}$. Let $X = vw^{\mathrm{T}}$. Then $AX = \lambda v w^{\mathrm{T}} = vw^{\mathrm{T}} B = XB$.

$\Rightarrow$ Assume $AX = XB$, where A and B have no common eigenvalue. By Jordan's theorem, there exists $S \in \mathrm{GL}(m, \mathbb{C})$ such that $S^{-1}BS = D + N$, where $D = \mathrm{diag}(b_1, \dots, b_m)$ and $N[kj] = 0$ for $k \geqslant j$. Then $AY = Y(D + N)$, where $Y = XS$. Let $y_1, \dots, y_m$ be the columns if Y, then

$$Ay_j = b_j y_j + \sum_{k<j} N[kj] y_k, \qquad j = 1, \dots, m. \tag{5.231}$$

In particular, $Ay_1 = b_1 y_1$. Since b_1 is not an eigenvalue of A, $y_1 = 0$. By induction on j, this means that $y_j = 0$, $j = 1, \dots, m$, so that $Y = 0$ and $X = YS^{-1} = 0$. □

In other words, the operator $L[A, B]$ is invertible if and only if A and B have no common eigenvalue.

Corollary 5.5. *If A has eigenvalues $a_1, \dots, a_n$ and B has eigenvalues $b_1, \dots, b_m$, then the eigenvalues of $L[A, B]$ are exactly*

$$\lambda_{ij} = a_i - b_j, \qquad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant m. \tag{5.232}$$

PROOF: λ is eigenvalue of L if and only if there exists $X \neq 0$ such that $[A - \lambda \mathbb{1}]X - XB = 0$. By Lemma 5.10, this is equivalent to $A - \lambda\mathbb{1}$ and B having a common eigenvalue, i.e., $a_i - \lambda = b_j$ for some i and j. □

We now show how to give an explicit formula for $L[A, B]^{-1}(C)$. We first consider the case where one matrix is contracting, and the other one expanding.

Lemma 5.11. *If A and $-B$ are contracting, the solution of $AX - XB = C$ is given by*

$$X = -\int_0^\infty \mathrm{e}^{As} C \, \mathrm{e}^{-Bs} \, \mathrm{d}s. \tag{5.233}$$

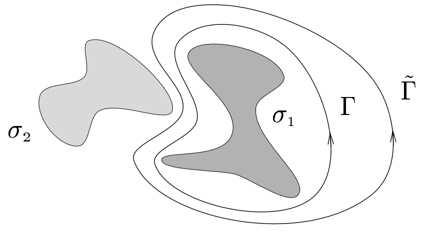

FIGURE 5.7. Let σ_1 and σ_2 be two regions of the complex plane, and Γ a closed path encircling only σ_1. If the eigenvalues of the matrix A are distributed over these sets, then the integral over Γ of $(A-z)^{-1}$ is $2\pi\,\mathrm{i}$ times the projector on the eigenspace associated with the eigenvalues of A contained in σ_1 (Lemma 5.12). If A and B are two matrices with eigenvalues in σ_1 and σ_2 respectively, then the integral over Γ of $(A-z)^{-1}C(B-z)^{-1}$ is the solution of the equation $AX - XB = -2\pi\,\mathrm{i}\,C$ (Lemma 5.13).

PROOF: The convergence of the integral follows from the fact that $\|\mathrm{e}^{As}\| \leqslant K\,\mathrm{e}^{-cs}$ (see Proposition 2.14). Since B commutes with e^{Bs}, we have

$$C = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{As}\,C\,\mathrm{e}^{-Bs})\,\mathrm{d}s = -\int_0^\infty [A\,\mathrm{e}^{As}\,C\,\mathrm{e}^{-Bs} - \mathrm{e}^{As}\,C\,\mathrm{e}^{-Bs}\,B]\,\mathrm{d}s = AX - XB. \qquad \square$$

A general formula for the inverse of $L[A,B](C)$ can be derived using complex analysis. Details on this approach can be found in [**Kr**].

Lemma 5.12. *Let A be a matrix in $\mathbb{M}_n(\mathbb{C})$, and Γ a closed path in $\mathbb{C}$, such that no eigenvalue of A lies on Γ. Then the operator*[8]

$$P := \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma (A-z)^{-1}\,\mathrm{d}z \tag{5.234}$$

is the projector on the invariant subspace of A associated with its eigenvalues contained in the interior of Γ.

PROOF: We first show that P is a projector. If $\tilde\Gamma$ is a closed path encircling Γ, but no other eigenvalue of A (Fig. 5.7), we may write

$$\begin{aligned}
P^2 &= \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \mathrm{d}z \frac{1}{2\pi\,\mathrm{i}}\int_{\tilde\Gamma} \mathrm{d}\tilde z (A-z)^{-1}(A-\tilde z)^{-1} \\
&= \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \mathrm{d}z (A-z)^{-1} \underbrace{\frac{1}{2\pi\,\mathrm{i}}\int_{\tilde\Gamma} \frac{\mathrm{d}\tilde z}{z-\tilde z}}_{1} - \frac{1}{2\pi\,\mathrm{i}}\int_{\tilde\Gamma} \mathrm{d}\tilde z (A-\tilde z)^{-1} \underbrace{\frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \frac{\mathrm{d}z}{z-\tilde z}}_{0} = P,
\end{aligned} \tag{5.235}$$

where we have used the relation $(A-z)^{-1}(z-\tilde z)(A-\tilde z)^{-1} = (A-z)^{-1} - (A-\tilde z)^{-1}$, and the fact that Γ is contained in the interior of $\tilde\Gamma$. By construction, P commutes with A, so that $P\mathbb{C}^n$ and $(\mathbb{1}-P)\mathbb{C}^n$ are invariant subspaces of A. If $v \in \mathbb{C}^n$ satisfies $Av = av$, then

$$Pv = \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma (A-z)^{-1} v\,\mathrm{d}z = \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \frac{\mathrm{d}z}{a-z} v = \begin{cases} v & \text{if } a \in \operatorname{int}\Gamma \\ 0 & \text{otherwise,} \end{cases} \tag{5.236}$$

which shows that P projects on the subspace associated with the eigenvalues in $\operatorname{int}\Gamma$. $\square$

[8] We write $(A-z)$ instead of $(A-z\mathbb{1})$.

Lemma 5.13. *Let A and B have distinct eigenvalues, and let Γ be a complex path encircling only the eigenvalues of A. Then the solution of $AX - XB = C$ is given by*

$$X = -\frac{1}{2\pi\,\mathrm{i}} \int_\Gamma (A-z)^{-1} C (B-z)^{-1} \,\mathrm{d}z. \tag{5.237}$$

PROOF: Since $(A-z)^{-1}(AC-CB)(B-z)^{-1} = C(B-z)^{-1} - (A-z)^{-1}C$, we have

$$AX - XB = \frac{1}{2\pi\,\mathrm{i}} \int_\Gamma (A-z)^{-1} \,\mathrm{d}z\, C - C\, \frac{1}{2\pi\,\mathrm{i}} \int_\Gamma (B-z)^{-1} \,\mathrm{d}z = C, \tag{5.238}$$

since, by the previous lemma, the first integral is $\mathbb{1}$ and the second is 0. □

5.A.2 Smooth Diagonalization

Lemma 5.14. *Let I be a real interval, and $A(\tau) \in \mathcal{C}^k(I, \mathbb{M}_n(\mathbb{C}))$ a smooth matrix–valued function. Assume that eigenvalues of $A(\tau)$ can be split into two groups $\{a_1(\tau), \dots, a_p(\tau)\}$ and $\{a_{p+1}(\tau), \dots, a_n(\tau)\}$ with strictly positive gap γ. Then there exists a matrix function $S(\tau) \in \mathcal{C}^k(I, \mathrm{GL}(n,\mathbb{C}))$ such that $D(\tau) = S(\tau)^{-1}A(\tau)S(\tau)$ is bloc–diagonal, with blocs respectively of size $p \times p$ and $(n-p)\times(n-p)$.*

PROOF: For any $\tau_0 \in I$, there exists $S_0 \in \mathrm{GL}(n,\mathbb{C})$ such that $S_0^{-1}A(\tau_0)S_0$ is bloc–diagonal, with blocs of the indicated size. Thus

$$B(\tau) := S_0^{-1}A(\tau)S_0 = \begin{pmatrix} B_{11}(\tau) & B_{12}(\tau) \\ B_{21}(\tau) & B_{22}(\tau) \end{pmatrix}, \qquad \text{where } B_{ij}(\tau_0) = 0 \text{ when } i \neq j. \tag{5.239}$$

Moreover, the matrices $B_{ii}(\tau_0)$ have exactly the abovementionned groups of eigenvalues, respectively for $i = 1, 2$. If we find matrices $T(\tau)$ and $D(\tau)$ of the form

$$T(\tau) := \begin{pmatrix} \mathbb{1}_p & T_{12}(\tau) \\ T_{21}(\tau) & \mathbb{1}_{n-p} \end{pmatrix}, \qquad D(\tau) = \begin{pmatrix} D_{11}(\tau) & 0 \\ 0 & D_{22}(\tau) \end{pmatrix}, \tag{5.240}$$

where $T_{ij}(\tau_0) = 0$, $i \neq j$, and such that $BT = TD$, then the matrix $S(\tau) := S_0T(\tau)$ satisfies $S^{-1}AS = T^{-1}BT = D$. But the equation $BT = TD$ is equivalent to the system

$$\begin{aligned} B_{11} + B_{12}T_{21} &= D_{11} \\ B_{21}T_{12} + B_{22} &= D_{22} \\ B_{11}T_{12} + B_{12} &= T_{12}D_{22} \\ B_{21} + B_{22}T_{21} &= T_{21}D_{11} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} B_{12} + B_{11}T_{12} - T_{12}B_{22} - T_{12}B_{21}T_{12} &= 0 \\ B_{21} + B_{22}T_{21} - T_{21}B_{11} - T_{21}B_{12}T_{21} &= 0. \end{aligned} \tag{5.241}$$

Consider for instance the equation for T_{21}. It is of the form $\Phi(T_{21}, \tau) = 0$, with $\Phi(0, \tau_0) = 0$. The Fréchet derivative with respect to T_{21} of Φ at $(0, \tau_0)$ is the operator $L[B_{22}(\tau_0), B_{11}(\tau_0)]$. Since these two matrices have no common eigenvalue, Lemma 5.10 shows that this operator does not admit 0 as an eigenvalue, and is thus invertible. The implicit function theorem then implies that $T(\tau)$ exists and is $\mathcal{C}^k$ in a neighborhood of τ_0. Since $T(\tau_0) = \mathbb{1}_n$ is invertible, by continuity T remains invertible in this neighborhood. Since this is true for every τ_0, the lemma is proved. □

Remark 5.12.

1. If all eigenvalues of $A(\tau)$ are different, Lemma 5.14 implies that $A(\tau)$ can be completely diagonalized by a matrix depending smoothly on τ.
2. Assume that $A(\varepsilon)$ is a $\mathcal{C}^1$ matrix function of ε, such that $A(0)$ is bloc–diagonal, and $A(\varepsilon) = A(0) + \mathcal{O}(\varphi(\varepsilon))$, where $\varphi(\varepsilon)$ is a continuous function vanishing at $\varepsilon = 0$. The proof shows that for small ε, $A(\varepsilon)$ can be bloc–diagonalized by a matrix $S(\varepsilon) = \left(\begin{smallmatrix} 1\!\!1 & \mathcal{O}(\varphi(\varepsilon)) \\ \mathcal{O}(\varphi(\varepsilon)) & 1\!\!1 \end{smallmatrix}\right)$.
3. A straightforward calculation shows that if $|\varepsilon|^2 < \|S_1S_2\|$, then

$$\begin{aligned} \begin{pmatrix} 1\!\!1 & \varepsilon S_1 \\ \varepsilon S_2 & 1\!\!1 \end{pmatrix}^{-1} &= \begin{pmatrix} 1\!\!1 + \varepsilon^2 S_1(1\!\!1 - \varepsilon^2 S_2S_1)^{-1}S_2 & -\varepsilon S_1(1\!\!1 - \varepsilon^2 S_2S_1)^{-1} \\ -\varepsilon S_2(1\!\!1 - \varepsilon^2 S_1S_2)^{-1} & 1\!\!1 + \varepsilon^2 S_2(1\!\!1 - \varepsilon^2 S_1S_2)^{-1}S_1 \end{pmatrix}, \\ (1\!\!1 - \varepsilon^2 S_1S_2)^{-1} &= \sum_{j \geqslant 0} (\varepsilon^2 S_1S_2)^j, \qquad \|(1\!\!1 - \varepsilon^2 S_1S_2)^{-1}\| \leqslant (1 - |\varepsilon|^2\|S_1S_2\|)^{-1}. \end{aligned} \tag{5.242}$$

5.B Proofs of Some Results

5.B.1 Proof of Proposition 5.2

We will prove that the equation

$$\begin{aligned} \varepsilon\dot{u} &= A_+(\tau)u + b_+(u,v,\tau,\varepsilon) + \varphi(\varepsilon)w_+(\tau,\varepsilon) \\ \varepsilon\dot{v} &= A_-(\tau)v + b_-(u,v,\tau,\varepsilon) + \varphi(\varepsilon)w_-(\tau,\varepsilon), \end{aligned} \tag{5.243}$$

admits a solution of order $\varphi(\varepsilon)$. All functions are assumed to be differentiable for $z = (u,v)$ in a neighborhood of the origin in $\mathbb{C}^n$, τ in a real interval $I = [\tau_0, \tau_1]$, and $0 < \varepsilon \leqslant \varepsilon_0$. The matrix $A_+(\tau)$ is expanding and $A_-(\tau)$ is contracting, $b_\pm(u,v,\tau,\varepsilon) = \mathcal{O}(\|u\|_2^2 + \|v\|_2^2) + \mathcal{O}(\varepsilon(\|u\|_2 + \|v\|_2))$, and $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = 0$.

1. Since A_- and $-A_+$ are contracting, we know by Lemma 5.2 that there exist ideal Lyapunov functions

$$V_+(u,u^*,\tau) = \langle u|Y_+(\tau)u\rangle, \qquad Y_+(\tau) = \int_0^\infty \mathrm{e}^{-A_+^*(\tau)s}\,\mathrm{e}^{-A_+(\tau)s}\,\mathrm{d}s, \tag{5.244a}$$

$$V_-(v,v^*,\tau) = \langle v|Y_-(\tau)v\rangle, \qquad Y_-(\tau) = \int_0^\infty \mathrm{e}^{A_-^*(\tau)s}\,\mathrm{e}^{A_-(\tau)s}\,\mathrm{d}s, \tag{5.244b}$$

satisfying the relations

$$\begin{aligned} &V_0^{-1}\|u\|_2 \leqslant V_+ \leqslant V_0\|u\|_2, && \partial_u V_+ A_+ u + \partial_{u^*} V_+ (A_+u)^* \geqslant a_0 V_+, \\ &V_0^{-1}\|v\|_2 \leqslant V_- \leqslant V_0\|v\|_2, && \partial_v V_- A_- v + \partial_{v^*} V_- (A_-v)^* \leqslant -a_0 V_-, \\ &\|\partial_\tau V_\pm\|_2 \leqslant V_0 V_\pm, && \|\partial_{u,u^*}V_+\|_2 \leqslant V_0, \quad \|\partial_{v,v^*}V_-\|_2 \leqslant V_0, \end{aligned} \tag{5.245}$$

for some constants $V_0, a_0 > 0$.
It then follows from (5.243) that

$$\begin{aligned} \varepsilon\dot{V}_+ &\geqslant a_0V_+ - M(\|u\|_2^2 + \|v\|_2^2 + \varepsilon\|u\|_2 + \varepsilon\|v\|_2 + \varphi(\varepsilon)) - \varepsilon V_0 V_+, \\ \varepsilon\dot{V}_- &\leqslant -a_0V_- + M(\|u\|_2^2 + \|v\|_2^2 + \varepsilon\|u\|_2 + \varepsilon\|v\|_2 + \varphi(\varepsilon)) + \varepsilon V_0 V_-, \end{aligned} \tag{5.246}$$

for some $M > 0$.

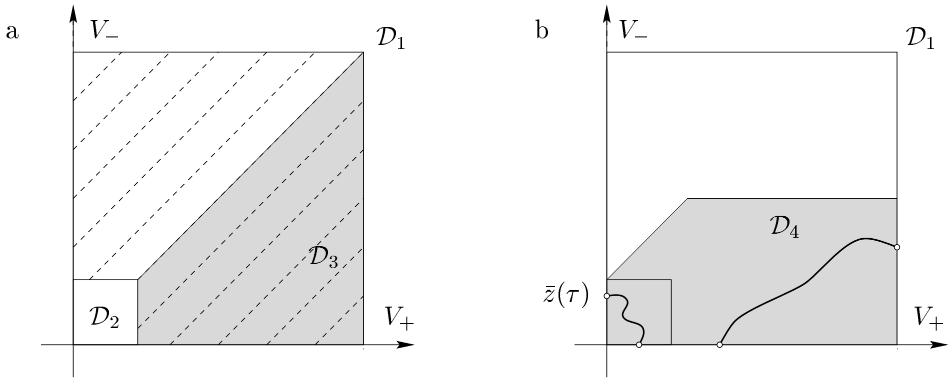

FIGURE 5.8. The sets occurring in the proof of Proposition 5.2. (a) The sets $\mathcal{D}_1$ (large square), $\mathcal{D}_2$ (small square) and $\mathcal{D}_3$ (shaded area). Dashed lines are level lines of the Lyapunov function V, which is shown to increase in the set $\mathcal{D}_1 \setminus \mathcal{D}_2$, while V_+ is increasing in $\mathcal{D}_3$. (b) Orbits such that $V_-(\tau_0) = 0$ are shown to lie in the shaded set $\mathcal{D}_4$. There exists in particular a solution such that $V_+(\tau_1) = 0$, which remains in a region of size $\mathcal{O}(\varphi(\varepsilon))$.

2. Let $c_1, c_2 > 0$ be constants. For any $\tau \in I$, we introduce three subsets of $\mathbb{C}^n$:

$$\begin{aligned}
\mathcal{D}_1(\tau) &= \big\{(u,v) \in \mathbb{C}^n \bigm| V_+, V_- \leqslant c_1\big\}, \\
\mathcal{D}_2(\tau) &= \big\{(u,v) \in \mathbb{C}^n \bigm| V_+, V_- \leqslant c_2^{-1}\varphi(\varepsilon)\big\}, \\
\mathcal{D}_3(\tau) &= \big\{(u,v) \in \mathcal{D}_1 \bigm| c_2^{-1}\varphi(\varepsilon) \leqslant V_+ \leqslant c_1, V_- \leqslant V_+\big\},
\end{aligned} \tag{5.247}$$

where $V_+ = V_+(u, u^*, \tau)$ and $V_- = V_-(v, v^*, \tau)$.
If $V(z, z^*, \tau) := V_+(u, u^*, \tau) - V_-(v, v^*, \tau)$, the above relations imply that for $(u,v) \in \mathcal{D}_1 \setminus \mathcal{D}_2$,

$$\varepsilon \dot{V} \geqslant [a_0 - \varepsilon_0 V_0 - 2MV_0(V_0 c_1 + \varepsilon_0)](V_+ + V_-) - 2M\varphi(\varepsilon) > 0, \tag{5.248}$$

(in fact, $\varepsilon\dot{V} \geqslant c_3\varphi(\varepsilon)$ with $c_3 > 0$), if c_1, c_2, ε_0 are small enough. Similarly, we can arrange that

$$\begin{aligned}
\varepsilon\dot{V}_+ &> 0, && \text{if } (u,v) \in \mathcal{D}_3, \\
\varepsilon\dot{V}_- &< 0, && \text{if } V_- \geqslant \tfrac{1}{2}c_1.
\end{aligned} \tag{5.249}$$

3. Let us denote the flow of (5.243) by $\phi_{\tau,\tau_0} = (\phi^u_{\tau,\tau_0}, \phi^v_{\tau,\tau_0})$. For any u such that $(u,0) \in \mathcal{D}_1(\tau_0)$ and $\tau \in I$, we define the function

$$\tau^\star(u,\tau) = \sup\big\{\tau' \in [\tau_0, \tau] \bigm| \phi_{\tau'',\tau_0} \in \mathcal{D}_1(\tau'') \text{ for } \tau_0 \leqslant \tau'' \leqslant \tau'\big\}. \tag{5.250}$$

Let us consider the orbit $\hat{z}(\tau) = \phi_{\tau^\star(u,\tau),\tau_0}((u,0))$. We show that

$$\hat{z}(\tau) \in \mathcal{D}_4(\tau) = \big\{(u,v) \in \mathcal{D}_1(\tau) \bigm| V_- \leqslant V_+ + c_2^{-1}\varphi(\varepsilon) \text{ and } V_- \leqslant \tfrac{1}{2}c_1\big\}. \tag{5.251}$$

Let $V_\pm(\tau)$, $V(\tau)$ be the Lyapunov functions evaluated along $\hat{z}(\tau)$. We have $V(\tau_0) = V_+(\tau_0) \geqslant 0$. If $\hat{z}(\tau) \in \mathcal{D}_2$, then $V(\tau) \geqslant -c_2^{-1}\varphi(\varepsilon)$ by (5.247). Otherwise, we have $\varepsilon\dot{V} > 0$ by (5.248), and thus $V(\tau)$ must remain larger than $-c_2^{-1}\varphi(\varepsilon)$, which proves the first bound in (5.251). The other bound follows from the fact that $V_-(\tau_0) = 0$, and V_- cannot grow larger than $\frac{1}{2}c_1$ because of (5.249). This shows in particular that if $\hat{z}(\tau)$ reaches $\partial\mathcal{D}_1(\tau)$, it does so at a point where $V_+ = c_1$ and $V_- \leqslant \frac{1}{2}c_1$.

4. We introduce the sets
$$\mathcal{E}(\tau) = \bigl\{u \in \mathbb{C}^{n_+} \bigm| V_+(u,u^*,\tau) \leqslant c_1\bigr\}. \tag{5.252}$$
Since the matrix $Y_+(\tau)$ in (5.244a) is hermitian positive definite, there exists a hermitian matrix $P(\tau)$ such that $c_1 P(\tau)^2 = Y_+(\tau)$. The linear application $u \mapsto P(\tau)u$ maps $\mathcal{E}(\tau)$ into the ball $\mathcal{B} = \{u \mid \|u\|_2 \leqslant 1\}$ and $\partial\mathcal{E}(\tau)$ on $\partial\mathcal{B}$, because $\langle Pu|Pu\rangle = c_1^{-1}V_+(u,u^*,\tau)$.
We define the map
$$\begin{array}{rccl} \Psi: & \mathcal{B} & \to & \mathcal{B} \\ & w & \mapsto & P(\tau^\star)\phi^u_{\tau^\star,\tau_0}((P(\tau_0)^{-1}w,0)), \end{array} \tag{5.253}$$
where $\tau^\star = \tau^\star(P(\tau_0)^{-1}w,\tau_1)$. Note that if $w \in \partial\mathcal{B}$, then $P(\tau_0)^{-1}w \in \partial\mathcal{E}(\tau_0)$, so that $\tau^\star = \tau_0$ and $\Psi(w) = w$.
Moreover, Ψ is continuous. This is due to the fact that the flow can leave $\mathcal{D}_1$ only on the smooth manifold where $V_+ = c_1$ and $V_- \leqslant \frac{1}{2}c_1$, and, because $\dot{V}_+ > 0$, it is transverse to the manifold at these points. The continuity of Ψ follows from the fact that the flow is differentiable, by the standard proof of the existence of a Poincaré map.
5. By a standard result of algebraic topology, Ψ is surjective. Indeed, it is known that there exists no continous map R from $\mathcal{B}$ to $\partial\mathcal{B}$ such that $R(w) = w$ for all w in $\partial\mathcal{B}$, i.e., $\partial\mathcal{B}$ is not a **retract** of $\mathcal{B}$ (see for instance Lemma 0.2 in [**Ro**]). Assume by contradiction that Ψ is not surjective. Then there exists $w^\star \notin \operatorname{im}\Psi$. Let r be the central projection on $\partial\mathcal{B}$ with center $w^\star$. Then the map $r\circ\Psi$ is a retract, contradicting the lemma.
Hence, there exists in particular $\bar{w} \in \mathcal{B}$ such that $\Psi(\bar{w}) = 0$. The associated orbit $\bar{z}(\tau)$ must satisfy $V_+(\tau) < c_2^{-1}\varphi(\varepsilon)$, since otherwise $u(\tau_1)$ could not be equal to 0 because of (5.249). By (5.251), this implies that $V_-(\tau) \leqslant 2c_2^{-1}\varphi(\varepsilon)$, and thus $\bar{z}(\tau)$ is the orbit we have been looking for. □

5.B.2 Proof of Theorem 5.2

Let $\Gamma(D) := \{\tau \in \mathbb{C} \mid |\operatorname{Im}\tau| < D, |\operatorname{Re}\tau - s| < D\,\forall s \in I\}$, and let $D_N := D_1 - (N-1)d\varepsilon_0$, where d and ε_0 are positive constants. We assume by induction that for $\tau \in \Gamma(D_N)$, the matrix $A_N(\tau,\varepsilon)$ in (5.32) is analytic, and satisfies the bounds
$$\begin{aligned} &|A^0_{11}|, |A^0_{22}| \leqslant K, \qquad |L[A^0_{11},A^0_{11}]^{-1}| \leqslant K, \\ &|A^N_{12}|, |A^N_{21}| \leqslant W_N = \frac{W_1}{(2\varepsilon_0)^{N-1}}, \\ &|A^N_{11}|, |A^N_{11}| \leqslant M_N = \overline{M}\Bigl(1 - \frac{1}{4^{N-1}}\Bigr), \end{aligned} \tag{5.254}$$
where $L[A^0_{11},A^0_{11}]$ is the operator (5.230). We can thus define the matrix $S_N(\tau,\varepsilon)$ in (5.33), with $|S^N_{ij}| \leqslant KW_N$. Using Remark 5.12 of Appendix 5.A, we know that its inverse exists and can be written
$$S_N(\tau,\varepsilon)^{-1} = \begin{pmatrix} \mathbb{1} + \varepsilon^{2N}T^N_{11} & \varepsilon^N T^N_{12} \\ \varepsilon^N T^N_{21} & \mathbb{1} + \varepsilon^{2N}T^N_{22} \end{pmatrix}, \tag{5.255}$$

where

$$|T_{11}^N|, |T_{22}^N| \leqslant 2(KW_N)^2,$$
$$|T_{12}^N|, |T_{21}^N| \leqslant 2KW_N, \qquad |T_{12}^N + S_{12}^N|, |T_{21}^N + S_{21}^N| \leqslant 2\varepsilon_0^{2N}(KW_N)^3. \tag{5.256}$$

A straightforward calculation shows that $A_{N+1} = S_N^{-1}A_N S_N - \varepsilon S_N^{-1}\dot{S}_N$ satisfies the bounds (5.254) for $N+1$, provided d^{-1} and ε_0 are sufficiently small, proving (5.254) by induction. We may thus take $N(\varepsilon) = [D_1/d\varepsilon_0]$, which makes the off–diagonal terms exponentially small. Moreover, since $|\varepsilon^N S_{ij}^N| \leqslant c_0 2^{-N}$, the total transformation satisfies

$$\ln|S_N \dots S_1| \leqslant \sum_{j \geqslant 1} \ln|S_j| \leqslant \sum_{j \geqslant 1} \ln(1 + c_0 2^{-j}) \leqslant \sum_{j \geqslant 1} c_0 2^{-j} \leqslant c_0, \tag{5.257}$$

so that $|S^{(N)}| \leqslant \mathrm{e}^{c_0}|S_0|$. □

5.B.3 Proof of Theorem 5.4

We prove the theorem by induction on the size n of the matrix. For $n = 1$, the proof is trivial, so we assume that $n > 1$, and that the result is proved for matrices of size $n-1$.

After a first static diagonalization, the system is put in the form

$$\varepsilon\dot{y}_1 = A_1 y_1, \qquad A_1 = \begin{pmatrix} a_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix} \in \mathcal{C}^{k-1}, \tag{5.258}$$

where $a_{11}(\tau,\varepsilon) = \mathrm{i}\,\omega_1(\tau) + \mathcal{O}(\varepsilon)$, and A_{12}, A_{21}, A_{22} are matrices of size $(n-1)\times 1$, $1 \times (n-1)$ and $(n-1)\times(n-1)$ respectively. We introduce matrices

$$S_1 = \begin{pmatrix} 1 & \varepsilon S_{12} \\ \varepsilon S_{21} & \mathbb{1}_{n-1} \end{pmatrix}, \qquad D = \begin{pmatrix} a_{11} + \varepsilon^2 A_{12}S_{21} & 0 \\ 0 & A_{22} + \varepsilon^2 A_{21}S_{12} \end{pmatrix} \tag{5.259}$$

with blocs of the same size. We know by equation (5.51) that if $S_{21}(\tau,\varepsilon)$ satisfies

$$\varepsilon\dot{S}_{21} = A_{21}(\tau,\varepsilon) + [A_{22}(\tau,\varepsilon) - a_{11}(\tau,\varepsilon)\mathbb{1}]S_{21} - \varepsilon^2 S_{21}A_{12}(\tau,\varepsilon)S_{21}, \tag{5.260}$$

and S_{12} satisfies a similar equation, then the change of variables $y_1 = S_1(\tau,\varepsilon)z$ transforms (5.258) into $\varepsilon\dot{z} = D(\tau,\varepsilon)z$. The matrix $L(\tau,\varepsilon) := A_{22}(\tau,\varepsilon) - a_{11}(\tau,\varepsilon)\mathbb{1}$ has eigenvalues $\lambda_j = \mathrm{i}(\omega_{j+1} - \omega_1) + \mathcal{O}(\varepsilon)$, and is invertible. We may thus introduce the change of variables $S_{21} = -L^{-1}A_{21} + u$, which yields the equation

$$\varepsilon\dot{u} = Lu + \varepsilon^2 L_1 u + \varepsilon^2 b_1(u,\tau,\varepsilon) + \varepsilon w_1(\tau,\varepsilon), \tag{5.261}$$

which is of class $\mathcal{C}^{k-2}$, and where $b_1 = \mathcal{O}(|u|^2)$. A further transformation of the form $u = -\varepsilon L(\tau,\varepsilon)^{-1}w_1(\tau,\varepsilon) + v$ then gives a similar equation of class $\mathcal{C}^{k-3}$,

$$\varepsilon\dot{v} = Lu + \varepsilon^2 L_2 v + \varepsilon^2 b_2(v,\tau,\varepsilon) + \varepsilon^2 w_2(\tau,\varepsilon). \tag{5.262}$$

Let $U_L(\tau,\varepsilon)$ be the principal solution of $\varepsilon\dot{u} = Lu$. By induction hypothesis, we have $|U_L|, |U_L^{-1}| \leqslant c\,\mathrm{e}^{aT}$. If we write $v = U_L v_1$, we obtain

$$\varepsilon\dot{v}_1 = \varepsilon^2 U_L^{-1}L_2 U_L v_1 + \varepsilon^2 U_L^{-1}b_2(U_L v_1,\tau,\varepsilon) + \varepsilon^2 U_L^{-1}w_2(\tau,\varepsilon). \tag{5.263}$$

Then the Lyapunov function $V = \langle v_1 | v_1 \rangle^{1/2}$ satisfies

$$\varepsilon \dot{V} \leqslant \varepsilon^2 a_0 V + \varepsilon^2 b_0 V^2 + \varepsilon^2 w_0, \tag{5.264}$$

where a_0, b_0 and w_0 are uniform in τ and ε. Lemma 4.7 with $q = 2$ and $\varphi(\varepsilon) = \varepsilon$ shows the existence of a solution of order ε. Going back to the original variables, we have proved existence of a solution $S_{21} = -L^{-1} A_{21} + \mathcal{O}(\varepsilon)$, so that S_1 is close to identity, and the theorem is proved. □

5.B.4 Proof of Lemma 5.3

Let us look for a matrix $S(\tau)$ of the form

$$S(\tau;\varepsilon) = \begin{pmatrix} \dot{\chi}(\tau)^{-1/2} & 0 \\ \varepsilon\xi(\tau)\dot{\chi}(\tau)^{-1/2} & \dot{\chi}(\tau)^{1/2} \end{pmatrix}, \tag{5.265}$$

satisfying the equation

$$\varepsilon \dot{S} = \begin{pmatrix} 0 & 1 \\ h(\tau) & 0 \end{pmatrix} S - \dot{\chi} S \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}. \tag{5.266}$$

Then the transformation $y = S(\tau) z$, $\sigma = \chi(\tau)$ satisfies the lemma. Equation (5.266) is equivalent to the system

$$\begin{aligned} \ddot{\chi} &= -2\xi\dot{\chi} \\ \varepsilon^2(\xi^2 + \dot{\xi}) &= h(\tau) - \dot{\chi}^2 \chi. \end{aligned} \tag{5.267}$$

We assume that the transformation (5.77) has already been carried out, so that $h(\tau) = \tau + \varepsilon^2 h_1(\tau)$, and we look for a solution of the form $\chi(\tau) = \tau + \varepsilon^2 \chi_1(\tau)$. Then (5.267) becomes

$$\begin{aligned} \varepsilon \dot{\chi}_1 &= \eta \\ \varepsilon \dot{\eta} &= -2\xi(1 + \varepsilon\eta) \\ \varepsilon \dot{\xi} &= \varepsilon(h_1(\tau) - \xi^2) - \varepsilon\chi_1(1 + \varepsilon\eta)^2 - \tau(2\eta + \varepsilon^2\eta^2). \end{aligned} \tag{5.268}$$

This equation has a fixed point at $u^\star = (h_1(\tau), 0, 0)$, around which the linearization of (5.268) is the matrix

$$A(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ -\varepsilon & -2\tau & 0 \end{pmatrix}, \tag{5.269}$$

which is hyperbolic for $\tau = \mathcal{O}(\varepsilon^{2/3})$, but with eigenvalues of order $\varepsilon^{1/3}$, so that Theorem 5.1 cannot be applied directly. However, we can use the following transformation[9], writing $\omega := \mathrm{e}^{\mathrm{i}\,2\pi/3}$:

$$\begin{aligned} U &= \begin{pmatrix} -2^{-1/3}\varepsilon^{2/3} & -2^{1/3}\omega\varepsilon^{1/3} & 1 \\ -2^{-2/3}\varepsilon^{1/3} & 1 & 2^{1/3}\omega^*\varepsilon^{1/3} \\ 1 & 2^{-2/3}\varepsilon^{1/3} & -2^{-1/3}\omega\varepsilon^{2/3} \end{pmatrix} \\ \Rightarrow T := U^{-1} A U &= \begin{pmatrix} (2\varepsilon)^{1/3} + \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon^{2/3}) & \mathcal{O}(\varepsilon) \\ -2 & (2\varepsilon)^{1/3}\omega + \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon^{4/3}) \\ \mathcal{O}(\varepsilon^{1/3}) & 1 & (2\varepsilon)^{1/3}\omega^* \end{pmatrix}. \end{aligned} \tag{5.270}$$

[9] This transformation (which is unitary up to corrections of order ε) is obtained using the Gram-Schmidt procedure on the eigenvectors of A.

If $(\chi_1, \eta, \xi) =: u = u^\star + Uv$, we obtain for v the equation $\varepsilon\dot{v} = Tv + \mathcal{O}(\varepsilon|v|^2) + \mathcal{O}(\varepsilon)$. We now introduce the Lyapunov functions $V_j := |v_{[j]}|$. They satisfy in particular

$$\varepsilon\dot{V}_1 \geqslant (2\varepsilon)^{1/3}V_1 - 2^{1/3}c(\varepsilon^{2/3}V_2 + \varepsilon V_3 + \varepsilon) \tag{5.271}$$

for some $c > 0$. We introduce the sets

$$\begin{aligned} \mathcal{D} &= \{v \mid V_1 > c(\varepsilon^{1/3}V_2 + \varepsilon^{2/3}V_3 + \varepsilon^{2/3})\} \\ \mathcal{L} &= \{v \mid V_1 \leqslant c\varepsilon^{2/3}, V_2 = V_3 = 0\}. \end{aligned} \tag{5.272}$$

Then, using a similar argument as in the proof of Proposition 5.2 (see Appendix 5.B.1), we can show existence of a solution $v^\star(\tau)$ such that $v^\star(0) \in \mathcal{L}$ and $v^\star(\tau) \notin \mathcal{D}$ $\forall\tau$, which remains at a bounded distance from the origin. □

5.B.5 Proof of Lemma 5.5

Since the eigenvalues are continuous functions of ε, we have $\operatorname{Re} a_j(\tau,\varepsilon) \leqslant \hat{a}(\tau,\varepsilon)$ for all j, where $\hat{a}(\tau,\varepsilon)$ goes continuously to $\hat{a}(\tau)$ when $\varepsilon \to 0$. Thus we may forget about the ε–dependence of A. We first prove the upper bound in (5.130).

Let $a_1(\tau)$ be the eigenvalue with largest real part of $A(\tau,0)$. As long as this real part is isolated from all other eigenvalues' real parts, we may apply a transformation

$$y = \exp\Bigl[\frac{1}{\varepsilon}\int_{\tau_0}^{\tau} a_1(s)\,\mathrm{d}s\Bigr]\,S(\tau,\varepsilon)z \qquad \Rightarrow \qquad \varepsilon\dot{z} = \begin{pmatrix} 1+\mathcal{O}(\varepsilon) & 0 \\ 0 & A_2(\tau,\varepsilon) \end{pmatrix} z, \tag{5.273}$$

where A_2 is contracting. Thus the result follows immediately from Proposition 5.3.

We now assume that some eigenvalues' real parts cross, say at $\tau = 0$. Then it may happen that $\hat{a}(\tau) \approx |\tau|^p$ near the crossing. We consider the worst case, when $0 < p < 1$. For some $\mu > 0$ to be determined later, we define the functions

$$\tilde{a}(\tau) := \max\{\hat{a}(\tau), \varepsilon^{p\mu}\}, \qquad \tilde{\alpha}(\tau) := \int_0^\tau \tilde{a}(s)\,\mathrm{d}s = \hat{\alpha}(\tau,0) + \mathcal{O}(\varepsilon^{(p+1)\mu}). \tag{5.274}$$

Let $\delta > 0$. If $y = \exp\frac{1}{\varepsilon}[\tilde{\alpha}(\tau) + \delta\tau]\,z$, then

$$\varepsilon\dot{z} = B_\delta z, \qquad B_\delta := A - (\delta + \tilde{a})\mathbb{1}. \tag{5.275}$$

We define the Lyapunov function

$$V_\delta(y,\tau) := \int_0^\infty \|\mathrm{e}^{B_\delta(\tau)s}\,y\|_2^2\,\mathrm{d}s. \tag{5.276}$$

By construction, B_δ has eigenvalues with real part not exceeding $-\delta$. Since by Schur's Lemma, there exists a unitary matrix U such that $U^*B_\delta U = T$ is triangular, we obtain

$$\|\mathrm{e}^{B_\delta(\tau)s}\,z\|_2^2 = \|\mathrm{e}^{T(\tau)s}\,z\|_2^2 \leqslant \mathrm{e}^{-2\delta s}\,p(s)\|z\|_2^2, \tag{5.277}$$

where $p(s)$ is a polynomial of degree $2(n-1)$ with positive coefficients. Integrating this bound, and using a similar lower bound, we obtain that

$$0 < \nu\|z\|_2^2 \leqslant V_\delta(z,\tau) \leqslant P(1/\delta)\|z\|_2^2, \tag{5.278}$$

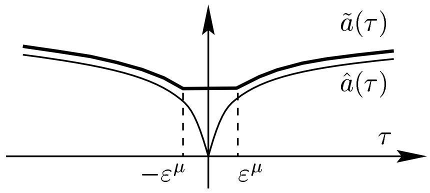

FIGURE 5.9. The functions $\hat{a}(\tau)$ and $\tilde{a}(\tau) = \max\{\hat{a}(\tau), \varepsilon^{p\mu}\}$ near an eigenvalue crossing.

where P is a polynomial of degree $2n-1$. Moreover, since $|\dot{\tilde{a}}| = \mathcal{O}(\varepsilon^{\mu(p-1)})$, we find that $|\partial_\tau V_\delta| \leqslant K\varepsilon^{\mu(p-1)}V_\delta$. Thus we obtain (see Lemma 5.2)

$$\varepsilon\dot{V}_\delta \leqslant -\|z\|_2^2 + K\varepsilon^{\mu(p-1)}V_\delta \leqslant K\varepsilon^{\mu(p-1)}V_\delta, \tag{5.279}$$

$$\|y(\tau)\|_2^2 \leqslant \exp\frac{2}{\varepsilon}\big[\tilde{\alpha}(\tau) + \delta\tau\big]\frac{P(1/\delta)}{\nu}\exp\frac{\tau}{\varepsilon}\big[K\varepsilon^{\mu(p-1)+1}\big]\|y(0)\|_2^2. \tag{5.280}$$

If we take $\mu = \frac{1}{2}$, we obtain $\tilde{\alpha} + \frac{\tau}{2}K\varepsilon^{\mu(p-1)+1} = \hat{\alpha} + \mathcal{O}(\varepsilon^{(p+1)/2})\tau$. Moreover, the minimum over δ of $P(1/\delta)\,\mathrm{e}^{\delta\tau/\varepsilon}$ can be bounded by a new polynomial $P'(\tau/\varepsilon)$. Thus we obtain

$$\begin{aligned}\|y(\tau)\|_2 &\leqslant P'(\tau/\varepsilon)\exp\frac{1}{\varepsilon}\big[\hat{\alpha}(\tau,0) + \mathcal{O}(\varepsilon^{(p+1)/2})\tau\big]\|y(0)\|_2\\ &= \exp\frac{1}{\varepsilon}\big[\hat{\alpha}(\tau,0) + \mathcal{O}(\varepsilon^{(p+1)/2})\tau + \mathcal{O}(\varepsilon|\ln\varepsilon|)\big]\|y(0)\|_2.\end{aligned} \tag{5.281}$$

A similar analysis can be done for $\tau < 0$. This proves the upper bound of (5.130).

To prove the lower bound, we simply change the direction of time, yielding the equation $\varepsilon\partial_{-\tau}y = -A(\tau)y$. This gives an upper bound on $\|U(\tau_0,\tau)\|$, an thus a lower bound on $\|U(\tau,\tau_0)\|$. □

5.B.6 Proof of Lemma 5.6

The change of variables is constructed by Iterative Scheme 5.4. The key point is to control, at each step, the norm of the instantaneous unstable manifold, solution of (5.148).

Lemma 5.15. *Consider the analytic system*

$$\begin{aligned}\mathrm{d}_t x &= A_+ x + b_+(x,y)\\ \mathrm{d}_t y &= A_- y + b_-(x,y).\end{aligned} \tag{5.282}$$

We assume that for $x \in \Delta \subset \mathbb{C}^{n_+}$, where Δ is a neighborhood of 0, and for $|y| \leqslant \delta$,

$$\begin{aligned}|b_+(x,y)| &\leqslant M_0(|x|^2 + |y|^2) & |b_-(x,y)| &\leqslant M_1|x|^2 + M_2|y|(|x| + |y|)\\ |\partial_x b_+(x,y)| &\leqslant M_0(|x| + |y|) & |\partial_x b_-(x,y)| &\leqslant M_1|x| + M_2|y|\\ |\partial_y b_+(x,y)| &\leqslant M_0(|x| + |y|), & |\partial_y b_-(x,y)| &\leqslant M_2(|x| + |y|).\end{aligned} \tag{5.283}$$

Furthermore, we assume that A_+ is an expanding matrix and A_- is a contracting matrix, i.e., there exist positive constants K_0, K_1 such that

$$|\mathrm{e}^{A_+ t}| \leqslant K_0\,\mathrm{e}^{t/K_1}, \qquad |\mathrm{e}^{-A_- t}| \leqslant K_0\,\mathrm{e}^{t/K_1}, \qquad \textit{for } t \leqslant 0. \tag{5.284}$$

Then (5.282) *admits an unstable manifold $h(x)$ which is analytic in a neighborhood Δ_1 of the origin (given by equation* (5.299) *below), and such that*

$$|h(x)| \leqslant K_2 M_1 |x|^2, \qquad |\partial_x h(x)| \leqslant K_3 M_1 |x|, \tag{5.285}$$

where K_2 and K_3 are constants depending only on K_0 and K_1.

PROOF: Let $h(x) = \mathcal{O}(|x|^2)$ be an analytic function on Δ. We denote by $\xi(t, x_0, h)$ the solution of the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_+ x + b_+(x, h(x)), \qquad x(0) = x_0. \tag{5.286}$$

A solution of (5.282) remaining on the manifold $y = h(x)$ satisfies

$$\mathrm{e}^{-A_- t}\, y(t) = h(x_0) + \int_0^t \mathrm{e}^{-A_- s}\, b_-\big(\xi(s, x_0, h), h(\xi(s, x_0, h))\big)\, \mathrm{d}s. \tag{5.287}$$

Letting $t \to -\infty$, we thus obtain that h should be a fixed point of

$$(Th)(x_0) := \int_{-\infty}^0 \mathrm{e}^{-A_- s}\, b_-\big(\xi(s, x_0, h), h(\xi(s, x_0, h))\big)\, \mathrm{d}s. \tag{5.288}$$

We know, by the Stable Manifold Theorem, that this operator admits a unique, analytic fixed point, in some neighborhood of the origin. There also exists a global stable manifold, but it cannot necessarily be represented by a graph $y = h(x)$. Here we would like to obtain an estimate on the domain Δ_1 where this function $h(x)$ exists, and we will do that with the help of the Fixed Point Theorem. Then we will derive a bound on the norm of h.

1. **Domain of analyticity.**
   We show that T is a contraction on an appropriate function space. Let $\delta_0 \in (0, \delta]$ be a constant and $\Delta_0 := \Delta \cap \mathcal{B}(0, \delta_0)$, where $\mathcal{B}(x, r)$ denotes the open ball $\{y \,|\, |y - x| < r\}$. The functions $b_\pm(x, y)$ admit $\mathcal{C}^\infty$–continuations $\tilde{b}_\pm(x, y)$ to the whole x–space, which coincide with $b_\pm(x, y)$ on Δ_0, and such that

$$|\tilde{b}_\pm| \leqslant \widetilde{M} \delta_0^2, \qquad |\partial_{x,y} \tilde{b}_\pm| \leqslant \widetilde{M} \delta_0, \qquad x \in \mathbb{C}^{n_+}, |y| \leqslant \delta. \tag{5.289}$$

   For some $\gamma > 0$, we assume that δ_0 satisfies[10]

$$\delta_0 < \frac{1}{K_0 K_1 \widetilde{M}} \min\Big\{\frac{1}{\gamma + 1}, \frac{\gamma}{K_0(1 + \gamma)}, \frac{1}{1 + K_1^{-1}}\Big\}. \tag{5.290}$$

   Let $\mathcal{E}$ be the space of Lipschitz continuous functions $h(x) : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$ such that $|h(x)| \leqslant \delta$ and $|h(x_2) - h(x_1)| \leqslant \gamma |x_2 - x_1|$. We define operators $\widetilde{\xi}$ and $\widetilde{T}$ obtained by replacing $b_\pm$ by $\tilde{b}_\pm$ in (5.286) and (5.288). We prove that $\widetilde{T}$ is a contraction on $\mathcal{E}$.

   (a) **Bounds on the function $\widetilde{\xi}(t, x, h)$.**
   Let $x_0, x_0' \in \mathbb{C}^{n_+}$ and $h \in \mathcal{E}$. If $x_1(t) = \widetilde{\xi}(t, x_0, h)$ and $x_2(t) = \widetilde{\xi}(t, x_0', h)$, the difference $z(t) = x_2(t) - x_1(t)$ satisfies the relation

$$z(t) = (x_0 - x_0')\, \mathrm{e}^{A_+ t} - \int_t^0 \mathrm{e}^{A_+(t-s)} \big[\tilde{b}_+(x_2, h(x_2)) - \tilde{b}_+(x_1, h(x_1))\big]\, \mathrm{d}s. \tag{5.291}$$

[10] Since $\widetilde{M}$ is a non–decreasing function of δ_0 with a finite limit when $\delta_0 \to 0$, (5.290) is always satisfied for sufficiently small δ_0.

Using the bounds on $\tilde{b}_{\pm}$, we obtain that the term in brackets is bounded by $\widetilde{M}\delta_0(1+\gamma)|z|$, so that

$$|z(t)| \leqslant |x_0 - x_0'| K_0 \,\mathrm{e}^{t/K_1} + \int_t^0 K_0 \,\mathrm{e}^{(t-s)/K_1}\, \widetilde{M}\delta_0(1+\gamma)|z(s)|\,\mathrm{d}s =: V(t). \quad (5.292)$$

We thus have, using (5.290),

$$\frac{\mathrm{d}V}{\mathrm{d}(-t)} = -\frac{1}{K_1}V + K_0\widetilde{M}\delta_0(1+\gamma)|z| \leqslant -\kappa V, \qquad \kappa > 0, \quad (5.293)$$

so that we obtain[11]

$$|\widetilde{\xi}(t, x_0', h) - \widetilde{\xi}(t, x_0, h)| \leqslant K_0 |x_0' - x_0| \,\mathrm{e}^{\kappa t}. \quad (5.294)$$

In a similar way, if $x_0 \in \mathbb{C}^{n+}$ and $h_1, h_2 \in \mathcal{E}$, we obtain

$$|\widetilde{\xi}(t, x_0, h_2) - \widetilde{\xi}(t, x_0, h_1)| \leqslant [K_1(1+\gamma)]^{-1}|h_2 - h_1|. \quad (5.295)$$

(b) **$\mathcal{E}$ is invariant under $\widetilde{T}$.**
If $h \in \mathcal{E}$, we have

$$|\widetilde{T}h| \leqslant \int_{-\infty}^0 K_0 \,\mathrm{e}^{s/K_1}\, \widetilde{M}\delta_0^2 \,\mathrm{d}s \leqslant K_0 K_1 \widetilde{M}\delta_0^2 \leqslant \delta. \quad (5.296)$$

Moreover, it follows from (5.294) that

$$\frac{|(\widetilde{T}h)(x_0') - (\widetilde{T}h)(x_0)|}{|x_0' - x_0|} \leqslant \int_{-\infty}^0 K_0 \,\mathrm{e}^{s/K_1}\, \widetilde{M}\delta_0(1+\gamma)K_0 \,\mathrm{d}s \leqslant K_0^2 K_1 \widetilde{M}\delta_0(1+\gamma) \leqslant \gamma, \quad (5.297)$$

and thus $\mathcal{E}$ is invariant under $\widetilde{T}$.

(c) **$\widetilde{T}$ is a contraction.**
If $h_1, h_2 \in \mathcal{E}$ and $x_i(t) = \xi(t, x_0, h_i)$, we have, using (5.295),

$$\begin{aligned}|(\widetilde{T}h_2)(x_0) - (\widetilde{T}h_1)(x_0)| &\leqslant \int_{-\infty}^0 K_0 \,\mathrm{e}^{s/K_1}\, \widetilde{M}\delta_0\big[(1+\gamma)|x_2 - x_1| + |h_2 - h_1|\big] \\ &\leqslant K_0 K_1 \widetilde{M}\delta_0(1 + K_1^{-1})|h_2 - h_1| < |h_2 - h_1|, \end{aligned} \quad (5.298)$$

so that $\widetilde{T}$ is a contraction.

We conclude by the fixed point theorem that $\widetilde{T}$ admits a Lipschitz continuous fixed point $h^\star(x)$ in $\mathcal{E}$. We define the set

$$\Delta_1 := \big\{x \in \Delta_0 \,\big|\, \xi(t, x, h^\star) \in \Delta_0, t \leqslant 0\big\}. \quad (5.299)$$

If $x \in \Delta_1$, the operators T and $\widetilde{T}$ coincide, so that $h^\star(x) = \widetilde{T}h^\star(x) = Th^\star(x)$ is a fixed point of T as well. But we know by the Stable Manifold Theorem that this fixed point is analytic in a neighborhood of the origin. Since the stable manifold exists globally, and $h^\star(x)$ is an invariant manifold on Δ_1, we have proved that $h^\star(x)$ is analytic in Δ_1.

[11] The procedure leading from (5.292) to (5.294) is known as Gronwall's inequality.

2. **Bounds on the fixed point $h^\star(x)$.**
Since $h(0) = 0$ implies $(Th)(0) = 0$, we have $h^\star(0) = 0$, and thus $|h(x)| \leqslant \gamma|x|$. Introducing this in the equation $h^\star(x) = (Th^\star)(x)$, and using the fact that, because of (5.294), $|\xi(t, x, h^\star)| \leqslant K_0|x|$, we obtain
$$|h^\star(x)| \leqslant c_0|x|^2, \qquad c_0 = K_0^3 K_1(M_1 + \gamma(1+\gamma)M_2). \tag{5.300}$$
Inserting this information into the fixed point equation again, we get
$$|h^\star(x)| \leqslant c_1|x|^2, \qquad c_1 = K_0^3 K_1(M_1 + c_0(1+\gamma)\delta_0 M_2). \tag{5.301}$$
Iterating this argument, we obtain that for all $N \geqslant 1$, $|h^\star(x)| \leqslant c_N|x|^2$ where the numbers c_N are defined recursively by
$$c_{N+1} = K_0^3 K_1 M_1 + K_0^3 K_1(1+\gamma)\delta_0 M_2 c_N. \tag{5.302}$$
If $\lambda := K_0^3 K_1(1+\gamma)\delta_0 M_2 < 1$, this iteration converges to
$$c^\star = \frac{K_0^3 K_1 M_1}{1-\lambda}, \tag{5.303}$$
and we have thus $|h^\star(x)| \leqslant c^\star|x|^2$. A similar argument can be applied to the derivative of $h^\star$. This proves the lemma. □

Proof of Lemma 5.6. We consider the system
$$\begin{aligned} \mathrm{d}_t x &= A_+(\tau,\varepsilon)x + b_+(x, u^{(N-1)} + y_N, \tau, \varepsilon) \\ \mathrm{d}_t y_N &= \bigl[A_-(\tau,\varepsilon) + B_N(x, y_N, \tau, \varepsilon)\bigr] y_N + \varepsilon^N w_N(x, \tau, \varepsilon), \end{aligned} \tag{5.304}$$
where τ and ε are considered as fixed parameters. If $h_N(x;\tau,\varepsilon)$ is a function from $\mathbb{C}^{n_+}$ to $\mathbb{C}^{n_-}$, we denote by $\xi_N(t, x_0, h_N;\tau,\varepsilon)$ the solution of the initial value problem
$$\mathrm{d}_t x = A_+(\tau,\varepsilon)x + b_+(x, u^{(N-1)} + h_N, \tau, \varepsilon), \qquad x(0) = x_0. \tag{5.305}$$
We assume that $A_\pm$ satisfy the bounds of (5.284) uniformly in τ, ε. Applying Lemma 5.15 to the original system (5.140), we obtain that for $N = 1$, (5.304) is analytic for $|y| \leqslant \delta_1$, and $x \in \Delta_1$. The neighborhood of the origin Δ_1 is defined by (5.299). It is negatively invariant under the flow of (5.304), and non–empty because ξ satisfies an identity of the form (5.294).

1. We assume that $|\varepsilon| \leqslant \varepsilon_0$. Let $D_N = D_1 - (N-1)d\varepsilon_0$ and $\delta_N = \delta_1 - (N-1)d\varepsilon_0$, where $d > 0$ is a constant to be determined later. For $N \geqslant 1$, we define the sets
$$\begin{aligned} \Gamma(D_N) &:= \bigl\{\tau \in \mathbb{C} \bigm| |\operatorname{Im}\tau| \leqslant D_N, |\operatorname{Re}\tau - s| \leqslant D_N \ \forall s \in I\bigr\}, \\ \Delta'_{N+1} &:= \bigl\{x \in \Delta_N \bigm| \mathcal{B}(x, d'\varepsilon_0) \subset \Delta_N\bigr\}, \\ \Delta_{N+1} &:= \bigl\{\xi_N(t, x, h_N) \bigm| x \in \Delta'_{N+1}, t \leqslant 0\bigr\}, \end{aligned} \tag{5.306}$$
where $d' > d$ is a constant to be determined. We assume by induction that (5.304) is analytic for $x \in \Delta_N$, $|y_N| \leqslant \delta_N$ and $\tau \in \Gamma(D_N)$, and satisfies the bounds
$$\begin{aligned} |b_+(x, u^{(N-1)} + y_N, \tau, \varepsilon)| &\leqslant M\bigl(1 - \tfrac{1}{2^N}\bigr)\bigl(|x|^2 + |y_N|^2\bigr), \\ |\partial_{x,y_N} b_+(x, u^{(N-1)} + y_N, \tau, \varepsilon)| &\leqslant M\bigl(1 - \tfrac{1}{2^N}\bigr)\bigl(|x| + |y_N|\bigr), \\ |B_N(x, y_N, \tau, \varepsilon)| &\leqslant M\bigl(1 - \tfrac{1}{2^N}\bigr)\bigl(|x| + |y_N|\bigr), \\ |\partial_{x,y_N} B_N(x, y_N, \tau, \varepsilon)| &\leqslant M\bigl(1 - \tfrac{1}{2^N}\bigr), \\ |w_N(x,\tau,\varepsilon)| \leqslant \tfrac{W}{(2\varepsilon_0)^{N-1}}|x|^2, &\qquad |\partial_x w_N(x,\tau,\varepsilon)| \leqslant \tfrac{W}{(2\varepsilon_0)^{N-1}}|x|. \end{aligned} \tag{5.307}$$

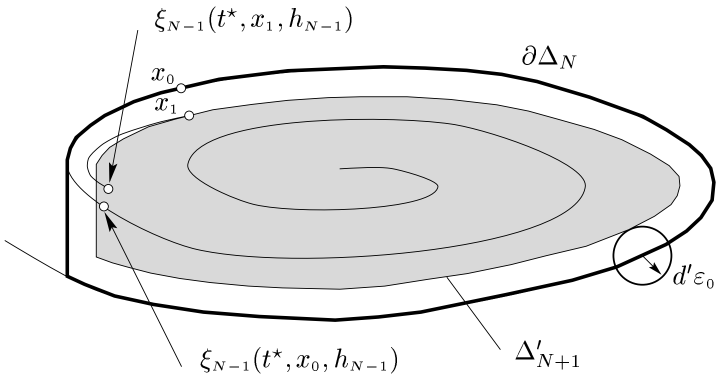

FIGURE 5.10. The set Δ_N is negatively invariant under the flow of the expanding part (indicated by the thin line). The set Δ'_{N+1} is the set of points in Δ_N at a distance larger than $d'\varepsilon_0$ from its boundary, and Δ_{N+1} is the union of preimages under the flow of Δ'_{N+1}. We show that for sufficiently large d', this last set is at a distance larger than $d\varepsilon_0$ from $\partial\Delta_N$.

2. **Domains of analyticity:** By Lemma 5.15, (5.304) admits a local unstable manifold $y_N = h_N(x;\tau,\varepsilon) = \varepsilon^N u_N(x;\tau,\varepsilon)$, which coincides with the manifold u_N of the iterative scheme. By the induction hypothesis, the constant δ_0 appearing in the proof of this lemma can be taken uniform in N. Thus, by construction h_N is analytic in Δ_{N+1} and satisfies the bound $|h_N| \leqslant K_2 W\varepsilon_0 2^{-(N-1)}|x|^2$.
By definition, Δ_{N+1} contains Δ'_{N+1}. We now show that this set is not too large. Let $x_0 \in \partial\Delta_N$ and $x_1 \in \Delta'_{N+1}$, so that $|x_1 - x_0| \geqslant d'\varepsilon_0$. We know that h_{N-1} is analytic in Δ_N, and that $|\xi_{N-1}(t,x_0,h_{N-1})| \leqslant K_0\,\mathrm{e}^{\kappa t}|x_0|$. Thus there exists a time $t^\star < 0$ (of order 1) such that $\xi_{N-1}(t^\star,x_0,h_{N-1}) \in \Delta'_{N+1}$. Since Δ_N is negatively invariant, we also have $\xi_{N-1}(t^\star,x_1,h_{N-1}) \in \Delta_N$. For large enough d', we should have
$$|\xi_{N-1}(t,x_0,h_{N-1}) - \xi_{N-1}(t,x_1,h_{N-1})| \geqslant d''\varepsilon_0 \quad \forall t \in [t^\star, 0], \tag{5.308}$$
for any constant $d'' > d$. If this were not the case, by Proposition 5.7 we would contradict the fact that $|x_1 - x_0| \geqslant d'\varepsilon_0$. On the other hand, since, by the iterative scheme, we have
$$\xi_N(t,x_1,h_N) = \xi_{N-1}(t,x_1,h_{N-1}+h_N), \tag{5.309}$$
we get, using the estimate (5.295),
$$\begin{aligned}|\xi_N(t,x_1,h_N) - \xi_{N-1}(t,x_1,h_{N-1})| &= \\ = |\xi_{N-1}(t,x_1,h_{N-1}+h_N) - \xi_{N-1}(t,x_1,h_{N-1})| &\leqslant c|h_N| \leqslant cW\varepsilon_0,\end{aligned} \tag{5.310}$$
for a constant c independent of N. Combining this with (5.308), we have proved that for sufficiently large d', the orbit $\xi_N(t,x_1,h_N)$ remains at a distance larger than $d\varepsilon_0$ from $\partial\Delta_N$. Thus,
$$\bigl\{x \in \Delta_N \bigm| \mathcal{B}(x,d'\varepsilon_0) \subset \Delta_N\bigr\} =: \Delta'_{N+1} \subset \Delta_{N+1} \subset \bigl\{x \in \Delta_N \bigm| \mathcal{B}(x,d\varepsilon_0) \subset \Delta_N\bigr\}. \tag{5.311}$$

3. **Induction relations:** Using the expressions (5.150) for B_{N+1} and w_{N+1}, and using Cauchy's formula to bound the derivatives, it is straightforward to show that for large enough d, the induction relations (5.307) are true at the step $N+1$, which proves the induction.

4. **Exponential bounds:** Relation (5.311) implies that we may carry out a number of changes of variables $N(\varepsilon)$ of order $1/\varepsilon_0$, and the domains of analyticity will still be non–empty. For this value of N, the bounds on the drift term $\varepsilon^N w_N$ are exponentially small. This proves the lemma. $\square$

5.B.7 Proof of Lemma 5.7

The proof is similar to the one of Lemma 5.15 in Appendix 5.B.6, so we only give the main steps.

Let us rewrite (5.147) under the form

$$\begin{aligned} \varepsilon\dot{x} &= A_+(\tau,\varepsilon)x + g(x,y,\tau,\varepsilon) \\ \varepsilon\dot{y} &= \bigl[A_-(\tau,\varepsilon) + B(x,y,\tau,\varepsilon)\bigr]y + \varepsilon^N w(x,\tau,\varepsilon). \end{aligned} \tag{5.312}$$

We may assume that for sufficiently small $\delta > 0$ there exists a neighborhood $\mathcal{D}(\delta)$ of the origin in $\mathbb{C}^{n_+}$ such that the functions g, B, w admit a $\mathcal{C}^1$–continuation to the whole x–space (resp. a $\mathcal{C}^\infty$–continuation if the system is analytic), coinciding with the original functions when $x \in \mathcal{D}(\delta)$, and satisfying the bounds

$$\begin{aligned} &|g(x,y,\tau,\varepsilon)| \leqslant M\delta^2 && |B(x,y,\tau,\varepsilon)| \leqslant M\delta && |w(x,\tau,\varepsilon)| \leqslant W\delta^2 \\ &|\partial_{x,y}g(x,y,\tau,\varepsilon)| \leqslant M\delta, && |\partial_{x,y}B(x,y,\tau,\varepsilon)| \leqslant M, && |\partial_x w(x,\tau,\varepsilon)| \leqslant W\delta, \end{aligned} \tag{5.313}$$

where $M, W > 0$. By Lemma 5.5, we may assume that for for $\tau_0 \leqslant s \leqslant \tau \leqslant \tau_1$, the propagators of $\varepsilon\dot{x} = A_+(\tau,\varepsilon)x$ and $\varepsilon\dot{y} = A_-(\tau,\varepsilon)y$ satisfy respectively the bounds

$$|U_+(s,\tau)| \leqslant K\,\mathrm{e}^{(s-\tau)/K_1\varepsilon}, \qquad |U_-(\tau,s)| \leqslant K\,\mathrm{e}^{(s-\tau)/K_1\varepsilon}, \tag{5.314}$$

with $K, K_1 > 0$. Using these continued functions, we define the operator

$$\begin{aligned} (T\eta)(x) &= U_-(\tau,\tau_0)\eta^0(\xi_{\tau_0},\varepsilon) \\ &\quad + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} U_-(\tau,s)\bigl[B(\xi_s,\eta(\xi_s,s,\varepsilon),s,\varepsilon)\eta(\xi_s,s,\varepsilon) + \varepsilon^N w(\xi_s,s,\varepsilon)\bigr]\,\mathrm{d}s, \end{aligned} \tag{5.315}$$

where ξ_s denotes the solution of the initial value problem $\varepsilon\dot{x} = A_+(s,\varepsilon)x + g(x,\eta,s,\varepsilon)$ with $x(\tau) = x$.

1. T **is a contraction:** For a constant $\gamma > 0$, we call $\mathcal{E}$ the space of Lipschitz continuous functions $\eta(x,\tau,\varepsilon)$ such that $|\eta(x,\tau,\varepsilon)| \leqslant \delta$ and $|\eta(x_2,\tau,\varepsilon) - \eta(x_1,\tau,\varepsilon)| \leqslant \gamma|x_2 - x_1|$ for $x \in \mathbb{C}^{n_+}$ and $\tau_0 \leqslant \tau \leqslant \tau_1$. Using the above bounds, one finds by a similar method than in the proof of Lemma 5.15 that for $\tau_0 \leqslant s \leqslant \tau \leqslant \tau_1$, for $\eta, \eta_1, \eta_2 \in \mathcal{E}$, and sufficiently small δ,

$$|\xi(s,\tau,x_2,\eta) - \xi(s,\tau,x_1,\eta)| \leqslant K\,\mathrm{e}^{(s-\tau)/2K_1\varepsilon}|x_2 - x_1| \tag{5.316a}$$

$$|\xi(s,\tau,x,\eta_2) - \xi(s,\tau,x,\eta_1)| \leqslant |\eta_2 - \eta_1|. \tag{5.316b}$$

Moreover, we have

$$\begin{aligned} |(T\eta)(x,\tau)| &\leqslant K\delta_0 + KK_1(M + \varepsilon^N W)\delta^2, \\ |(T\eta)(x_2,\tau) - (T\eta)(x_1,\tau)| &\leqslant \bigl[K\gamma_0 + KK_1[M(1+2\gamma) + \varepsilon^N W]\delta\bigr]|x_2 - x_1|. \end{aligned} \tag{5.317}$$

Thus, if $\delta > \delta_0$ and $\gamma > \gamma_0$ are small enough, the space $\mathcal{E}$ is invariant under T. Finally, one obtains that

$$|T\eta_2 - T\eta_1| \leqslant \bigl[K\gamma_0 + KK_1((3+2\gamma)M + \varepsilon^N W)\delta\bigr]|\eta_2 - \eta_1|, \tag{5.318}$$

which shows that T is a contraction for sufficiently small δ and γ_0. We conclude that T admits a unique Lipschitz continuous fixed point $\eta^\star(x,\tau,\varepsilon)$, with $\eta^\star(x,\tau_0,\varepsilon) = \eta^0(x,\varepsilon)$, and which coincides with the adiabatic manifold of (5.312) in the vicinity of $x = 0$.

2. **Bounds on the fixed point:** We now assume that in the domain $\mathcal{D}(\delta)$, the bounds

$$|B(x,y,\tau,\varepsilon)| \leqslant M(|x| + |y|), \quad |w(x,\tau,\varepsilon)| \leqslant W|x|^2, \quad |\eta^0(x,\varepsilon)| \leqslant H|x|^2 \tag{5.319}$$

are satisfied. We consider the fixed point $\eta^\star(x,\tau;\varepsilon)$ in the domain $\mathcal{D}(\delta) \cap \mathcal{B}(0,\delta_0)$. Let us recall that $\eta^\star(x,\tau;\varepsilon)$ satisfies the equation

$$\begin{aligned}\eta^\star(x,\tau;\varepsilon) &= U_-(\tau,\tau_0)\eta^0(\xi_{\tau_0},\varepsilon)\\ &\quad + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} U_-(\tau,s)\bigl[B(\xi_s,\eta^\star(\xi_s,s,\varepsilon),s,\varepsilon)\eta^\star(\xi_s,s,\varepsilon) + \varepsilon^N w(\xi_s,s,\varepsilon)\bigr]\,\mathrm{d}s,\end{aligned} \tag{5.320}$$

Since $\eta^\star(0,\tau) = 0$, we have $|\eta^\star(x,\tau)| \leqslant \gamma|x|$. Moreover, we know by equation (5.316a) that $|\xi(s,\tau,x,\eta^\star)| \leqslant K\,\mathrm{e}^{(s-\tau)/2K_1\varepsilon}|x|$. Replacing this in the fixed point equation, we get

$$|\eta^\star(x,\tau)| \leqslant \bigl[P_0\,\mathrm{e}^{-2(\tau-\tau_0)/K_1\varepsilon} + Q_0\bigr]|x|^2, \tag{5.321}$$

with $P_0 = HK^3$ and $Q_0 = \frac{1}{2}K^3K_1[M\gamma(1+\gamma) + \varepsilon^N W]$. This implies in particular that $|B(\xi_s,\eta^\star,s)| \leqslant Mc_0|\xi_s|$, where $c_0 = 1 + (P_0+Q_0)K\delta_0$. Hence,

$$\begin{aligned}|\eta^\star(x,\tau)| &\leqslant P_0\,\mathrm{e}^{-2(\tau-\tau_0)/K_1\varepsilon}|x|^2\\ &\quad + \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} K\,\mathrm{e}^{(s-\tau)/K_1\varepsilon}|\xi_s|^2\bigl[Mc_0(|\xi_s|P_0\,\mathrm{e}^{-2(s-\tau_0)/K_1\varepsilon} + Q_0K\delta_0) + \varepsilon^N W\bigr]\,\mathrm{d}s.\end{aligned} \tag{5.322}$$

The integral is bounded by

$$\frac{1}{2}K^3K_1\bigl[\varepsilon^N W + MK\delta_0c_0Q_0\bigr]|x|^2 + MK^3K_1^2\delta_0c_0P_0\,\mathrm{e}^{-2(\tau-\tau_0)/K_1\varepsilon}|x|^2. \tag{5.323}$$

Hence we obtain

$$|\eta^\star(x,\tau)| \leqslant \bigl[P_1\,\mathrm{e}^{-2(\tau-\tau_0)/K_1\varepsilon} + Q_1\bigr]|x|^2, \tag{5.324}$$

with $P_1 = P_0 + 2MK^3K_1^2\delta_0c_0P_0$, $Q_1 = \frac{1}{2}K^3K_1W\varepsilon^N + \frac{1}{2}MK^3K_1^2c_0\delta_0Q_0$. Let us assume that $8MK^3K_1^2c_0\delta_0 \leqslant 1$. Then the above argument can be repeated, with constants P_k, Q_k satisfying $P_{k+1} = P_0 + \frac{1}{2}P_k$ and $Q_{k+1} = \frac{1}{2}K^3K_1W\varepsilon^N + \frac{1}{2}Q_k$. Since these geometric sequences are convergent, we obtain finally

$$|\eta^\star(x,\tau)| \leqslant \bigl[2HK^3\,\mathrm{e}^{-2(\tau-\tau_0)/K_1\varepsilon} + K^3K_1W\varepsilon^N\bigr]|x|^2, \tag{5.325}$$

so that the lemma is proved. □

5.B.8 Proof of Lemma 5.8

We define the operator

$$F(\chi(\cdot)) := A\chi - \partial_z\chi Az + b_N(z+\chi;\tau,\varepsilon), \tag{5.326}$$

where $b_N = \mathcal{O}(|z|^N)$. We know by the Sternberg–Chen theorem that for fixed $(\tau^\star,\varepsilon^\star)$, the equation $F(\chi)=0$ admits a solution $\chi^\star(z) = \chi(z,\tau^\star,\varepsilon^\star) = \mathcal{O}(|z|^N)$. To apply the implicit function theorem, we need to show that the Fréchet derivative of F at $\chi^\star$ is invertible. This derivative is given by the linear operator

$$L\zeta(\cdot) = \frac{\mathrm{d}}{\mathrm{d}\lambda}F(\chi^\star+\lambda\zeta)|_{\lambda=0} = [A^\star + B^\star(z)]\zeta - \partial_z\zeta A^\star z, \tag{5.327}$$

where $A^\star = A(\tau^\star,\varepsilon^\star)$ and $B^\star(z) = \partial_z b(z+\chi^\star(z),\tau^\star,\varepsilon^\star)$. To show that L is invertible amounts to proving that the equation

$$[A^\star + B^\star(z)]\zeta(z) - \partial_z\zeta(z)A^\star z = g(z) \tag{5.328}$$

admits a solution for every smooth $g(z) = \mathcal{O}(|z|^N)$. Let us consider the system

$$\begin{aligned} \dot z &= A^\star z \\ \dot u &= A^\star u + B^\star(z)u - g(z). \end{aligned} \tag{5.329}$$

If this system admits a solution $u = \zeta(z) = \mathcal{O}(|z|^N)$, then $\zeta(z)$ is also a solution of (5.328). But (5.329) satisfies the hypotheses of the Sternberg–Chen theorem, so that there exists a transformation $u = v + h(v,z)$, $h = \mathcal{O}(|v|^N + |z|^N)$, such that $\dot v = A^\star v$. But this implies that (5.329) admits the solution $u = h(0,z)$, so that $\zeta(z) = h(0,z)$ is the function we have been looking for. □

5.B.9 Proof of Lemma 5.9

Any function $h(z) = \mathcal{O}(|z|^N)$ in $\mathcal{C}^N(\mathbb{C}^n,\mathbb{C}^n)$ can be decomposed in a unique way on the basis $\bigl\{e_{\mathrm{p},j}(z) \bigm| |\mathrm{p}| = N, j = 1,\dots,n\bigr\}$ by the rule

$$\begin{aligned} h(z) &= \sum_{|\mathrm{p}|=N,j} h_{\mathrm{p},j}(z)e_{\mathrm{p},j}(z), \qquad h_{\mathrm{p},j}(z) = P_{\mathrm{p},j}h(z), \\ P_{\mathrm{p},j}h(z) &:= \frac{N!}{\mathrm{p}!}\int_0^1 \mathrm{d}t_1\, t_1^{N-1}\int_0^1 \mathrm{d}t_2\, t_2^{N-2}\cdots\int_0^1 \mathrm{d}t_N\, \partial_{\mathrm{p}}h[j](t_1t_2\cdots t_N z), \end{aligned} \tag{5.330}$$

see Proposition 2.8. If $\eta_{\mathrm{p},j}(z,\tau,\varepsilon) = P_{\mathrm{p},j}\eta(z,\tau,\varepsilon)$, equation (5.183) is equivalent to the set of equations

$$\varepsilon\dot\eta_{\mathrm{p},j} = \bigl[a_j - \langle\mathrm{p}|a\rangle\bigr]\eta_{\mathrm{p},j} - \partial_z\eta_{\mathrm{p},j}Az + \varepsilon^k P_{\mathrm{p},j}w_k(z+\eta,\tau,\varepsilon). \tag{5.331}$$

The solution with initial condition $\eta_{\mathrm{p},j}(z,\tau_0,\varepsilon) = \eta^0_{\mathrm{p},j}(z,\varepsilon)$ is a fixed point of the operator

$$\begin{aligned} T_{\mathrm{p},j}\eta(z,\tau,\varepsilon) &:= \mathrm{e}^{\alpha_{\mathrm{p},j}(\tau,\tau_0)/\varepsilon}\,\eta^0_{\mathrm{p},j}(U(\tau_0,\tau)z,\varepsilon) \\ &\quad + \varepsilon^{k-1}\int_{\tau_0}^{\tau}\mathrm{e}^{\alpha_{\mathrm{p},j}(\tau,s)/\varepsilon}\,P_{\mathrm{p},j}w_k\bigl(U(s,\tau)z + \eta(U(s,\tau)z,\tau,\varepsilon),\tau,\varepsilon\bigr)\,\mathrm{d}s, \end{aligned} \tag{5.332}$$

where

$$\begin{aligned}\alpha_{\mathrm{p},j}(\tau,\tau_0) &:= \int_{\tau_0}^{\tau} a_j(s;\varepsilon) - \langle \mathrm{p}|a(s;\varepsilon)\rangle \,\mathrm{d}s, \\ U(s,\tau) &:= \mathrm{diag}(\mathrm{e}^{-\alpha_{0,1}(\tau,s)/\varepsilon},\dots,\mathrm{e}^{-\alpha_{0,n}(\tau,s)/\varepsilon}).\end{aligned} \tag{5.333}$$

Let us consider the case where the eigenvalues have positive real parts (the case with negative real parts is discussed by inverting direction of time). Hypothesis 5.2 implies that for $s \leqslant \tau$,

$$\begin{aligned}\mathrm{Re}\,\alpha_{\mathrm{p},j}(\tau,s) &\leqslant -c(\tau-s) \qquad \text{if } |\mathrm{p}| = N, \\ \mathrm{Re}\,\alpha_{0,j}(\tau,s) &\geqslant a_0(\tau-s),\end{aligned} \tag{5.334}$$

where $c, a_0 > 0$. It is then easy to show that $T = \{T_{\mathrm{p},j}\}$ is a contraction on a space of Lipschitz continuous functions η which are close enough to 0, for the norm

$$\|\eta(z,\tau,\varepsilon)\| = \sup_{|\mathrm{p}|=N,j,\tau,\varepsilon} |P_{\mathrm{p},j}\eta(z,\tau,\varepsilon)|. \tag{5.335}$$

T admits thus a fixed point corresponding to the desired solution. Using the fixed point equation, we obtain that if η^0 is of order ε^k, then η is of order ε^k. □

Remark 5.13. If A is not diagonal, the linearization of (5.183) is no more diagonal in the basis $\{e_{\mathrm{p},j}\}$, but still has eigenvalues $a_j - \langle \mathrm{p}|a\rangle$. Thus $T = \{T_{\mathrm{p},j}\}$ is still a contraction, so that the lemma still holds.

5.B.10 Proof of Theorem 5.8

We would like to prove that the system

$$\begin{aligned}\varepsilon\dot{y} &= By + f_1(y,z,\tau) \\ \varepsilon\dot{z} &= Cz + f_2(y,z,\tau)\end{aligned} \tag{5.336}$$

admits an invariant adiabatic center manifold $y = \eta(z,\tau;\varepsilon)$. The proof is similar to the proof of Lemma 5.7 in Appendix 5.B.7, thus we only give the main steps.

For sufficiently small $\delta > 0$, the functions $f_{1,2}$ admit smooth continuations to the whole z–space, coinciding with the original functions for $|z| \leqslant \delta$, and such that

$$|f_{1,2}(y,z,\tau)| \leqslant M\delta^2, \qquad |\partial_{x,z} f_{1,2}(y,z,\tau)| \leqslant M\delta \tag{5.337}$$

for $z \in \mathbb{C}^m$, and $|y|, |\tau| \leqslant \delta$. Since B is contracting, we have

$$|\mathrm{e}^{B\tau/\varepsilon}| \leqslant K_0\,\mathrm{e}^{-\tau/K_1\varepsilon}, \qquad \tau \geqslant 0, \quad K_0, K_1 > 0. \tag{5.338}$$

Since C is elliptic, it satisfies

$$|\mathrm{e}^{C\tau/\varepsilon}| \leqslant P(\kappa)\,\mathrm{e}^{\kappa|\tau|/\varepsilon} \quad \forall\kappa > 0, \tag{5.339}$$

where $P(\kappa)$ may diverge as $\kappa \to 0$. [12]

[12] We have $\mathrm{e}^{C\tau/\varepsilon} = \mathrm{e}^{D\tau/\varepsilon}\,\mathrm{e}^{N\tau/\varepsilon}$ where D is diagonalizable, so that the norm of $\mathrm{e}^{D\tau/\varepsilon}$ is independent of τ, ε, and N is nilpotent, so that $\mathrm{e}^{N\tau/\varepsilon} = \sum_{k=0}^{m} \frac{1}{k!}(\tau/\varepsilon)^k N^k$. Since $\mathrm{e}^{-\kappa x}\,x^k \leqslant (k/\mathrm{e}\,\kappa)^k$ for all positive x, $P(\kappa)$ diverges at most as κ^{-m}.

Solutions of (5.336) satisfy

$$y(\tau) = \mathrm{e}^{B\tau/\varepsilon}\, y(0) + \frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{B(\tau-s)/\varepsilon}\, f_1(y(s), z(s), s)\,\mathrm{d}s. \tag{5.340}$$

For any function $\eta(z,\tau;\varepsilon)$, we denote by $\zeta(\tau,\tau_1,z_1,\eta)$ the solution of the initial value problem

$$\varepsilon\dot{z} = Cz + f_2(\eta(z,\tau;\varepsilon), z, \tau), \qquad z(\tau_1) = z_1. \tag{5.341}$$

The invariant manifold $\eta(z,\tau;\varepsilon)$ should thus be a fixed point of the operator

$$(T\eta)(z,\tau;\varepsilon) = \mathrm{e}^{B\tau/\varepsilon}\, \eta^0(\zeta_0) + \frac{1}{\varepsilon}\int_0^\tau \mathrm{e}^{B(\tau-s)/\varepsilon}\, f_1(\eta(\zeta_s, s), \zeta_s, s)\,\mathrm{d}s, \tag{5.342}$$

where $\zeta_s := \zeta(s,\tau,z,\eta)$ and $\eta^0(z)$ is an initial condition, which we may take equal to the instantaneous center manifold of (5.336) at $\tau = 0$.

We now assume that for $\delta > \delta_0 > 0$ and $\gamma > \gamma_0 > 0$, we have $|\eta^0| \leqslant \delta_0$ and $|\partial_z \eta^0| \leqslant \gamma_0$. We show that when these constants are sufficiently small, T is a contraction on the space $\mathcal{E}$ of functions $\eta(z,\tau;\varepsilon)$ such that $|\eta| \leqslant \delta$ and $|\eta(z_2) - \eta(z_1)| \leqslant \gamma|z_2 - z_1|$. By similar arguments than in Appendix 5.B.6, we obtain that

$$\begin{aligned} |\zeta(\tau,\tau_1,z_2,\eta) - \zeta(\tau,\tau_1,z_1,\eta)| &\leqslant P(\kappa)\,\mathrm{e}^{\kappa_1(\tau_1-\tau)/\varepsilon}|z_2 - z_1| \\ |\zeta(\tau,\tau_1,z,\eta_2) - \zeta(\tau,\tau_1,z,\eta_1)| &\leqslant \frac{1}{\kappa_1} P(\kappa) M\delta |\eta_2 - \eta_1|, \end{aligned} \tag{5.343}$$

where $\kappa_1 = \kappa + P(\kappa)M\delta(1+\gamma)$. Using these bounds, we get

$$\begin{aligned} |T\eta| &\leqslant K_0\delta_0 + K_0K_1M\delta^2 \\ |T\eta(z_2,\tau) - T\eta(z_1,\tau)| &\leqslant \bigl[K_0\gamma_0P(\kappa) + K_0K_2M\delta(1+\gamma)P(\kappa)\bigr]|z_2 - z_1|, \end{aligned} \tag{5.344}$$

provided

$$\frac{1}{K_1} - \kappa_1 \geqslant \frac{1}{K_2} > 0. \tag{5.345}$$

The function space $\mathcal{E}$ is thus invariant if

$$\begin{aligned} K_0\delta_0 + K_0K_1M\delta^2 &\leqslant \delta \\ \bigl[K_0\gamma_0 + K_0K_2M\delta(1+\gamma)\bigr]P(\kappa) &\leqslant \gamma. \end{aligned} \tag{5.346}$$

Finally, we obtain that

$$|T\eta_2(z,\tau) - T\eta_1(z,\tau)| \leqslant \lambda|\eta_2 - \eta_1|, \tag{5.347}$$

$$\lambda = K_0K_1M\delta + \frac{1}{\kappa_1}P(\kappa)M\delta K_0\bigl[\gamma_0 + K_1(1+\gamma)M\delta\bigr]. \tag{5.348}$$

The conditions for $\mathcal{E}$ being invariant and T being a contraction can be fulfilled in the following way. We fix $\kappa = (4K_1)^{-1}$, $\delta_0 = \delta(2K_0)^{-1}$ and $\gamma_0 = \gamma[2K_0P(\kappa)]^{-1}$. Then for sufficiently small δ, we have $\kappa_1 = (2K_1)^{-1}$, $K_2 = 2K_1$, and the conditions (5.346) are satisfied, as well as the condition $\lambda < 1$.

We obtain the conclusion as in Appendix 5.B.7. The operator T admits a fixed point, which coincides with the desired manifold in the neighborhood of the origin. If the order of the drift term in (5.208) has been decreased to ε^k, an analysis of the fixed point equation shows that η is also of order ε^k. □

Chapter 6

Nonlinear Oscillators

"The study of non-linear physics is like the study of non-elephant biology."

Nonlinear oscillators belong to the most popular dynamical systems. Consider for instance the motion of a particle in a potential (with or without damping). If the potential is confining, the system may admit oscillating solutions. In the case of a quadratic potential, we obtain a harmonic oscillator, the solutions of which can be computed explicitly, even when the system is subject to a time–dependent force.

Nonlinear oscillators are characterized by a potential having other than quadratic terms. Famous examples are the pendulum, with a cosine potential, and the Dufing oscillator, which has a quartic potential [**GH**]. Such systems may become chaotic when they are subject to a time–dependent perturbation. One frequently considers a periodic external forcing of the form

$$\ddot{q} + \gamma\dot{q} + \Phi'(q) = f(t). \tag{6.1}$$

Another possibility is to consider the effect of a modulated potential amplitude, which yields a nonlinear Mathieu equation of the form

$$\ddot{q} + \alpha(t)\Phi'(q) = 0. \tag{6.2}$$

Let us also mention the case of an amplitude–dependent friction, as in the Van der Pol oscillator,

$$\ddot{q} + \gamma(q)\dot{q} + q = f(t). \tag{6.3}$$

More generally, one can also consider the dynamics of several coupled nonlinear oscillators.

If such a system depends adiabatically on time, we may use the tools developed in previous chapters to study its dynamics. In fact, we may as well consider general equations with a potential of the form $\Phi(q, \lambda(\varepsilon t))$, which include the external forcing and the nonlinear Mathieu equation as special cases, but allow for a large number of alternative forcings. The above systems are known to display chaotic behaviour under certain conditions [**GH**]. It is thus interesting to examine how such systems behave in the adiabatic limit, where their dynamics should be analytically more tractable. One could guess that chaotic behaviour is impossible in this limit. We will show in an example that this guess is incorrect.

In this chapter, we apply the methods of Chapter 5 to analyse the following nonlinear oscillators.

- **Section 6.1** We discuss in detail the dynamics of a mathematical pendulum, which is attached to a table rotating with angular frequency $\Omega(\varepsilon t)$. This system, which is equivalent to the motion of a particle in a potential oscillating slowly between a simple and a double well, displays several important phenomena. The first one is a bifurcation delay leading to hysteresis. The second one is related to the sequence of visited equilibrium positions: we show that in general, the pendulum follows a hysteresis cycle with either once, or *twice* the driving period. However, for some values of the adiabatic parameter, which may be arbitrarily small, the orbits of the system are chaotic.
- **Section 6.2** We present two examples illustrating eigenvalue crossings discussed in the previous chapter. The first example deals with an overdamped, two–dimensional system, which exhibits a diagonal eigenvalue crossing. The second one concerns a system of two coupled oscillators, displaying an eigenvalue cruising.

6.1 The Rotating Pendulum

6.1.1 Description of the System

Let us consider a pendulum oscillating in a vertical plane, which is rotating with angular frequency Ω (Fig. 6.1). We assume that the pendulum is characterized by a linear density $\rho(\xi)$, where $\xi \in [0, L]$ is the distance to the suspension point P.

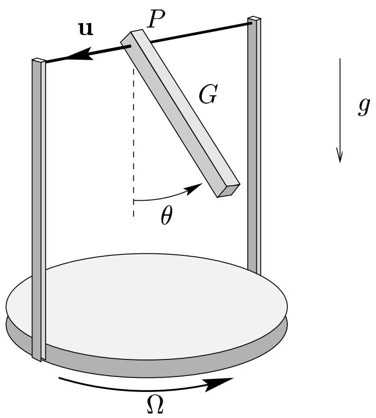

FIGURE 6.1. The rotating pendulum consists in a table, rotating with angular frequency Ω, and supporting a pendulum, subject to gravity and friction, which is constrained to oscillate in a vertical plane attached to the table.

Let $\mathbf{u}$ be a vector of unit length perpendicular to the rotating plane. When the pendulum makes an angle θ with the vertical, it experiences three different torques parallel to $\mathbf{u}$:

- the torque due to gravity g,

$$\mathbf{M}_{\mathrm{G}} = \int_0^L -\xi\rho(\xi)\,\mathrm{d}\xi\, g \sin\theta\, \mathbf{u} = -L_G M g \sin\theta\, \mathbf{u}, \tag{6.4}$$

  where M is the total mass, and L_G is the distance between suspension point P and center of mass G;

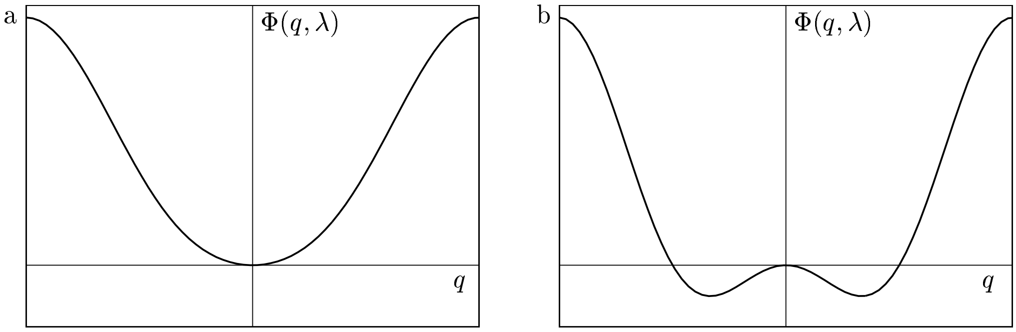

FIGURE 6.2. The potential $\Phi(q,\lambda)$ which describes the motion of the rotating pendulum. (a) When $\lambda < 0$ (i.e., Ω is smaller than its critical value), the potential has a single minimum at the origin. (b) When $\lambda > 0$ (Ω is larger than its critical value), the origin becomes unstable and two new symmetric wells are formed, so that Φ becomes a double well potential.

- the torque due to the centrifugal forces of intensity $\Omega^2\xi\sin\theta\,\rho(\xi)\,\mathrm{d}\xi$,

$$\mathbf{M}_{\mathrm{C}} = \int_0^L \Omega^2\xi^2\rho(\xi)\,\mathrm{d}\xi\;\sin\theta\cos\theta\;\mathbf{u} = I_P\Omega^2\sin\theta\cos\theta\;\mathbf{u}, \tag{6.5}$$

  where I_P is the moment of inertia with respect to P;
- the torque due to friction, which we assume to be visquous,

$$\mathbf{M}_{\mathrm{F}} = -\Gamma\dot\theta\;\mathbf{u}. \tag{6.6}$$

From the angular momentum theorem, we get the equation of motion

$$\ddot\theta = \Omega^2\sin\theta\cos\theta - \alpha\frac{g}{L}\sin\theta - 2\gamma\dot\theta, \tag{6.7}$$

where $\alpha = LL_GM/I_P$ is a characteristic of the pendulum (e.g. $\alpha = 1$ for a point mass, $\alpha = 3/2$ for a homogeneous bar) and $\gamma = \Gamma/2I_P$. Let us introduce the frequency

$$\Omega_{\mathrm{cr}} = \sqrt{\alpha g/L}. \tag{6.8}$$

It corresponds both to the frequency of small oscillations when $\Omega = 0$, and to the critical value of Ω beyond which the vertical position is unstable.

Taking $\Omega_{\mathrm{cr}}^{-1}$ as new time unit, we obtain the dimensionless equation of motion

$$\ddot\theta = \Omega^2\sin\theta\cos\theta - \sin\theta - 2\gamma\dot\theta, \tag{6.9}$$

where Ω and γ have been scaled by a factor $\Omega_{\mathrm{cr}}^{-1}$. Introducing the parameter $\lambda = \Omega^2 - 1$, the equation can be written in the variables $x = (q,p) :=(\theta,\dot\theta)$ as

$$\begin{aligned} \dot q &= p \\ \dot p &= -2\gamma p - \Phi'(q,\lambda), \end{aligned} \tag{6.10}$$

which is exactly the equation of motion of a damped particle in a potential $\Phi(q,\lambda)$, defined by

$$\Phi'(q,\lambda) = \sin q\bigl[1 - (\lambda+1)\cos q\bigr]. \tag{6.11}$$

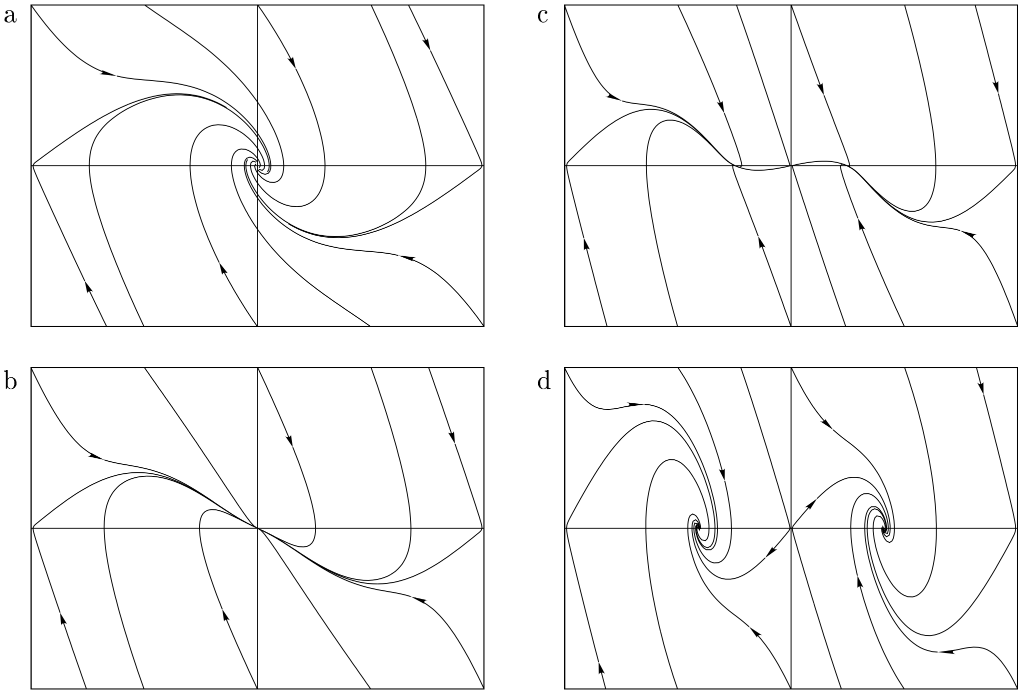

FIGURE 6.3. Four typical phase portraits of the rotating pendulum. (a) If $\lambda < \lambda_-(\gamma)$, the origin is a stable focus. (b) When $\lambda_-(\gamma) < \lambda < 0$, the origin becomes a stable node. For positive λ, the origin gets unstable, and two new symmetric stable equilibria are formed, which are nodes when $0 < \lambda < \lambda_+(\gamma)$ (c), and focuses when $\lambda > \lambda_+(\gamma)$ (d).

This potential has the shape of a simple or double well, depending on the value of λ (Fig. 6.2): if $\lambda < 0$ (i.e., $\Omega < 1$), it has a maximum at $q = \pi \equiv -\pi$ and a minimum at $q = 0$. When $\lambda > 0$ (i.e., $\Omega > 1$), the origin becomes a maximum, and two new minima are formed at $\pm q^\star(\lambda)$, where $q^\star(\lambda) := \operatorname{Arccos} \frac{1}{1+\lambda}$. Physically, this means that beyond the critical rotation frequency, the vertical position becomes unstable, and two new symmetric equilibria are formed. In the language of dynamical systems, we have a **pitchfork bifurcation** at $\lambda = 0$.

The linear analysis (see Section 3.1) shows that

- the origin is a saddle for $\lambda > 0$, a stable node for $\lambda_-(\gamma) < \lambda < 0$ and a stable focus for $\lambda < \lambda_-(\gamma)$, where

$$\lambda_-(\gamma) = -\gamma^2; \tag{6.12}$$

- the points $\pm q^\star(\lambda)$ are stable nodes for $0 < \lambda < \lambda_+(\gamma)$ and stable focuses for $\lambda > \lambda_+(\gamma)$, where

$$\lambda_+(\gamma) = \frac{1}{2}\Big[\gamma^2 - 2 + \sqrt{\gamma^2 + 4}\Big]. \tag{6.13}$$

Our analysis will apply to more general even potentials $\Phi(q, \lambda)$ with the same stability properties, but possibly different functions $\lambda_\pm(\gamma)$. For instance, the **Ginzburg–Landau potential**

$$\Phi_{\mathrm{GL}}(q, \lambda) := -\frac{1}{2}\lambda q^2 + \frac{1}{4}q^4 \tag{6.14}$$

a b

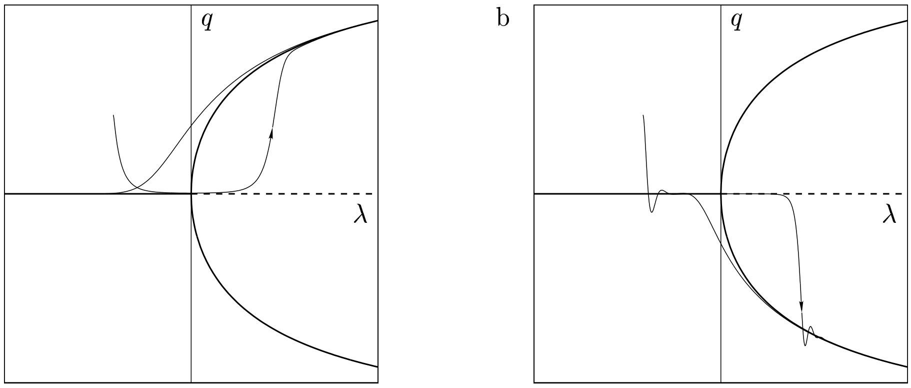

FIGURE 6.4. Numerical solutions of equation (6.15). (a) In the overdamped case, the solution follows the origin, before leaving it some time after the bifurcation has occurred. It then follows the upper branch until merging again with the origin, thus displaying hysteresis (note that this solution is still a transient one, the delay time will be different at the next cycle). (b) At lower friction, the behaviour is similar, but due to the fact that the trajectory may oscillate around the equilibria, it may jump on either symmetric equilibrium.

satisfies these properties with $\lambda_-(\gamma) = -\gamma^2$ and $\lambda_+(\gamma) = \gamma^2/2$.

For such a system, we may have four qualitatively different phase portraits, depending on the value of parameters, see Fig. 6.3.[1]

Let us now consider what happens if we impose a slow periodic variation of Ω in time. For simplicity, let us take a "sine–like" function $\lambda(\tau)$, which has exactly one (positive) maximum and one (negative) minimum. The adiabatic equations of motion are

$$\begin{aligned} \varepsilon \dot{q} &= p \\ \varepsilon \dot{p} &= -2\gamma p - \Phi'(q, \lambda(\tau)), \end{aligned} \tag{6.15}$$

where $\dot{x} := \mathrm{d}x/\mathrm{d}\tau$.

Fig. 6.4 shows numerical solutions of (6.15). Starting with $\lambda < 0$, the pendulum relaxes to the equilibrium $q = 0$ (after some oscillations if the friction is low enough). When λ becomes positive, the pendulum does not react immediately to the bifurcation, but rather remains in unstable equilibrium near the origin for some finite time. This is the phenomenon of **bifurcation delay** that we encountered for instance in Sections 4.3 and 5.4. When the pendulum finally reaches the stable branch, it remains in its vicinity until λ has reached 0 again, and the branch merges with the origin. Thus the bifurcation delay leads to **hysteresis**.

Another remarkable phenomenon is related to the sequence of visited wells. When, after some bifurcation delay, the pendulum departs from the origin, it may choose between the left and right equilibrium. Fig. 6.5 shows numerical solutions for different values of the adiabatic parameter ε (all other quantities being fixed). The time scale has been contracted in order to show a large number of cycles; each peak corresponds to the particle falling into one of the wells. One observes that most of the time, either the particle always chooses the same side, or alternatively falls into one well and the other one. But for some special

[1] At sufficiently low friction, the unstable manifold of the saddle at $(\pi, 0)$ in Fig. 6.3d may have a more complicated shape, winding around the origin. This, however, will not affect our analysis of the motion which never stays in these regions of phase space.

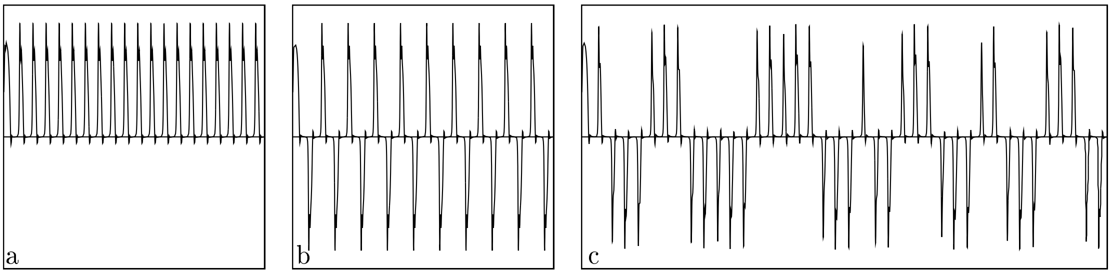

FIGURE 6.5. Solutions $q(\tau)$ for slightly different values of the adiabatic parameter ε. The time scale has been contracted in such a way as to show 20, resp. 40 periods of λ. One observes (a) solutions with the same period then $\lambda(\tau)$, (b) solutions with twice the period of $\lambda(\tau)$, going alternatively to one side and the other one, and (c), if ε is carefully adjusted, solutions which have no apparent period.

values of ε, the sequence of visited wells is apparently random, so that we called this phenomenon **chaotic hysteresis** [BK2].

In the remainder of this section, we apply the methods of Chapters 4 and 5 in order to give an analytic understanding of this phenomenon. We first compute an approximate expression of the Poincaré map in Subsection 6.1.2, before analysing it in Subsection 6.1.3.

6.1.2 Computation of the Poincaré Map

We assume that $\lambda(\tau)$ is a 1–periodic function which is positive for $0 < \tau < \check{\tau}$ and negative for $\check{\tau} < \tau < 1$ (Fig. 6.6). Our purpose is to compute the Poincaré map of (6.15) in the $x = (q,p)$–plane during one period, to leading order in the adiabatic parameter ε. The instant of the Poincaré section may in principle be chosen arbitrarily, but later on we will choose a specific time which simplifies the understanding of the map.

When $\lambda(\tau) < 0$, the static system has two fixed points, the origin $O = (0,0)$ and the point $P = (\pi,0) \equiv (-\pi,0)$. When $\lambda(\tau) > 0$, there are two additional fixed points $Q_\pm = (\pm q^\star(\lambda(\tau)),0)$. Let us denote by $a^0_\pm(\tau)$ and $a^\star_\pm(\tau)$ the eigenvalues of the linearization around O and $Q_\pm$ respectively. In the case of the rotating pendulum, we have

$$\begin{aligned} a^0_\pm(\tau) &= -\gamma \pm \sqrt{\gamma^2 + \lambda(\tau)} \\ a^\star_\pm(\tau) &= -\gamma \pm \sqrt{\gamma^2 - \tfrac{\lambda(\lambda+2)}{\lambda+1}}. \end{aligned} \tag{6.16}$$

Since the points O and P are independent of λ, the particular adiabatic solutions $x(\tau) = (0,0)$ and $x(\tau) = (\pi,0)$ define fixed points of the Poincaré map.[2]

On an interval $I \subset (0,\check{\tau})$, Theorem 5.1 shows that one can associate with the equilibria $Q_\pm$ the adiabatic solutions

$$\bar{x}_\pm(\tau) = \pm\left[\begin{pmatrix} q^\star(\lambda(\tau)) \\ 0 \end{pmatrix} + \varepsilon\partial_\lambda q^\star(\lambda(\tau))\dot{\lambda}\begin{pmatrix} -2\gamma/\Phi''(q^\star(\lambda)) \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2)\right]. \tag{6.17}$$

Note that these adiabatic solutions acquire a p–component of order ε, which is due to the small drift of the equilibria.

[2]The existence of the saddle point $(\pi,0)$ simplifies the discussion of invariant manifolds, but is not necessary for the main properties of the map to hold.

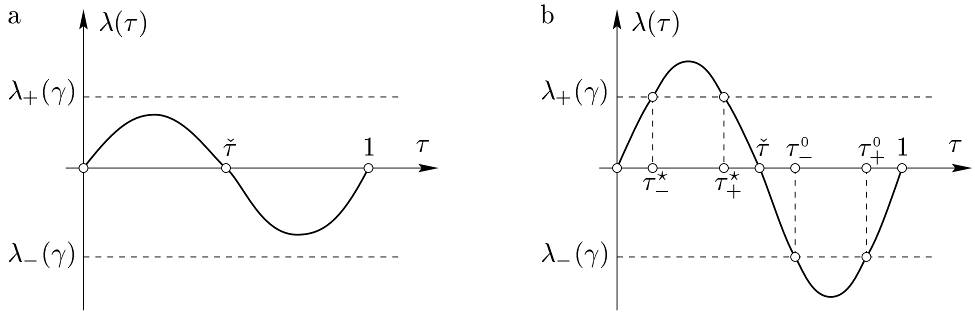

FIGURE 6.6. Functions $\lambda(\tau)$ which are considered in the analysis. (a) In the overdamped case, $\lambda(\tau)$ is assumed to remain within the interval $(\lambda_-(\gamma), \lambda_+(\gamma))$, so that stable equilibria are always nodes. (b) In the low friction case, we assume that $\lambda(\tau)$ has a larger amplitude and crosses the values $\lambda_\pm(\gamma)$. These points define new times $\tau_\pm^0$, $\tau_\pm^\star$ at which the equilibria change from nodes to focuses.

1. Overdamped Case

Let us first consider the case where the eigenvalues $a_\pm^{0,\star}(\tau)$ are always real, i.e., we assume that $\lambda(\tau)$ always remains within the interval $(\lambda_-(\gamma), \lambda_+(\gamma))$ (Fig. 6.6a).

By Proposition 3.1, there exists for each λ an instantaneous invariant manifold $p = h(q, \lambda)$ with $0 = h(0, \lambda) = h(\pi, \lambda) = h(\pm q^\star(\lambda), \lambda)$. It satisfies the equation

$$\partial_q h(q,\lambda) h(q,\lambda) + 2\gamma h(q,\lambda) + \Phi'(q,\lambda) = 0, \tag{6.18}$$

and admits an expansion of the form

$$h(q,\lambda) = -\frac{1}{2\gamma}\Phi'(q,\lambda)\big[1 + \mathcal{O}(\gamma^{-2}\Phi''(q,\lambda))\big]. \tag{6.19}$$

Thus the change of variables $p = h(q,\lambda) + p_1$ transforms (6.15) into

$$\begin{aligned} \varepsilon\dot{q} &= h(q,\lambda(\tau)) + p_1 \\ \varepsilon\dot{p} &= a_1(q,\tau)p_1 + \varepsilon w_1(q,\tau), \end{aligned} \tag{6.20}$$

where

$$\begin{aligned} a_1(q,\tau) &= -2\gamma - \partial_q h(q,\lambda) = \frac{\Phi'(q,\lambda)}{h(q,\lambda)} < 0 \\ w_1(q,\tau) &= -\partial_\lambda h(q,\lambda)\dot{\lambda}. \end{aligned} \tag{6.21}$$

We would like to show that (6.20) admits an invariant manifold of the form $p_1 = \eta_1(q,\tau;\varepsilon)$. It should satisfy the equation

$$\varepsilon\partial_\tau\eta_1(q,\tau) = a_1(q,\tau)\eta_1(q,\tau) - \partial_q\eta_1[h(q,\lambda(\tau)) + \eta_1(q,\tau)] + \varepsilon w_1(q,\tau). \tag{6.22}$$

Lemma 6.1. *Equation* (6.22) *admits a solution* $\eta_1(q,\tau) = \mathcal{O}(\varepsilon)$, *such that* $\eta_1(0,\tau) = \eta_1(\pi,\tau) = 0$. *Moreover, the initial condition* $\eta_1(q,0)$ *can be chosen in such a way that the manifold* $p_1 = \eta_1(q,\tau)$ *contains the adiabatic solutions* (6.17).[3]

[3] While $\eta(q,\tau)$ is analytic at hyperbolic points, it will in general only be $\mathcal{C}^\infty$ at nodes.

PROOF: The proof is based on the following remarks:

1. Theorem 5.6 shows that (6.22) admits a solution in the neighborhood of hyperbolic points, which is tangent to the dynamic eigenspaces of the linearization around these points. By Theorem 5.8, we also know that this solution can be continued beyond the bifurcation points of the origin.
2. Using the Lyapunov function
$$V(\tau) := \frac{1}{2}\int_0^\pi \eta_1(q,\tau)^2\,\mathrm{d}q, \tag{6.23}$$
one obtains by integration by parts (and using the Cauchy-Schwartz relation) that $\varepsilon\dot{V} \leqslant -c_1 V + c_2\varepsilon\sqrt{V}$, which implies that (6.22) admits a solution of order ε in the L^2–norm.
3. Let ϕ_t be the flow of the vector field
$$\begin{aligned} \dot{q} &= h(q,\lambda(\mu)) + p_1 \\ \dot{p} &= a_1(q,\mu)p_1 + \varepsilon w_1(q,\mu) \\ \dot{\mu} &= \varepsilon. \end{aligned} \tag{6.24}$$
We define the manifold
$$\mathcal{W} := \bigl\{(q,p_1,\mu) = \phi_t(q,g(q),0)\,\big|\, t \geqslant 0, 0 \leqslant q \leqslant \pi\bigr\}. \tag{6.25}$$
By construction, this manifold coincides with the manifold of equation $p_1 = \eta_1(q,\mu)$, with initial condition $\eta_1(q,0) = g(q)$. This initial condition can be chosen in such a way that $\mathcal{W}$ contains the adiabatic solutions (6.17). Using a similar argument than in Proposition 3.1, one shows that $\eta_1(q,\mu) = \mathcal{O}(\varepsilon)$. □

The change of variables $p_1 = \eta_1(q,\tau) + \bar{p}$ transforms (6.20) into
$$\begin{aligned} \varepsilon\dot{q} &= \eta(q,\tau) + \bar{p} \\ \varepsilon\dot{\bar{p}} &= \bar{a}(q,\tau)\bar{p}, \end{aligned} \tag{6.26}$$
where $\eta(q,\tau) = h(q,\lambda(\tau)) + \eta_1(q,\tau)$ and $\bar{a} = a_1 - \partial_q\eta_1 < 0$ (for sufficiently small ε). It is easy to show that solutions of this equation are such that $\bar{p}$ decreases exponentially fast. Indeed, the Lyapunov function $V = \frac{1}{2}\bar{p}^2$ satisfies $\varepsilon\dot{V} = 2\bar{a}V$. Thus, after some transient motion, the dynamics will be governed by the one–dimensional equation
$$\varepsilon\dot{q} = \eta(q,\tau). \tag{6.27}$$
We know that the function $\eta(q,\tau)$ satisfies $\eta(0,\tau) = \eta(\pi,\tau) = 0$, that $\partial_q\eta(\pi,\tau) > 0$ and that $\partial_q\eta(0,\tau) = a_+^0(\tau) + \mathcal{O}(\varepsilon)$. Thus, (6.27) undergoes direct pitchfork bifurcation at $\tau = \mathcal{O}(\varepsilon)$, and indirect pitchfork bifurcation at $\tau = \check{\tau} + \mathcal{O}(\varepsilon)$. In other words, it behaves in exactly the same way as the equation in Example 4.18 (see Fig. 4.26). We have seen in this example that if the average of $a_+^0(\tau)$ over one period is negative, then $q(\tau)$ decreases exponentially fast to 0. If this average is positive, we may define a bifurcation delay time by
$$\int_{\check{\tau}}^{\hat{\tau}+1} a_+^0(\tau)\,\mathrm{d}\tau = 0, \qquad 0 < \hat{\tau} < \check{\tau}. \tag{6.28}$$

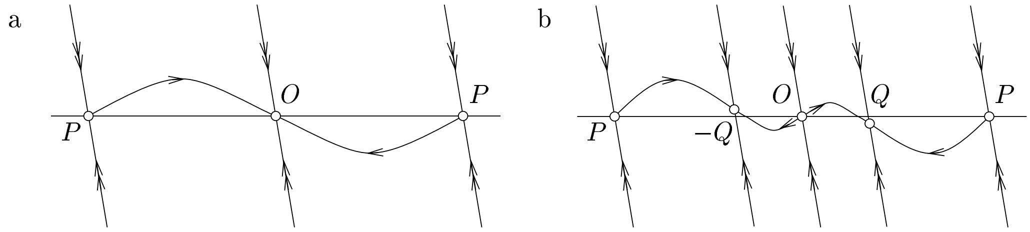

FIGURE 6.7. Schematic representation of the Poincaré map, (a) when $\langle a_+^0 \rangle \leqslant 0$ and (b) when $\langle a_+^0 \rangle > 0$. The curved lines represent unstable manifolds of the hyperbolic points. In the vertical direction, there is exponentially strong contraction. Compare these Poincaré maps with the phase portraits in Fig. 6.3b and c.

After a transient cycle, the trajectory will follow the origin during the time intervals $[\check{\tau}+n-1, \hat{\tau}+n]$, $n \geqslant 1$, and a branch $\pm q^\star(\lambda(\tau))$ during the remaining time. The Poincaré map has exactly 4 fixed points, corresponding to the unstable solutions $x(\tau) = (0,0)$ and $(\pi, 0)$, and to two stable symmetric hysteresis cycles. Due to the fact that the point $q = 0$ cannot be crossed, these cycles have the same period than $\lambda(\tau)$, i.e., we are in the situation of period–1 hysteresis (Fig. 6.5a). Finally, we also know that the area of this cycle scales as

$$\mathcal{A}(\varepsilon) - \mathcal{A}(0) \approx \varepsilon^{3/4}. \tag{6.29}$$

Let us summarize these results in the following way:

Proposition 6.1. *Assume the system* (6.15) *is overdamped, and let* $\langle a_+^0 \rangle$ *denote the average of* $a_+^0(\tau)$ *over one period. Then*

1. *If* $\langle a_+^0 \rangle \leqslant 0$, *the Poincaré map admits the origin* O *as asymptotically stable fixed point; all trajectories, except the one remaining on the hyperbolic point* P, *are attracted by* O.
2. *If* $\langle a_+^0 \rangle > 0$, *the origin is unstable, and there exist two stable fixed points* $\pm Q$ *corresponding to hysteresis cycles* $x_\pm(\tau; \varepsilon)$, *satisfying*

$$\lim_{\varepsilon \to 0} x_\pm(\tau, \varepsilon) = \begin{cases} (\pm q^\star(\lambda(\tau)), 0) & \text{if } \hat{\tau} + n < \tau < \check{\tau} + n \\ (0,0) & \text{otherwise.} \end{cases} \tag{6.30}$$

Remark 6.1.

1. The linearized map around its fixed points can be computed as in Subsection 5.5.1. The hyperbolic points admit unstable manifolds, which correspond to the intersection of $\eta(q, \tau)$ with the Poincaré section[4]. When the origin is unstable, its stable manifold delimits the basins of attraction of the two hysteresis cycles.
2. The position of Q in Fig. 6.7 depends on the choice of the section. If it is taken at a time $\tau \in (\hat{\tau} + n, \check{\tau} + n)$, Q will be close to Q_+, if $\tau \in (\check{\tau} + n, \hat{\tau} + n + 1)$, it will be exponentially close to the origin.

[4]More precisely, the solution of (6.22) can be chosen in such a way that $\eta(q, \tau)$ is periodic, so that its intersection with the Poincaré section is an invariant manifold of the map.

2. General Case

Let us now return to the general case, when the equilibrium points may also be focuses. We introduce the notations

$$\begin{aligned}\alpha^{0,\star}(\tau_2,\tau_1) &:= \int_{\tau_1}^{\tau_2} \operatorname{Re} a_+^{0,\star}(\tau)\,\mathrm{d}\tau \\ \phi^{0,\star}(\tau_2,\tau_1) &:= \int_{\tau_1}^{\tau_2} \operatorname{Im} a_+^{0,\star}(\tau)\,\mathrm{d}\tau \\ \delta^{0,\star}(\tau_2,\tau_1) &:= \int_{\tau_1}^{\tau_2} \operatorname{Re}\bigl[a_+^{0,\star}(\tau) - a_-^{0,\star}(\tau)\bigr]\,\mathrm{d}\tau.\end{aligned} \tag{6.31}$$

We will assume that the origin is a focus (i.e., $\lambda(\tau) < \lambda_-(\gamma)$) for $\tau_-^0 < \tau < \tau_+^0$, and that the points $Q_\pm$ are focuses (i.e., $\lambda(\tau) > \lambda_+(\gamma)$) for $\tau_-^\star < \tau < \tau_+^\star$ (see Fig. 6.6b and Fig. 6.8). Moreover, we are interested in the most complicated situation, when the bifurcation delay time $\hat{\tau}$ (defined by (6.28)) lies in the interval (τ_-^0, τ_+^0). In particular, this means that $\langle a_+^0\rangle = \alpha^0(1,0)$ is positive.

We will do this analysis in two steps:

1. On the interval $[\check{\tau}, \hat{\tau}+1]$, the trajectory remains close to the origin, and the flow is dominated by its linear part.
2. On the interval $[\hat{\tau}, \check{\tau}]$, when the trajectory jumps on one of the branches $Q_\pm$, the flow is strongly nonlinear.

Let us first consider the linearization of (6.15) around the origin,

$$\varepsilon\dot{x} = A^0(\tau)x, \qquad A^0(\tau) = \begin{pmatrix} 0 & 1 \\ \lambda(\tau) & -2\gamma \end{pmatrix}. \tag{6.32}$$

We study this equation with the methods of Section 5.3. Note that the eigenvalues of the matrix $A^0(\tau)$ cross at times $\tau_\pm^0$. In fact, the change of variables

$$x = S_0(\tau)y_0, \qquad S_0(\tau) = \mathrm{e}^{-2\gamma\tau/\varepsilon}\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}, \tag{6.33}$$

which was introduced in Subsection 5.3.5 to perform a local analysis near the crossing point, can be carried out globally in this case, with the result

$$\varepsilon\dot{y}_0 = A_1^0(\tau)y_0, \qquad A_1^0(\tau) = \begin{pmatrix} 0 & 1 \\ \gamma^2+\lambda(\tau) & 0 \end{pmatrix}. \tag{6.34}$$

By Theorem 5.5, the solution can be written, for $\tau \in (\tau_+^0, \check{\tau}+1)$, as

$$\begin{aligned} x(\tau) &= U(\tau,\check{\tau})x(\check{\tau}), \\ U(\tau,\check{\tau}) &= S(\tau)U^+(\tau,\tau_+^0)T_2U^-(\tau_+^0,\tau_-^0)T_1U^+(\tau_-^0,\check{\tau})S(\check{\tau})^{-1}, \end{aligned} \tag{6.35}$$

with the following notations. $S(\tau)$ is a matrix which diagonalizes the system dynamically (when $\lambda > -\gamma^2$). It has the form

$$S(\tau) = \begin{pmatrix} 1 & -1 \\ a_+^0(\tau) & a_-^0(\tau) \end{pmatrix} + \mathcal{O}(\varepsilon), \tag{6.36}$$

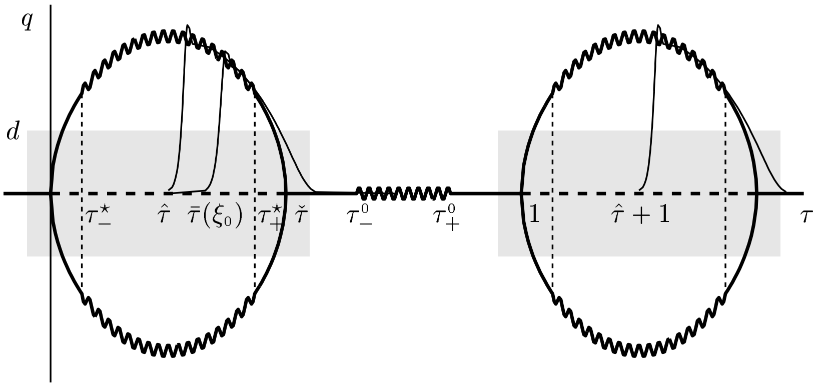

FIGURE 6.8. Geometry of equilibrium branches in the (τ, q)–plane, for the general case. Thick full lines represent nodes, dashed lines represent saddles, and wavy lines stand for focuses (the hyperbolic point P is not shown on this picture). Thin lines are orbits of the system. After a first transient cycle (not shown), most orbits remain exponentially close to the origin until the bifurcation delay time $\hat\tau$. Some solutions, however, are further delayed until a time $\bar\tau(\xi_0)$. Shaded areas represent the domains where the origin admits an adiabatic unstable manifold.

where we have chosen the eigenvectors in such a way that S is orientation–preserving (see Remark 5.3). The matrices $U^\pm$ are diagonal propagators, which can be written, using the notations (6.31),

$$\begin{aligned} U^+(\tau_2,\tau_1) &= \mathrm{e}^{\alpha^0(\tau_2,\tau_1)/\varepsilon}\begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{-\delta^0(\tau_2,\tau_1)/\varepsilon}\end{pmatrix}, \\ U^-(\tau_+^0,\tau_-^0) &= \mathrm{e}^{\alpha^0(\tau_+^0,\tau_-^0)/\varepsilon}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\,\phi^0/\varepsilon} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\,\phi^0/\varepsilon}\end{pmatrix}, \end{aligned} \tag{6.37}$$

where $\phi^0 := \phi^0(\tau_+^0,\tau_-^0)$ is the **dynamic phase** due to oscillations around the origin. Finally, T_1 and T_2 are matrices describing the crossing, given by

$$T_2 = \begin{pmatrix}1 & \mathrm{i}\\ \frac{\mathrm{i}}{2} & \frac12\end{pmatrix} + \mathcal{O}(\varepsilon^{2/3}), \qquad T_1 = \begin{pmatrix}\frac12 & -\mathrm{i}\\ -\frac{\mathrm{i}}{2} & 1\end{pmatrix} + \mathcal{O}(\varepsilon^{2/3}). \tag{6.38}$$

If we choose $\tau = \hat\tau + 1$ as final time, we have by definition $\alpha^0(\hat\tau+1,\check\tau) = 0$ (see (6.28)). Thus, multiplying the above matrices, we can write (6.35) in the form

$$\begin{aligned} U(\hat\tau+1,\check\tau) &= S(\hat\tau)U_0(\hat\tau+1,\check\tau)S(\check\tau)^{-1} \\ U_0(\hat\tau+1,\check\tau) &= \begin{pmatrix} \mathrm{e}^{\rho_1^0}\cos\Bigl(\frac{\phi^0}{\varepsilon}+\theta_1^0\Bigr) & 2\,\mathrm{e}^{-\delta_3^0/\varepsilon+\rho_3^0}\sin\Bigl(\frac{\phi^0}{\varepsilon}+\theta_3^0\Bigr) \\ -\frac12\,\mathrm{e}^{-\delta_2^0/\varepsilon+\rho_2^0}\sin\Bigl(\frac{\phi^0}{\varepsilon}+\theta_2^0\Bigr) & \mathrm{e}^{-\delta_4^0/\varepsilon+\rho_4^0}\cos\Bigl(\frac{\phi^0}{\varepsilon}+\theta_4^0\Bigr) \end{pmatrix}, \end{aligned} \tag{6.39}$$

where

$$\begin{aligned} \phi^0 &= \phi^0(\tau_+^0,\tau_-^0), \\ \delta_2^0 &= \delta^0(\hat\tau+1,\tau_+^0), \\ \delta_3^0 &= \delta^0(\tau_-^0,\check\tau), \\ \delta_4^0 &= \delta_2^0+\delta_3^0 \end{aligned} \tag{6.40}$$

are dynamic terms. The factors 2 and $\frac{1}{2}$, as well as the phase shifts $\pm\frac{\pi}{2}$ between the sin and cos are geometric terms due to the crossing, which can be considered as small corrections to the dynamic terms. The terms θ_i^0 and ρ_i^0 are even smaller corrections of order $\mathcal{O}(\varepsilon^{2/3})$.

The matrix (6.39) describes a rotation of angle ϕ^0/ε, where ϕ^0 is a constant depending only on the function $\lambda(\tau)$, composed with exponential contractions along the unstable dynamic eigenvectors of A^0.

Let us now consider the effect of nonlinear terms (which are of order $|x|^3$ because of the symmetry $x \mapsto -x$). We first note that Proposition 5.7 shows that if $|x(\check{\tau})|$ is small enough, then $|x(\tau)|$ remains small for $\check{\tau} \leqslant \tau \leqslant \hat{\tau}+1$. This means that the bifurcation is delayed from $\tau = 1$ to $\tau = 1 + \hat{\tau}$.

When the origin is hyperbolic, it admits local stable and unstable adiabatic manifolds. By Lemma 6.1, the unstable manifold can be continued to those regions where O is a node. Similar properties hold for the stable manifold. Let us introduce two intermediate times $\tau_1 \in (\check{\tau}, \tau_-^0)$ and $\tau_2 \in (\tau_+^0, 1)$. For x in a neighborhood of the origin, and for $\tau \in [\tau_2+n, \tau_1+n+1]$, there exists a transformation, which we will denote $x = \zeta(z) = \zeta(\xi,\eta)$, changing (6.15) into

$$
\begin{aligned}
\varepsilon\dot{\xi} &= \big[a_+^0(\tau,\varepsilon) + \beta_+(z,\tau,\varepsilon)\big]\xi \\
\varepsilon\dot{\eta} &= \big[a_-^0(\tau,\varepsilon) + \beta_-(z,\tau,\varepsilon)\big]\eta,
\end{aligned}
\tag{6.41}
$$

where $a_\pm^0(\tau,\varepsilon) = a_\pm^0(\tau) + \mathcal{O}(\varepsilon)$ and $\beta_\pm(z,\tau,\varepsilon) = \mathcal{O}(|z|^2)$. Moreover, we know that $\zeta(z) = Sz + \mathcal{O}(|z|^3)$ where $S(\tau)$ is the matrix (6.36) which diagonalizes the linear part.

On the other hand, in the interval $[\tau_1,\tau_2]$ the system can be diagonalized dynamically using the normal form theory of Subsection 5.4.3. Combining these two changes of variables, we obtain the following result for the essentially linear part of the Poincaré map.

Lemma 6.2. *Assume $z(\check{\tau}) = \mathcal{O}(\varepsilon^{1/4})$. Then the solution of the nonlinear equation (6.15) can be written as*

$$
z(\hat{\tau}+1) = \widetilde{U}^0(z(\check{\tau}))z(\check{\tau}),
\tag{6.42}
$$

where $\widetilde{U}^0(z)$ is obtained by adding corrections of order $\mathcal{O}(\varepsilon^{1/2})$ to the geometric amplitudes ρ_i^0 of the propagator $U_0(\hat{\tau}+1,\check{\tau})$ in (6.39).

Proof: Since by Proposition 5.7, $z(\tau) = \mathcal{O}(\varepsilon^{1/4}\,\mathrm{e}^{\alpha^0(\tau,\check{\tau})/\varepsilon})$, we have by (6.41)

$$
z(\tau_1) = \widetilde{U}_1 z(\check{\tau}), \qquad \widetilde{U}_1(z) = \begin{pmatrix} \mathrm{e}^{\tilde{\alpha}_+/\varepsilon} & 0 \\ 0 & \mathrm{e}^{\tilde{\alpha}_-/\varepsilon} \end{pmatrix},
\tag{6.43}
$$

where $\tilde{\alpha}_+ = \alpha^0(\tau_1,\check{\tau}) + \int_{\check{\tau}}^{\tau_1} \mathcal{O}(\sqrt{\varepsilon}\,\mathrm{e}^{2\alpha^0/\varepsilon})\,\mathrm{d}s = \alpha^0(\tau_1,\check{\tau}) + \mathcal{O}(\varepsilon^{3/2})$, and a similar relation holds for $\tilde{\alpha}_-$.

The discussion of Subsection 5.4.3 shows that in the interval $[\tau_1,\tau_2]$, the equation can be diagonalized dynamically, by a change of variables that we denote $x \mapsto u$. We have to analyse the matching between z and u at τ_1. Since z and u have the same linear part, we can write $u(\tau_1) = z(\tau_1) + h(z(\tau_1),\tau_1)$, where h satisfies an equation of the form (5.165) (at $\tau = \tau_1$). Since the nonlinear term in (6.41) has the particular form $b(z,\tau) = \beta(z,\tau)z$, where β is a diagonal matrix, we can construct the solution h of the form $h(z,\tau) = \mu(z,\tau)z$ where μ is also diagonal. A similar relation holds at $\tau = \tau_2$. This shows that

$$
\begin{aligned}
u(\tau_1) &= [\mathbb{1} + \mu_1]z(\tau_1) \\
u(\tau_2) &= U_0(\tau_2,\tau_1)[\mathbb{1} + \mu_1]z(\tau_1) = [\mathbb{1} + \mu_2]z(\tau_2),
\end{aligned}
\tag{6.44}
$$

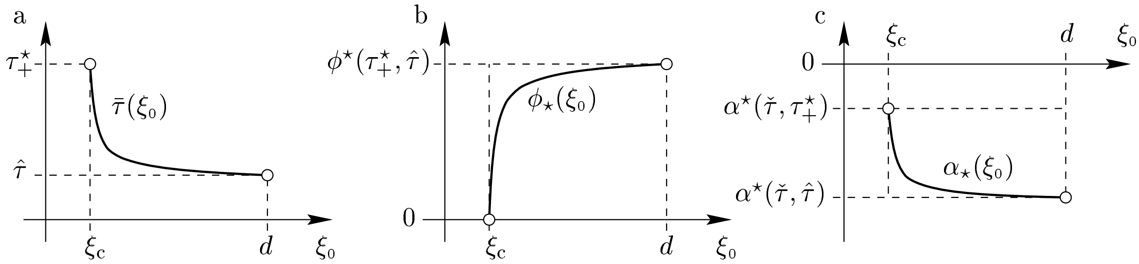

FIGURE 6.9. (a) The function $\bar\tau(\xi_0)$, which represents the time at which a trajectory starting with $\xi(\hat\tau)=\xi_0$ arrives at $\xi=d$, see (6.46). The functions $\phi^\star(\xi_0)$ (b) and $\alpha^\star(\xi_0)$ (c) of equation (6.59) correspond to the rotation phase and amplitude acquired by a trajectory starting at ξ_0.

where $\mu_i = \mu(z(\tau_i),\tau_i) = \mathcal{O}(|z|^2/\sqrt{\varepsilon})$ (because of the possible resonances). Since z is exponentially small at τ_1 and τ_2, we get

$$z(\tau_2) = \widetilde{U}_2 z(\tau_1), \tag{6.45}$$

where $\widetilde{U}_2$ is obtained by multiplying the matrix elements of $U_0(\tau_2,\tau_1)$ by $1+\mathcal{O}(\mathrm{e}^{-1/C\varepsilon})$. A similar analysis between τ_2 and $\hat\tau+1$ yields the result. □

This result shows that the influence of nonlinear terms on the flow is negligible at leading order, since it can be absorbed in the geometric amplitudes.

We now proceed to the computation of the other part of the Poincaré map, between $\hat\tau$ and $\check\tau$. We may assume that for $|\xi|\leqslant d$, equation (6.41) is defined and satisfies $a_-^0+\beta_-<0$ (see (6.21)). Let us assume that $\xi_0 := \xi(\hat\tau) = \mathcal{O}(\varepsilon^{1/4})$ and $\eta_0 := \eta(\hat\tau) = \mathcal{O}(\varepsilon)$. Then the second equation of (6.41) tells us that η will remain of order ε.

For $\xi_0>0$, we denote by $\bar\tau(\xi_0,\eta_0,\varepsilon)$ the first time such that $\xi(\bar\tau)=d$. Using Propositions 4.4 and 4.7, we find that $\bar\tau(\xi_0,\eta_0,\varepsilon)=\bar\tau(\xi_0)+\mathcal{O}(\varepsilon)$, where

$$\alpha^0(\bar\tau(\xi_0),\hat\tau) = -\varepsilon\ln(\xi_0/d). \tag{6.46}$$

The function $\bar\tau(\xi_0)$ is decreasing in ξ_0, with $\bar\tau(d)=\hat\tau$, and $\bar\tau(\xi_c)=\tau_+^\star$, where we introduced

$$\xi_c := \exp\Bigl[-\frac{1}{\varepsilon}\alpha^0(\tau_+^\star,\hat\tau)\Bigr] \tag{6.47}$$

(see Fig. 6.9a). It may also be written as

$$\bar\tau(\xi_0) = \hat\tau - \varepsilon c(\xi_0)\ln(\xi_0/d), \tag{6.48}$$

where $c(\xi_0)$ is a smooth function of ξ_0 and ε, satisfying

$$\int_0^{c(\xi_0)} a_+^0(\hat\tau - \varepsilon\ln(\xi_0/d)s)\,\mathrm{d}s = 1. \tag{6.49}$$

For most trajectories, $\bar\tau(\xi_0)$ is very close to $\hat\tau$. But solutions which are exponentially close to 0 at $\hat\tau$ may be appreciably delayed (this is why the peaks have different heights on Fig. 6.5c).

Let us assume that $\bar\tau(\xi_0)<\tau_+^\star$, i.e., $\xi_0>\xi_c$. Since $z(\bar\tau(\xi_0))=(d,\mathcal{O}(\varepsilon))$, $x(\bar\tau(\xi_0)) = \zeta(z(\bar\tau(\xi_0)))$ is close to the unstable manifold of the origin. We now consider the adiabatic solution $\bar{x}_+(\tau)$ in (6.17). We know that it remains at a distance of order ε from the equilibrium branch $Q_+(\lambda(\tau))$ as long as it does not bifurcate. Combining the center manifold reduction of Theorem 5.8 with the methods of Chapter 4, we find that $\bar{x}_+(\tau)$ can be constructed in such a way that $\bar{z}_+(\check{\tau}) := \zeta^{-1}(\bar{x}_+(\check{\tau})) = (C\varepsilon^{1/4}, 0)$. In fact, the first component of $\bar{x}_+(\tau)$ scales as

$$\bar{q}(\tau) - q^\star(\lambda(\tau)) \asymp \begin{cases} \frac{\varepsilon}{\lambda^{3/2}} & \text{for } \lambda \succcurlyeq \varepsilon^{1/2} \\ \varepsilon^{1/4} & \text{for } \lambda \preccurlyeq \varepsilon^{1/2}. \end{cases} \tag{6.50}$$

Let us write $x = \bar{x}_+(\tau) + y$. Then

$$\varepsilon \dot{y} = \bar{A}(\tau) y + b(y, \tau), \tag{6.51}$$

where $b(y,\tau) = \mathcal{O}(|y|^2)$ and $\bar{A}(\tau) := \partial_x f(\bar{x}_+(\tau), \tau)$ has eigenvalues

$$\bar{a}_\pm(\tau) = a^\star_\pm(\tau) + \mathcal{O}(\varepsilon^{1/2}). \tag{6.52}$$

This is due to the fact that $\Phi'''(q^\star) = \mathcal{O}(q^\star) = \mathcal{O}(\sqrt{\lambda})$, so that $\bar{a} - a^\star = \mathcal{O}(\sqrt{\lambda}(\bar{q} - q^\star)) = \mathcal{O}(\sqrt{\varepsilon})$.

The linearization of (6.51) can again be studied with the help of Theorem 5.5, since the eigenvalues of $\bar{A}$ cross at $\tau^\star_+$. By a similar calculation than around the origin, we obtain that the principal solution has the form $U(\check{\tau}, \bar{\tau}) = S^+(\check{\tau}) U_0(\check{\tau}, \bar{\tau})$, where

$$\begin{aligned} U_0(\check{\tau}, \bar{\tau}) &= U^+(\check{\tau}, \tau^\star_+) T U^-(\tau^\star_+, \bar{\tau}) S^-(\bar{\tau})^{-1} \\ &= \mathrm{e}^{\alpha^\star/\varepsilon} \begin{pmatrix} \mathrm{e}^{\rho^\star_1} \cos\Bigl(\frac{\phi^\star}{\varepsilon} + \theta^\star_1\Bigr) & \mathrm{e}^{\rho^\star_3} \cos\Bigl(\frac{\phi^\star}{\varepsilon} + \theta^\star_3\Bigr) \\ \mathrm{e}^{-\delta^\star/\varepsilon + \rho^\star_2} \sin\Bigl(\frac{\phi^\star}{\varepsilon} + \theta^\star_2\Bigr) & \mathrm{e}^{-\delta^\star/\varepsilon + \rho^\star_4} \sin\Bigl(\frac{\phi^\star}{\varepsilon} + \theta^\star_4\Bigr) \end{pmatrix}. \end{aligned} \tag{6.53}$$

We have used the notations

$$\begin{aligned} \alpha^\star &= \alpha^\star(\check{\tau}, \bar{\tau}) + \mathcal{O}(\sqrt{\varepsilon}), \\ \phi^\star &= \phi^\star(\tau^\star_+, \bar{\tau}) + \mathcal{O}(\sqrt{\varepsilon}), \\ \delta^\star &= \delta^\star(\check{\tau}, \tau^\star_+) + \mathcal{O}(\sqrt{\varepsilon}), \end{aligned} \tag{6.54}$$

while the $\rho^\star_i$ and $\theta^\star_i$ are geometric terms of order 1, stemming form the matrix $S^-(\bar{\tau})^{-1}$ (which can be chosen with some freedom).

Again, we consider the effect of nonlinear terms.

Lemma 6.3. *Using the variables $z = (\xi, \eta)$, the solution of (6.51) can be written as*

$$z(\check{\tau}) = \bar{z}_+(\check{\tau}) + \widetilde{U}^\star(y(\bar{\tau})) y(\bar{\tau}), \tag{6.55}$$

where $\widetilde{U}^\star$ is a matrix obtained by adding corrections of order 1 to the geometric terms in the propagator $U_0(\check{\tau}, \bar{\tau})$ of equation (6.53).

PROOF: When $\bar{x}_+$ is a focus, equation (6.51) can be written in polar coordinates $y = S^-(\tau)(r\,\mathrm{e}^{\mathrm{i}\varphi}, r\,\mathrm{e}^{-\mathrm{i}\varphi})$ as

$$\begin{aligned} \varepsilon \dot{r} &= -\gamma r + \mathcal{O}(r^2) \\ \varepsilon \dot{\varphi} &= \omega(\tau) + \mathcal{O}(r). \end{aligned} \tag{6.56}$$

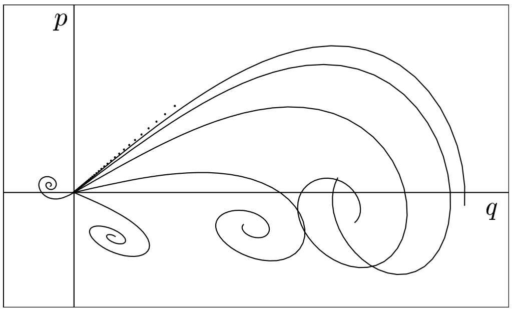

FIGURE 6.10. Evolution of the unstable manifold of the origin. The dots represent initial conditions at $\tau = \hat{\tau}$. They have been taken more dense near the origin, in order to counteract the exponential contraction at $\bar{x}_+$. Their evolution for $\hat{\tau} < \tau < \tau_-^0$ shows that the unstable manifold transforms into a squeezed spiral described by equation (6.57). For later times, this spiral begins to rotate around the origin while being further contracted.

One checks that $x(\bar{\tau})$ is sufficiently close to $\bar{x}_+$ so that $\varepsilon\dot{r} \leqslant -\gamma r/2$. Thus, any $\mathcal{O}(1)$–neighborhood of $\bar{x}_+$ can be reached in a time of order ε. During this time, φ rotates by an angle $\mathcal{O}(1)$, so that only geometric terms of order 1 are created.

Once a sufficiently small $\mathcal{O}(1)$–neighborhood of $\bar{x}_+$ is reached, we may apply dynamic normal form theory and a similar analysis as in Lemma 6.2. When $\bar{x}_+$ has become a node, the important point to note is that the adiabatic unstable manifold of the origin is tangent to the dynamic eigenspaces associated with $\bar{x}_+$. □

The key observation to be made is that $z(\check{\tau})$ depends on $z(\hat{\tau})$ in two ways: once through the coordinates of $y(\bar{\tau})$, and once through the dynamic terms $\alpha^\star$ and $\phi^\star$, which depend on $z(\hat{\tau})$ via the function $\bar{\tau}(\xi_0)$. But since $y(\bar{\tau}) = \mathcal{O}(1)$ varies in some bounded set, it only influences geometric terms, which are small compared to the dynamic terms depending on $\bar{\tau}(\xi_0)$. We can thus write

$$\begin{aligned} \xi(\check{\tau}) &= C\varepsilon^{1/4} + \mathrm{e}^{\alpha_\star/\varepsilon}\cos\Big(\frac{\phi_\star}{\varepsilon}\Big) \\ \eta(\check{\tau}) &= \mathrm{e}^{(\alpha_\star-\delta_\star)/\varepsilon}\sin\Big(\frac{\phi_\star}{\varepsilon}+\theta^\star\Big), \end{aligned} \tag{6.57}$$

where the terms $\alpha_\star$, $\phi_\star$, $\delta_\star$ are given by

$$\begin{aligned} \alpha_\star(\xi_0;\eta_0,\varepsilon) &= \alpha^\star(\check{\tau},\bar{\tau}(\xi_0)) + \mathcal{O}(\sqrt{\varepsilon}), \\ \phi_\star(\xi_0;\eta_0,\varepsilon) &= \phi^\star(\tau_+^\star,\bar{\tau}(\xi_0)) + \mathcal{O}(\sqrt{\varepsilon}), \\ \delta_\star(\xi_0,\eta_0,\varepsilon) &= \delta^\star(\check{\tau},\tau_+^\star) + \mathcal{O}(\sqrt{\varepsilon}). \end{aligned} \tag{6.58}$$

At leading order in ε, $\delta_\star$ is a constant, while $\alpha_\star$ and $\phi_\star$ depend only on ξ_0. In view of (6.48), they can be written (see Fig. 6.9)

$$\begin{aligned} \alpha_\star(\xi_0;\eta_0,\varepsilon) &= \alpha^\star(\check{\tau},\hat{\tau}) - \varepsilon f(\xi_0)\ln(\xi_0/d) + \mathcal{O}(\sqrt{\varepsilon}), \qquad & \alpha_\star(\xi_c) &= \alpha^\star(\check{\tau},\tau_+^\star), \\ \phi_\star(\xi_0;\eta_0,\varepsilon) &= \phi^\star(\tau_+^\star,\hat{\tau}) + \varepsilon g(\xi_0)\ln(\xi_0/d) + \mathcal{O}(\sqrt{\varepsilon}), \qquad & \phi_\star(\xi_c) &= 0. \end{aligned} \tag{6.59}$$

Geometrically speaking, (6.57) is the parametric equation of an exponentially squeezed spiral (Fig. 6.10). Is is by far not a unit speed parametrization. Most points, which are

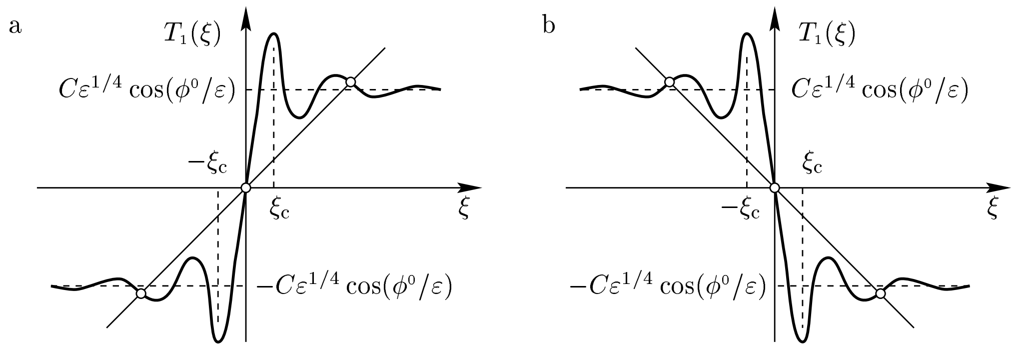

FIGURE 6.11. Schematic shape of the function $T_1(\xi)$ of equation (6.61) (a) for $\cos(\phi^0/\varepsilon) > 0$ and (b) for $\cos(\phi^0/\varepsilon) < 0$ (to compare with the numerical result in Fig. 6.12). In the first case, there are two symmetric stable fixed points. In the second case, there is a stable orbit of period 2.

initially not exponentially close to the origin, will gather exponentially close to $\bar{x}_+$ at the center of the spiral.

We are now able to combine these steps, in particular Lemmas 6.2 and 6.3, in order to compute the Poincaré map. We choose to make the section at $\tau = \hat{\tau}$, since at this time the trajectories are not all exponentially close to one of the adiabatic solutions, so that the behaviour of the map will become easier to visualize.

Proposition 6.2. *Assume $z(\hat{\tau}) = (\xi_0, \eta_0) = (\mathcal{O}(\varepsilon^{1/4}), \mathcal{O}(\varepsilon))$. We denote the Poincaré map by*

$$z(\hat{\tau}+1) = Tz(\hat{\tau}) = (T_1(\xi_0, \eta_0), T_2(\xi_0, \eta_0)). \tag{6.60}$$

Then $T_{1,2}$ have the following properties:

- *$T_2(\xi_0, \eta_0) = \mathcal{O}(\mathrm{e}^{-\delta_2^0/\varepsilon})$ is exponentially small.*
- *$T_1(\xi_0, \eta_0)$ is an odd function of ξ_0 and η_0. For $|\xi_0| \leqslant \xi_c$, it is monotonous in ξ_0 and has an exponentially large slope at the origin.*
- *For $\xi > \xi_c$, it can be written*

$$\begin{aligned} T_1(\xi_0; \eta_0, \varepsilon) = &\cos\Big(\frac{\phi^0}{\varepsilon}\Big)\Big[C\varepsilon^{1/4} + \mathrm{e}^{\alpha_\star/\varepsilon}\cos\Big(\frac{\phi_\star}{\varepsilon}\Big)\Big] \\ &+ \mathrm{e}^{(\alpha_\star-\delta)/\varepsilon}\sin\Big(\frac{\phi^0}{\varepsilon}+\theta^0\Big)\sin\Big(\frac{\phi_\star}{\varepsilon}+\theta^\star\Big), \end{aligned} \tag{6.61}$$

where $\delta = \delta_\star + \delta_3^0$ and $\theta^0 = \mathcal{O}(\varepsilon^{2/3})$. The variables $\phi^0, \phi_\star, \alpha_\star, \delta, \theta^0$ and $\theta^\star$ all depend on ξ_0 and η_0, while C depends on ε. However, at leading order in ε, only $\phi_\star$ and $\alpha_\star$ are functions of ξ_0, given by (6.58).

Remark 6.2. The variables ξ_0 and η_0 appearing in (6.61) measure respectively the distance from the adiabatic stable and unstable manifolds of the origin. Since η_0 is small, ξ_0 is close to q. However, the manifold $\xi_0 = 0$ being invariant, it has the nice property to delimit the initial conditions which are attracted by either equilibrium $Q_\pm$.

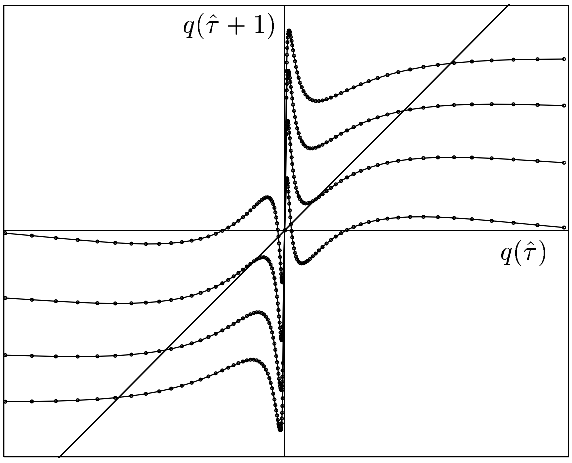

FIGURE 6.12. Numerically computed Poincaré maps, showing $q(\hat{\tau}+1)$ as a function of $q(\hat{\tau})$, where the initial condition has been taken on the unstable manifold of the origin. The values of ε are ranging from 0.0181 to 0.0187, which includes the transition zone of Fig. 6.15a.

6.1.3 Properties of the Poincaré Map – Chaotic Hysteresis

The discussion of the previous Subsection allowed us to establish the following properties of the Poincaré map:

- The map has at least two fixed points, the origin O and the hyperbolic point $P = (\pi, 0) \equiv (-\pi, 0)$.
- Let $\langle \operatorname{Re} a_+^0 \rangle = \alpha^0(1,0)$ be the average of the eigenvalue with largest real part of the linearization around the origin. If $\langle \operatorname{Re} a_+^0 \rangle$ is negative, the origin is asymptotically stable and all orbits but the one starting on P are attracted by O. If $\langle \operatorname{Re} a_+^0 \rangle$ is positive, orbits leave the origin after a bifurcation delay time $\hat{\tau}$ defined by $\alpha^0(\hat{\tau}+1, \check{\tau}) = 0$.
- If the system is overdamped, the map admits an attracting invariant manifold. The map restricted to this manifold is monotonous, and admits two additional stable fixed points $Q_\pm$ when $\langle \operatorname{Re} a_+^0 \rangle$ is positive. These points correspond to hysteresis cycles.
- When the system is not overdamped, there is no more invariant manifold, but orbits are attracted by an exponentially thin layer. The map restricted to this layer is almost one–dimensional, but not necessarily invertible. There are two effects:
  - If the origin is a focus in the interval (τ_-^0, τ_+^0), the map acquires a prefactor $\cos \phi^0/\varepsilon$, where ϕ^0 is the dynamic phase of oscillations around the origin.
  - If the equilibria $Q_\pm$ are focuses in the interval $(\tau_-^\star, \tau_+^\star)$ *and* if the delay time $\hat{\tau}$ belongs to this same interval, then the map acquires a non–monotonous part depending on oscillations around $Q_\pm$, given by (6.61).

Regarding hysteretic properties of the pendulum, we have thus obtained the following classification:

1. $\langle \operatorname{Re} a_+^0 \rangle < 0$: there is no hysteresis, since all orbits asymptotically follow O or P.
2. $\langle \operatorname{Re} a_+^0 \rangle > 0$:
   (a) If $Q_\pm$ is never a focus, there exists a periodic orbit displaying hysteresis:
      i. If the origin is never a focus, this orbit has the same period than $\lambda(\tau)$.
      ii. If the origin is a focus in some interval,

A. this orbit has the same period if $\cos(\phi^0/\varepsilon) > 0$,
B. but *twice* the period of the parameter variation if $\cos(\phi^0/\varepsilon) < 0$ (Fig. 6.5b).

(b) If $Q_\pm$ is a focus in some domain, the map becomes non–invertible, and allows in principle for non–periodic orbits.

Let us study this last, most complicated case in more detail. The Poincaré map (6.61) is of course invertible as it should be, but it is exponentially close to a one–dimensional, non–invertible one. This is due to the fact that the dynamics transforms the unstable manifold of the origin into a spiral, which is squeezed exponentially, rotated by an angle ϕ^0, and squeezed again. The essential properties of the dynamics can thus be understood by studying the 1D map $\xi \mapsto T_1(\xi; 0, \varepsilon)$.

For $\xi > \xi_c$, the function $T_1(\xi)$ is oscillating around $C\varepsilon^{1/4}\cos(\phi^0/\varepsilon)$, with increasing amplitude and frequency as $\xi \searrow \xi_c$. Indeed, we have for instance by (6.58)

$$\mathrm{e}^{\alpha_\star(\xi)/\varepsilon} = \mathrm{e}^{\alpha^\star(\check\tau,\hat\tau)/\varepsilon}\left(\frac{\xi}{d}\right)^{-f(\xi)}. \tag{6.62}$$

The dynamics of this map can be understood as follows. If $C\varepsilon^{1/4}\cos(\phi^0/\varepsilon)$ is larger than the amplitude of the oscillations, there will be one positive fixed point, which will be stable if T is flat enough. On the other hand, if $C\varepsilon^{1/4}\cos(\phi^0/\varepsilon)$ is sufficiently negative, there will be an orbit of period 2 (Fig. 6.11). In the intermediate region, the behaviour may be more complicated, even chaotic.

Let us start by proving the existence of periodic orbits. To do this, we just have to control the amplitude of oscillations.

Proposition 6.3. *There exists a constant $\mu > 0$ such that*
- *if $\varepsilon^{1/4}\cos(\phi^0/\varepsilon) > \mathrm{e}^{-\mu/\varepsilon}$, T_1 admits a stable fixed point at $\xi^\star \approx \varepsilon^{1/4}\cos(\phi^0/\varepsilon)$ (and a symmetric one at $-\xi^\star$);*
- *if $\varepsilon^{1/4}\cos(\phi^0/\varepsilon) < -\mathrm{e}^{-\mu/\varepsilon}$, T_1 admits a stable periodic orbit of period 2 through $\pm\xi^\star$.*

PROOF: We consider the case $\varepsilon^{1/4}\cos(\phi^0/\varepsilon) > \mathrm{e}^{-\mu/\varepsilon}$. Then

$$T_1(\xi) \geqslant \mathrm{e}^{-\mu/\varepsilon}\,\varepsilon^{-1/4}[C\varepsilon^{1/4} - \mathrm{e}^{\alpha_\star/\varepsilon}] - \mathrm{e}^{(\alpha_\star-\delta)/\varepsilon} > C'\,\mathrm{e}^{-\mu/\varepsilon} - \mathrm{e}^{(\alpha_\star-\delta)/\varepsilon} \tag{6.63}$$

where C' is a constant. We impose that

$$T_1(\xi_c) > C'\,\mathrm{e}^{-\mu/\varepsilon} - \mathrm{e}^{(\alpha^\star(\check\tau,\tau_+^\star)-\delta)/\varepsilon} > \xi_c = \mathrm{e}^{-\alpha^0(\tau_+^\star,\hat\tau)/\varepsilon}, \tag{6.64}$$

which is clearly equivalent to

$$\mu < \min\{\alpha^0(\tau_+^\star,\hat\tau), \delta - \alpha^\star(\check\tau,\tau_+^\star)\} + \mathcal{O}(\varepsilon). \tag{6.65}$$

Since, on the other hand, $T_1(d) = \mathcal{O}(\varepsilon^{1/4}) < d$, continuity implies that T_1 has at least one fixed point $\xi^\star \in (\xi_c, d)$. Let us now examine the derivative of T_1. Differentiating (6.58) and (6.46), we obtain

$$\begin{aligned}
\frac{\mathrm{d}\alpha_\star}{\mathrm{d}\xi} &= -\operatorname{Re} a_+^\star(\bar\tau(\xi))\frac{\mathrm{d}\bar\tau}{\mathrm{d}\xi} + \mathcal{O}(\sqrt{\varepsilon}),\\
\frac{\mathrm{d}\phi_\star}{\mathrm{d}\xi} &= -\operatorname{Im} a_+^\star(\bar\tau(\xi))\frac{\mathrm{d}\bar\tau}{\mathrm{d}\xi} + \mathcal{O}(\sqrt{\varepsilon}),\\
\frac{\mathrm{d}\bar\tau}{\mathrm{d}\xi} &= -\frac{\varepsilon}{\xi\operatorname{Re} a_+^0(\bar\tau(\xi))} + \mathcal{O}(\varepsilon),
\end{aligned} \tag{6.66}$$

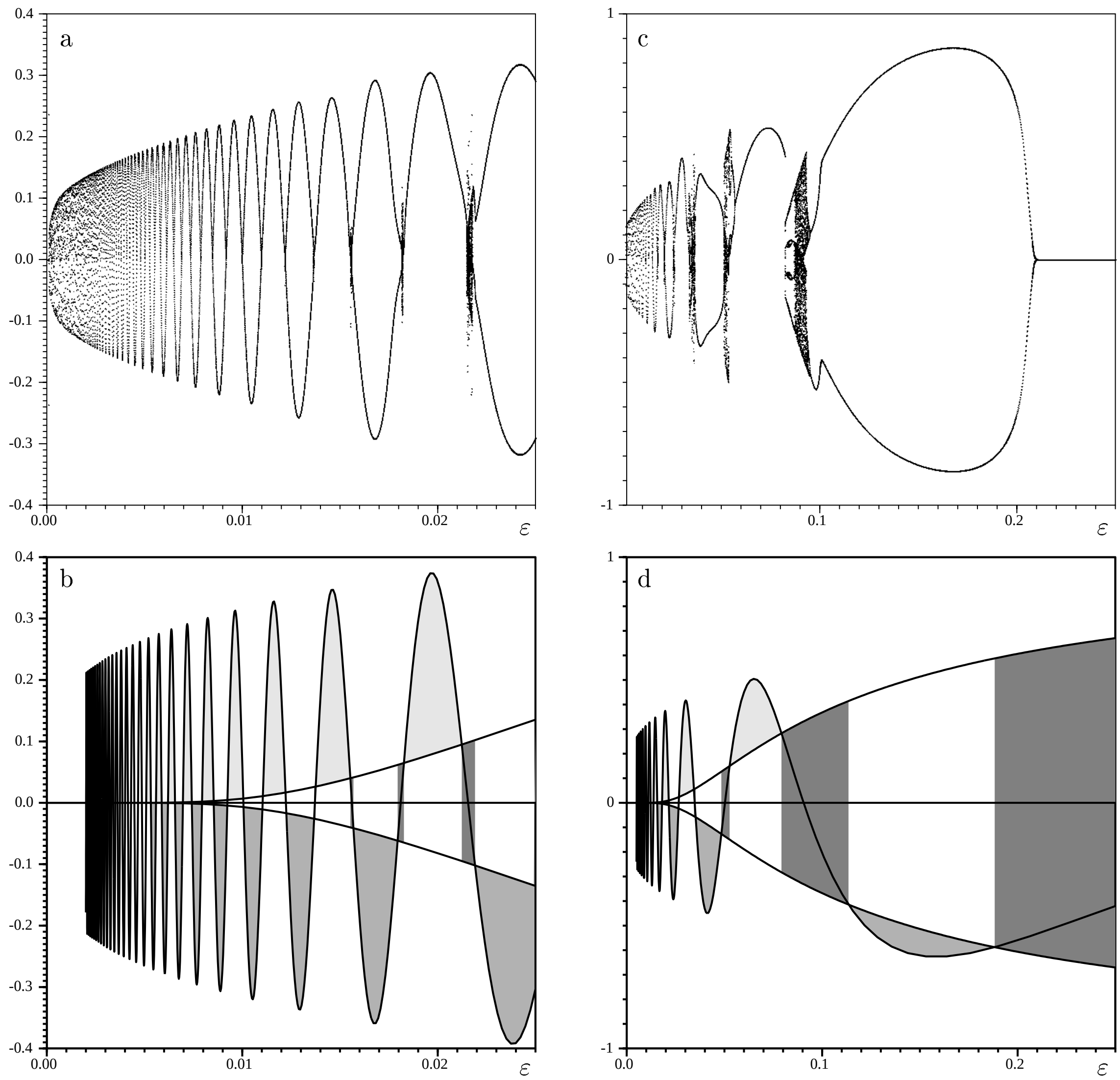

FIGURE 6.13. (a) Numerically computed bifurcation diagrams of the Poincaré map. For each value of ε, we have plotted the asymptotic value of $q(\hat{\tau}+n)$, for *one* initial condition. On the domain $0 < \varepsilon < 0.025$, the diagram clearly shows the alternates of regions with a period–1 and a period–2 cycle, separated by small chaotic zones. (b) Plots of the functions $\varepsilon^{1/4}\cos(\phi^0/\varepsilon)$ and $\pm\,\mathrm{e}^{-\mu/\varepsilon}$. Light gray zones are those where Proposition 6.3 predicts existence of a period–1 cycle, medium gray zones those with a period–2 cycle. Dark gray zone are those where chaotic hysteresis is possible, and, indeed, observed. (c) Same as (a) on a larger scale, where ε goes up to 0.25. We observe that this behaviour subsists until the origin becomes stable in an inverse flip bifurcation. Results are still compatible with Proposition 6.3 (d). We point out that in figures (b) and (d), the dynamic phase $\phi^0(0)$ has been computed using (6.31). Only the next-to-leading-order correction to $\phi^0(\varepsilon)$ (which results in a phase shift) has been chosen in order to fit the numerical results.

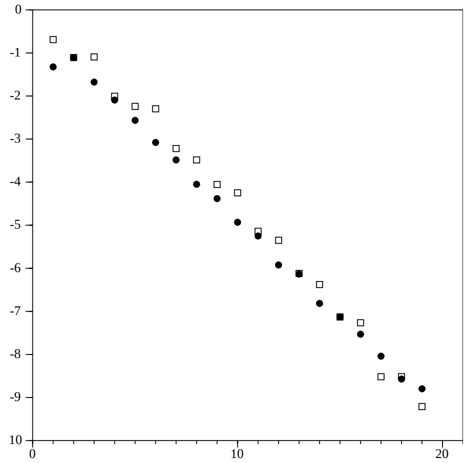

FIGURE 6.14. Logarithm of height (•) and width (□) of the n^{th} chaotic zone as a function of n. As predicted, the dimensions of the chaotic zones decrease exponentially, with the same exponent $\mu\pi/\phi^0$.

so that for $\xi \in (\xi_c, d)$,

$$\left|\frac{\mathrm{d}T_1}{\mathrm{d}\xi}(\xi)\right| \leqslant \mathrm{e}^{\alpha_\star/\varepsilon}\left[\mathcal{O}\Big(\frac{1}{\xi}\Big) + \mathcal{O}\Big(\frac{1}{\sqrt{\varepsilon}}\Big)\right] = \mathcal{O}\big(\mathrm{e}^{(\alpha_\star+\mu)/\varepsilon}\big) < 1, \tag{6.67}$$

provided $\mu < -\alpha^\star(\check\tau, \tau_+^\star)$. Thus T_1 is a contraction, and the fixed point is unique and stable. The case $\varepsilon^{1/4}\cos(\phi^0/\varepsilon) < \mathrm{e}^{-\mu/\varepsilon}$ is treated in a similar way. □

Remark 6.3. The existence of a fixed point of T_1 implies the existence of a fixed point of T. Indeed, orbits starting close to $(\xi^\star, 0)$ arrive close to $(\xi^\star, 0)$. Using (5.219), we can compute the derivative of T at these orbits, and prove that it is contracting, so that it admits a fixed point.

Proposition 6.3 shows that zones with hysteresis cycles of period 1 and 2 alternate more and more frequently as ε goes to zero. These zones are separated by exponentially small transition zones, where the behaviour may be more complicated. The n^{th} transition zone occurs near $\varepsilon_n := \phi^0/(n+\frac{1}{2})\pi$, and both its width and height should be proportional to $\mathrm{e}^{-(\mu\pi/\phi^0)n}$.

In our perturbative analysis, we have always assumed that ε is small enough. Thus the behaviour described above should only be valid for $\varepsilon < c_0$, where the constant c_0 might be very small, so that the transition zones would be difficult to see. Numerical simulations show that this is not so (Fig. 6.13). In fact, Proposition 6.3 is satisfied up to $\varepsilon \simeq 0.1$, and above this value there is a sharp transition to the high–frequency behaviour, where the orbits are attracted by the origin. Thus the adiabatic analysis happens to be accurate for fairly high frequencies. On the other hand, transition zones can easily be tracked numerically down to $n = 20$ (Fig. 6.14).

Up to now, we have not shown that the dynamics is chaotic in the transition zones. Fig. 6.15 shows that in these zones, the system displays indeed a large variety of behaviours, including period doubling cascades and chaotic orbits. To gain some understanding, let us try to apply some theory of iterated maps on the interval. We first recall a few elements of symbolic dynamics, which are proved in [**MS**].

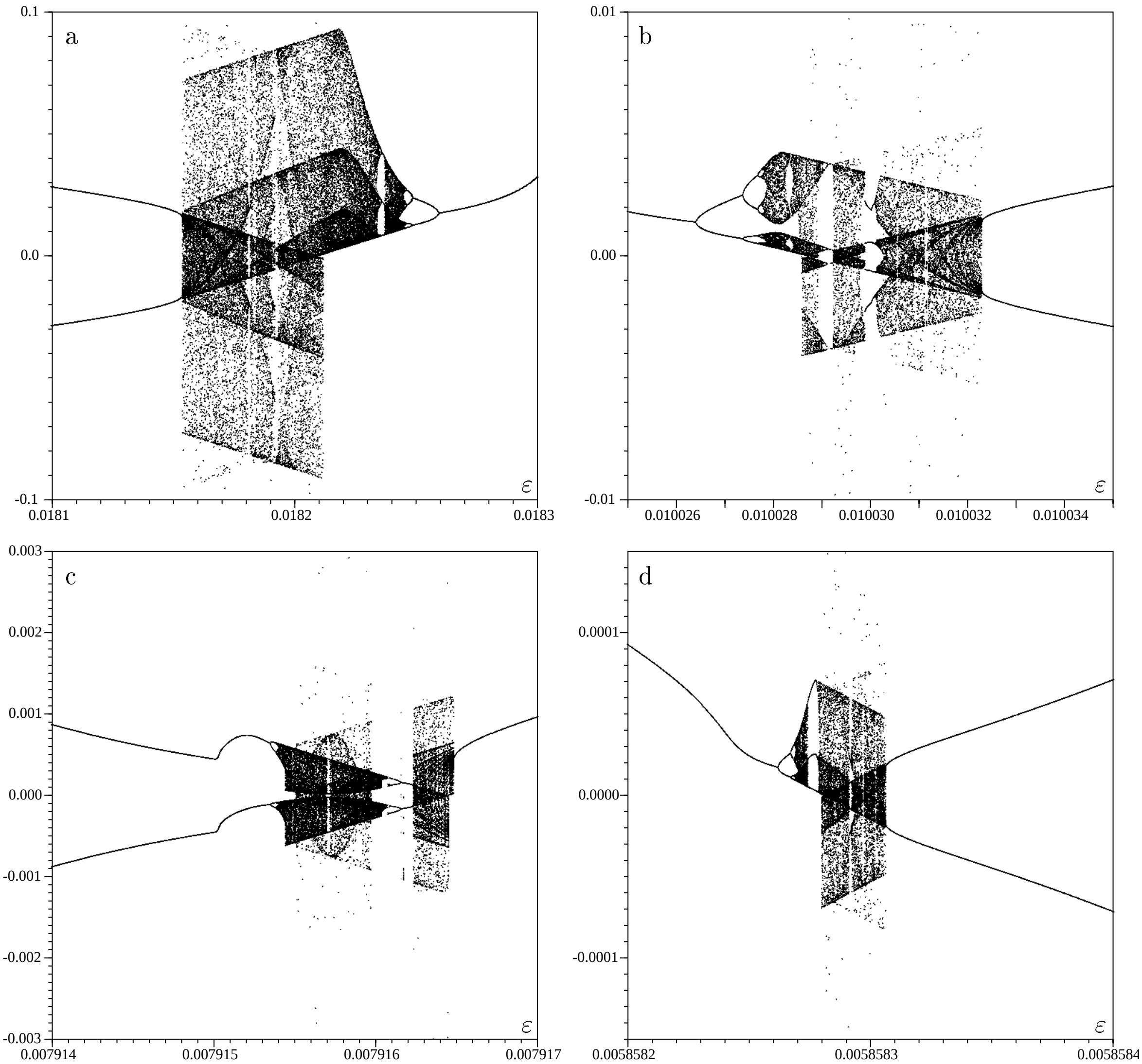

FIGURE 6.15. Magnifications of some transition zones of Fig. 6.13a. We show respectively the $6^{\rm th}$ (a), $11^{\rm th}$ (b), $14^{\rm th}$ (c) and $19^{\rm th}$ transition zone (d). Chaotic orbits seem to appear generally either after a saddle–node bifurcation, or a period doubling cascade, (although a pitchfork bifurcation is also observed in (c)). The chaotic zones display "veils", which we believe are related to the oscillations of the Poincaré map T_1. Orbits have a higher probability to remain close to the origin because these oscillations quickly decrease in amplitude.

Lemma 6.4. *Let I be an interval and $T : I \to I$ a continuous map. Then*

1. *If $J \subset I$ is an interval such that $T(J) \supset J$, T has a fixed point in the closure of J.*
2. *If $\{I_k \,|\, k \geqslant 0\}$ is a family of closed intervals such that $T(I_k) \supset I_{k+1}$, there exists a nested sequence of intervals $\dots \subset J_k \subset \dots \subset J_2 \subset J_1 \subset I_0$ such that $T^k(J_k) = I_k \ \forall k$. In particular, there exists $x \in I_0$ such that $T^k(x) \in I_k$ for every $k \geqslant 0$.*

Definition 6.1.

1. A **partition** of the interval I is a collection of closed subintervals $\{I_k \in I\}$ such that the interiors of the I_k are pairwise disjoint.
2. The **Markov graph** of a map $T : I \to I$ associated with the partition $\{I_k\}$ is the graph with vertices $\{I_k\}$, and with (oriented) edges $I_j \to I_k$ for each (j,k) such that $T(I_j) \supset I_k$.

Lemma 6.4 implies that for every path of the Markov graph, there exists an orbit visiting the sequence of edges of this path. In particular, if the path $I_{k_0} \to I_{k_1} \to \dots \to I_{k_{n-1}} \to I_{k_0}$ is closed, there exists a periodic orbit starting at $x \in I_{k_0}$ such that $T^j(x) \in I_{k_j}$ for all j. However, since the closed intervals I_k have only disjoint interiors, this orbit may have a period smaller than n.

Symbolic dynamics can be used to prove Sarkovskii's theorem.

Definition 6.2. The following ordering on the set of natural numbers is called **Sarkovskii ordering**:

$$\begin{array}{ccccccccccccc} & 3 & \succ & 5 & \succ & 7 & \succ & \dots & \succ & 2n+1 & \succ & \dots \\ \succ & 2\times 3 & \succ & 2\times 5 & \succ & 2\times 7 & \succ & \dots & \succ & 2\times(2n+1) & \succ & \dots \\ \succ & 2^m\times 3 & \succ & 2^m\times 5 & \succ & 2^m\times 7 & \succ & \dots & \succ & 2^m\times(2n+1) & \succ & \dots \\ \succ & \dots & \succ & 2^n & \succ & \dots & \succ & 4 & \succ & 2 & \succ & 1 \end{array} \tag{6.68}$$

Theorem 6.1 (Sarkovskii). *Let $T : [0,1] \to [0,1]$ be a continuous map having a periodic orbit of period n. Then T has a periodic orbit of period m for every m such that $n \succ m$ in the Sarkovskii ordering.*

Let us now examine the map (6.61) in the middle of a transition zone, that is when $\varepsilon = \varepsilon_n := \phi^0/(n+\frac12)\pi$. Then it can be written as

$$T_1(\xi) = (-1)^n \, \mathrm{e}^{(\alpha_\star(\xi)-\delta)/\varepsilon} \sin\Big(\frac{\phi_\star(\xi)}{\varepsilon} + \theta^\star\Big), \tag{6.69}$$

where the phase θ^0 has been absorbed in δ. Near the origin, this map is oscillating as in Fig. 6.16a. We now construct a Markov partition to analyse this map.

Proposition 6.4. *Let $T : I \to I$ be an odd map, continuous on an interval $I \ni 0$.*

1. *Assume there exist points $0 < \xi_1 < \xi_2$ such that $T(\xi_1) = T(\xi_2) = 0$ and $\pm\xi_2 \in I$. If*

$$\max_{0 \leqslant \xi \leqslant \xi_1} T(\xi) \geqslant \xi_2 \qquad \text{and} \qquad \min_{\xi_1 \leqslant \xi \leqslant \xi_2} T(\xi) \leqslant -\xi_1, \tag{6.70}$$

then T admits periodic orbits of every period, except possibly 3.

2. *Assume that instead of (6.70),*

$$\min_{0 \leqslant \xi \leqslant \xi_1} T(\xi) \leqslant -\xi_2 \qquad \text{and} \qquad \max_{\xi_1 \leqslant \xi \leqslant \xi_2} T(\xi) \geqslant \xi_1. \tag{6.71}$$

Then T admits periodic orbits of every period.

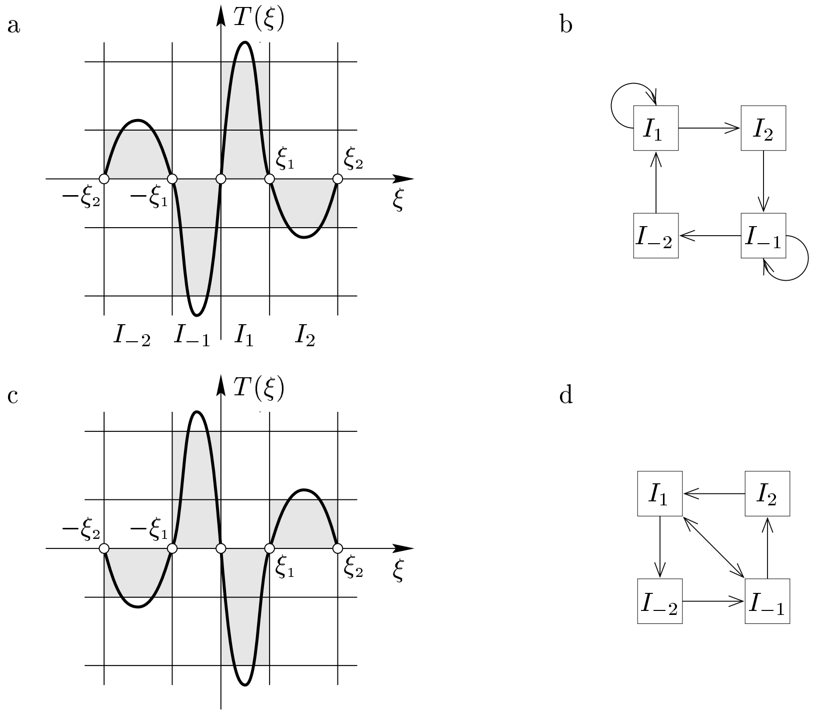

FIGURE 6.16. Interval maps to which Proposition 6.4 applies. (a) The map satisfying (6.70) admits the Markov subgraph (b), which allows for periodic orbits of all periods, except possibly 3. (c) If the sign of the map is reversed, the Markov graph (d) shows existence of orbits of all periods.

PROOF: We define the intervals $I_1 = [0, \xi_1]$, $I_2 = [\xi_1, \xi_2]$, and $I_{-j} = -I_j$, $j = 1, 2$. In the first case, the hypotheses imply that the Markov graph contains the subgraph of Fig. 6.16b. Consider the closed path $I_1 \to I_1 \to I_2 \to I_{-1} \to I_{-2} \to I_1$. By Lemma 6.4, there exists $x \in I_1$ such that $T^5 x = x$, and the iterates of x visit the intervals of this path. We show that this orbit has period 5. If it had a smaller period, one of the iterates of x should belong to $I_1 \cap I_2$ or $I_{-2} \cap I_{-1}$. But then the image of this point would be 0, so that we must have $x = 0$. This contradicts the fact that $T^2(x) \in I_2$. By Sarkovskii's theorem, the map has periodic orbits of all periods succeeding 5 in the Sarkovskii ordering.

In the second case, we obtain the Markov graph of Fig. 6.16d, which contains the closed path $I_1 \to I_{-1} \to I_2 \to I_1$. By a similar argument, the map admits an orbit of period 3, and the conclusion follows. $\square$

Remark 6.4. In fact, Lemma 6.4 shows that there exist orbits for every symbol sequence compatible with the Markov graph. In the first case, these sequences are of the form $I_1^{n_1} I_2 \; I_{-1}^{n_2} I_{-2} I_1^{n_3} \dots$, where $n_1, n_2, n_3, \dots \geqslant 1$ are arbitrary integers. In the second case, allowed symbol sequences are arbitrary combinations of the words $I_1 I_{-1}$, $I_1 I_{-1} I_2$ and $I_1 I_{-2} I_{-1}$.

We now apply Proposition 6.4 to the expression (6.69) of the Poincaré map.

Proposition 6.5. *Assume that*

$$\alpha^0(\tau_+^\star, \hat{\tau}) + \alpha^\star(\check{\tau}, \tau_+^\star) > \delta^0(\tau_-^0, \check{\tau}) + \delta^\star(\check{\tau}, \tau_+^\star). \tag{6.72}$$

Then the 1D map (6.69) *admits periodic orbits of all periods except possibly 3.*

PROOF: Let us write $\xi = \mathrm{e}^{-\alpha_0/\varepsilon+\psi}$, where $\alpha_0 := \alpha^0(\tau_+^\star, \hat\tau)$ (in particular, for $\psi = 0$, $\xi = \xi_c$). If we assume that ψ is of order 1, it follows from (6.59) that

$$\begin{aligned} \sin\Bigl(\frac{\phi_\star}{\varepsilon}\Bigr) &= \sin\Bigl(\frac{\phi_1}{\alpha_0}\psi\Bigr) + \mathcal{O}(\sqrt{\varepsilon}), \qquad \phi_1 = \phi^\star(\tau_+^\star, \hat\tau), \\ \mathrm{e}^{\alpha_\star/\varepsilon} &= \mathrm{e}^{\alpha_2/\varepsilon}\exp\Bigl(\frac{\alpha_1-\alpha_2}{\alpha_0}\,\psi\Bigr)\bigl(1+\mathcal{O}(\sqrt{\varepsilon})\bigr), \qquad \alpha_1 = \alpha^\star(\check\tau,\hat\tau),\ \alpha_2 = \alpha^\star(\check\tau, \tau_+^\star). \end{aligned} \tag{6.73}$$

Assume n is even. If $c := \alpha_0/\phi_1$, the map vanishes for $\psi = c\pi/2, 3c\pi/2$. We also have $T_1(\psi = 0) = \mathrm{e}^{\alpha_2-\delta/\varepsilon}$, and $T_1(\psi = c\pi) = -T_1(\psi = 0)\mathcal{O}(1)$. Thus, condition (6.70) is fulfilled if $\alpha_2 - \delta > -\alpha_0$, which is equivalent to (6.72). The case n odd is similar. □

We have thus proved that, for some specific values of the parameters, the 1D map which approximates the dynamics of the rotating pendulum admits an infinite number of periodic orbits. In fact, according to Remark 6.4, this map also admits orbits with every symbolic sequence satisfying some simple grammar rules. This does not automatically imply existence of orbits with similar properties for the real 2D map of Proposition 6.2, but nevertheless yields strong evidence for chaotic hysteresis.

To summarize, we have shown that the rotating pendulum displays the following asymptotic behaviours, depending on the value of parameters:

- absence of hysteresis, the pendulum asymptotically remaining in vertical position;
- a hysteresis cycle with the same period than the variation of Ω;
- hysteresis with twice the driving period;
- for some particular values of the parameters, assuming the 1D map provides an accurate description of the dynamics, hysteresis cycles with arbitrarily complicated symbolic sequence.

Let us point out that the conditions on parameters delimiting these different regimes depend, on leading order in ε, only on the linearization of the vector field around its equilibria (via the functions $\alpha^{0,\star}$, $\phi^{0,\star}$ and $\delta^{0,\star}$). Although our adiabatic analysis assumes ε to be sufficiently small, it gives fairly accurate predictions for appreciably large values of ε.

Finally, we make a remark on the limit $\varepsilon \to 0$. Let $x(\tau, \varepsilon)$ be the solution for some given initial condition. The above discussion shows that $\lim_{\varepsilon\to 0} x(\tau,\varepsilon) = 0$ for all times in intervals $(\check\tau + n, \hat\tau + n + 1)$ (during these intervals, the solution stays close to the origin, either because the origin is stable, or because of bifurcation delay). For the remaining time, the situation is more complicated. In fact, we can extract subsequences $x(\tau, \varepsilon_n)$, with $\lim_{n\to\infty}\varepsilon_n = 0$, that admit a well defined limit: it is sufficient to choose ε_n in the intervals of period–1 hysteresis (the fact that such a sequence exists is natural since x is bounded). But for other choices of ε_n, the sequence $x(\tau, \varepsilon_n)$ may have a very complicated behaviour.

6.2 Examples of Eigenvalue Crossings

6.2.1 Overdamped System and Diagonal Crossing

Let us consider the overdamped system

$$\varepsilon\dot{x} = -\partial_x\Phi(x,\lambda(\tau)), \tag{6.74}$$

where $\Phi(x,\lambda)$ is a quartic potential of the form

$$\Phi(x,\lambda) = -\frac{1}{2}\langle x|A(\lambda)x\rangle + \frac{1}{4}\langle x|x\rangle^2, \qquad A = A^{\mathrm{T}}, \tag{6.75}$$

so that equation (6.74) becomes

$$\varepsilon\dot{x} = A(\lambda(\tau))x - \langle x|x\rangle x. \tag{6.76}$$

The matrix $A = \left(\begin{smallmatrix} b & c \\ c & d \end{smallmatrix}\right)$ being symmetric, it can be diagonalized statically by a matrix

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \operatorname{tg} 2\theta = \frac{2c}{b-d}. \tag{6.77}$$

The change of variables $x = Sy$, taking the principal axes of A as rotating reference frame, transforms (6.76) into

$$\varepsilon\dot{y} = A_1 y - \langle y|y\rangle y, \qquad A_1 = \begin{pmatrix} a_1 & \varepsilon\dot{\theta} \\ -\varepsilon\dot{\theta} & a_2 \end{pmatrix}, \tag{6.78}$$

where a_1, a_2 are the eigenvalues of A. The off–diagonal terms are due to the rotation of the reference frame. We would now like to construct a transformation $y = S_1 z$ which diagonalizes the system completely. This can be done, for instance, by solving the system

$$\varepsilon\dot{S}_1 = A_1 S_1 - S_1 D, \qquad S_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_2 \\ \sin\theta_1 & \cos\theta_2 \end{pmatrix}, \tag{6.79}$$

where $D = \operatorname{diag}(d_1, d_2)$, which is equivalent to

$$d_1 = a_1\cos^2\theta_1 + a_2\sin^2\theta_1, \qquad \varepsilon\dot{\theta}_1 = \frac{a_2 - a_1}{2}\sin 2\theta_1 - \varepsilon\dot{\theta}, \tag{6.80a}$$

$$d_2 = a_1\sin^2\theta_2 + a_2\cos^2\theta_2, \qquad \varepsilon\dot{\theta}_2 = \frac{a_1 - a_2}{2}\sin 2\theta_2 - \varepsilon\dot{\theta}. \tag{6.80b}$$

If these equations admit bounded solutions, the system (6.78) becomes

$$\varepsilon\dot{z} = Dz - \langle z|S_1^{\mathrm{T}}S_1 z\rangle z. \tag{6.81}$$

We know by the theory of Chapter 4 that the equations (6.80) admit solutions of order ε if the eigenvalues a_1 and a_2 do not cross. In this case, we obtain two invariant subspaces, on which the motion is governed by a cubic one–dimensional equation.

Let us now consider the effect of eigenvalue crossings in the special case $-a_2(\tau) = a_1(\tau) = a(\tau)$. In this case, the origin is hyperbolic, and there are two potential minima, at $y = (0, \pm\sqrt{a})$ if a is positive, and at $(\pm\sqrt{-a}, 0)$ if a is negative. (6.80) reduces to

$$d_1 = a\cos 2\theta_1, \qquad \varepsilon\dot{\theta}_1 = -a\sin 2\theta_1 - \varepsilon\dot{\theta}, \tag{6.82a}$$

$$d_2 = -a\cos 2\theta_2, \qquad \varepsilon\dot{\theta}_2 = a\sin 2\theta_2 - \varepsilon\dot{\theta}. \tag{6.82b}$$

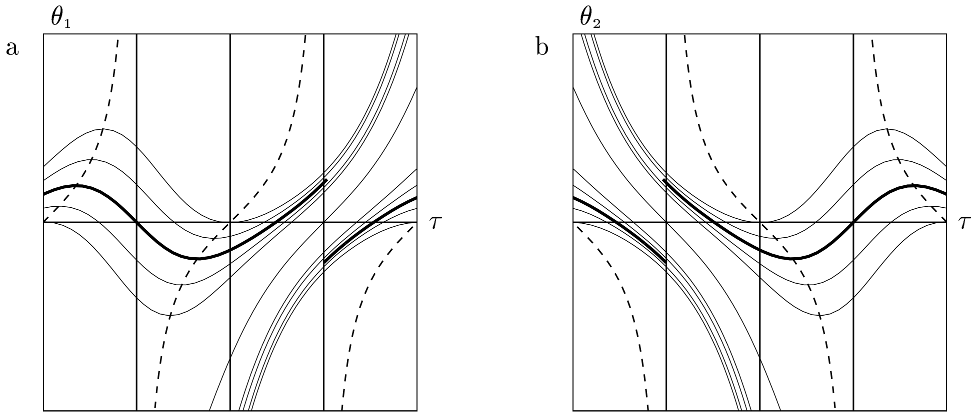

FIGURE 6.17. Solutions of equation (6.82) in the particular case (6.83). We choose the particular solutions defined by $\theta_1(\frac{1}{4}) = 0$ and $\theta_2(\frac{3}{4}) = 0$ (thick lines), which necessarily admit a gap of order $\sqrt{\varepsilon}$. This gap leads to the transition amplitude.

Let us begin with the particular case

$$a(\tau) = \theta(\tau) = -\frac{1}{2\pi}\cos(2\pi\tau). \tag{6.83}$$

The static equilibria of (6.82) are given by

$$\theta_1^\star = \frac{1}{2}\operatorname{Arcsin}\Big(-\frac{\varepsilon\dot{\theta}}{a}\Big) = \varepsilon\pi\,\mathrm{tg}(2\pi\tau) + \mathcal{O}(\varepsilon^2), \qquad \theta_2^\star = -\theta_1^\star. \tag{6.84}$$

and diverge at $\tau = \frac{1}{4}, \frac{3}{4}$. The discussion of Subsection 5.3.6 (see Fig. 5.4) shows that we can construct periodic solutions of (6.82) which remain of order $\sqrt{\varepsilon}$, but with discontinuities of the same order. For instance, we can choose a solution $\theta_1(\tau)$ such that $\theta_1(\frac{1}{4}) = 0$, with a discontinuity $-d\sqrt{\varepsilon}$ at $\tau = \frac{3}{4}$, and a solution $\theta_2(\tau)$ such that $\theta_2(\frac{3}{4}) = 0$, with a discontinuity $d\sqrt{\varepsilon}$ at $\tau = \frac{1}{4}$ (Fig. 6.17).

We can now construct the propagator $U(\tau_2, \tau_1)$ of the linear equation $\varepsilon\dot{y} = A_1(\tau)y$. According to Proposition 5.4, it will contain transition matrices of the form

$$\begin{aligned}
T_1 = S_1^+\big(\tfrac{1}{4}\big)^{-1} S_1^-\big(\tfrac{1}{4}\big) &= \frac{1}{\cos\theta_2^+(\frac{1}{4})}\begin{pmatrix}\cos\theta_2^+(\frac{1}{4}) & \sin\big(\theta_2^+(\frac{1}{4}) - \theta_2^-(\frac{1}{4})\big) \\ 0 & \cos\theta_2^-(\frac{1}{4})\end{pmatrix} \approx \begin{pmatrix}1 & 2d\sqrt{\varepsilon} \\ 0 & 1\end{pmatrix}, \\
T_1 = S_1^+\big(\tfrac{3}{4}\big)^{-1} S_1^-\big(\tfrac{3}{4}\big) &= \frac{1}{\cos\theta_1^+(\frac{3}{4})}\begin{pmatrix}\cos\theta_1^-(\frac{3}{4}) & 0 \\ \sin\big(\theta_1^-(\frac{3}{4}) - \theta_1^+(\frac{3}{4})\big) & \cos\theta_1^+(\frac{3}{4})\end{pmatrix} \approx \begin{pmatrix}1 & 0 \\ 2d\sqrt{\varepsilon} & 1\end{pmatrix}.
\end{aligned} \tag{6.85}$$

Taking into account the diagonal part of evolution, we find for instance that at leading order in ε,

$$U(\tfrac{1}{2}, 0) = \begin{pmatrix}1 & 2d\sqrt{\varepsilon}\,\mathrm{e}^{2/\varepsilon} \\ 0 & 1\end{pmatrix}, \qquad U(1, \tfrac{1}{2}) = \begin{pmatrix}1 & 0 \\ 2d\sqrt{\varepsilon}\,\mathrm{e}^{2/\varepsilon} & 1\end{pmatrix}. \tag{6.86}$$

The first part of dynamics leaves the horizontal axis invariant, while the vertical axis rotates clockwise by almost $\pi/2$. The second transformation leaves the vertical axis invariant and rotates the horizontal one by almost $\pi/2$ in the opposite direction. For the nonlinear system, this implies that the particle will oscillate periodically between the minima $(0, \sqrt{a})$ and $(\sqrt{-a}, 0)$ or their symmetric images (Fig. 6.19a).

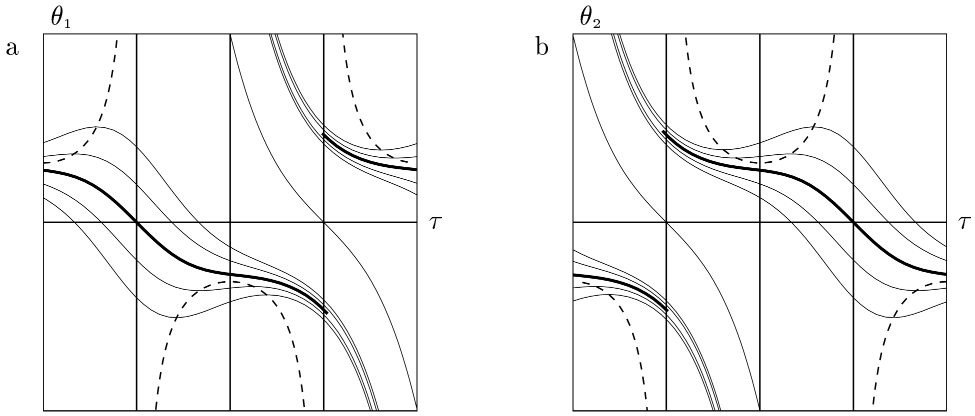

FIGURE 6.18. Solutions of equation (6.82) in the particular case (6.87). We choose again the solution marked by a thick line.

Let us now consider another situation,

$$a(\tau) = -\frac{1}{2\pi}\cos(2\pi\tau), \qquad \dot\theta(\tau) = 1. \tag{6.87}$$

The solutions of (6.80) behave in a similar way as before, with the difference that the discontinuity of θ_1 changes sign (Fig. 6.18), so that we obtain

$$U(\tfrac{1}{2},0) = \begin{pmatrix} 1 & 2d\sqrt{\varepsilon}\,\mathrm{e}^{2/\varepsilon} \\ 0 & 1 \end{pmatrix}, \qquad U(1,\tfrac{1}{2}) = \begin{pmatrix} 1 & 0 \\ -2d\sqrt{\varepsilon}\,\mathrm{e}^{2/\varepsilon} & 1 \end{pmatrix}. \tag{6.88}$$

This time, both transformations rotate coordinate axes in the same direction, so that the particle will visit the sequence of minima $(0,\sqrt{a})$, $(\sqrt{-a},0)$, $(0,-\sqrt{a})$ and $(-\sqrt{-a},0)$, in an orbit with *twice* the period of $a(\tau)$ (Fig. 6.19b). This is due to the fact that the system is not of the form $\varepsilon\dot x = f(x,\lambda(\tau))$, because θ rotates with constant angular frequency.

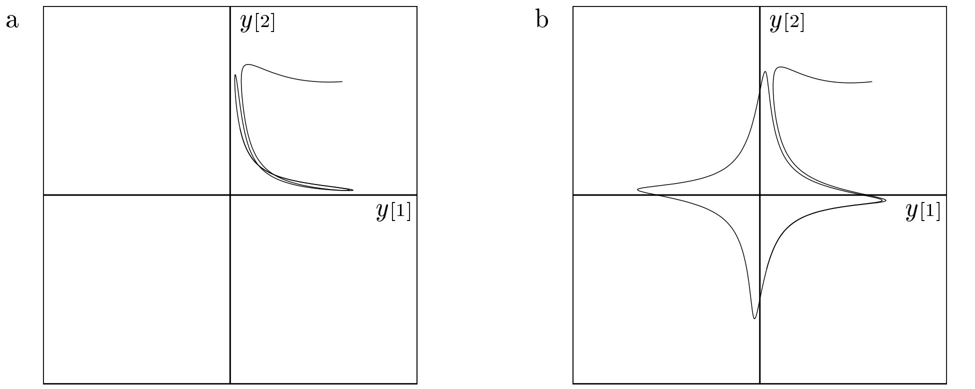

FIGURE 6.19. Solutions of (6.78): (a) in the case (6.83), the particle oscillates back and forth between two potential wells, while (b) in the case (6.87) it visits all four wells in a row.

6.2.2 Coupled Oscillators and Eigenvalue Cruising

Let us consider two coupled oscillators of the form

$$\begin{aligned}\ddot{q}_1 + 2\gamma_1\dot{q}_1 + \Phi_1'(q_1) - \mu_1 q_2 &= 0,\\ \ddot{q}_2 + 2\gamma_2\dot{q}_2 + \Phi_2'(q_2) - \mu_2 q_1 &= 0,\end{aligned} \tag{6.89}$$

where Φ_1 and Φ_2 are analytic. These equations are equivalent to the first order system

$$\begin{aligned}\dot{q}_1 &= p_1,\\ \dot{q}_2 &= p_2,\\ \dot{p}_1 &= -2\gamma_1 p_1 - \Phi_1'(q_1) + \mu_1 q_2,\\ \dot{p}_2 &= -2\gamma_2 p_2 - \Phi_2'(q_2) + \mu_2 q_1.\end{aligned} \tag{6.90}$$

We assume that $\Phi_1'(0) = \Phi_2'(0) = 0$. Then the origin is an equilibrium point with linearization

$$A = \begin{pmatrix} 0 & \mathbb{1} \\ -H & -2\Gamma \end{pmatrix}, \qquad H := \begin{pmatrix} -\Phi_1''(0) & \mu_1 \\ \mu_2 & -\Phi_2''(0) \end{pmatrix}, \quad \Gamma := \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}. \tag{6.91}$$

If the coupling constants μ_1, μ_2 are zero, A has eigenvalues

$$a_{\pm,j} = -\gamma_j \pm \sqrt{\gamma_j^2 - \Phi_j''(0)}, \qquad j = 1, 2. \tag{6.92}$$

If these eigenvalues are distinct, they will change smoothly with μ_1 and μ_2 when these couplings are small.

We are interested in the following special case:

- The first oscillator is overdamped, i.e., $\Phi_1''(0) < \gamma_1^2$. Thus it has two real eigenvalues, which we denote $\nu < \lambda$, where ν is always negative. We assume that $\Phi_1''(0)$ depends adiabatically on time.
- The second oscillator is underdamped, i.e., $\Phi_2''(0) > \gamma_2^2$. Thus it has eigenvalues $-\gamma \pm \mathrm{i}\,\Omega$, where $\gamma = \gamma_2 + \mathcal{O}(\mu_1\mu_2)$ and $\Omega = \sqrt{\Phi_2''(0) - \gamma_2^2} + \mathcal{O}(\mu_1\mu_2)$.

If ν is always smaller than $-\gamma$, this mode can be decoupled, at least near the origin, from the others, so that we obtain an effective 3–dimensional dynamics

$$\begin{aligned}\varepsilon\dot{q}_1 &= -\Phi_1'(q_1) + \mu_1 q_2\\ \varepsilon\dot{q}_2 &= p_2\\ \varepsilon\dot{p}_2 &= -2\gamma_2 p_2 - \Phi_2'(q_2) + \mu_2 q_1.\end{aligned} \tag{6.93}$$

The linearization of this system around the origin is

$$\varepsilon\dot{x} = A(\tau)x, \qquad A(\tau) = \begin{pmatrix} -\Phi_1''(0) & \mu_1 & 0 \\ 0 & 0 & 1 \\ \mu_2 & -\Phi_2''(0) & -2\gamma_2 \end{pmatrix}, \tag{6.94}$$

where $A(\tau)$ has eigenvalues $\lambda(\tau)$ and $-\gamma(\tau) \pm \mathrm{i}\,\Omega(\tau)$.

If we assume that $\lambda(\tau)$ is initially smaller than $-\gamma$ and then grows larger, we are in a situation of eigenvalue cruising, between the eigenvalues $\lambda(\tau)$ and $-\gamma(\tau) \pm \mathrm{i}\,\Omega(\tau)$. Suppose that this cruising occurs at $\tau = 0$. By Proposition 5.6, there exists a buffer point $\tau^+ > 0$, such that if $0 < \tau < \tau^+$, the solution of the linearized system (6.94) can be written as

$$x(\tau) = S(\tau)U_0(\tau, \tau_0)S(\tau_0)^{-1}x(\tau_0), \tag{6.95}$$

where $S(\tau)$ is the matrix diagonalizing A statically (plus corrections of order ε), and

$$U_0(\tau, \tau_0) = \begin{pmatrix} \mathrm{e}^{\alpha(\tau,\tau_0)/\varepsilon} & 0 & 0 \\ 0 & \mathrm{e}^{-\beta(\tau,\tau_0)/\varepsilon}\,\mathrm{e}^{\mathrm{i}\,\Phi(\tau,\tau_0)/\varepsilon} & 0 \\ 0 & 0 & \mathrm{e}^{-\beta(\tau,\tau_0)/\varepsilon}\,\mathrm{e}^{-\mathrm{i}\,\Phi(\tau,\tau_0)/\varepsilon} \end{pmatrix}, \tag{6.96}$$

where α, β and Φ denote respectively the integrals of λ, γ and Ω between τ_0 and τ. This means that there exists an invariant subspace, given by the equation $x = S(\tau)(0, p_2, q_2)$, which implies $q_1 = \mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)$. This subspace corresponds to a stable position of the overdamped oscillator, which is unaffected by the underdamped one. But when τ becomes larger than the buffer time τ^+, the matrix $U_0(\tau, \tau_0)$ necessarily acquires a nondiagonal component, and the overdamped oscillator no longer behaves as if it were isolated.

Let us illustrate these phenomena on a simple special case. We take $\mu_2 = \gamma_2 = 0$, $-\Phi_1''(0, \tau) = \tau$ and $\Phi_2''(0) = 1$, so that the eigenvalues of A become τ and $\pm\,\mathrm{i}$. The linearized system (6.94) becomes

$$\varepsilon\dot{x} = A(\tau)x, \qquad A(\tau) = \begin{pmatrix} \tau & \mu_1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{6.97}$$

To diagonalize this equation, we have to solve an equation of the form

$$\varepsilon\dot{s} = 1 + (\tau - \mathrm{i})s, \tag{6.98}$$

see (5.115) in Subsection 5.3.7. The fact that this equation is solvable is due to the fact that we have taken $\mu_2 = 0$. But this equation has been discussed in Example 4.13, see Subsection 4.4.1. We know that it admits buffer points $(\tau_-, \tau_+) = (-1, 1)$. Thus our system admits stable solutions up to $\tau = 1$.

Now we remark that the second oscillator is decoupled from the first, and evolves according to $q_2(\tau) = A\cos(\tau/\varepsilon + \varphi)$. Let us take the case $q_2(\tau) = \cos(\tau/\varepsilon)$. Then (6.97) can be written as

$$\varepsilon\dot{q}_1 = \tau q_1 + \mu_1\cos(\tau/\varepsilon), \tag{6.99}$$

which is the equation of a forced oscillator, but note that this forcing is *not* adiabatic. The solution is given by

$$q_1(\tau) = \mathrm{e}^{(\tau^2-\tau_0^2)/2\varepsilon}\,q_1(\tau_0) + \frac{\mu_1}{\varepsilon}\int_{\tau_0}^{\tau} \mathrm{e}^{(\tau^2-s^2)/2\varepsilon}\cos(s/\varepsilon)\,\mathrm{d}s. \tag{6.100}$$

By deforming the integration path into the complex plane as in Example 4.13, we obtain that the integral is of order μ as long as $\tau < 1$, and then it becomes exponentially large.

Assume for instance that the first oscillator starts at $q_1(-2) = 1$. If it were not coupled to the second one, it would stay close to the origin until $\tau = 2$. But when the coupling is present, it will lose stability already at the buffer time $\tau = 1$.

Chapter 7

Magnetic Hysteresis

"Magnet, n.: Something acted upon by magnetism
Magnetism, n.: Something acting upon a magnet.
The two definitions immediately foregoing are condensed from the works of one thousand eminent scientists, who have illuminated the subject with a great white light, to the inexpressible advancement of human knowledge."
Ambrose Bierce, "The Devil's Dictionary"

Physical and mathematical models trying to describe hysteresis in ferromagnets are numerous. Some of them, like the Preisach model that we described in Section 3.4, are purely phenomenological. More recently, there has been a number of attempts to derive hysteretic properties from microscopic interactions. Such derivations are often based on a lattice model (for instance one of the models introduced in Section 3.2), with a Glauber spin–flip or similar stochastic dynamics (see Section 3.3).

In particular, the following models have been frequently considered:

- Ising models with Glauber dynamics, studied with Monte–Carlo simulations;
- Ordinary differential equations obtained from a mean field approximation;
- Partial differential Langevin equations, with a Ginzburg–Landau type of potential, and an additive white noise (this kind of model is more elaborate, and tries to capture the spatial structure of the magnet; we will not consider it here).

One of the features analysed in these works is the dependence of the hysteresis loop shape, in particular its area, on amplitude h_0 and frequency ε of the oscillating magnetic field. Various scaling laws have been proposed for the area, in the limit of small h_0 and ε. We have summarized some of them in the introduction, see Table 1.2. There appear to be two kinds of models:

- Models in which hysteresis is only possible for sufficiently large driving amplitude; in this case, the area tends to a nonzero limit when the driving frequency ε goes to zero, and increases with ε according to some power law $\mathcal{A} \approx \mathcal{A}_0 + \varepsilon^\alpha h_0^\beta$.
- Models in which hysteresis cycles exist for arbitrarily small h_0, with an area following an approximate scaling law $\mathcal{A} \approx \varepsilon^\alpha h_0^\beta$.

The difference between these two types of models is clearly related to the mechanism of magnetization reversal.

It is out of question to analyse all the above models at this place. Our more modest aim is to apply the theory of adiabatic Dynamical Systems, which we developed in previous chapters, to some of those models which can be described by finite–dimensional ODEs. Most of the mentioned articles are based on numerical simulations, even those dealing with ODEs, and some of them contain mistakes. We believe that such errors could have been avoided by trying to understand the *qualitative* properties of the flow, before (or even instead of) using numerics. We would like to demonstrate this on a number of selected examples.

The following models are considered in this chapter:

- **Section 7.1** We examine lattice models with infinite range interaction, like the Curie–Weiss model, which can be described by a finite–dimensional ODE in the thermodynamic limit. In the 1D case, we analyse in detail the “dynamic phase transition” and scaling of hysteresis area. In the 2D case, we consider the influence of anisotropy on magnetization reversal and shape of hysteresis cycles.
- **Section 7.2** We show why the evolution equations are much more complicated for the Ising model. We examine hysteresis cycles for a few simple approximations.

7.1 Curie–Weiss Model

We consider in this section a lattice model with infinite range interaction, defined by the Hamiltonian

$$H(\sigma) = -\frac{1}{2N} \sum_{i \neq j \in \Lambda} \langle \sigma_i | J \sigma_j \rangle - \sum_{i \in \Lambda} \langle h | \sigma_i \rangle. \tag{7.1}$$

Here Λ is a finite subset of $\mathbb{Z}^d$ with N sites.[1] The classical spins $\sigma_i \in \mathbb{S}^{n-1}$ are vectors of unit length in $\mathbb{R}^n$. The magnetic field h is a vector in $\mathbb{R}^n$, and the coupling J is a symmetric matrix of size n.

In Section 3.3, we have considered the **Curie–Weiss model**, where the spins are one–dimensional ($n = 1$). We have shown how to derive a deterministic evolution equation for the magnetization in the thermodynamic limit, and that fluctuations around this limit obey a Langevin equation.

In Subsection 7.1.1, we propose a derivation of a deterministic evolution equation for vectorial spins, starting with an appropriate master equation generalizing Glauber dynamics. We obtain that for small magnetic field and magnetization, this equation reduces to a Ginzburg–Landau model.

In Subsection 7.1.2, we discuss in detail the one–dimensional case. We give a critical analysis of the phenomenon of **dynamic phase transition** discussed in [TO], and of the scaling of hysteresis cycles, which has been studied in [HL&].

In Subsection 7.1.3, we discuss some aspects of the two–dimensional case, for different coupling matrices J. We analyse in particular the influence of anisotropy on the shape of hysteresis cycles.

[1] More precisely, we consider a sequence of such subsets with $N \to \infty$. Since the interaction is uniform, the geometry of these sets, as well as the dimensionality d of the lattice, have no influence.

7.1.1 Evolution Equation

Let us assume that the configurations $\sigma \in (\mathbb{S}^{n-1})^\Lambda$ evolve according to a stochastic process, with transition probability

$$w(\sigma'|\sigma) = \sum_{i\in\Lambda} \prod_{j\neq i} \delta(\sigma'_j - \sigma_j) w_i(\sigma'|\sigma), \tag{7.2}$$

which means that transitions are allowed only between configurations differing by one spin. This probability should satisfy the detailed balance condition

$$w(\sigma|\sigma')p_0(\sigma') = w(\sigma'|\sigma)p_0(\sigma), \tag{7.3}$$

where $p_0(\sigma) = \mathrm{e}^{-\beta H(\sigma)}/Z$ is the equilibrium distribution at temperature $T = 1/\beta$. Let σ' be a configuration such that $\sigma'_j = \sigma_j$ for $j \neq i$. Then

$$H(\sigma') - H(\sigma) = \langle \sigma_i - \sigma'_i | h_i(\sigma)\rangle, \qquad \text{where } h_i(\sigma) := h + \frac{1}{N} J \sum_{j\neq i} \sigma_j = h_i(\sigma') \tag{7.4}$$

is the **local field** at site i. The detailed balance condition (7.3) is thus satisfied by a transition probability of the form

$$w_i(\sigma'|\sigma) = \mathrm{e}^{\beta\langle \sigma'_i | h_i(\sigma)\rangle} g(h_i(\sigma)), \tag{7.5}$$

where the function $g(h)$ is arbitrary. The associated master equation reads

$$\begin{aligned} \frac{\partial}{\partial t} p(\sigma,t) &= \int \mathrm{d}\mu(\sigma') \bigl[w(\sigma|\sigma')p(\sigma',t) - w(\sigma'|\sigma)p(\sigma,t)\bigr] \\ &= \sum_i \int \mathrm{d}\mu_i(\sigma'_i) \bigl[w_i(\sigma|\sigma')p(\sigma',t) - w_i(\sigma'|\sigma)p(\sigma,t)\bigr], \end{aligned} \tag{7.6}$$

where $\sigma'_j = \sigma_j$ for $j \neq i$. We now consider a sequence of systems with N sites, $N \to \infty$, with probability measures $p_N(\sigma,t)$. We are interested in the magnetization density

$$m_N(t) = \Bigl\langle \sum_j \frac{\sigma_j}{N} \Bigr\rangle_N = \int \mathrm{d}\mu(\sigma) p_N(\sigma,t) \sum_j \frac{\sigma_j}{N}. \tag{7.7}$$

In order that the thermodynamic limit be well defined, we assume that for a given time t, the sequence $(p_N(\sigma,t))_N$ is **macroscopic at** $m(t)$, i.e., for any differentiable function $\varphi(m)$,

$$\lim_{N\to\infty} \Bigl\langle \varphi\Bigl(\sum_j \frac{\sigma_j}{N}\Bigr)\Bigr\rangle_N = \varphi(m(t)). \tag{7.8}$$

The evolution equation for $m_N(t)$ is obtained by inserting (7.6) into the derivative of (7.7). Just as in Subsection 3.3.3, we can interchange the integration variables σ_i and σ'_i in one of the terms, with the result

$$\frac{\mathrm{d}m_N}{\mathrm{d}t} = \Bigl\langle \frac{1}{N} \sum_i \int \mathrm{d}\mu_i(\sigma'_i)(\sigma_i - \sigma'_i)\, \mathrm{e}^{\beta\langle \sigma'_i | h_i(\sigma)\rangle} g(h_i)\Bigr\rangle_N. \tag{7.9}$$

In the case of one–dimensional spins, this equation reduces in the thermodynamic limit to

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 2\bigl(\operatorname{sh}\beta(Jm+h) - m\operatorname{ch}\beta(Jm+h)\bigr)g(Jm+h). \tag{7.10}$$

A similar calculation can be done for general functions $\varphi(m)$, showing that the sequence $p_N(\sigma,t)$ will remain macroscopic during its future evolution. The particular case of Glauber dynamics corresponds to $g(x) = 1/(2\operatorname{ch} x)$, and yields the well–known equation

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -m + \operatorname{th}\beta(Jm+h). \tag{7.11}$$

Let us now consider the case of two–dimensional spins. If φ_i' denotes the angle between the i^{th} spin and the local field h_i, we have

$$\begin{aligned} \int \mathrm{d}\mu_i(\sigma_i')\, \mathrm{e}^{\beta\langle\sigma_i'|h_i\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi_i'\, \mathrm{e}^{\beta\cos\varphi_i'\|h_i\|_2} = f_1(\beta\|h_i\|_2) \\ \int \mathrm{d}\mu_i(\sigma_i')\sigma_i'\, \mathrm{e}^{\beta\langle\sigma_i'|h_i\rangle} &= \beta h_i f_2(\beta\|h_i\|_2), \end{aligned} \tag{7.12}$$

where the functions f_1 and f_2 are given by

$$\begin{aligned} f_1(x) &= \sum_{k\geqslant 0}\frac{x^{2k}}{4^k(k!)^2} = 1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots \\ f_2(x) &= \frac{1}{x}f_1'(x) = \frac{1}{2} + \frac{1}{16}x^2 + \frac{1}{384}x^4 + \cdots \end{aligned} \tag{7.13}$$

Taking the thermodynamic limit, we thus obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \bigl[-mf_1(\|\beta\bar{h}\|_2) + \beta\bar{h}f_2(\|\beta\bar{h}\|_2)\bigr]g(\beta\bar{h}), \qquad \bar{h} := Jm+h. \tag{7.14}$$

Note that an equilibrium point is given by the equation

$$m^\star(h) = \beta(Jm^\star(h)+h)f_3(\|\beta(Jm^\star(h)+h)\|_2) \tag{7.15}$$

independently of g, where

$$f_3(x) = \frac{f_1(x)}{f_2(x)} = \frac{1}{2} - \frac{1}{16}x^2 + \cdots \tag{7.16}$$

Equation (7.15) is nothing but the self–consistency equation (3.40) of the mean field approximation. If $g \neq 0$, there are no other equilibria.

If we take, as in the 1D case, $g(x) = 1/f_1(\|x\|_2)$, the evolution equation (7.14) reduces to

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -m + \beta\bar{h}f_3(\|\beta\bar{h}\|_2) \tag{7.17}$$

The essential features of this equation are preserved by the following simplifications. We may take $\beta = 2$, and absorb the temperature–dependence into J and h. If we retain only terms of order up to 3, we get

$$\frac{\mathrm{d}m}{\mathrm{d}t} = (J - \mathbb{1})m + h - \frac{1}{2}(Jm+h)\|Jm+h\|_2^2, \tag{7.18}$$

which is a **Ginzburg–Landau** type of equation.

Using the same procedure, we can obtain similar evolution equations for higher dimensional spins. As in the 1D case (see Subsection 3.3.3), under appropriate hypotheses on the $p_N(\sigma,t)$, fluctuations around these limit values should be of order $1/\sqrt{N}$, and obey a Langevin equation.

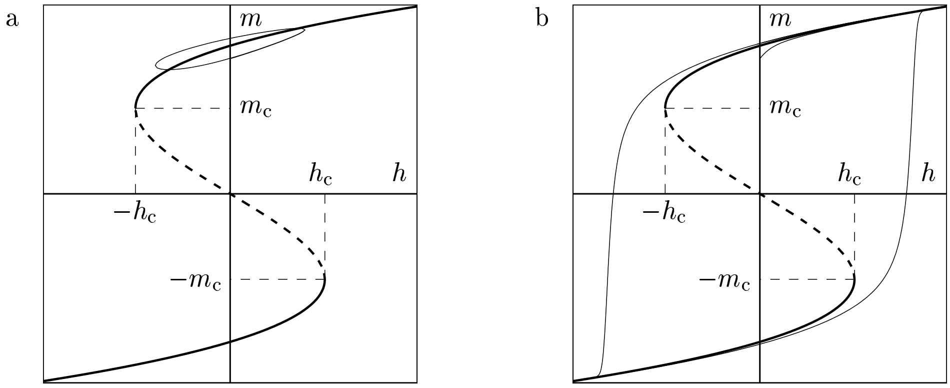

FIGURE 7.1. Solutions of (7.19) illustrating the phenomenon of "dynamic phase transition". When the amplitude h_0 of the magnetic field is smaller than the critical field h_c, the magnetization oscillates around a nonzero average, and encloses an area of order εh_0 (a). When h_0 is larger than h_c, the average magnetization is zero, and the periodic solution encloses an area $\mathcal{A}(\varepsilon) = \mathcal{A}(0) + \mathcal{O}(\varepsilon^{2/3})$ (b).

7.1.2 One-Dimensional Spins

We consider the Curie–Weiss model with a slowly varying magnetic field $h(\varepsilon t)$. In the thermodynamic limit, its magnetization obeys the equation

$$\varepsilon \dot{m} = -m + \operatorname{th} \beta(Jm + h(\tau)). \tag{7.19}$$

Strictly speaking, we should take into account the slow time dependence of h when deriving the evolution equation from (7.7). This will produce a term of order ε in (7.19), but its influence should be small, because the bifurcation diagram of (7.19) is in general stable against small perturbations, so we neglect this term.

If we work at finite temperature, we may absorb it into J and h and put $\beta = 1$. This amounts to saying that $J = T_c/T$. The bifurcation diagram of (7.19) has the following structure: When $J < 1$, it has a unique stable equilibrium branch $m^\star(h)$. When $J > 1$, three equilibrium branches exist for small magnetic field (Fig. 7.1). They meet at two bifurcation points with coordinates $(\pm h_c, \mp m_c)$, where

$$m_c = \sqrt{\frac{J-1}{J}}, \qquad h_c = \operatorname{Argth} \sqrt{\frac{J-1}{J}} - \sqrt{J(J-1)}. \tag{7.20}$$

The qualitative features of this diagram are shared by a Ginzburg–Landau type equation, obtained by retaining only the first terms in the Taylor expansion of (7.19):

$$\varepsilon \dot{m} = (J-1)m + h(\tau) - \frac{J^3}{3} m^3. \tag{7.21}$$

In this case, the bifurcation points have coordinates

$$m_c = \sqrt{\frac{J-1}{J^3}}, \qquad h_c = \frac{2}{3}\Big(\frac{J-1}{J}\Big)^{3/2}. \tag{7.22}$$

We consider here a periodic variation of the magnetic field, given by

$$h(\tau) = h_0 \sin(2\pi\tau) \tag{7.23}$$

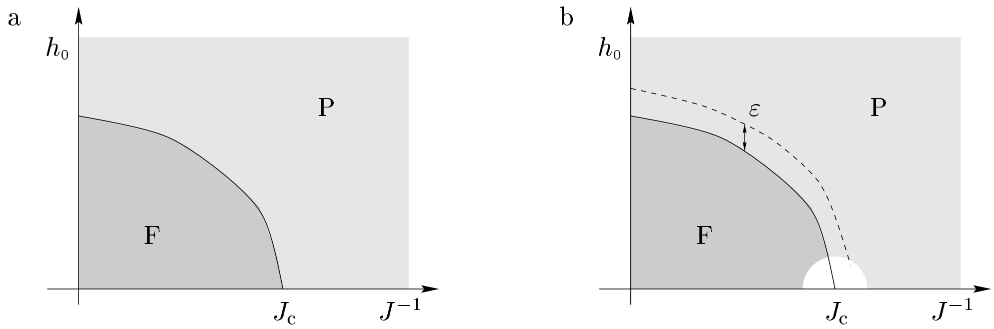

FIGURE 7.2. Phase diagram associated with equation (7.20). The F and P–regions correspond respectively to an average magnetization $M \neq 0$ and $M = 0$. In the adiabatic limit (a), these regions are separated by the line $h = h_\mathrm{c}(J)$. We show that for small positive ε, the F–region grows by a distance of order ε (b). In the analysis, we have excluded a small neighborhood of the point $J_\mathrm{c} = (1, 0)$.

We would like to discuss two related phenomena associated with these equations. The first one is a "dynamic phase transition", which has been studied for equation (7.19) in [TO]. The second one is the scaling of the hysteresis cycle area with ε and h_0, which was analysed for (7.21) in [HL&].

The phenomenon of "dynamic phase transition" can be understood as follows: Assume $J > 1$, the adiabatic parameter ε is sufficiently small, and the system is observed for different magnetic field amplitudes h_0.

- If $h_0 < h_\mathrm{c}$, the system always sees two stable and one unstable equilibrium branch, without encountering any bifurcation. Thus there are two symmetric stable periodic orbits, on which the magnetization remains respectively positive and negative (Fig. 7.1a).
- If $h_0 > h_\mathrm{c}$, the magnetic field will cross both bifurcation points, and the magnetization may change sign, just as in Example 4.16. It reaches an asymptotic hysteresis cycle, which is symmetric with respect to $(h, m) = (0, 0)$ (Fig. 7.1b).

We can use two diagnostics to distinguish these behaviours: the average magnetization over one cycle M, and the area $\mathcal{A}$ of a hysteresis cycle. They are defined by

$$M = \int_0^1 m(\tau)\,\mathrm{d}\tau, \qquad \mathcal{A} = \oint m\,\mathrm{d}h, \tag{7.24}$$

where $m(\tau)$ is a given stable periodic orbit. The average magnetization is used to define two regions in the (J, h_0) parameter plane. In [TO], they are called "ferromagnetic" or F–region if $M \neq 0$, and "paramagnetic" or P–region if $M = 0$. This terminology, as well as the name "dynamic phase transition", may of course be discussed. In particular, it is not clear whether it is justified to speak of a phase transition, which is usually associated with divergence of some thermodynamic variable. But let us stick to this vocabulary, and try to determine the precise shape of the transition line.

In the adiabatic limit $\varepsilon \to 0$, the F and P–region are separated by the line $h = h_\mathrm{c}(J)$.

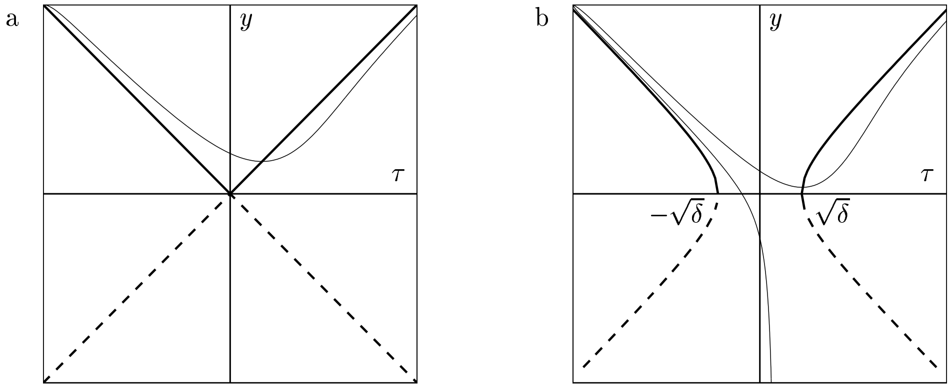

FIGURE 7.3. If the amplitude of the magnetic field is equal to $h_0 = h_c + \delta$, the motion near the turning point is governed by the Ricatti equation (7.28). If $\delta = 0$, it describes a transcritical bifurcation, and adiabatic solutions follow the upper branch (a). This means that we are still in the F–region. For positive δ, we show that this behaviour subsists as long as $\delta = \mathcal{O}(\varepsilon)$. In (b), trajectories are shown for two different values of ε. If $\varepsilon < \delta$, the solution escapes from below after a delay of order $\varepsilon^{2/3}\delta^{-1/6}$, and we have reached the P–region.

The average magnetization and hysteresis area are given by

$$\text{F–region:} \qquad M = \int_0^1 m_+^\star(h(\tau))\,\mathrm{d}\tau \neq 0, \qquad \mathcal{A} = 0, \tag{7.25a}$$

$$\text{P–region:} \qquad M = 0, \qquad \mathcal{A} = 2\int_{-h_c}^{h_c} m_+^\star(h)\,\mathrm{d}h, \tag{7.25b}$$

where $m_+^\star(h)$ denotes the upper stable equilibrium branch.

Our aim is to discuss how these properties change when ε becomes positive. In [HL&], the following scaling law has been derived for the area:

$$\mathcal{A}(\varepsilon, h_0) \approx \mathcal{A}(0, h_0) + \varepsilon^{2/3}(h_0^2 - h_c^2)^{1/3}. \tag{7.26}$$

Obviously, this relation can only be true in the P–region. We already know from Example 4.16 that the ε–dependence is correct in that region. However, we claim that the h_0–dependence in (7.26) is incorrect.

To see this, we have to understand what happens in the neighborhood of a bifurcation point when h_0 is close to h_c. By translating the variables to the point $(-h_c, m_c)$ and scaling them in the proper way, we obtain at leading order the equation

$$\varepsilon\dot{y} = -y^2 + h(\tau) + h_c, \tag{7.27}$$

If $h_0 = h_c + \delta$, we have for small δ that $h(\tau) + h_c = -\delta + \tau^2$. Thus we have to analyse the equation

$$\varepsilon\dot{y} = -y^2 - \delta + \tau^2. \tag{7.28}$$

The discussion in Chapter 4 shows that the neglected higher order terms have no influence on qualitative properties like scaling laws. This Ricatti equation is very interesting by itself, and can be analysed in various ways.[2] We adopt here the point of view of bifurcation theory, as developed in Chapter 4. There are three cases:

[2] One can carry out the transformation $y = \varepsilon\dot{\psi}/\psi$, which yields the Weber equation $\varepsilon^2\ddot{\psi} = (\tau^2 - \delta)\psi$.

1. $\delta = 0$: In this case, the amplitude h_0 is exactly equal to h_c, and the equation $\varepsilon\dot{y} = -y^2+\tau^2$ describes a transcritical bifurcation (Fig. 7.3a). The transformation $y = z-\tau$ yields

$$\dot{z} = 2\tau z - z^2 + \varepsilon, \tag{7.29}$$

so that we know, by the results of Section 4.3, that $y(0) \asymp \sqrt{\varepsilon}$. The trajectory follows the stable upper branch and encloses an area $\mathcal{A} = \mathcal{O}(\varepsilon|\ln\varepsilon|)$. Thus we are still in the F–region.
2. $\delta < 0$: There is an avoided crossing between a stable and an unstable branch, and solutions follow the stable one.
3. $\delta > 0$: There are four equilibrium branches undergoing pairwise saddle–node bifurcation at $\tau = \pm\sqrt{\delta}$ (Fig. 7.3b). If we fix an initial condition and a final time and increase δ monotonically, the comparison Lemma 4.1 shows that the final state will decrease monotonically. We know that when $\delta \to 0$, the trajectory follows the upper branch. On the other hand, when δ is of order 1, we know that the trajectory will leave the bifurcation region from below at time $-\sqrt{\delta} + \mathcal{O}(\varepsilon^{2/3})$. In this case, the trajectory of the initial system (7.19) will reach the lower branch, which means that we are in the P–region. By monotonicity, there must be a sharp transition between these two behaviours at an intermediate scale of δ.
To analyse this transition, we write $\delta = \varepsilon^{2\nu}$. There are two cases:
   - $\nu > 1/2$: We first consider an outer region $\tau \leqslant -2\varepsilon^\nu$. Then $\frac{3}{4}\tau^2 \leqslant \tau^2 - \delta \leqslant \tau^2$ and thus $\varepsilon\dot{y} \approx -y^2 + \tau^2$. As in the case $\delta = 0$, we have $y(-2\varepsilon^\nu) \asymp \sqrt{\varepsilon}$.
   In the inner region $|\tau| \leqslant 2\varepsilon^\nu$, we carry out the scalings $\tau = \varepsilon^\nu\sigma$ and $y = \sqrt{\varepsilon}z$, giving

   $$\dot{z} = -\tilde{\varepsilon}z^2 + \tilde{\varepsilon}^3(\sigma^2 - 1), \qquad \tilde{\varepsilon} = \varepsilon^{\nu-1/2}. \tag{7.30}$$

   Since $\tilde{\varepsilon}$ is small, z does not move very much during a time of order 1, an cannot change sign. We conclude that in this regime, the trajectory still follows the upper branch, and we are in the F–region.
   - $\nu < 1/2$: In the same outer region as before, we still have $\varepsilon\dot{y} \approx -y^2 + \tau^2$ so that $y \approx |\tau| + \varepsilon/|\tau|$, and thus $y(-2\varepsilon^\nu) \asymp \varepsilon^\nu$.
   In the inner region $|\tau| \leqslant 2\varepsilon^\nu$, we translate the origin of time to the bifurcation point $\tau = -\varepsilon^\nu$. After the scalings $\tau = \varepsilon^\nu(\sigma - 1)$ and $y = \varepsilon^\nu z$, we get

   $$\tilde{\varepsilon}\dot{z} = -z^2 - \sigma + \sigma^2, \qquad \tilde{\varepsilon} = \varepsilon^{1-2\nu}. \tag{7.31}$$

   This is exactly the equation studied in Example 4.7, which tells us that $z(0) \asymp \tilde{\varepsilon}^{1/3}$ and that the trajectory escapes the bifurcation region from below at a time $\sigma \asymp \tilde{\varepsilon}^{2/3}$.
   We conclude that we are in the P–region, and as in Example 4.7, the main contribution to the hysteresis cycle area comes from the bifurcation delay $\sigma \asymp \tilde{\varepsilon}^{2/3}$. Returning to original variables, we obtain

   $$\begin{aligned} y(0) &\asymp \varepsilon^\nu\tilde{\varepsilon}^{1/3} \asymp \varepsilon^{1/3+\nu/3} \asymp \varepsilon^{1/3}\delta^{1/6}, \\ \mathcal{A}(\varepsilon, h_0) &\asymp \varepsilon^\nu\tilde{\varepsilon}^{2/3} \asymp \varepsilon^{2/3-\nu/3} \asymp \varepsilon^{2/3}\delta^{-1/6}. \end{aligned} \tag{7.32}$$

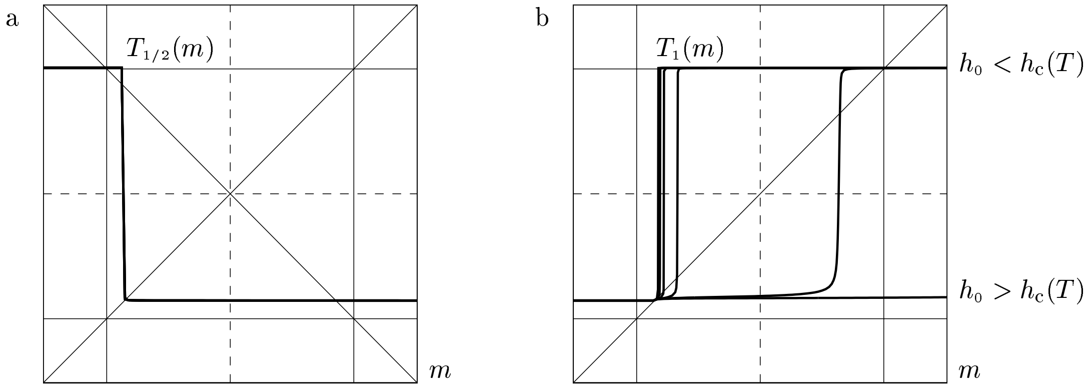

FIGURE 7.4. Plots of the Poincaré maps $T_{1/2}(m)$ and $T_1(m)$ for $\tau_0 = \frac{1}{4}$ and $\varepsilon = 0.05$. The field amplitude is kept fixed, while the temperature is varied through the transition line. One can observe the creation of asymmetric orbits in a saddle–node bifurcation.

We have thus obtained the following scaling relations for the hysteresis loop area:

$$\text{F–region:} \qquad \mathcal{A}(\varepsilon, h_0) \approx h_0\varepsilon \quad \text{if } h_0 < h_c, \quad \varepsilon|\ln\varepsilon| \quad \text{if } h_0 = h_c, \tag{7.33a}$$

$$\text{P–region:} \qquad \mathcal{A}(\varepsilon, h_0) \approx \mathcal{A}(0, h_0) + \varepsilon^{2/3}(h_0 - h_c)^{-1/6}. \tag{7.33b}$$

The boundary between these regions is located, for $J > 1$, at $h_0 \approx h_c + \varepsilon$. The term $\mathcal{A}(0, h_0)$ is given by (7.25b). Note that the correction to this term goes to $\varepsilon^{2/3}$ when $h_0 - h_c$ becomes of order 1, and to $\varepsilon^{1/2}$ when approaching the P–F transition line. The h_0–dependence is very different from the formula (7.26) of [HL&]. The reason is that the authors overlooked the fact that as h_0 approaches h_c, the velocity with which the bifurcation point is crossed goes to zero (they describe the transition by the equation $\varepsilon\dot{y} = -y^2 + \tau$).

In (7.33), we have excluded a neighborhood of the critical point $(J, h_0) = (1, 0)$. It is possible to do a similar analysis in this situation.

Let us finally comment on the transition between the P and F–region. Our analysis shows that in the adiabatic limit, this transition is sharp, since the average magnetization M jumps discontinuously from a nonzero to a zero value. A different behaviour is reported in [TO]: For ε of order 0.1 and sufficiently large temperature, the transition between both zones becomes continuous. Indeed, M goes to zero when the boundary is approached. In fact, when moving along the P–F transition line from low to high temperature, the jump of M changes continuously from positive to zero, and the authors postulate the existence of a "tricritical point" on which $M = 0$ for the first time. Moreover, when the transition is discontinuous, there is a small domain in the (T, h_0)–plane where symmetric and asymmetric cycles coexist.

To understand this behaviour, it is useful to know some properties of the Poincaré map. Symmetries play an important role. Both evolution equations (7.20) and (7.22) are of the form $\varepsilon\dot{m} = f(m, h)$, where $f(m, h) = -f(-m, -h)$. Since $h(\tau + \frac{1}{2}) = -h(\tau)$, this implies that periodic solutions belong to one of the following two categories:

1. Symmetric solutions of the form $m(\tau) = -m(\tau + \frac{1}{2})$, for which the average magnetization M is zero;
2. Asymmetric solutions $m_1(\tau)$ with nonzero average magnetization; they necessarily admit a conjugate solution $m_2(\tau) = -m_1(\tau + \frac{1}{2})$.

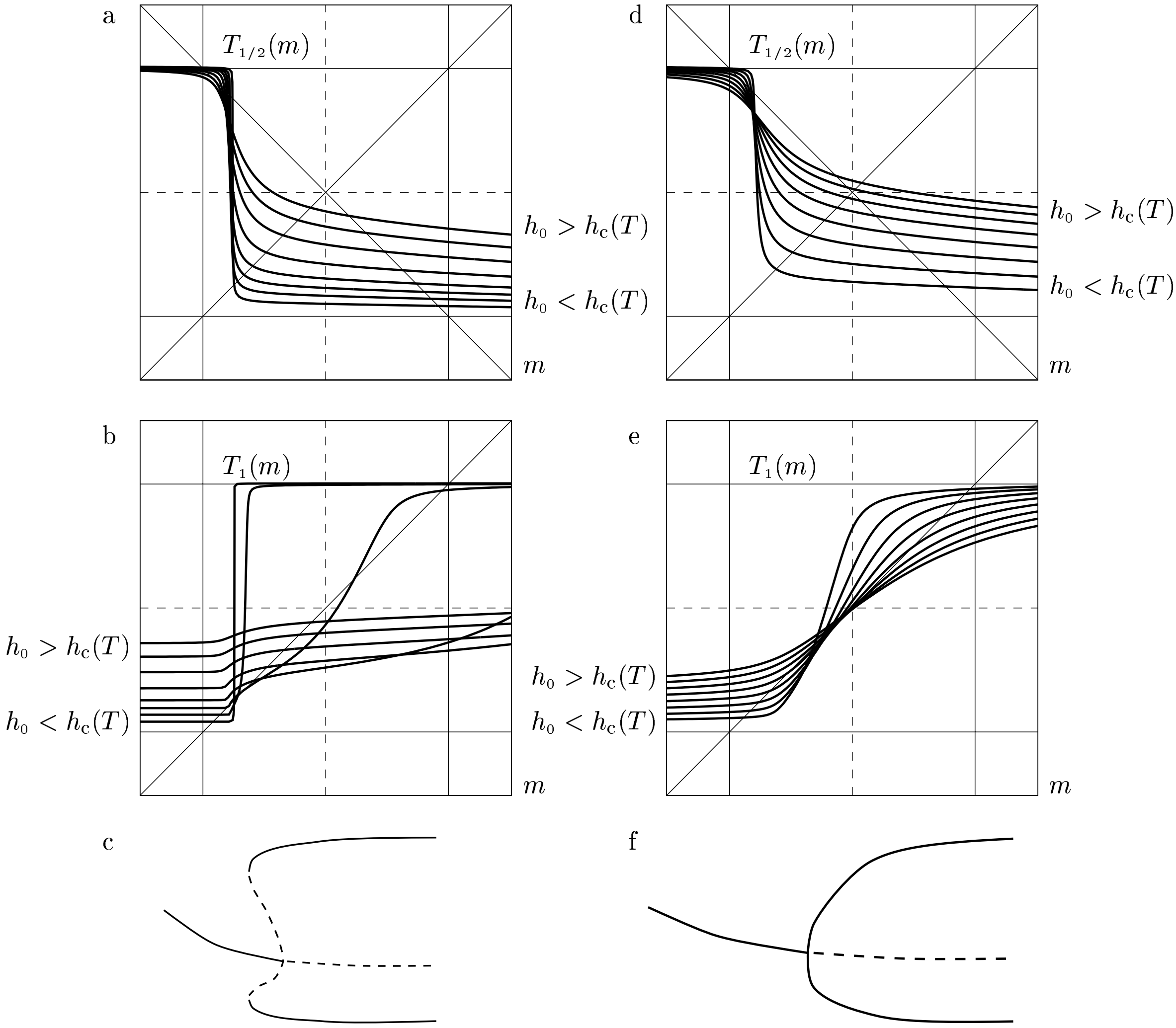

FIGURE 7.5. Same as Fig. 7.4, but for $\varepsilon = 0.1$ (a,b) and $\varepsilon = 0.2$ (d,e). For $\varepsilon = 0.1$, the bifurcation diagram of T_1 contains two saddle–node bifurcations and an indirect pitchfork bifurcation (c), so that the average magnetization M varies discontinously when the transition line is crossed. For $\varepsilon = 0.2$, there is only a direct pitchfork bifurcation, and M varies continuously (f).

These solutions are best characterized by the Poincaré map T_1, and the "semi–Poincaré map" $T_{1/2}$, defined by

$$m(\tau_0 + 1) = T_1(m(\tau_0)), \qquad m(\tau_0 + \tfrac{1}{2}) = -T_{1/2}(m(\tau_0)), \tag{7.34}$$

where τ_0 is an arbitrary initial time. All periodic orbits are fixed points of T_1. Symmetric ones are also fixed points of $T_{1/2}$, while conjugate asymmetric orbits are such that $T_{1/2}(m_1) = m_2$ and $T_{1/2}(m_2) = m_1$.

These symmetry properties have to be conserved during bifurcations. A symmetric periodic orbit cannot bifurcate with a single asymmetric one, but has to interact with the conjugate orbit as well. Fig. 7.4 shows the Poincaré maps for a relatively small value of ε. The map $T_{1/2}$ always admits a fixed point, corresponding to a symmetric orbit. The plot of T_1 shows the appearance of a pair of stable and unstable asymmetric orbits in a saddle–node bifurcation. These orbits necessarily admit two conjugate ones.

The succession of bifurcations is better understood for slightly larger ε (Fig. 7.5). The map $T_{1/2}$ still admits a single fixed point, which clearly undergoes period doubling bifurcation. It corresponds to a pitchfork bifurcation of T_1. For $\varepsilon = 0.1$, one observes again the creation of two asymmetric orbits in a saddle–node bifurcation. Their conjugate partners are more difficult to see, but they do exist. The unstable orbits participate in the pitchfork bifurcation, which is indirect. Thus we observe indeed coexistence of a symmetric and an asymmetric stable orbit for a small interval of parameter values. Independently of which stable orbit we choose to compute the average magnetization M, this quantity is discontinuous when the bifurcation is crossed.

For larger values of ε, however, the saddle–node bifurcations may disappear, and the pitchfork bifurcation becomes direct. Thus, M varies continuously. It appears that this behaviour is only possible for sufficiently large ε and temperature. Hence we believe that the "tricritical point", if it exists, will move towards $(J_c, 0)$ in the adiabatic limit.

7.1.3 Two-Dimensional Spins

We have shown in Subsection 7.1.1 that a lattice model with two–dimensional spins and infinite range interaction could be described by the approximate evolution equation

$$\varepsilon\dot{m} = (J - \mathbb{1})m + h - \frac{1}{2}(Jm + h)\|Jm + h\|_2^2, \tag{7.35}$$

where J is a real symmetric coupling matrix. There are two characteristic directions in this problem: the orthogonal eigenvectors of J, and the magnetic field h. We consider here the case of a periodic magnetic field pointing in a fixed direction, $h(\tau) = (h_{\shortparallel}, 0) = (h_0 \sin(2\pi\tau), 0)$. This defines a basis, in which we write the magnetization as $m(\tau) = (m_{\shortparallel}(\tau), m_{\perp}(\tau))$. For simplicity, we will mainly examine the cases when the eigenvectors of J coincide with these basis vectors, i.e., J is diagonal. The dynamics depend in an essential way on the eigenvalues of J.

Isotropic case

Let us start with an isotropic coupling matrix $J = \begin{pmatrix} J_{\shortparallel} & 0 \\ 0 & J_{\shortparallel} \end{pmatrix}$.

In this case, we can further simplify (7.35) by neglecting terms of order $\|m\|_2^2 h$ and $m\|h\|_2^2$, so that we obtain the evolution equation

$$\varepsilon\dot{m} = am - bm\|m\|_2^2 + h, \tag{7.36}$$

where $a = J_{\shortparallel} - 1$ and $b = \frac{1}{2}J_{\shortparallel}^3$. This equation also describes the overdamped motion of a particle in a sombrero–shaped Ginzburg–Landau potential $\Phi(m) = -\frac{a}{2}\|m\|_2^2 + \frac{b}{4}\|m\|_2^4$, with a superimposed linear potential $\langle m|h\rangle$: just imagine that the sombrero is periodically slanted to one side and the other.

If $h = 0$, equation (7.36) admits the origin and the whole circle $\|m\|_2 = \sqrt{b/a}$ as equilibrium points. If $h \neq 0$, the stationary points satisfy $m_{\perp} = 0$ and $am_{\shortparallel} - bm_{\shortparallel}^3 + h_{\shortparallel} = 0$. Just as in the 1D case, we have one equilibrium when $\|h\|_2 > h_c$, and three equilibria when $\|h\|_2 < h_c$ and $a > 0$, where $h_c = \sqrt{4a^3/27b}$ (Fig. 7.6).

We first note that the axis $m_{\perp} = 0$ is invariant by the flow of (7.36), and the motion on this axis is governed by the same equation as in the 1D case.

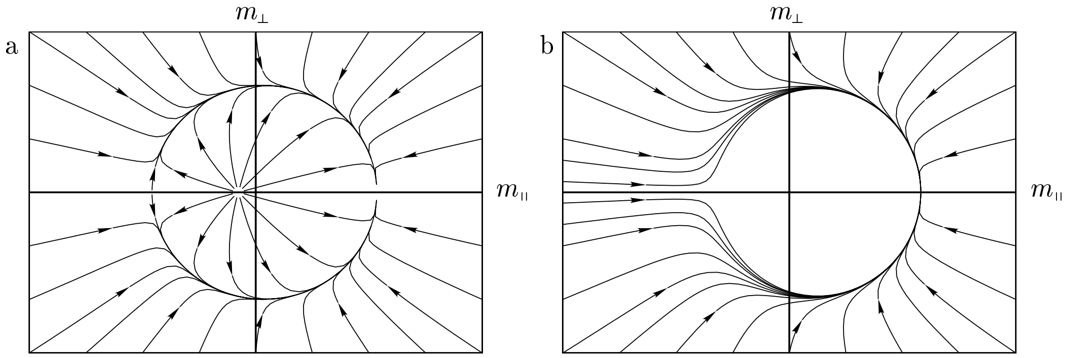

FIGURE 7.6. Phase portraits of the isotropic equation (7.36), (a) for $0 < h_{\shortparallel} < h_c$ and (b) for $h_{\shortparallel} > h_c$.

Assume now that the magnetization starts close to the right equilibrium for positive magnetic field, and with a finite transverse component. It will approach this equilibrium and follow it adiabatically. When the magnetic field turns negative, this equilibrium becomes unstable. It admits an unstable adiabatic manifold, which is locally close to the instantaneous unstable manifold. The motion will thus be quasi–one–dimensional for some time, with a bounded distance to the origin.

To be more precise, let us introduce polar coordinates for m, to transform equation (7.36) into

$$\begin{aligned} \varepsilon\dot{r} &= ar - br^3 + h\cos\varphi \\ \varepsilon\dot{\varphi} &= -\frac{1}{r}h(\tau)\sin\varphi. \end{aligned} \tag{7.37}$$

If the amplitude of $h(\tau)$ is smaller than the critical field h_c, the first equation shows that r is bounded away from 0 and infinity. The equation for φ has fixed points at $0, \pi$. Even when the origin has become unstable, φ will remain for some time in its vicinity because of the phenomenon of **bifurcation delay**. The delay time is given by the formula

$$\int_{\tau_0}^{\Psi(\tau_0)} \frac{h(\tau)}{r_+(\tau)}\,\mathrm{d}\tau = 0, \tag{7.38}$$

where $r_+(\tau)$ defines the location of the right equilibrium branch. For $\tau > \Psi(\tau_0)$, φ will quickly leave the origin, following the unstable adiabatic manifold. In other words, the magnetization will *rotate* instead of flip, in order to align with the magnetic field.

Once φ has reached π, the same scenario will repeat itself. Observe, however, that the next bifurcation delay is determined by the previous delay time $\Psi(\tau_0)$: the longer $\varphi(\tau)$ has stayed close to 0, the shorter it will stay near π. In fact, the sequence of successive bifurcation delays is determined by the map $\tau \mapsto \Psi(\tau) - \frac{1}{2}$. Since this map has a slope close to -1, there is in principle no reason for this sequence to be periodic, although in our case, the delay appears to stabilize at a constant value (Fig. 7.7). The resulting hysteresis cycle has an area $\mathcal{A}(\varepsilon) = \mathcal{A}(0) + \mathcal{O}(\varepsilon)$.

We have thus found two important effects, which are characteristic for two–dimensional spins. The first one is a competition between two mechanisms of magnetization reversal: spin rotation and spin flip. For small magnetic field amplitude, spin rotation is the most common mechanism, while spin flip may occur if the amplitude h_0 is larger than h_c. In the case of spin rotation, a second important effect is bifurcation delay. The magnetization

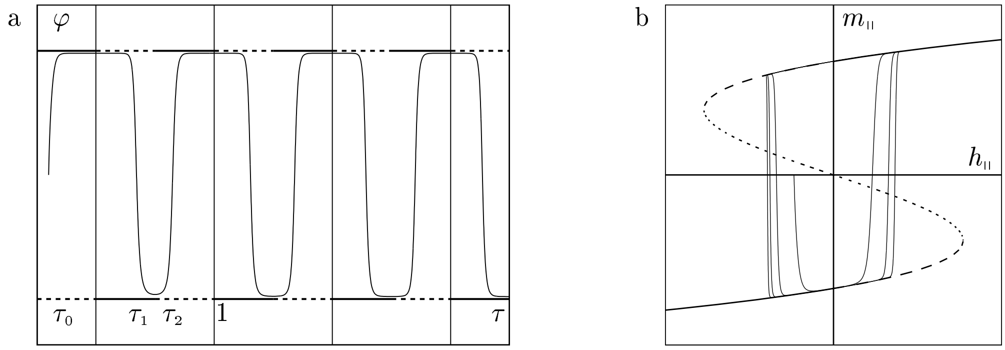

FIGURE 7.7. (a) Evolution of $\varphi(\tau)$, solution of equation (7.37). We consider here the case when φ reaches π at time τ_0. It will leave this branch at the delay time $\tau_1 = \Psi(\tau_0)$, and reach the stable origin in a time of order $\varepsilon|\ln\varepsilon|$. The origin is followed until τ_2, which is given, by symmetry, by the formula $\tau_2 = \frac{1}{2} + \Psi(\tau_1 - \frac{1}{2})$. The magnetization thus quickly rotates at times τ_n, determined recursively by the relation $\tau_{n+1} = \frac{n}{2} + \Psi(\tau_n - \frac{n}{2})$. (b) The plot of $m_{\shortparallel}$ as a function of $h_{\shortparallel}$ shows the asymptotic hysteresis cycle, which is determined solely by the delay times. Thick full lines, dashed and dotted lines represent respectively sinks, saddles and sources of the static system. Due to bifurcation delay, the magnetization follows the hyperbolic branch for some time, but ultimately rotates around the unstable origin.

does not reverse immediately when the magnetic field changes sign, but remains for some time in metastable equilibrium. The delay times are self–determined by a one–dimensional map.

Since bifurcation delay is a relatively fragile effect due to symmetries, it is important to understand the effect of anisotropy.

Weakly anisotropic case

We consider now an anisotropic coupling matrix $J = \begin{pmatrix} J_{\shortparallel} & 0 \\ 0 & J_{\perp} \end{pmatrix}$, where $J_{\shortparallel}$ and $J_{\perp}$ are both larger than 1.

One can carry out the same simplifications as in the isotropic case. If we scale, moreover, $m_{\shortparallel}$ and $m_{\perp}$ by a factor $J_{\shortparallel}$ and $J_{\perp}$, we get the evolution equation

$$\begin{aligned} \varepsilon\dot{m}_{\shortparallel} &= (J_{\shortparallel} - 1)m_{\shortparallel} - \tfrac{1}{2}J_{\shortparallel}m_{\shortparallel}(m_{\shortparallel}^2 + m_{\perp}^2) + h_{\shortparallel} \\ \varepsilon\dot{m}_{\perp} &= (J_{\perp} - 1)m_{\perp} - \tfrac{1}{2}J_{\perp}m_{\perp}(m_{\shortparallel}^2 + m_{\perp}^2). \end{aligned} \tag{7.39}$$

This system admits two types of equilibrium branches:

- A longitudinal branch, determined by $m_{\perp} = 0$ and $(J_{\shortparallel} - 1)m_{\shortparallel} - \frac{1}{2}J_{\shortparallel}m_{\shortparallel}^3 + h_{\shortparallel} = 0$;
- A transversal branch, such that $m_{\shortparallel}^2 + m_{\perp}^2 = 2(1 - 1/J_{\perp})$ and $(1 - J_{\shortparallel}/J_{\perp})m_{\shortparallel} = h_{\shortparallel}$.

In polar coordinates, (7.39) becomes

$$\begin{aligned} \varepsilon\dot{r} &= (J_{\shortparallel}\cos^2\varphi + J_{\perp}\sin^2\varphi - 1)r - \tfrac{1}{2}(J_{\shortparallel}\cos^2\varphi + J_{\perp}\sin^2\varphi)r^3 + h_{\shortparallel}\cos\varphi \\ \varepsilon\dot{\varphi} &= (1 - \tfrac{1}{2}r^2)(J_{\perp} - J_{\shortparallel})\sin\varphi\cos\varphi - \frac{h_{\shortparallel}(\tau)}{r}\sin\varphi. \end{aligned} \tag{7.40}$$

There are two cases, depending on the relative value of $J_{\shortparallel}$ and $J_{\perp}$.

If $J_{\shortparallel} > J_{\perp}$, the longitudinal equilibria are stable for small magnetic field, while the transverse equilibria are saddles. Thus, the magnetization tends to align with the magnetic

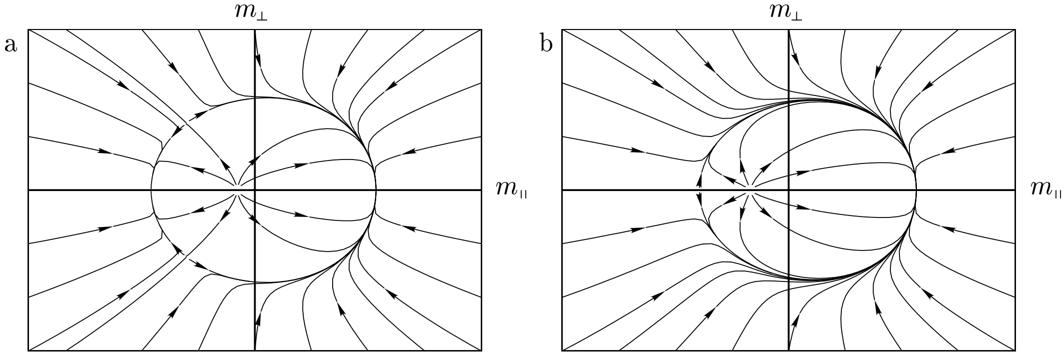

FIGURE 7.8. Phase portraits of the anisotropic equation (7.39) with $J_\shortparallel > J_\perp$. (a) For $0 < h_\shortparallel < h_{\rm b}$, two equilibria are sinks, two are saddles, and one is a source. (b) For $h_{\rm b} < h_\shortparallel < h_{\rm c}$, the transverse equilibria have disappeared, and there remain only a saddle and a sink. In the case $h_\shortparallel > h_{\rm c}$, the phase portrait is similar to Fig. 7.6b.

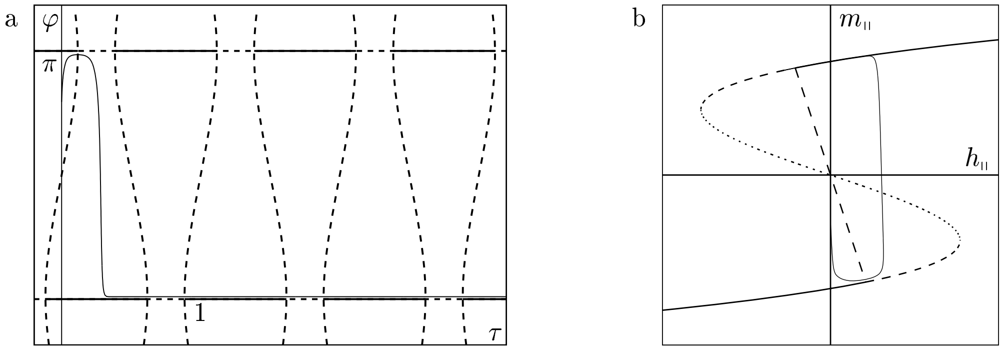

FIGURE 7.9. (a) Evolution of φ in the case $J_\shortparallel > J_\perp$, as determined by equation (7.40). Due to bifurcation delay, the magnetization tends to stabilize in one direction. (b) The corresponding plot in the $(h_\shortparallel, m_\shortparallel)$–plane shows that after a first transient cycle, there is no hysteresis. In this picture, the curved thick lines represent longitudinal equilibria (those with $m_\perp = 0$), and the dashed straight line corresponds to the transverse equilibrium (with $m_\perp \neq 0$).

field. At the same time, it is more difficult to reverse the magnetization, since the saddles act as a barrier. In fact, there exists a threshold field

$$h_{\rm b} = J_\perp^{-3/2}(J_\shortparallel - J_\perp)\sqrt{2(J_\perp - 1)} \tag{7.41}$$

with the following property: if $\|h\|_2 < h_{\rm b}$, two longitudinal equilibria are stable, and there exists a transverse saddle (Fig. 7.8a). If $\|h\|_2 > h_{\rm b}$, the saddle has collapsed with one of the longitudinal equilibria (Fig. 7.8b).

The amplitude h_0 of the magnetic field must be larger than $h_{\rm b}$ to allow for magnetization reversal. The dynamics of φ is then governed by the bifurcation diagram of Fig. 7.9a. Here the bifurcation delay phenomenon comes into play again: due to the fact that the points $\varphi = 0, \pi$ are more often stable than unstable, the magnetization tends to choose one of the longitudinal equilibria, and stays there. Thus there is no hysteresis.

Only when the amplitude h_0 becomes larger than the critical field

$$h_{\rm c} = \tfrac{2}{3}(J_\shortparallel - 1)^{3/2} J_\shortparallel^{-1}, \tag{7.42}$$

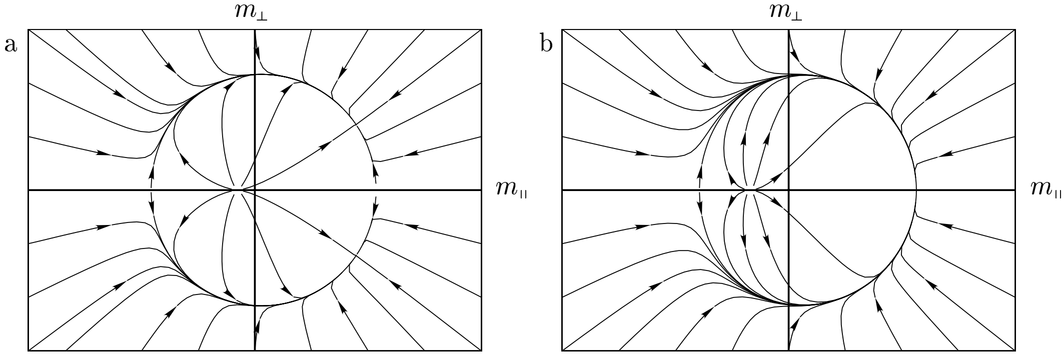

FIGURE 7.10. Phase portraits of the anisotropic equation (7.39) with $J_\shortparallel < J_\perp$, (a) for $0 < h_\shortparallel < h_{\rm b}$ and (b) for $h_{\rm b} < h_\shortparallel < h_{\rm c}$. In the case $h_\shortparallel > h_{\rm c}$, the phase portrait is similar to Fig. 7.6b.

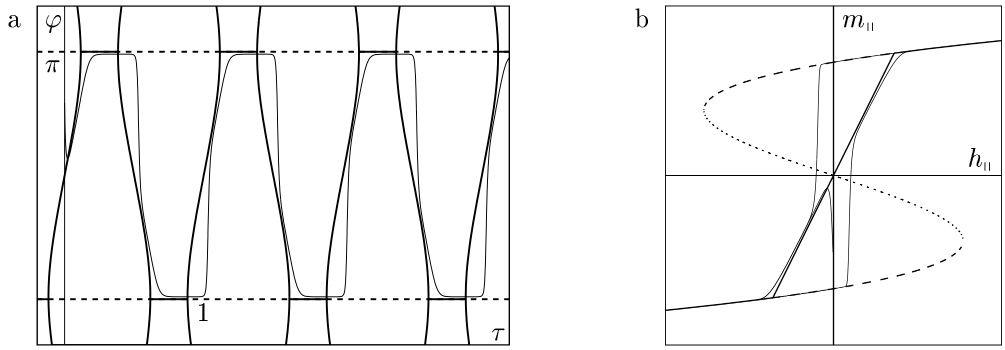

FIGURE 7.11. (a) Evolution of φ in the case $J_\shortparallel < J_\perp$. Due to bifurcation delay, φ spends some time near 0 or π, even when these points are unstable. It always drops back, however, to the transverse stable position. (b) The resulting hysteresis loop looks triangular. As in Fig. 7.9, curved lines are longitudinal equilibria, and the straight line represents transverse branches.

will the magnetization be forced to reverse again, because two of the longitudinal equilibria disappear completely. In this case, it is possible that the spins reverse by flipping, rather than rotating.

The situation is completely different when $J_\shortparallel < J_\perp$. In this case, the longitudinal equilibria are saddles for small magnetic field, while the transverse ones are stable (Fig. 7.10). Thus, the magnetization has a tendency to spend more time in a transverse position. When $\|h\|_2$ becomes larger than $h_{\rm b}$, m will align with the magnetic field. The bifurcation delay will maintain the magnetization in this position even when h becomes small again, but is not sufficient to keep it in this metastable position forever. Thus, the magnetization finally acquires a transverse component again (Fig. 7.11a). The resulting hysteresis cycle is composed of two triangular loops (Fig. 7.11b).

To complete this discussion, we should look at a case with still lower symmetry, namely when the eigenvectors of J do not coincide with the magnetic field. If the magnetization makes an angle α with one of these eigenvectors, the angle φ between m and h satisfies the equation

$$\varepsilon\dot\varphi = \bigl[1 - \tfrac{1}{2}r^2\bigr](J_2 - J_1)\sin(\varphi+\alpha)\cos(\varphi+\alpha) - \frac{h_\shortparallel(\tau)}{r}\sin\varphi, \tag{7.43}$$

where J_1, J_2 are eigenvalues of J.

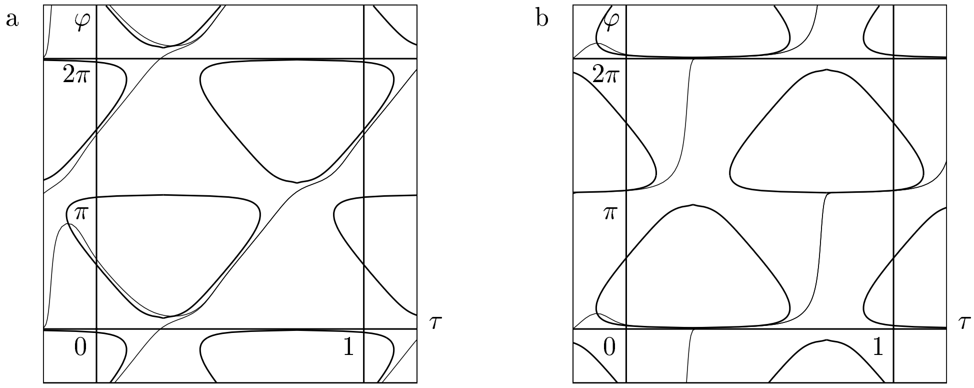

FIGURE 7.12. Evolution of φ in the case where the field makes an angle α with the eigenvectors J, (a) for $\alpha = 0.05\frac{\pi}{2}$ and (b) for $\alpha = 0.95\frac{\pi}{2}$. The pitchfork bifurcations have disappeared in favour of saddle–nodes. As a result, the magnetization performs full circles.

When α is different from 0 and $\frac{\pi}{2}$, the bifurcation diagrams of Fig. 7.9a or Fig. 7.11a will be deformed in such a way that the pitchfork bifurcations are replaced by saddle–nodes (Fig. 7.12). Thus, there is no more bifurcation delay. The effect on the dynamics is that instead of oscillating back and forth between $\varphi = 0$ and π, the magnetization rotates in a given direction, performing a full circle for each cycle of h.

Strongly anisotropic case

We briefly discuss here the extreme situation when one of the couplings $J_{\shortparallel}$ or $J_{\perp}$ vanishes. If $J_{\perp} = 0$, one obtains that the transverse magnetization relaxes to zero exponentially fast, while the longitudinal magnetization obeys the equation of the one–dimensional case.

Let us examine the case $J_{\shortparallel} = 0$. Then the general equation of motion (7.35) reduces to

$$\begin{aligned} \varepsilon \dot{m}_{\shortparallel} &= -m_{\shortparallel} - b(m_{\perp}^2 + h_{\shortparallel}^2)h_{\shortparallel} + h_{\shortparallel} \\ \varepsilon \dot{m}_{\perp} &= am_{\perp} - b(m_{\perp}^2 + h_{\shortparallel}^2)m_{\perp}, \end{aligned} \tag{7.44}$$

where $a = J_{\perp} - 1$, $b = \frac{1}{2}J_{\perp}^3$, and we have scaled the magnetic field by a factor $J_{\perp}$.

This system is equivalent to the "Ising model in transverse field" considered in [ACS]. It admits two branches of fixed points:

- a longitudinal branch $m_{\perp} = 0$, $m_{\shortparallel} = h_{\shortparallel} - bh_{\shortparallel}^3$;
- a transverse branch $m_{\shortparallel} = h_{\shortparallel}(1 - a)$, $m_{\perp}^2 + h_{\shortparallel}^2 = \frac{a}{b}$.

For small magnetic field, the longitudinal branch is unstable, and the magnetization tends to acquire a transverse component (Fig. 7.13). When $\|h\|_2 > h_{\mathrm{c}} = \sqrt{a/b}$, only the longitudinal branch exists, and it is stable.

In fact, the equation for $m_{\perp}$ is closed, and involves pitchfork bifurcations. If the magnetic field oscillates with an amplitude $h_0 < h_{\mathrm{c}}$, the transverse magnetization remains different from zero (Fig. 7.14b). If $h_0 > h_{\mathrm{c}}$, it returns to zero, and tends to remain there because of bifurcation delay. If the amplitude is sufficiently large, this delay may become infinite, and $m_{\perp}$ remains equal to zero, even at small magnetic field. We have thus a kind of "dynamic phase transition" between a regime with zero and nonzero transverse magnetization.

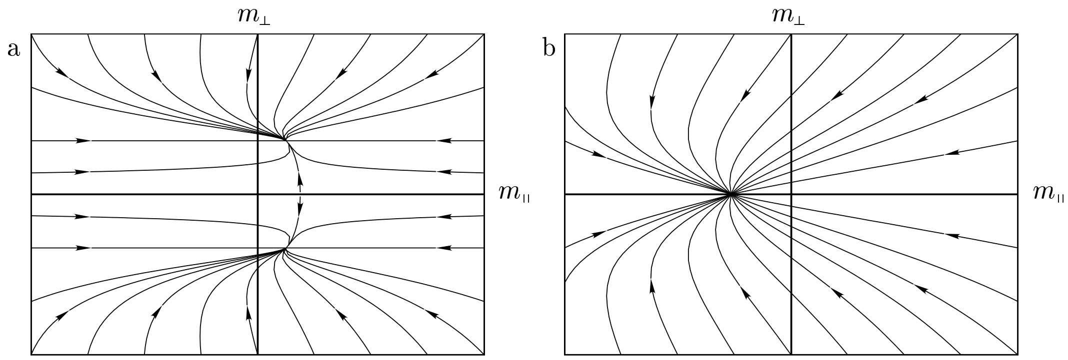

FIGURE 7.13. Phase portraits of the strongly anisotropic equation (7.44), (a) for $0 < h_\shortparallel < h_c$ and (b) for $h_\shortparallel > h_c$.

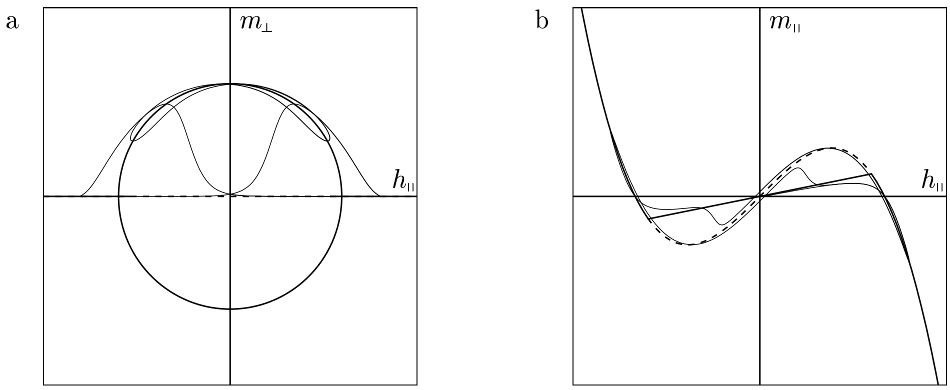

FIGURE 7.14. Hysteresis cycles in the case $J_\shortparallel = 0$. (a) If the amplitude h_0 is smaller than h_c, the transverse magnetization $m_\perp$ oscillates around a nonzero value. If h_0 is larger than h_c, it tends to follow the origin, and only jumps on a nonzero value after some delay if h_0 is not too large. (b) The corresponding plots of the longitudinal magnetization.

The linearized equation around 0 is

$$\varepsilon \dot{m}_\perp = (a - b h_\shortparallel(\tau)^2) m_\perp. \tag{7.45}$$

This implies that the condition for a zero transverse magnetization is

$$\int_0^1 h_\shortparallel(\tau)^2 \, \mathrm{d}\tau \geqslant \frac{a}{b}. \tag{7.46}$$

For a sinusoidal field, it reduces to $h_0^2 \geqslant 2\frac{a}{b}$.

We may thus introduce a similar characterization as for the 1D Curie–Weiss model. The average of $m_\perp$ over one cycle is zero by symmetry. But we may define the observables

$$M_\perp := \int_0^1 m_\perp(\tau) \, \mathrm{d}\tau, \qquad \mathcal{A}_\perp := \oint |m_\perp \, \mathrm{d}\, h_\shortparallel|. \tag{7.47}$$

Then we have three regions in parameter space:

$$h_0 < h_c(J_\perp): \qquad M_\perp \neq 0, \qquad \mathcal{A}_\perp(\varepsilon) \approx h_0 \varepsilon, \tag{7.48a}$$

$$h_c(J_\perp) < h_0 < \sqrt{2}\, h_c(J_\perp): \qquad M_\perp \neq 0, \qquad \mathcal{A}_\perp(\varepsilon) \approx \mathcal{A}_\perp(0) + \varepsilon^{3/4}, \tag{7.48b}$$

$$h_0 > \sqrt{2}\, h_c(J_\perp): \qquad M_\perp = 0, \qquad \mathcal{A}_\perp(\varepsilon) = 0. \tag{7.48c}$$

7.2 Ising Model

We consider now an Ising Hamiltonian

$$H(\sigma) = -\frac{1}{2}\sum_{<ij>} J\sigma_i\sigma_j - \sum_i h\sigma_i, \tag{7.49}$$

where i, j belong to a finite lattice $\Lambda \in \mathbb{Z}^d$ with N sites, $d = 1$ or 2, $\sigma \in \{-1,1\}^\Lambda$ is any configuration, and $< ij >$ denotes nearest neighbours.

This system is much more difficult to study, since it is not possible to obtain a low–dimensional evolution equation as for the Curie–Weiss model. In Subsection 7.2.1, we explain why, and show that the dynamics are governed by an infinite hierarchy of differential equations. It is out of question to study these complicated equations here. We merely introduce some very gross approximations, in order to try to understand where to look for differences between Ising and Curie–Weiss models. A first approximation, presented in Subsection 7.2.2, consists in neglecting all correlations. Despite its simplicity, it preserves an essential feature of the model, namely the absence of phase transition for a 1D lattice, and its presence for a 2D lattice. In Subsection 7.2.3, we present a slightly more accurate approximation, which takes into account some correlations.

7.2.1 Evolution Equation

In the case of an Ising model, the Glauber dynamics give transition probabilities

$$w(\sigma^i|\sigma) = \frac{1}{2}\bigl[1 - \sigma_i \operatorname{th}\beta h_i(\sigma)\bigr], \qquad h_i(\sigma) := h + \sum_{\|j-i\|_1=1} J\sigma_j, \tag{7.50}$$

where σ^i is obtained by flipping the i^{th} spin of σ. The associated master equation reads

$$\frac{\partial}{\partial t}p(\sigma,t) = \frac{1}{2}\sum_i \bigl[1 + \sigma_i \operatorname{th}\beta h_i(\sigma)\bigr]p(\sigma^i,t) - \bigl[1 - \sigma_i \operatorname{th}\beta h_i(\sigma)\bigr]p(\sigma,t). \tag{7.51}$$

Let A be a subset of Λ. We are interested in observables of the form

$$\sigma_A := \prod_{i\in A}\sigma_i, \qquad \sigma_\emptyset := 1, \qquad \langle\sigma_A\rangle(t) := \sum_\sigma p(\sigma,t)\sigma_A. \tag{7.52}$$

We will assume here that Λ has the topology of a torus (i.e., a square set with periodic boundary conditions), and that the initial distribution $p(\sigma,0)$ has all the translation and reflection symmetries of the lattice. Then these symmetries are preserved by the evolution, and also appear in the averaged observables (for instance, $\langle\sigma_A\rangle = \langle\sigma_B\rangle$ whenever B is a translate of A).

Using the fact that $\sigma_A^i - \sigma_A = -2\sigma_A$ if $i \in A$ and 0 otherwise, we obtain from (7.51) and (7.52) the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_A\rangle = -|A|\langle\sigma_A\rangle + \sum_{i\in A}\langle\sigma_{A\setminus\{i\}} \operatorname{th}\beta h_i(\sigma)\rangle, \tag{7.53}$$

where $|A|$ denotes the cardinality of A. The main difference with the Curie–Weiss model is that the average of the hyperbolic tangent has no reason to be equal to the hyperbolic tangent of the average. We can use instead the following decomposition rule:

$$f(\sigma) = \sum_{A\subset\Lambda} c_A \sigma_A, \qquad c_A = \frac{1}{2^N}\sum_{\sigma} \sigma_A f(\sigma). \tag{7.54}$$

Consider for instance the one–dimensional case. The local field at site 0 is given by $h_0(\sigma) = h + J(\sigma_{-1} + \sigma_1)$. Then we have the expansion

$$\begin{aligned}
\operatorname{th} h_0(\sigma) &= \sum_{A\subset\Lambda} c_A\sigma_A = b_0 + b_1(\sigma_{-1}+\sigma_1) + b_2\sigma_{-1}\sigma_1, \\
b_0 = c_\emptyset &= \frac{1}{2^N}\sum_\sigma h_0(\sigma) = \frac{1}{4}\big[\operatorname{th}(h+2J) + 2\operatorname{th}(h) + \operatorname{th}(h-2J)\big], \\
b_1 = c_{\{-1\}} &= c_{\{1\}} = \frac{1}{4}\big[\operatorname{th}(h+2J) - \operatorname{th}(h-2J)\big], \\
b_2 = c_{\{-1,1\}} &= \frac{1}{4}\big[\operatorname{th}(h+2J) - 2\operatorname{th}(h) + \operatorname{th}(h-2J)\big].
\end{aligned} \tag{7.55}$$

It is easy to show that the coefficients c_A are zero for all subsets A that are not included in $\{-1, 1\}$. Using translation invariance, we obtain from equation (7.53) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_0\rangle = (2b_1 - 1)\langle\sigma_0\rangle + b_0 + b_2\langle\sigma_{-1}\sigma_1\rangle. \tag{7.56}$$

Hence the dynamics of magnetization (which is equal to $\langle\sigma_0\rangle$ by translation invariance) depends on the two–point function $\langle\sigma_{-1}\sigma_1\rangle = \langle\sigma_0\sigma_2\rangle$. This function satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_0\sigma_2\rangle = -2\langle\sigma_0\sigma_2\rangle + 2b_0\langle\sigma_0\rangle + 2b_1\big(\langle\sigma_0\sigma_1\rangle + \langle\sigma_0\sigma_3\rangle\big) + 2b_2\langle\sigma_0\sigma_1\sigma_3\rangle. \tag{7.57}$$

Continuing in this way, we will obtain an infinite hierarchy of equations, where the dynamics of each n–point function depends on m–point functions with m ranging from 1 to $n+1$. This hierarchy can only be solved using approximations.

7.2.2 Mean Field Approximation

A first approximation, which is similar to the mean field approximation, is obtained by neglecting all correlations. Thus, if $m = \langle\sigma_0\rangle$ is the magnetization, we approximate $\langle\sigma_0\sigma_i\rangle$ by m^2, and, more generally, $\langle\sigma_A\rangle$ by $m^{|A|}$. In this case, the evolution equation (7.56) of the 1D Ising model reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}m = (2b_1 - 1)m + b_0 + b_2 m^2, \tag{7.58}$$

and the equation (7.57) for $\langle\sigma_0\sigma_2\rangle$ becomes

$$2m\frac{\mathrm{d}}{\mathrm{d}t}m = -2m^2 + 2b_0 m + 4b_1 m^2 + 2b_2 m^3, \tag{7.59}$$

which happens to be equivalent to (7.58). So far, the approximation is consistent. A similar calculation shows that the same is true for all two–point functions $\langle\sigma_0\sigma_i\rangle$ with $i \geqslant 2$, but not for $\langle\sigma_0\sigma_1\rangle$. Thus, equation (7.58) is only approximate.

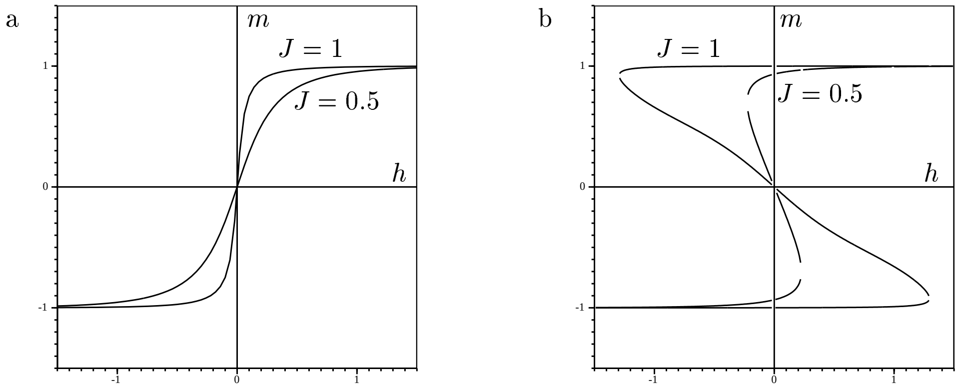

FIGURE 7.15. Equilibrium curves of the "mean field" Ising model as a function of h, for $J = 0.5$ and $J = 1$. (a) For a 1D lattice, there is a single equilibrium curve, and no phase transition. (b) For a 2D lattice and sufficiently large J, three equilibria coexist for small magnetic field, indicating the presence of a phase transition.

Let us examine the solutions of (7.58). For $h = 0$, we have $b_1 = \frac{1}{2}\operatorname{th}(2J)$ and $b_0 = b_2 = 0$, so that is reduces to $\dot{m} = (\frac{1}{2}\operatorname{th}(2J) - 1)m$, and the origin is the unique, asymptotically stable fixed point. For $h \gg 1$, we have $b_0 \simeq \operatorname{th}(h)$ and $b_1 \simeq b_2 \simeq 0$, so that $\dot{m} \simeq \operatorname{th}(m) - m$. In fact, we can show that there is a unique, stable equilibrium branch for all values of h and J (Fig. 7.15a).

We conclude that there is no phase transition in this situation, which is correct for the 1D Ising model. Thus, our very gross approximation has at least preserved this essential feature of the model, which distinguishes it from the Curie–Weiss model. When h is varied periodically in time, with frequency ε, we obtain the existence of a periodic orbit, enclosing an area of order ε.

We now turn to the two–dimensional case. Here we have to expand the transition probabilities involving four nearest neighbours. From the decomposition rule (7.54), we obtain that

$$\operatorname{th}[h + J(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] = b_0 + b_1 \sum_{\{i\}}^{4} \sigma_{\{i\}} + b_2 \sum_{\{ij\}}^{6} \sigma_{\{ij\}} + b_3 \sum_{\{ijk\}}^{4} \sigma_{\{ijk\}} + b_4 \sigma_{\{1234\}}, \tag{7.60}$$

where the weights b_i are given by

$$\begin{aligned}
b_0 &= \tfrac{1}{16}\big[\operatorname{th}(h+4J) + 4\operatorname{th}(h+2J) + 6\operatorname{th}(h) + 4\operatorname{th}(h-2J) + \operatorname{th}(h-4J)\big], \\
b_1 &= \tfrac{1}{16}\big[\operatorname{th}(h+4J) + 2\operatorname{th}(h+2J) - 2\operatorname{th}(h-2J) - \operatorname{th}(h-4J)\big], \\
b_2 &= \tfrac{1}{16}\big[\operatorname{th}(h+4J) - 2\operatorname{th}(h) + \operatorname{th}(h-4J)\big], \\
b_3 &= \tfrac{1}{16}\big[\operatorname{th}(h+4J) - 2\operatorname{th}(h+2J) + 2\operatorname{th}(h-2J) - \operatorname{th}(h-4J)\big], \\
b_4 &= \tfrac{1}{16}\big[\operatorname{th}(h+4J) - 4\operatorname{th}(h+2J) + 6\operatorname{th}(h) - 4\operatorname{th}(h-2J) + \operatorname{th}(h-4J)\big].
\end{aligned} \tag{7.61}$$

In the "mean field" approximation, we obtain the evolution equation

$$\dot{m} = b_0 + (4b_1 - 1)m + 6b_2 m^2 + 4b_3 m^3 + b_4 m^4. \tag{7.62}$$

For $h = 0$, it reduces to

$$\dot{m} = (4b_1 - 1)m + 4b_3 m^3, \qquad b_{1,3} = \frac{1}{8}\operatorname{th}(4J) \pm \frac{1}{4}\operatorname{th}(2J), \tag{7.63}$$

and thus the origin is an unstable equilibrium point for sufficiently large J. When the magnetic field is very large, we have $\dot{m} \simeq \mathrm{th}(h) - m$ as in the 1D case, so that there is a single stable equilibrium curve (Fig. 7.15b). We conclude that our simple approximation predicts the existence of a phase transition, as it should for the 2D Ising model.

The critical coupling J_c is given by the relation

$$\frac{1}{2}\,\mathrm{th}(4J) + \mathrm{th}(2J) - 1 = 0, \qquad \Rightarrow \qquad x^3 - 3x^2 - x - 5 = 0, \quad x := \mathrm{e}^{4J_c}. \tag{7.64}$$

It can be solved, with the result

$$x = 1 + \left[4 + \tfrac{4}{3}\sqrt{\tfrac{23}{3}}\right]^{1/3} + \left[4 - \tfrac{4}{3}\sqrt{\tfrac{23}{3}}\right]^{1/3} \simeq 3.6469, \qquad \Rightarrow \qquad J_c = \frac{1}{4}\ln x \simeq 0.3236, \tag{7.65}$$

to be compared with the exact result $J_c = \frac{1}{2}\ln(1+\sqrt{2}) \simeq 0.4407$.

Note, however, that the bifurcation diagram of (7.63) has the same structure than for the Curie–Weiss model, so that it also predicts a scaling law for the hysteresis cycle area of the form $\mathcal{A}(\varepsilon) \approx \mathcal{A}(0) + \varepsilon^{2/3}$. Furthermore, it predicts a "dynamic phase transition", since a minimal field amplitude is necessary to reverse magnetization.

7.2.3 Beyond Mean Field

Better approximations can be obtained by including some correlation functions in the dynamics. A systematic way to do this relies on the following expansion of n–point functions. We define the **truncated functions** $\langle\sigma_B\rangle^T$ by the recursive relations

$$\langle\sigma_A\rangle =: \sum_{\{B_i\}\text{partition of } A} \prod_i \langle\sigma_{B_i}\rangle^T. \tag{7.66}$$

The first relations are

$$\begin{aligned}
\langle\sigma_1\rangle &= \langle\sigma_1\rangle^T, \\
\langle\sigma_1\sigma_2\rangle &= \langle\sigma_1\sigma_2\rangle^T + \langle\sigma_1\rangle^T\langle\sigma_2\rangle^T, \\
\langle\sigma_1\sigma_2\sigma_3\rangle &= \langle\sigma_1\sigma_2\sigma_3\rangle^T + \langle\sigma_1\sigma_2\rangle^T\langle\sigma_3\rangle^T + \langle\sigma_1\sigma_3\rangle^T\langle\sigma_2\rangle^T + \langle\sigma_2\sigma_3\rangle^T\langle\sigma_1\rangle^T + \langle\sigma_1\rangle^T\langle\sigma_2\rangle^T\langle\sigma_3\rangle^T.
\end{aligned} \tag{7.67}$$

In particular, $\langle\sigma_1\sigma_2\rangle^T$ is equal to the two–point correlation function. We define an **approximation of order** $k-1$ by setting $\langle\sigma_1\dots\sigma_k\rangle^T = 0$. The mean field approximation of the previous subsection was of order 1.

We examine here an approximation of order 2, obtained by neglecting all three–point correlations. Moreover, we keep only the two leading two–point functions, so that the dynamics are expressed in function of the magnetization m, and the correlation functions c_1 and c_2 between nearest and next–to–nearest neighbours. They are given by

$$\begin{aligned}
c_1 &= \langle\sigma_{(0,0)}\sigma_{(1,0)}\rangle - \langle\sigma_{(0,0)}\rangle\langle\sigma_{(1,0)}\rangle, \\
c_2 &= \langle\sigma_{(0,0)}\sigma_{(1,1)}\rangle - \langle\sigma_{(0,0)}\rangle\langle\sigma_{(1,1)}\rangle.
\end{aligned} \tag{7.68}$$

We assume again that these functions have all the symmetries of the lattice. Moreover, we neglect correlations between spins which are further apart, i.e., $\langle\sigma_i\sigma_j\rangle = m^2$ whenever

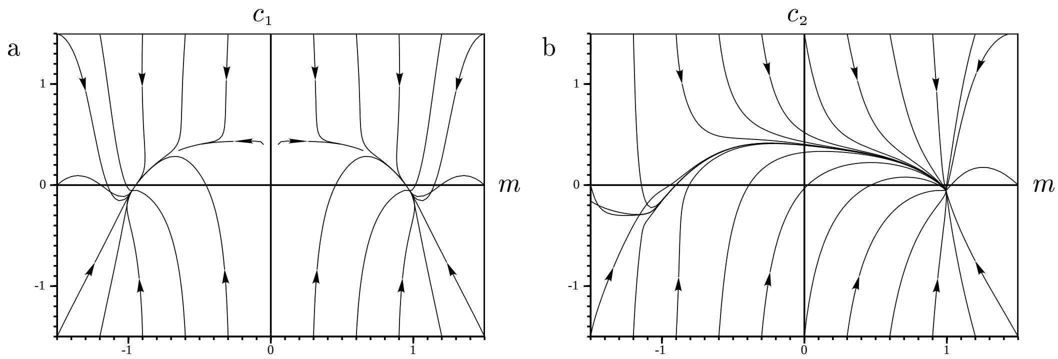

FIGURE 7.16. Phase portraits of the second order approximation, given by (7.69) for $J = 0.5$ and (a) $h = 0$, (b) $h = 0.5$. Three equilibria coexist for small magnetic field, and two of them undergo pitchfork bifurcation for larger h.

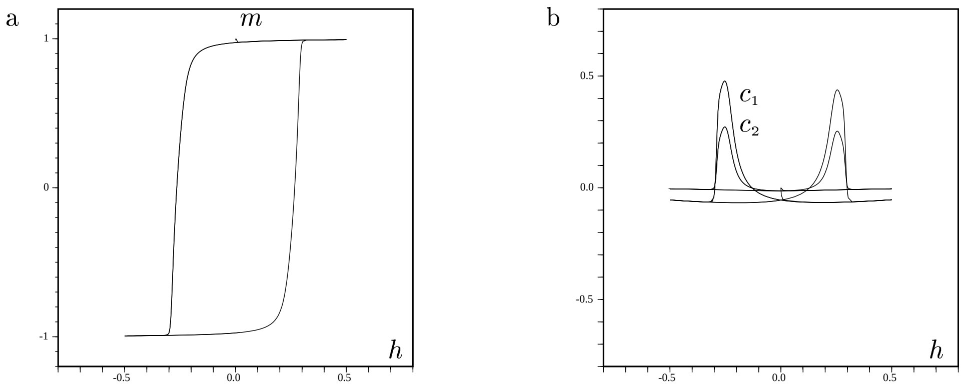

FIGURE 7.17. Hysteresis cycles of the second order approximation. (a) The cycle of the magnetization has a familiar shape. (b) The correlation functions c_1 and c_2 are in general small, but for intermediate values of h, they reach values up to 0.5.

$\|i-j\|_1 \leqslant 2$. In this way, we obtain the following closed three–dimensional system.

$$\begin{aligned}\dot{m} =&(4b_1-1)m + b_0 + 2b_2(3m^2+2c_2) + 4b_3 m(m^2+2c_2) + b_4(m^4+4m^2c_2+2c_2^2)\\ \dot{c}_1 =&-2c_1 + 2b_1(1-m^2+2c_2) + 2b_2(3m(1-m^2)+2c_2)\\ &+2b_3(3m^2+2c_2-2(m^2+c_2)^2) + 2b_4 m(m^2+2c_2-m^4-4m^2c_2-2c_2^2)\\ \dot{c}_2 =&-2c_2 + \big[4b_1+12b_2m+4b_3(3m^2+2c_2)+4b_4m(m^2+2c_2)\big]c_1\end{aligned} \tag{7.69}$$

For large h, this system reduces to $\dot{m} = \mathrm{th}(h) - m$, $\dot{c}_1 = -2c_1$, $\dot{c}_2 = -2c_2$, so that the correlations relax to 0, while m tends to the asymptotic value $\mathrm{th}(h)$ (Fig. 7.16b). For small h and sufficiently large J, one can show the existence of two stable equilibria, and one which is hyperbolic, with a one–dimensional unstable manifold (Fig. 7.16a).

When h is varied adiabatically, the magnetization follows an asymptotic hysteresis cycle, which has the same qualitative features as for the Curie–Weiss model (Fig. 7.17a). Thus, our slightly improved approximation has not really brought any new insights. The only additional information is that the correlation functions remain small most of the time, but can reach appreciable values for intermediate magnetic field (Fig. 7.17b). In fact, the correlations reach their maximum just before magnetization reversal, and then drop almost to zero again. Similar behaviours of the correlations have been reported for the $O(N)$ model [DT].

7.3 Summary and Conclusion

Properties of hysteresis cycles produced by dynamical lattice models depend in an essential way on the dimension of spins and lattice, as well as on range and anisotropy of interactions.

We have examined two models with one–dimensional spins: the Curie–Weiss model (where the spins have infinite range interaction), and the Ising model (with nearest neighbour interaction). These models have in common the property that a potential barrier has to be overcome in order to flip spins. The Curie–Weiss model can be accurately described, in the thermodynamic limit, by an effective mean field equation. It predicts that a minimal field amplitude h_c is necessary to reverse magnetization; when the amplitude is larger than h_c, the hysteresis cycle area goes to a finite limit when the adiabatic parameter goes to zero. For positive driving frequency, it follows the scaling law

$$\mathcal{A}(\varepsilon, h_0) - \mathcal{A}(0, h_0) \approx (h_0 - h_c)^{-1/6}\varepsilon^{2/3}, \qquad h_0 \geqslant h_c + \mathcal{O}(\varepsilon). \tag{7.70}$$

This discussion, however, applies only to infinite systems. When the number of spins is finite, noise has to be included into the evolution equation, which can help to reverse spins by thermic activation.[3]

The Ising model is incomparatively more difficult to analyse, since there is no natural way to truncate the infinite hierarchy of equations governing its evolution. We have examined some very gross approximations, obtained by neglecting (almost) all correlations. These approximations predict a behaviour of hysteresis cycles which is similar to the Curie–Weiss model. It is difficult to compare this result with numerical simulations found in the literature, since most of them fit the area on a scaling law without constant term $\mathcal{A}(0, h_0)$ (even, as in [LP], when a "dynamic phase transition" has been observed).

We believe that the key property is the detailed dynamics of magnetization reversal. It is conceivable that by including more correlation functions, new intermediate equilibrium branches will appear, which allow for a more gradual change of m. But it is also possible that an infinite number of correlation functions is necessary to explain this phenomenon accurately. In fact, correlation functions are probably not the appropriate variables to describe the reversal, and a finite–dimensional description, should it be possible, would have to include variables describing the growth of droplets.

We would like to advocate the point of view that several distinct scales govern magnetization reversal, depending on frequency and amplitude of the magnetic field. A recent, very interesting study by Schonmann and Shlosman [SS] proves in particular the following property. Assume an Ising model with Glauber dynamics is in equilibrium, with negative magnetization at negative magnetic field. When the field is abruptly "quenched" to a small, positive value h, the system keeps a negative magnetization for all times of order $e^{\lambda/h}$ for $\lambda < \lambda_c$, and has already relaxed to the equilibrium state with positive magnetization at times of order $e^{\lambda/h}$ for $\lambda > \lambda_c$. The critical value λ_c is explicitly known, and depends only on equilibrium properties of the system (the spontaneous magnetization and integrated surface tension of the Wulff droplet). Thus, for very small magnetic field, reversal is possible but takes an exponentially long time. This time will decrease with increasing h, and should become almost zero for sufficiently large magnetic field. When this field is varied periodically, with amplitude h_0 and frequency ε, the reversal dynamics

[3]The noise–induced escape of a particle from a well has been observed experimentally in [SL].

will be governed by an interplay between the velocities of droplet growth and variation of h. It is, however, not clear why a power law should arise for the hysteresis area. Recent extensive numerical simulations [RTMS, SRN] indicate that different reversal mechanisms predominate, depending on field amplitude and system size. When the amplitude becomes small, there is a "crossover" to a logarithmic scaling law.

The dynamics of vectorial spins is very different. In this case, the dominant mechanism of magnetization reversal is spin rotation, rather than spin flip. The dynamics depend essentially on the anisotropy of coupling. Bifurcation delay plays an important role here, and is a major source of hysteresis. It is, however, questionable whether this delay will survive the noise appearing in finite–dimensional systems. The most robust situation is the one with lowest symmetry, i.e., when the coupling is anisotropic and not aligned with the magnetic field, in which case the magnetization tends to rotate in a fixed direction, instead of oscillating back and forth.

Chapter 8

Iterated Maps

"If you only have a hammer, you tend to see every problem as a nail."
Maslow

In this last section, we give a short presentation of a few results on adiabatic iterated maps. We will distinguish between adiabatic maps, which depend on a slowly varying parameter, and slow–fast maps, which involve two dynamical variables evolving on different time scales.

Maps are often derived from differential equations, for instance when taking a Poincaré section near a periodic orbit. If such an ODE depends on a slowly varying parameter, the dynamics is in general more complicated than the adiabatic map, which is obtained by introducing the drifting parameter into the family of static Poincaré maps. In some cases, however, for instance when the time of flight between intersections becomes small, can the dynamics be described by a slow–fast mapping.

Adiabatic maps are nonetheless interesting to study in their own right, since other systems than ODEs are naturally described by iterations, for instance population dynamics in a slowly varying climate.

Since the properties discussed here are either similar to those in Chapter 5 or have been presented elsewhere, we give only a brief an rather informal description.

- **Section 8.1** We extend a few basic properties, that we derived for adiabatic ODEs, to iterated maps which depend on a slowly time–dependent parameter. We show that the results on existence of adiabatic solutions, and on the linearized dynamics near these solutions, admit natural extensions to such adiabatic maps.
- **Section 8.2** We summarize some results on existence of adiabatic invariants for slow–fast iterated maps. These results are illustrated on some billiard systems.

8.1 Adiabatic Systems

The natural analogue of adiabatic ordinary differential equations considered in Chapters 4 and 5 is the map

$$x_{m+1} = F(x_m, \lambda(\varepsilon m)), \tag{8.1}$$

where m is an integer index, and $\lambda(\varepsilon m)$ is the slowly varying parameter. As before, we would like to introduce a **slow time** $\mu = \varepsilon m$. The variable μ is now a discrete time taking real values, instead of only integer ones, and approaching a continuum in the adiabatic limit $\varepsilon \to 0$. The adiabatic map can be written

$$x_{\mu+\varepsilon} = f(x_\mu, \mu) := F(x_\mu, \lambda(\mu)). \tag{8.2}$$

Let us examine how a few basic properties of adiabatic ODEs translate to this situation. As usual, we assume that $f(x,\mu)$ is of class $\mathcal{C}^k$ in some open subset of $\mathbb{R}^n \times \mathbb{R}$ or $\mathbb{C}^n \times \mathbb{R}$, with $k \geqslant 2$.

8.1.1 Adiabatic Solutions

Assume that the adiabatic map (8.2) admits a branch of fixed points $x^\star(\mu) = f(x^\star(\mu), \mu)$. We call **adiabatic solution** of order ε an orbit of (8.2) remaining at a distance of order ε from $x^\star(\mu)$. The difference $y = x - x^\star(\mu)$ obeys the equation

$$y_{\mu+\varepsilon} = A(\mu) y_\mu + b(y_\mu, \mu) + \varepsilon w(\mu, \varepsilon), \tag{8.3}$$

where

$$\begin{aligned} A(\mu) &:= \partial_x f(x^\star(\mu), \mu), \\ b(y,\mu) &:= f(x^\star(\mu) + y, \mu) - x^\star(\mu) - A(\mu) y = \mathcal{O}(\|y\|^2), \\ w(\mu,\varepsilon) &:= \frac{x^\star(\mu) - x^\star(\mu+\varepsilon)}{\varepsilon} = -\int_0^1 \partial_\mu x^\star(\mu + \varepsilon s)\, \mathrm{d}s. \end{aligned} \tag{8.4}$$

The order of the drift term $\varepsilon w(\mu,\varepsilon)$ can be decreased by an iterative scheme. The first step consists in the change of variables

$$y_\mu = \varepsilon[\mathbb{1} - A(\mu)]^{-1} w(\mu, \varepsilon) + z_\mu, \tag{8.5}$$

which yields an equation of the form

$$z_{\mu+\varepsilon} = A(\mu) z_\mu + b_1(z_\mu, \mu, \varepsilon) + \varepsilon^2 w_1(\mu, \varepsilon), \tag{8.6}$$

where $b_1(z,\mu,\varepsilon) = \mathcal{O}(\|z\|^2) + \mathcal{O}(\varepsilon\|z\|)$. This procedure can be repeated as long as the map is sufficiently differentiable. We now prove that equation (8.3) admits adiabatic solutions. The extension of this result to similar maps with a smaller drift term is straightforward.

Proposition 8.1. *Let $A(\mu)$ be a differentiable matrix–valued function, whose eigenvalues have a module strictly smaller than 1. Assume that $w(\mu,\varepsilon)$ is differentiable and $b(y,\mu,\varepsilon) = \mathcal{O}(\|y\|^2) + \mathcal{O}(\varepsilon\|y\|)$ is continuous. Then, for sufficiently small ε,* (8.3) *admits an adiabatic solution of order ε.*

PROOF: Let N be a given positive integer (uniform in ε). We obtain by induction that

$$y_{\mu+N\varepsilon} = A(\mu)^N y_\mu + b_N(y_\mu, \mu, \varepsilon) + \varepsilon w_N(\mu, \varepsilon), \tag{8.7}$$

where the functions b_N and w_N satisfy bounds of the form

$$\begin{aligned} \|b_N(y,\mu,\varepsilon)\| &\leqslant M(N)(\|y\|^2 + \varepsilon\|y\|), \\ \|w_N(\mu,\varepsilon)\| &\leqslant W(N). \end{aligned} \tag{8.8}$$

For sufficiently small ε and y, the bounds $M(N)$ and $W(N)$ grow at most exponentially fast with N. Let a be the largest module of the eigenvalues of A. It follows from the Jordan decomposition of A (see Theorem 2.4 and equation (2.69)) that

$$\|A^N\| \leqslant K(N)a^{N-N_0}, \tag{8.9}$$

where N_0 is not larger than the size n of the matrix, and $K(N)$ is a polynomial of degree N_0. Since we assume that $a < 1$, there exists an N such that $\|A^N\| = \lambda < 1$. If $V_m :=\|y_{\mu+mN\varepsilon}\|$, we obtain from (8.7) that

$$V_{m+1} \leqslant \big[\lambda + M(N)(\varepsilon + V_m)\big]V_m + \varepsilon W(N). \tag{8.10}$$

For sufficiently small ε and y_0, the term in brackets is smaller than 1, and the sequence of V_m is of order ε. Using (8.7), we find that all terms y_μ are of order ε. □

We have obtained that for each stable branch of fixed points $x^\star(\mu)$, the adiabatic map (8.2) admits an adiabatic solution

$$\bar{x}(\mu) = x^\star(\mu) + \varepsilon[\mathbb{1} - A(\mu)]^{-1}\partial_\mu x^\star(\mu) + \mathcal{O}(\varepsilon^2). \tag{8.11}$$

In the analytic case, this adiabatic solution can be represented by an asymptotic series in ε.

Let us say a word on the particular case of a periodic system, $f(x,\mu) = f(x,\mu+1)$. Then there exists a periodic adiabatic solution, which can be interpreted as an **invariant curve** of the map. The proposition shows that this curve can be constructed as the limit of the sequence $x_\mu^{(N+1)} = f(x_{\mu-\varepsilon}^{(N)}, \mu - \varepsilon)$, taking as initial condition a periodic function sufficiently close to $x^\star(\mu)$.

8.1.2 Linear Systems

In the neighborhood of an adiabatic solution, the dynamics is governed by an equation of the form $y_{\mu+\varepsilon} = A(\mu)y_\mu + \mathcal{O}(\|y_\mu\|^2)$. Here we restrict the discussion to the linearized equation. Let us start with the one–dimensional case:

$$y_{\mu+\varepsilon} = a(\mu)y_\mu \qquad \Rightarrow \qquad y_{\mu_0+N\varepsilon} = a(\mu_0 + (N-1)\varepsilon)\dots a(\mu_0+\varepsilon)a(\mu_0)y_{\mu_0}. \tag{8.12}$$

Although the product (8.12) is easy to write down, it is not straightforward to compute. We are interested in its value for small ε, and $N\varepsilon$ or order 1. Let us start by transforming the product into a sum, using its logarithm.

$$\ln\frac{y_{\mu_0+N\varepsilon}}{y_{\mu_0}} = \ln a(\mu_0 + (N-1)\varepsilon) + \dots + \ln a(\mu_0+\varepsilon) + \ln a(\mu_0). \tag{8.13}$$

For small ε, we can transform this sum into an integral. Using the fact that if $f(s)$ is a differentiable function with $|f'(s)| \leqslant M$,

$$\frac{1}{\varepsilon}\int_0^\varepsilon f(s)\,\mathrm{d}s = f(0) + r(\varepsilon), \qquad |r(\varepsilon)| \leqslant \frac{1}{2}M\varepsilon, \tag{8.14}$$

we obtain that the solution of (8.12) can be written as

$$y_\mu = \mathrm{e}^{\alpha(\mu,\mu_0,\varepsilon)/\varepsilon}y_{\mu_0}, \qquad \alpha(\mu,\mu_0,\varepsilon) = \int_{\mu_0}^{\mu}\ln a(s)\,\mathrm{d}s + \varepsilon R(\mu,\mu_0,\varepsilon), \tag{8.15}$$

where the remainder satisfies the bound

$$|R(\mu,\mu_0,\varepsilon)| \leqslant \frac{1}{2}\,|\mu-\mu_0| \sup_{[\mu_0,\mu]}\left|\frac{a'(s)}{a(s)}\right|. \tag{8.16}$$

Thus solutions of linear adiabatic iterated maps depend exponentially on time, in a similar way than solutions of linear adiabatic ODEs.

We examine now the n–dimensional case. We would like to solve the system

$$y_{\mu+\varepsilon} = A(\mu) y_\mu, \tag{8.17}$$

using a dynamic diagonalization, similar to the procedure we developed for differential equations. The basic observation to be made is that if $S(\mu)$ is any nonsingular matrix function, the change of variables $y_\mu = S(\mu) z_\mu$ will transform (8.17) into

$$z_{\mu+\varepsilon} = S(\mu+\varepsilon)^{-1} A(\mu) S(\mu) z_\mu. \tag{8.18}$$

For instance, if $S(\mu)$ diagonalizes $A(\mu)$ statically, (8.18) will contain off–diagonal terms of order ε. If we manage to construct a bounded solution $S(\mu)$ of the equation

$$S(\mu+\varepsilon) D(\mu) = A(\mu) S(\mu), \tag{8.19}$$

then z_μ will obey the equation $z_{\mu+\varepsilon} = D(\mu) z_\mu$. In particular, if $D(\mu)$ is diagonal, the solution of (8.17) can be written as

$$y_\mu = S(\mu) U_0(\mu,\mu_0) S(\mu_0)^{-1} y_{\mu_0}, \tag{8.20}$$

where $U_0(\mu,\mu_0)$ is a diagonal matrix, with diagonal elements of the form (8.15) obtained by solving (8.12) for each diagonal entry of $D(\mu)$. To lowest order, we have

$$U_0(\mu,\mu_0)_{[jj]} = \exp\frac{1}{\varepsilon}\left[\int_{\mu_0}^{\mu} \ln a_j(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon)\right], \tag{8.21}$$

where the $a_j(\mu)$ are eigenvalues of $A(\mu)$.

To show that (8.19) admits a bounded solution, we proceed as usual by induction on the size of the matrix. Assume that the eigenvalues of $A(\tau)$ can be split into two groups, with module respectively smaller than a constant c_1 and larger than $c_2 > c_1$. We may assume that a first static bloc–diagonalization has already been carried out, so that we have to solve (8.19) for matrices of the form

$$A(\mu) = \begin{pmatrix} A_{11}(\mu) & \varepsilon A_{12}(\mu) \\ \varepsilon A_{21}(\mu) & A_{22}(\mu) \end{pmatrix}, \qquad S(\mu) = \begin{pmatrix} \mathbb{1} & \varepsilon S_{12}(\mu) \\ \varepsilon S_{21}(\mu) & \mathbb{1} \end{pmatrix}, \tag{8.22}$$

and $D(\mu)$ a bloc–diagonal matrix with blocs D_1 and D_2. Then (8.19) reduces to

$$\begin{aligned} D_1 &= A_{11} + \varepsilon^2 A_{12} S_{21}, \\ D_2 &= A_{22} + \varepsilon^2 A_{21} S_{12}, \\ S_{21}(\mu+\varepsilon) &= \left[A_{21}(\mu) + A_{22}(\mu) S_{21}(\mu)\right]\left[A_{11}(\mu) + \varepsilon^2 A_{12}(\mu) S_{21}(\mu)\right]^{-1}, \\ S_{12}(\mu+\varepsilon) &= \left[A_{12}(\mu) + A_{11}(\mu) S_{12}(\mu)\right]\left[A_{22}(\mu) + \varepsilon^2 A_{21}(\mu) S_{12}(\mu)\right]^{-1}. \end{aligned} \tag{8.23}$$

Let us consider the equation for S_{21}, which is of the form

$$\begin{aligned} S_{21}(\mu+\varepsilon) &= f(S_{21}(\mu),\mu,\varepsilon), \\ f(S,\mu,\varepsilon) &= \left[A_{21}(\mu) + A_{22}(\mu) S\right]\left[A_{11}(\mu) + \varepsilon^2 A_{12}(\mu) S\right]^{-1}. \end{aligned} \tag{8.24}$$

Proposition 8.2. *Assume that $A_{11}(\mu)$ has eigenvalues $a_1(\mu),\dots,a_p(\mu)$ and $A_{22}(\mu)$ has eigenvalues $b_1(\mu),\dots,b_q(\mu)$, with $|b_j(\mu)/a_i(\mu)| > 1$ for all i,j and uniformly in μ. Then* (8.24) *admits a bounded solution, and thus* (8.17) *can be dynamically bloc–diagonalized.*

PROOF: The proof is similar to that of Theorem 5.3, so that we only outline the differences. For $\varepsilon = 0$, the map f admits a fixed point $S^\star(\mu,0)$, given by

$$S^\star = f(S^\star,\mu,0) \qquad \Leftrightarrow \qquad S^\star A_{11} - A_{22}S^\star = A_{21}. \tag{8.25}$$

We know (see Corollary 5.5) that this equation admits a solution, since A_{11} and A_{22} have no common eigenvalue. Next we compute the Fréchet derivative $\partial_S f$, which is given by the map

$$X \mapsto A_{22}X[A_{11}+\varepsilon^2 A_{12}S^\star]^{-1} - \varepsilon^2[A_{21}+A_{22}S^\star][A_{11}+\varepsilon^2 A_{12}S^\star]^{-2}A_{12}X. \tag{8.26}$$

In particular, we have

$$\partial_S f(S^\star,\mu,0) : X \mapsto A_{22}XA_{11}^{-1}. \tag{8.27}$$

This operator has eigenvalues b_j/a_i. This shows that $\partial_S f(S^\star,\mu,0) - \mathbb{1}$ is invertible, and by the implicit function theorem, the fixed point equation admits solutions for small positive ε. Moreover, by continuity of the eigenvalues, $f(S,\mu,\varepsilon)$ satisfies the hypotheses of Proposition 8.1, which shows that (8.24) admits an adiabatic solution $S^\star(\mu,\varepsilon)+\mathcal{O}(\varepsilon)$. □

8.2 Slow–Fast Systems

In the introduction, we have distinguished between adiabatic and slow–fast systems. A slow–fast iterated map can be defined by the equation

$$\begin{aligned} x_{m+1} &= F(x_m,y_m,\varepsilon) \\ y_{m+1} &= y_m + \varepsilon G(x_m,y_m,\varepsilon), \end{aligned} \tag{8.28}$$

where x_m is the fast variable, and y_m is the slow one. In particular, when $G \equiv 1$, we recover the adiabatic system (8.2). For $\varepsilon = 0$, (8.28) reduces to the "autonomous" system

$$\begin{aligned} x_{m+1} &= F(x_m,y_m,0) \\ y_{m+1} &= y_m. \end{aligned} \tag{8.29}$$

In this case, y_m is constant, and acts as a parameter on the system governing the fast variable x_m. When the adiabatic parameter ε is switched on, the slow variable y_m changes by small amounts, which may accumulate in such a way as to become large after a number of iterations of order ε^{-1}. An **adiabatic invariant** on a time ε^{-k} is a function of x and y which changes only by a small amount on this time interval.

Different cases are of interest. If F is dissipative, x_m will converge to some attractor depending on y, and a small positive ε will cause the fast variable to follow this attractor adiabatically.

Another important case is the conservative one (for simplicity, we consider both x and y to be real numbers). In this situation, the map (8.29) is called **integrable**. To be area–preserving, its Jacobian $|\partial_x F(x,y,0)|$ should be equal to 1, so that we necessarily have $F(x) = x + \Omega(y)$. Thus, a perturbed integrable map can be written as[1]

$$\begin{aligned} x_{m+1} &= x_m + \Omega(y_m) + \varepsilon f(x_m, y_m, \varepsilon) \pmod 1 \\ y_{m+1} &= y_m + \varepsilon g(x_m, y_m, \varepsilon), \end{aligned} \tag{8.30}$$

where the notation (mod 1) indicates that x is a periodic variable, which is by far the most common situation in Hamiltonian dynamics.

8.2.1 Adiabatic Theorems

One of the ways to analyse systems of the form (8.30) is by Kolmogorov–Arnol'd–Moser (KAM) theory. In this case, one of the relevant results is Moser's theorem [**Mo**].

Theorem 8.1 (Moser). *Assume that the area–preserving map* (8.30) *satisfies the twist condition* $\Omega'(y) \geqslant W > 0$, *that* $f, g, \Omega \in \mathcal{C}^k$, $k \geq 5$, *and let* $\|f\|_{\mathcal{C}^k}$ *be the norm*

$$\|f\|_{\mathcal{C}^k} = \sup_{i+j \leq k} \left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right|. \tag{8.31}$$

Let ω *be in the range of* Ω *and satisfy the Diophantine conditions*

$$|n\omega - m| > \frac{\gamma}{n^\tau} \qquad \text{for all coprime } n, m, \text{ and some } \tau \in (1, (k-1)/2). \tag{8.32}$$

Then, for any $\delta > 0$, *there exists a positive* ε_0 *such that if* $\varepsilon(\|f\|_{\mathcal{C}^k} + \|g\|_{\mathcal{C}^k}) < \varepsilon_0 W \gamma^2$, *the map* (8.30) *has an invariant curve of the form*

$$\begin{aligned} x_m &= \xi_m + V(\xi_m) \\ y_m &= \Omega^{-1}(\omega) + U(\xi_m), \end{aligned} \tag{8.33}$$

with $\xi_m = \xi_0 + m\omega$ *and* $\|U\|_{\mathcal{C}^1} + \|V\|_{\mathcal{C}^1} < \delta$.

Remark 8.1.
- Area preservation is not required: it is sufficient that any curve sufficiently close to a horizontal intersect its image.
- The map (8.30) is assumed to be defined on the annulus $a < y < b$, but this annulus is not required to be invariant.
- It is known that the values of ω satisfying the Diophantine condition (8.32) form a Cantor set with measure approaching 1 when $\gamma \to 0$.
- The condition $\Omega'(y) \geqslant W > 0$ is a particular case of twist condition. The map (8.28) is called a **twist map** if $\partial_y F(x, y, \varepsilon) \geqslant W > 0$. Properties of such maps are discussed in [Me].

Equation (8.33) can be interpreted as a change of variables transforming the perturbed integrable map (8.30) into the integrable map $\xi_{m+1} = \xi_m + \omega_m$, $\omega_{m+1} = \omega_m \equiv \omega$. This transformation exists, however, only on a Cantor set of initial conditions (the so–called **KAM tori**). Where it is defined, the **rotation number** $\omega = \omega(x, y)$ is an exact adiabatic

[1] We keep here the parameter ε for convenience, to indicate the smallness of the perturbation. Moser's theorem is often written without ε, with appropriate smallness assumptions on f and g.

invariant. The relation $\omega(x, y) = \omega(x_0, y_0)$ is the implicit equation of the invariant curve originating in (x_0, y_0).

Another interesting situation arises when considering a slowly varying Hamiltonian of the form $H(q, p, \tau)$, depending periodically on the slow time $\tau = \varepsilon t$. Under appropriate hypotheses, it can be reduced to a perturbed integrable map [Ar3]. The idea is to carry out first a canonical transformation to action–angle variables (I, ϕ), giving the Hamiltonian

$$H(I, \phi, \tau) = H_0(I, \tau) + \varepsilon H_1(I, \phi, \tau). \tag{8.34}$$

The time–dependence of H_0 can be removed by solving this relation with respect to I, and introducing the new Hamiltonian

$$\varepsilon I(H, \tau, \phi) = \varepsilon I_0(H, \tau) + \varepsilon^2 I_1(H, \tau, \phi). \tag{8.35}$$

Here H is considered as a momentum variable, τ as a position, and ϕ as time. Another transformation to action–angle variables (y, x) yields the Hamiltonian

$$J(y, x, \phi) = \varepsilon J_0(y) + \varepsilon^2 J_1(y, x, \phi). \tag{8.36}$$

Writing down the equations of motion, and integrating them on a period of ϕ yields the near–integrable map

$$\begin{aligned} x_{m+1} &= x_m + \varepsilon\Omega(y_m) + \varepsilon^2 f(x_m, y_m, \varepsilon) \pmod 1 \\ y_{m+1} &= y_m + \varepsilon^2 g(x_m, y_m, \varepsilon). \end{aligned} \tag{8.37}$$

One can apply a variant of Theorem 8.1 to show the existence of adiabatic invariants [Ar3, Zh]. We present here a complementary result, which shows only the existence of approximate invariants, but without the twist and area preservation hypotheses. Moreover, this result works for any initial condition, not only on a Cantor set.

Theorem 8.2. *Consider the differentiable iterated map*

$$\begin{aligned} x_{m+1} &= x_m + \varepsilon c(y_m)\big[\Omega(y_m, \varepsilon) + \varepsilon f(x_m, y_m, \varepsilon)\big] \pmod 1 \\ y_{m+1} &= y_m + \varepsilon^2 c(y_m)^2 g(x_m, y_m, \varepsilon), \end{aligned} \tag{8.38}$$

where $c(0) = c(1) = 0$ *and* $\Omega(y, 0) \neq 0$*. For sufficiently small* ε*, there exists a transformation*

$$\begin{aligned} \xi &= x + \varepsilon V(x, y, \varepsilon) \\ \eta &= y + \varepsilon U(x, y, \varepsilon) \end{aligned} \tag{8.39}$$

with the following properties:

1. *It preserves the unit square* $[0, 1] \times [0, 1]$.
2. *It transforms the map* (8.38) *into*

$$\begin{aligned} \xi_{m+1} &= \xi_m + \varepsilon c(\eta_m)\big[\bar\Omega(\eta_m, \varepsilon) + r(\varepsilon)\bar f(\xi_m, \eta_m, \varepsilon)\big] \pmod 1 \\ \eta_{m+1} &= \eta_m + \varepsilon c(\eta_m)^2\big[\varepsilon\Theta(\eta_m, \varepsilon) + r(\varepsilon)\bar g(\xi_m, \eta_m, \varepsilon)\big], \end{aligned} \tag{8.40}$$

where $r(\varepsilon) = \varepsilon^{k+1}$ *if* $\Omega, f, g \in \mathcal{C}^k$ *and* $c \in \mathcal{C}^{k+1}$*, and* $r(\varepsilon) = \mathrm{e}^{-1/C|\varepsilon|}$ *if all functions are analytic in a complex neighborhood of the annulus.*

3. *If* (8.38) *preserves the measure* $c(y)\rho(x,y,\varepsilon)\,\mathrm{d}x\,\mathrm{d}y$*, where* $\rho(x,y,0)=1$*, then* $\Theta(\eta,\varepsilon)=0$*, and thus*

$$\eta_{m+1}=\eta_m+\varepsilon r(\varepsilon)c(\eta_m)^2\bar{g}(\xi_m,\eta_m,\varepsilon). \tag{8.41}$$

The proof is given in [BK1]. It relies on successive transformations which decrease the order of the terms depending on x_m. The exponential bounds in the analytic case are obtained in a similar way than in Lemma 4.2.

This result has the following interpretation. During a number of iterations of order $r(\varepsilon)^{-1}$, the terms depending on ξ_m have a negligible influence, and the map can be approximated by

$$\begin{aligned}\xi_{m+1}&\simeq\xi_m+\varepsilon c(\eta_m)\bar{\Omega}(\eta_m,\varepsilon)\\ \eta_{m+1}&\simeq\eta_m+\varepsilon^2c(\eta_m)^2\Theta(\eta_m,\varepsilon).\end{aligned} \tag{8.42}$$

In this system, η_m is the slow variable, which follows a self–determined evolution, while the fast variable ξ_m rotates with a frequency depending slowly on time. In particular, when the map is integrable, $\Theta=0$ and η_m remains constant. When the terms of order $r(\varepsilon)$ are introduced, η becomes an adiabatic invariant on a time scale of order $r(\varepsilon)^{-1}$. During this period, ξ rotates by an angle $m\varepsilon c(\eta_0)\bar{\Omega}(\eta_0,\varepsilon)=\mathcal{O}(\varepsilon r(\varepsilon)^{-1})$.

8.2.2 Applications to Billiards

Billiards are popular models for various physical systems. Birkhoff introduced them in connection with the three–body problem, in order to illustrate a theorem on existence of periodic orbits [**Bi**]. They were used by Sinai to describe a hard sphere gas, and became an extensively studied paradigm for the problem of ergodic versus integrable behaviour of Hamiltonian systems [**KT**]. More recently, billiards have been used to study quantum chaos and mesoscopic systems. In particular, billiards with electromagnetic fields are relevant for the modeling of transport properties [Tr, KS].

We summarize here a few results on billiards in a magnetic field, and, in one case, an additional electric field, which have been presented in [BK1, BHHP, Berg].

A classical billiard is defined by specifying a connected domain Q of the plane. The boundary ∂Q is parametrized by its arclength, $\mathbf{x}(s)=(X(s),Y(s))$, with $X'(s)^2+Y'(s)^2=1$; the unit tangent vector and the curvature are given respectively by $\mathbf{t}(s)=(X'(s),Y'(s))$ and $\kappa(s)=X'(s)Y''(s)-X''(s)Y'(s)$.[2]

Inside Q, there is a particle of mass m and charge q, subject to a uniform magnetic field B perpendicular to the plane, and an in–plane potential $V(\mathbf{x})$. The billiard flow is defined by the Lagrangian

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}})=\frac{1}{2}m\dot{\mathbf{x}}^2+q\langle\dot{\mathbf{x}}|\mathbf{A}(\mathbf{x})\rangle-V(\mathbf{x}), \tag{8.43}$$

where $\mathbf{A}(\mathbf{x})=\frac{1}{2}B(-y,x)$ is the vector potential in symmetric gauge (we will adopt the sign convention $qB<0$, so that in absence of potential, the particle rotates counterclockwise).

The dynamics is defined in the following way. Assume that the billiard particle starts on the boundary at $\mathbf{x}(s_0)$, with a velocity $\dot{\mathbf{x}}_0$ making an angle θ_0 with $\mathbf{t}(s_0)$. It then evolves

[2] The direction of parametrization is chosen in such a way that κ is positive for a convex boundary.

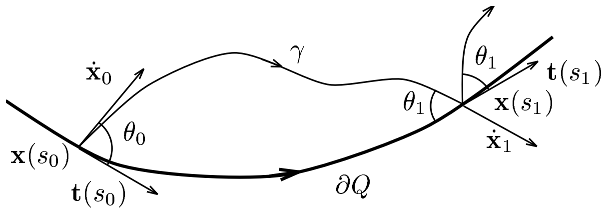

FIGURE 8.1. Geometry of a billiard trajectory. The abscissa s_0 of the starting point and the angle θ_0 uniquely define the trajectory γ, and, if the boundary ∂Q is reached again, the coordinates (s_1, θ_1) of the next collision.

in Q according to the Lagrange equations. If the particle returns to the boundary, at a point $\mathbf{x}(s_1)$, with a velocity $\dot{\mathbf{x}}_1$ making an angle $-\theta_1$ with $\mathbf{t}(s_1)$, it is reflected elastically, meaning that it leaves the boundary again with an angle θ_1 (Fig. 8.1). The i–th collision may thus be parametrized by the coordinates (s_i, θ_i), and as long as the particle returns to the boundary, the dynamics are characterized by the **bouncing map**

$$\begin{aligned} s_{m+1} &= f(s_m, \theta_m) \\ \theta_{m+1} &= g(s_m, \theta_m). \end{aligned} \tag{8.44}$$

It can be shown that this map preserves the measure $\mathrm{d}s\,\mathrm{d}(\|\dot{\mathbf{x}}\|_2 \cos\theta)$, where the velocity depends on the energy E according to $\|\dot{\mathbf{x}}\|_2^2 = \frac{2}{m}(E - V(\mathbf{x}(s))$. It is, in fact, possible to describe this map implicitly by a generating function, which contains all necessary information on existence and stability of periodic orbits [Berg].

The particular case when there is only the magnetic field ($V(\mathbf{x}) = 0$) was first considered by Robnik and Berry [RB]. It has been studied by several authors in recent years [BS, KS, Ta].

For a given energy E, the trajectories are arcs of Larmor radius $\mu = \sqrt{2mE}/|qB|$. If we take some characteristic length of the billiard domain as a unit, μ is the single dimensionless parameter on which the dynamics depends. For a generic, sufficiently smooth boundary, near–integrable motions appear in two situations: for orbits moving along the boundary, and in the strong–magnetic–field limit.

When orbits start with a velocity almost tangent to the boundary, the bouncing map takes the form

$$\begin{aligned} s_{m+1} &= s_m - \frac{2\mu \sin\theta_m}{1 - \mu\cos\theta_m \kappa(s_m)} + \mathcal{O}(\sin\theta_m) \quad (\mathrm{mod}\ |\partial Q|) \\ \theta_{m+1} &= \theta_m + \mathcal{O}(\sin\theta_m), \end{aligned} \tag{8.45}$$

provided we are in one of the following cases:[3]

- If θ is close to π, this map is correct if the curvature of the boundary is everywhere larger than $-1/\mu$; thus it may contain both convex and concave parts, but the concave parts should have a smaller curvature than the Larmor arcs. The map describes trajectories skipping backwards along the boundary.
- If θ is close to 0, the map is only correct under certain conditions on μ and κ:
  - if the boundary is convex, with $\kappa(s) \geqslant 1/\rho_{\max} \geqslant 1/\mu$; then it describes trajectories skipping forward along the boundary (Fig. 8.2d);

[3]These cases are determined by the requirements that the denominator in (8.45) should not vanish, and that the orbit should not encounter any other part of the boundary between intersections.

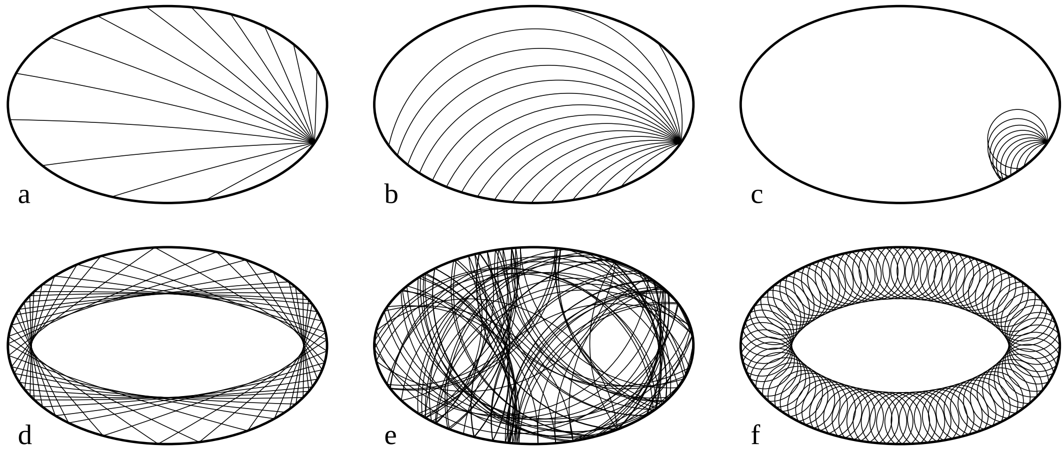

FIGURE 8.2. Curvature regimes for the magnetic billiard in a convex boundary, such that $1/\rho_{\max} \leqslant \kappa(s) \leqslant 1/\rho_{\min}$. Trajectories with fixed s_0 show that the bouncing map is (a) a continuous twist map when $\mu > \rho_{\max}$, (b) discontinuous when $\rho_{\min} < \mu < \rho_{\max}$, and (c) continuous but not twist when $\mu < \rho_{\min}$. Trajectories with $\theta_0 \sim 0$ may be quasiperiodic when (d) $\mu > \rho_{\max}$ or (f) $\mu < \rho_{\min}$, but not when (e) $\rho_{\min} < \mu < \rho_{\max}$.

- if the boundary has bounded curvature, $|\kappa(s)| \leqslant 1/\rho_{\min}$, and if μ is smaller than the radius $\mu^\star$ of the smallest circle inscribed in Q; then it describes backward skipping trajectories, performing almost complete circles (Fig. 8.2f);

In all these cases, Moser's Theorem 8.1 shows that for sufficiently smooth boundary, some orbits starting with a sufficiently small normal velocity will remain on invariant curves, and thus stay close to the boundary [BK1]. Such solutions are sometimes called "whispering gallery modes", and may be associated with caustics. Their existence was first proved by Lazutkin in the zero field case [La]. Mather showed that they disappear when the curvature of the boundary is allowed to vanish [Mat]. Our result shows that the magnetic field has a stabilizing effect, since it is only required that the curvature be larger than $-1/\mu$. These orbits correspond physically to diamagnetic currents along the boundary.

Let us point out that the first two types of skipping orbits (with a length approaching zero as $\sin\theta \to 0$) can be analysed by a different technique [Zh]. For simplicity, we consider the zero field case. In the neighborhood of the boundary, the dynamics can be described by the Hamiltonian

$$H = \frac{p_r^2}{2} + \frac{p_s^2}{2(1-\kappa(s)|r|)^2}, \tag{8.46}$$

where r is the distance to the boundary, and $|r|$ accounts for the reflections. The idea is to use the fact that for small $\sin\theta$, the particle sees a slowly varying curvature. Performing a canonical transformation, which uses $-p_s$ as new Hamiltonian and s as new time, and rescaling the variables in a proper way ($r = \varepsilon^2 x, p_r = \varepsilon y, s = \varepsilon t = \tau$), one obtains the Hamiltonian

$$H = \frac{y^2}{2} + \kappa(\tau)|x| + \mathcal{O}(\varepsilon^2), \tag{8.47}$$

which is of the form studied in the previous subsection, and thus reducible to (8.36). This method is easily generalized to billiards with external fields [Zh].

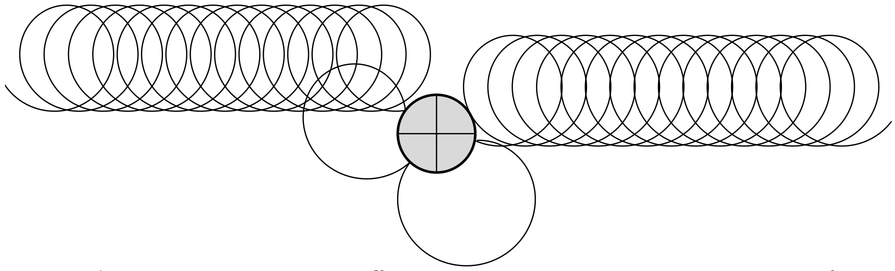

FIGURE 8.3. A trajectory scattered off the hard disc. Trajectories coming in from infinity leave the scatterer again with probability one. However, some orbits may form "bound states" which are indefinitely bouncing on the scatterer.

Another near–integrable limit is obtained for high magnetic field, $\mu \ll 1$. If the boundary has a bounded curvature and is of class $\mathcal{C}^k$, $k \geqslant 3$, one can show [BK1] that for sufficiently small μ, the bouncing map is $\mathcal{C}^{k-1}$ and of the form

$$\begin{aligned} s_{m+1} &= s_m - 2\mu \sin\theta_m + \mu^2 \sin\theta_m \, a(s_m, \theta_m, \mu) \pmod{|\partial Q|}, \\ \theta_{m+1} &= \theta_m + \mu^2 \sin^2\theta_m \, b(s_m, \theta_m, \mu). \end{aligned} \tag{8.48}$$

The functions $a \in \mathcal{C}^{k-2}$ and $b \in \mathcal{C}^{k-3}$ are uniformly bounded for $s \in \mathbb{R}, 0 \leq \theta \leq \pi$, $|\partial Q|$-periodic in s, and admit expansions in μ which can be explicitly computed. The first terms are

$$\begin{aligned} a(s,\theta,\mu) &= -2\cos\theta\,\kappa(s) - \tfrac{2}{3}\big[(1+2\cos 2\theta)\kappa(s)^2 - \sin 2\theta\,\kappa'(s)\big]\mu + \mathcal{O}(\mu^2), \\ b(s,\theta,\mu) &= \tfrac{2}{3}\kappa'(s) + \tfrac{2}{3}\big[2\cos\theta\,\kappa(s)\kappa'(s) - \sin\theta\,\kappa''(s)\big]\mu + \mathcal{O}(\mu^2). \end{aligned} \tag{8.49}$$

Note that θ is already an adiabatic invariant for a number of bounces of order μ^{-1}, but since the time between two bounces is of order μ, this means that θ varies by an amount of order μ in a time of order 1, which is not very interesting. But one can improve this result, using Theorem 8.2 to construct a better invariant $J(s,\theta)$, such that $J(s_{m+1},\theta_{m+1}) = J(s_m,\theta_m) + \mathcal{O}(\mu^{k+1})$. This quantity varies by $\mathcal{O}(\mu)$ in a time of order μ^{2-k}. In particular, if ∂Q is analytic, J varies only slightly during a time of order $\mathrm{e}^{1/C|\mu|} \approx \mathrm{e}^B$. This invariant admits the expansion

$$J(s,\theta;\mu) = \theta + \mu\sin\theta\big[\tfrac{1}{3}\kappa(s) + \tfrac{2}{9}\mu\cos\theta\,\kappa(s)^2 + \mathcal{O}(\mu^2)\big]. \tag{8.50}$$

The level curves of $J(s,\theta)$ are good approximations of the orbits during a time of order μ^{2-k} or $\mathrm{e}^{1/C|\mu|}$.

To conclude this summary, let us briefly mention an example with an external electric field. Consider the scattering system outside a disc of unit radius, with a small constant in–plane electric field, in addition to the homogeneous magnetic field perpendicular to the plane. When there is no electric field, trajectories starting close to the scatterer surround it on "rosette" orbits [KS]. When the electric field is switched on, the trajectories become cycloids drifting in a direction perpendicular to both fields (Fig. 8.3).

It follows from measure preservation that trajectories drifting in from "infinity" will leave the scatterer again with probability 1. The question is whether trajectories starting initially close to the scatterer will leave it after a finite number of bounces, or may collide with it forever, forming a classical "bound state". Trugman studied a similar case, with a horizontal segment instead of a circular scatterer, and showed that bound states form only resonances with zero measure [Tr].

For the circular scatterer, intuition might tell us that the electric field will ultimately manage to pull the particle away from the disc. In fact, for sufficiently small electric field, this is not necessarily the case. It turns out that orbits starting with a sufficiently *large* normal velocity will return to the boundary at least once, according to the map

$$\begin{aligned} s_{m+1} &= s_m + \Omega(\theta_m) + \varepsilon f(s_m, \theta_m, \varepsilon) \pmod{2\pi}, \\ \theta_{m+1} &= \theta_m + \varepsilon g(s_m, \theta_m, \varepsilon), \end{aligned} \tag{8.51}$$

where ε is proportional to the electric field. In [BHHP], we showed that this map is correct for very small magnetic field. In [Berg], we show that it actually works for all fields such that the Larmor radius is larger than the radius of the scatterer. Using Moser's Theorem 8.1, we obtain the existence of invariant curves, and since these curves act as a barrier, orbits trapped between them cannot escape the scatterer. This is true at sufficiently low electric field. By studying elliptic islands in the vicinity of orbits of period 2, existence of bound states could be proved for relatively large electric fields, up to a value corresponding roughly to a drift per cycle of the order of the scatterer radius.

Chapter 9

Conclusion and Outlook

"A conclusion is simply the place where someone got tired of thinking."

9.1 Summary of Main Results

9.1.1 Adiabatic Dynamical Systems

One of our goals in this work was to establish a mathematical framework to deal with adiabatic equations of the form

$$\varepsilon \dot{x} = F(x, \lambda(\tau)). \tag{9.1}$$

In order to do this, we chose to favour geometric methods, allowing to derive the main qualitative features of the solutions (such as existence and scaling properties of hysteresis cycles), with a minimum of analytic calculations. The main points of this procedure are the following:

1. **Determine the static bifurcation diagram:** The equilibrium branches are obtained by solving the algebraic equation $F(x, \lambda) = 0$, and bifurcation points are those for which, moreover, $\det \partial_x F(x, \lambda) = 0$. For many qualitative properties of the solutions, it is not necessary to know these equilibrium branches with the last precision. It is, in general, sufficient to know bifurcation points and the first terms of the Taylor series of F around them, as well as how these points are connected by equilibrium branches.
2. **Perform a local analysis around bifurcation points:** This analysis is done in two steps. The center manifold reduction described in Subsection 5.4.4 allows to describe the dynamics by an effective low–dimensional equation. In the most generic cases, this effective equation can be analysed with the methods of Sections 4.3 or 4.4. The most important is to determine whether the solutions, after crossing the bifurcation point, will follow a stable or unstable branch, or escape the neighborhood of the bifurcation to reach some other attractor.
3. **Determine future and past of some adiabatic solutions:** In particular, when $\lambda(\tau)$ is periodic, this is the first step of the construction of the Poincaré map. Solutions starting close to a stable equilibrium branch will follow it until the next bifurcation.

Since they are exponentially attracting, their evolution during one period allows to determine the value of the Poincaré map for a large set of initial conditions. Solutions starting near completely unstable equilibria can be followed backwards in time. With these informations, it may already be possible to determine some fixed points of the Poincaré map.

4. **Analyse solutions in a neighborhood of adiabatic ones:** This analysis is necessary to determine the stability of periodic orbits, and the boundary between basins of attraction. It is done by studying the linearized equation $\varepsilon\dot{y} = A(\tau,\varepsilon)y$, with the method of smooth diagonalization developed in Section 5.3. To leading order, its solutions are determined by eigenvalues and eigenspaces of A.
5. **Control the effect of nonlinear terms:** Our results on adiabatic manifolds and dynamic normal forms show that the orbits of the linearized system are in general a good approximation to the orbits of the nonlinear system, when they remain close to the origin. A more difficult situation arises when orbits leave this neighborhood, to jump (hopefully) on some other attractor. The example in Section 6.1 shows that adiabatic manifolds and normal forms can be helpful to control this jump, when they are defined in a fairly large neighborhood of the equilibrium branch.
6. **Find hysteresis cycles and determine their scaling properties:** Once the Poincaré map is sufficiently well known, it should be possible to determine its fixed points or periodic orbits. Using the informations of point 2., we immediately obtain the shape of the corresponding solutions in the (λ, x)–space, and we can determine whether they describe hysteresis cycles. More detailed properties, such as scaling laws, can be derived by computing the dependence on ε at leading order. In general, the dominant contributions to the hysteresis area are due to bifurcation points, and can be determined in a simple way, using Newton's polygon (see Section 4.3).

The method described here works fine when most solutions follow equilibrium branches most of the time. This corresponds to the rather common situation when fast variables are enslaved by slow ones. There are, of course, more complicated situations, arising when solutions follow periodic orbits or more complicated attractors of the instantaneous system. Some methods have already been developed to deal with such problems, but much work still needs to be done. We come back to this point in Section 9.2.

Up to now, we have discussed the problem of analysing the adiabatic system (9.1), when the static bifurcation diagram is known. In the introduction, we wondered whether a reliable approximation of the static bifurcation diagram could be obtained when the control parameter is varied slowly in time. Our analysis shows that in the adiabatic limit, the system indeed tends to follow certain equilibrium branches (mostly stable ones, but sometimes also unstable branches, due to the phenomenon of bifurcation delay). Bifurcations will cause the system to switch between different equilibrium branches, so that by varying the control parameter back and forth, one gets a chance of visualizing several, but not necessarily all asymptotic states of the static system.

9.1.2 Hysteresis

The second aim of this work was to apply our general method to some concrete examples involving hysteresis. Hysteresis is a common phenomenon in adiabatic differential equations, depending on a periodically varying parameter $\lambda(\tau)$. In phenomenological models, hysteresis is often considered as rate-independent. This is clearly not true for ODEs, but in the adiabatic limit, it is sometimes possible to obtain an asymptotic behaviour which can be considered as a first approximation to real, rate–dependent hysteresis. We found the following types of hysteresis, depending on the nature of equilibrium branches.

1. **Absence of hysteresis:** If $X^\star(\lambda)$ is an asymptotically stable equilibrium branch for all values of $\lambda(\tau)$, then any orbit starting in some neighborhood of this branch at τ_0 satisfies the relation

$$\lim_{\varepsilon\to 0} x(\tau;\varepsilon) = X^\star(\lambda(\tau)) \qquad \forall\tau > \tau_0. \tag{9.2}$$

This is the simplest situation, in which the asymptotic and adiabatic limit commute. The state of the fast system is entirely determined by the slow variable, thus we have reduction of variables[1] and no hysteresis. This behaviour may subsist in presence of certain bifurcations, for instance with transcritical bifurcations or with bifurcation delay.

2. **Periodic hysteresis:** When x is a scalar variable, and under very general conditions on F, any asymptotic solution is necessarily periodic and satisfies

$$\lim_{\varepsilon\to 0} x(\tau;\varepsilon) = X^\star_{j[\lambda(\cdot)](\tau)}(\lambda(\tau)), \tag{9.3}$$

where $X^\star_j(\lambda)$ are different equilibrium branches of F, and $j[\lambda(\cdot)](\tau)$ is some functional operator, associating an index $j(\tau)$ with the function $\lambda(\tau)$. This solution displays hysteresis if there exist times τ_1 and τ_2 such that $j(\tau_1) \neq j(\tau_2)$, although $\lambda(\tau_1) = \lambda(\tau_2)$. Such a situation may arise when F contains bifurcation points with a different number of branches on each side, such as saddle–node or pitchfork bifurcations. The function $j(\tau)$ may jump every time $\lambda(\tau)$ crosses a bifurcation point. This situation is a generalization of **relay hysteresis** described in Section 3.4. Relation (9.3) may remain true for vectorial x.

3. **Rate–dependent or chaotic hysteresis:** When x is a vectorial variable, however, asymptotic solutions do not always satisfy (9.3). We found such an example in Section 6.1, when studying the rotating pendulum. For certain functions $\lambda(\tau)$, the limit (9.3) is not defined. When ε is decreased, the system goes through an infinite sequence of alternating regimes, in which the asymptotic solution is sometimes periodic with the same period than λ, sometimes periodic with a larger period, and sometimes even chaotic. In this example, it was essential, for this behaviour to occur, that solutions could rotate around the equilibrium branches, so that the acquired phase would cause them to choose between different equilibrium branches at a later time.

When the adiabatic parameter ε is increased, the hysteresis cycle changes its shape. We have confirmed the observation made by several authors that its area follows a scaling law of the form

$$\mathcal{A}(\varepsilon) - \mathcal{A}(0) \approx \varepsilon^\mu, \tag{9.4}$$

and showed that the exponent μ can be related in a simple way to local properties of bifurcation points. The most common exponents are $\mu = \frac{2}{3}$ for saddle–node bifurcations and $\mu = \frac{3}{4}$ for pitchfork bifurcations. In fact, not only does the area obey these scaling

[1]In the case of the slow–fast system $\varepsilon\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, this reduction of variables leads to the effective system $\dot{y} = g(X^\star(y), y)$.

laws, but also do solutions near bifurcation points follow piecewise linear scaling laws of the form $x - x^\star(\tau) \sim |\tau - \tau^\star|^\alpha \varepsilon^\beta$.

We presented in Section 3.4 some phenomenological models of hysteresis. As we already pointed out, the hysteretic behaviour occurring in ODEs resembles, in the adiabatic limit, the so–called **relay hysteresis**. Other models have been introduced to describe **active hysteresis**, in which minor hysteresis loops can be produced by varying the input periodically back and forth between two intermediate values. Since these models are often used to describe hysteresis in real ferromagnets, the question arises whether such complicated systems can be described, even approximately, by adiabatic differential equations.

In some limiting cases, as for lattice models with infinite range interaction in the thermodynamic limit, such a reduction of variables is possible. The system can be described by an effective mean–field equation of Ginzburg–Landau type. The properties of hysteresis cycles are determined by a competition between two collective mechanisms of magnetization reversal, by rotation and spin flip. Taking into account the anisotropy of interactions, we could use the methods of adiabatic ODEs to classify the possible types of hysteresis, and to show that the hysteresis area obeys indeed some simple scaling laws.

More realistic models, with short range interactions and taking into account finite size effects, display a third mechanism of magnetization reversal, by thermic activation. Here it is not obvious that the system can be modeled by low–dimensional differential equations. Hysteresis depends in a more detailed way on the dynamics of droplet growth, and it seems probable that the scaling laws proposed in the literature are only artifacts, due to the finite range of amplitudes and frequencies on which the system was observed.

9.2 Some Extensions and Open Problems

To finish this conclusion, we would like to mention some problems which have not been treated in this work, some domains to which the results could be extended, and to cite some related work by other authors, as well as some connections between adiabatic theory and other domains of interest, to which it would be worth applying the techniques developed here.

Periodic solutions

The analysis of adiabatic systems relies mainly on a description of the dynamics in a neighborhood of attractors of the instantaneous system. In the present work, we have considered the simplest attractors, namely equilibrium points. This is the only situation when the dynamics of the fast variable becomes slow, since it follows adiabatically the slowly varying equilibrium of the instantaneous system. Another situation arises when the fast system admits a periodic orbit. Then the fast system displays rapid oscillations with slowly varying amplitude and frequency.

The fact that solutions of the adiabatic system will relax to a stable periodic orbit has already been proved in [PR]. In his first paper [Ne1], Neishtadt showed the existence of a bifurcation delay, when the Poincaré map of the periodic orbit undergoes Hopf bifurcation. The exact expression of the delay time is more complicated, and was obtained only recently in [NST]. The motion near a stable periodic orbit does also display geometric phase shifts, when the parameters follow a closed loop in parameter space [KK, KKE].

It would be worth looking at the effect of other bifurcations, such as period doubling. Informations on invariant manifolds would also be useful.

Infinite–dimensional systems

We have restricted our discussion to finite–dimensional equations of the form $\varepsilon\dot{x} = f(x,\tau)$, $x \in \mathbb{R}^n$. When dealing with adiabatic manifolds and dynamic normal forms, however, we encountered adiabatic partial differential equations of the form

$$\varepsilon\partial_\tau u(x,\tau,\varepsilon) = F(u, \partial_x u, \tau, \varepsilon), \tag{9.5}$$

see equations (5.146) and (5.165). We circumvented the difficulty of solving these equations by transforming them into an integral fixed point problem. Since the results, however, are similar to those of the finite–dimensional case,[2] it could be possible to simplify the proofs substantially by using Lyapunov functions, in an appropriate function space.[3]

Some properties of linear equations of the form $\varepsilon\dot{y} = A(\tau,\varepsilon)y$ are known to work for infinite–dimensional systems, when the eigenvalues of the operator A admit a strictly positive gap [JKP]. Situations involving continuous spectrum are far more difficult to analyse, already in the autonomous situation.

Complex bifurcations

We have provided a detailed method to analyse the dynamics near bifurcation points of real scalar equations $\varepsilon\dot{x} = f(x,\tau)$, $x \in \mathbb{R}$. In the case of a complex variable, we only discussed a particular type of bifurcation. It would be useful to do a more detailed analysis of the complex situation, since it also provides a method to deal with eigenvalue crossings in the time–dependent Schrödinger equation. It may be possible to use some results in [Hag] to estimate integrals of the form $\int \tau^q \exp(\mathrm{i}\,\tau^{p+1})\,\mathrm{d}\tau$ which appear in this problem.

Iterated maps

Similar remarks apply to iterated maps, for which we did not examine the dynamics near bifurcation points. It would be interesting to determine whether these dynamics are characterized by similar scaling laws. The phenomenon of bifurcation delay is known to exist for period doubling bifurcations, and has been analysed in [Bæ].

Effect of noise

The dynamics of the magnetization for Curie–Weiss type models can be described, in the thermodynamic limit, by a deterministic equation. Corrections due to the finite size of the system obey a Langevin equation. It is thus important to understand the effect of a small additive noise on the properties derived from deterministic equations. We believe that scaling laws are relatively robust to noise with an amplitude of order ε. More subtle phenomena, such as bifurcation delay, might not survive a noise which is not exponentially small. It is known, moreover, that the introduction of noise may induce new phenomena, like stochastic resonance [CFV, SRN], escape and synchronization [SL, BER].

[2] We showed that adiabatic manifolds admit asymptotic series in ε, which can be truncated so as to yield exponentially accurate approximations in the analytic case. We did not prove the existence of an exponential bound for normal forms.

[3] We used such a technique in a simpler situation in Lemma 6.1.

Multiple time scales

Physical models involving multiple time scales are often treated by a specific perturbation technique, in which solutions of the form $x(t, \varepsilon t, \varepsilon^2 t, \dots)$ are sought [**Pi**]. Since this method has not always been put on a rigorous footing, it would be interesting to examine this technique in the light of the new insights obtained recently for adiabatic systems.

Control theory

We have mainly discussed the response of a system subject to a slow parameter variation imposed from outside. One could ask the inverse question, of how to move the parameter in order that the system follow some prescribed trajectory. In the adiabatic limit, this problem translates into finding the output of the hysteresis operator (9.3) for a set of inputs $\lambda(\tau)$ in some function space. For instance, the properties of dynamics near bifurcations could be used to design a device, for which the slow motion of the input takes the system from an initial to one of several final states, with a possibility to switch between these final states by modifying the structure of the bifurcation.

Consider for example the equation

$$\varepsilon \dot{x} = \lambda(\tau) x - x^3 + \delta, \qquad \lambda(\tau) = \mathrm{th}(\tau), \tag{9.6}$$

which corresponds to the imperfect pitchfork bifurcation considered in [EM]. The asymptotic state $x(+\infty)$ will be equal to x_+ if $\delta > 0$ and to x_- if $\delta < 0$, where $x_\pm$ are respectively the largest and smallest solution of $x - x^3 + \delta = 0$.

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“Au commencement de ce siècle, et à peu près en même temps, (...) Lobatchevsky et Bolyai établirent d'une façon irréfutable que cette démonstration est impossible; (...) depuis lors l'Académie des Sciences ne reçoit plus guère qu'une ou deux démonstrations nouvelles par an.”

Henri Poincaré, “La Science et l'Hypothèse”

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"The reward of a thing well done is to have done it."

Emerson

"The world is coming to an end! Repent and return those library books!"

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