A "thermodynamic" characterisation of some regularity structures near the subcriticality threshold

1. Motivation

[N.B & C. Kuehn, EJP 2016]

FitzHugh-Nagumo eq: $\begin{cases} \partial_t u = \Delta u + u - u^3 + v + \xi \\ \partial_t v = a_1 u + a_2 v \end{cases}$ $\qquad x \in \mathbb{T}^d, t > 0, \; d \in \{2, 3\}$

Thm: $\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon + \underbrace{[1 + C(\varepsilon)] u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon}_{F_\varepsilon(u^\varepsilon, v^\varepsilon)} + \xi^\varepsilon \\ \partial_t v^\varepsilon = a_1 u^\varepsilon + a_2 v^\varepsilon \end{cases}$ ← mollified space-time white noise

admits, for appropriate $C(\varepsilon)$, local solutions $(u^\varepsilon, v^\varepsilon)$ cv to limit as $\varepsilon \to 0$.

New difficulty: $\begin{cases} u = G * (F_\varepsilon(u, v) + \xi^\varepsilon) \\ v_t = \int_0^t u_s \, e^{(t-s)a_2} a_1 \, ds \end{cases}$ ← $v = Qu$, only smoothing in time

$$\begin{array}{ccc} (u_0, Z^\varepsilon) & \longrightarrow & U \in \mathcal{D}^\gamma \\ \Psi \uparrow & & \downarrow \mathcal{R} \\ (u_0, \xi^\varepsilon) & \longrightarrow & u^\varepsilon \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{D}^\gamma & \xrightarrow{\mathcal{K}} & \mathcal{D}^{\gamma+2} \\ \mathcal{R} \downarrow & & \downarrow \mathcal{R} \\ \mathcal{C}_\mathfrak{s}^\alpha & \xrightarrow[G*]{} & \mathcal{C}_\mathfrak{s}^{\alpha+2} \end{array}$$

$\mathcal{K} = \mathcal{I} + \mathcal{J} + \mathcal{N}$ $\qquad \mathcal{J}, \mathcal{N}$ polynomial but depend on derivatives of G

Workaround:
1) define new symbols $\mathcal{E}\tau$ only for $|\tau|_\mathfrak{s} < 0$
2) $f = \sum_{|\tau|_\mathfrak{s} < 0} c_\tau \tau + \sum_{|\tau|_\mathfrak{s} \geq 0} c_\tau(t, x) \tau$

$\Rightarrow \mathcal{K}^Q f = \sum_{|\tau|_\mathfrak{s} < 0} c_\tau \mathcal{E}\tau + \sum_{|\tau|_\mathfrak{s} \geq 0} (Q * c_\tau) \tau$

Alternatives:
a) $\partial_t v = \delta \Delta v + a_1 u + a_2 v$ $\qquad \delta \to 0$: discont.

b) $\partial_t v = \Delta^{\rho/2} v + a_1 u + a_2 v$ $\qquad \rho \to 0$

2. Fractional Laplacian

kernel $\quad G_\rho(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t\|\xi\|^\rho} d\xi$

semigroup $\quad P_t f(x) = \int_{\mathbb{R}^d} G_\rho(t,x-y) f(y)\, dy \qquad$ Lévy process

inf. generator $\quad \underbrace{-(-\Delta)^{\rho/2}}_{=\Delta^{\rho/2}} f(x) = \varkappa(d,\rho) \lim_{\varepsilon\to 0+} \int_{\|y\|>\varepsilon} \frac{f(x+y)-f(x)}{\|y\|^{d+\rho}}\, dy$

Aim: Consider $\quad \partial_t u = \Delta^{\rho/2} u + \underbrace{F(u)}_{=\sum_{j=0}^N a_j u^j} + \xi \quad$ on $\mathbb{T}^d$ $\quad (N\geq 2,\ a_N \neq 0)$

- when can we apply the theory of regularity structures?
- how does the structure depend on ρ, N and d?

Easy Fact 1: For the scaling $\mathfrak{s} = (\rho, 1, \dots, 1)$, G_ρ is regularising of order ρ.
(i.e. $f \in \mathcal{C}^\alpha_\mathfrak{s}$, $\alpha+\rho \notin \mathbb{Z}$ $\Rightarrow$ $G_\rho * f \in \mathcal{C}^{\alpha+\rho}_\mathfrak{s}$)
(scaling argument)

Easy Fact 2: For $\mathfrak{s} = (\rho, 1, \dots, 1)$, space-time white noise $\xi \in \mathcal{C}^\alpha_\mathfrak{s}$
$\forall \alpha < -\frac{\rho+d}{2}$

Easy Fact 3: For space-time white noise, the equ. is locally subcritical $\iff \rho > \rho_c = d\,\frac{N-1}{N+1}$

$\rho_c < 2$? $\quad d=2:\ N<\infty$
$\quad d=3:\ N\leq 4$
$\quad d\in\{4,5\}:\ N\leq 2$
$\quad d\geq 6:\ N=1$

$|\Xi|_\mathfrak{s} = \alpha_0 = -\frac{\rho+d}{2} - \varkappa$

$|\mathcal{I}(\Xi)|_\mathfrak{s} = \alpha_0 + \rho = \frac{\rho-d}{2} - \varkappa$

$|\mathcal{I}(\Xi)|^N_\mathfrak{s} = \frac{N}{2}(\rho-d) - N\varkappa$

$> \alpha_0 \iff \rho > \rho_c$

3. Size of model space and index set

$$U = \mathcal{I}(\Xi + F(U)) + q\mathbf{1} + \dots$$

Inductive construction: $\mathcal{W}_0 = \mathcal{U}_0 = \emptyset$

$$\begin{cases} \mathcal{W}_m = \mathcal{W}_{m-1} \cup \mathcal{U}_{m-1} \cup \dots \cup \mathcal{U}_{m-1}^N \cup \{\Xi\} \\ \mathcal{U}_m = \{X^k\} \cup \mathcal{I}(\mathcal{W}_m) \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{F}_F = \bigcup_{m\geq 0} (\mathcal{W}_m \cup \mathcal{U}_m) : \text{ symbols to represent equation} \\ \mathcal{U}_F = \bigcup_{m\geq 0} \mathcal{U}_m \quad : \text{ symbols to represent solution} \end{cases}$$

$$A_F = \{ |\tau|_{\mathfrak{s}} : \tau \in \mathcal{F}_F \}$$

$$\begin{cases} h_F := \#(A_F \cap \mathbb{R}_-) \\ c_F := \# \{ \tau \in \mathcal{F}_F : |\tau|_{\mathfrak{s}} < 0 \} \end{cases}$$

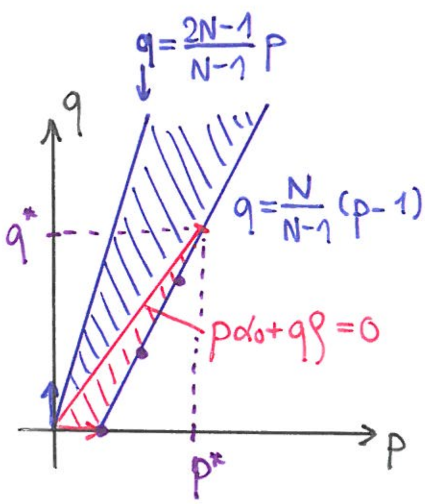
a) **Thm:** $$\frac{\rho+d}{N+1} \frac{1}{\rho - \rho_c} \leq h_F(N,d,\rho) \leq 1 + \frac{(\rho+d)^2 N}{N+1} \frac{1}{\rho-\rho_c}$$

$$\begin{cases} p(\tau) = \# \ \Xi\text{'s in } \tau \\ q(\tau) = \# \ \mathcal{I}\text{'s in } \tau \\ k(\tau) = \text{total exponent of poly terms} \end{cases}$$

$$|\tau|_{\mathfrak{s}} = p\alpha_0 + q\rho + |k|_{\mathfrak{s}}$$
$$\alpha_0 = -\frac{\rho+d}{2} - \varkappa$$

$$D_0(\mathcal{U}) = \{ (p(\tau), q(\tau)) : \tau \in \mathcal{U} \} \subset \mathbb{N}_0^2$$
($\mathcal{U} \subset \mathcal{F}_F$)

$$D_0(\mathcal{U}^n) = \text{convex envelope of } nD_0(\mathcal{U}) \cap \mathbb{N}_0^2$$



Then
- $\lim_{m\to\infty} D_0(\mathcal{U}_m) = \text{truncated cone} \Rightarrow$
- $|\tau|_{\mathfrak{s}} < 0 \Rightarrow p = 1 + \lfloor \frac{N-1}{N} q \rfloor$
- $h_F = \#$ points in 

where $$q^* = \frac{(\rho+d)N}{(N+1)(\rho-\rho_c)} + O(\varkappa)$$

b) **Thm:** $C_N^- (\rho-\rho_c)^{3/2} e^{\beta N d/(\rho-\rho_c)} \leq C_F(N, d, \rho) \leq C_N^+ (\rho-\rho_c)^{3/2} e^{\beta N d/(\rho-\rho_c)}$

$\tau \in \mathcal{F}_F \longleftrightarrow$ decorated tree (rooted)

$\mathcal{I}(\Xi)\mathcal{I}(\mathcal{I}(\Xi)^3)$

$\overset{1\text{ to }1}{\longleftrightarrow}$ bare tree

(noise additive)

q edges
p leaves

N=2 bare tree: degree ≤ 3

$d_i = \#$ vertices of deg i $\Rightarrow \begin{cases} d_1 + d_2 + d_3 = q+1 \\ d_1 + 2d_2 + 3d_3 = 2q \end{cases}$

$d_1 = p + 1_{\{\deg \rho = 1\}}$

Prop:
- $q = 2n \Rightarrow$ binary tree with $q+1$ vertices
- $q = 2n+1 \Rightarrow$ binary tree with $q+2$ vert. minus 1 edge

⚠  $\Rightarrow$ count non-isomorphic trees

$W_n = \#$ non-iso rooted binary trees with n leaves = Wedderburn-Etherington nb.

$W_n \sim C_2 \dfrac{(\alpha_2^{-1})^n}{n^{3/2}}$ $\quad \alpha_2 \cong 0.4027$ $\quad$ [R. Otter, 1948]

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4. Statistical prop.

Unif. prob. on $\mathcal{F}_F^- = \{\tau \in \mathcal{F}_F : |\tau|_\xi < 0\}$

$X : \mathcal{F}_F^- \to \mathbb{R}$ r.v. such as $Q, P, H_\xi, \dots$

Prop: $\mathbb{E}(H_\xi) = O(\rho-\rho_c)$

$\mathrm{Var}(H_\xi) = O(\rho-\rho_c)$

LDP $\quad -\lim_{\rho\to\rho_c} (\rho-\rho_c) \log \mathbb{P}(H_\xi \leq h) = \dfrac{2N}{N+1} \log(\alpha_N^{-1})(-h)$

$-\dfrac{d}{N+1} \leq h \leq 0$