

Probability Seminar, University of Warwick

An Eyring–Kramers formula for one-dimensional parabolic SPDEs with space-time white noise

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Warwick, 9 December 2015

Joint work with Barbara Gentz (Bielefeld)

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + f(u(x, t))$$

- ▷ $x \in [0, L]$
- ▷ $u(x, t) \in \mathbb{R}$
- ▷ Either periodic or zero-flux Neumann boundary conditions
- ▷ In this talk: $f(u) = u - u^3$ (results more general)

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Energy function:

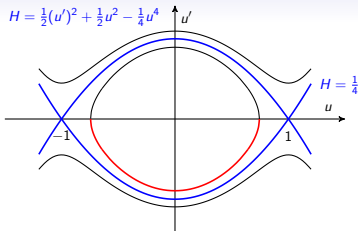
$$V[u] = \int_0^L \left[\frac{1}{2} u'(x)^2 - \frac{1}{2} u(x)^2 + \frac{1}{4} u(x)^4 \right] dx \quad \rightarrow \min$$

Stationary solutions: $u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

Stationary solutions

$$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$$

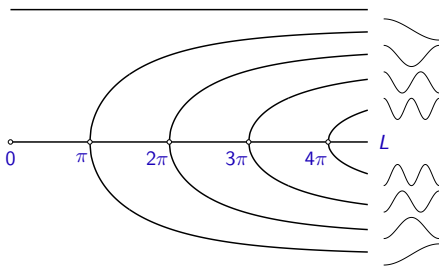
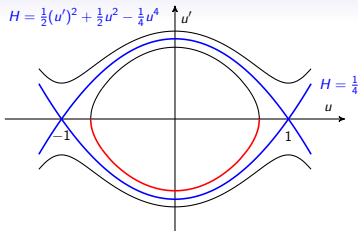
- ▷ $u_{\pm}(x) \equiv \pm 1$
- ▷ $u_0(x) \equiv 0$
- ▷ Nonconstant solutions satisfying b.c.
(expressible in terms of Jacobi elliptic fcts)



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(expressible in terms of Jacobi elliptic fcts)
- ▷ Neumann b.c: k nonconstant solutions when $L > k\pi$



- ▷ Periodic b.c: k families when $L > 2k\pi$

Stability of stationary solutions

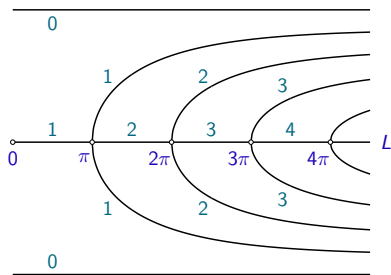
$$u_0''(x) = -u_0(x) + u_0(x)^3$$

Variational eq around u_0 : $\partial_t v_t(x) = v_t''(x) + [1 - 3u_0(x)^2]v_t(x)$

Sturm–Liouville spectrum of RHS determines stability of u_0

- ▷ $u_{\pm} \equiv \pm 1$: always stable (global minima of V)
- ▷ $u_0 \equiv 0$: always unstable, eigenvalues $\lambda_k = \left(\frac{\beta k \pi}{L}\right)^2 - 1$
(Neumann b.c.: $\beta = 1$, periodic b.c.: $\beta = 2$)

Neumann b.c.:
Number of positive
eigenvalues
(= unstable directions)



Coarsening dynamics

[Carr & Pego 89, Chen 04]

([Link to simulation](#))

Stochastic partial differential equations

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) + \sqrt{2\varepsilon} \ddot{W}_{tx} \quad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

\ddot{W}_{tx} : space-time white noise (formal derivative of Brownian sheet)

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Construction of mild solution via Duhamel formula:

$$\triangleright \dot{u}_t = \Delta u_t \quad \Rightarrow \quad u_t = e^{\Delta t} u_0$$

where $e^{\Delta t} \cos\left(\frac{k\pi x}{L}\right) = e^{-(k\pi/L)^2 t} \cos\left(\frac{k\pi x}{L}\right)$

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$$\triangleright \dot{u}_t = \Delta u_t + \sqrt{2\varepsilon} \ddot{W}_{tx} \quad \Rightarrow \quad u_t = e^{\Delta t} u_0 + \underbrace{\sqrt{2\varepsilon} \int_0^t e^{\Delta(t-s)} \dot{W}_x(ds)}_{=: w_t(x)}$$

$$w_t(x) = \sum_k \int_0^t e^{-(k\pi/L)^2(t-s)} dW_s^{(k)} \cos\left(\frac{k\pi x}{L}\right) \in H^s \cap C^\alpha \quad \forall s, \alpha < \frac{1}{2}$$

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$$\triangleright \dot{u}_t = \Delta u_t + \sqrt{2\varepsilon} \ddot{W}_{tx} + f(u_t)$$

$$\Rightarrow \quad u_t = e^{\Delta t} u_0 + \sqrt{2\varepsilon} w_t + \int_0^t e^{\Delta(t-s)} f(u_s) ds =: F_t[u]$$

\Rightarrow Existence and a.s. uniqueness [Faris & Jona-Lasinio 1982]

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Fourier variables: $u_t(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} z_k(t) e^{i\pi kx/L}$

$$\Rightarrow dz_k = -\lambda_k z_k dt - \frac{1}{L} \sum_{k_1+k_2+k_3=k} z_{k_1} z_{k_2} z_{k_3} dt + \sqrt{2\varepsilon} dW_t^{(k)}$$

Itô SDE, $dW_t^{(k)}$: independent Wiener processes

$$\lambda_k = -1 + (\pi k/L)^2$$

Energy functional:

$$V[u] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \lambda_k |z_k|^2 + \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} z_{k_1} z_{k_2} z_{k_3} z_{k_4}$$

$$\Rightarrow dz_t = -\nabla V(z_t) dt + \sqrt{2\varepsilon} dW_t$$

Coarsening dynamics with noise

([Link to simulation](#))

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Question: transition time from $u_- \equiv -1$ to $u_+ \equiv +1$?

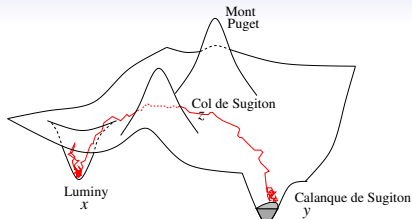
Metastability and Eyring–Kramers law

$$dx_t = -\nabla V(x_t) dt + \sqrt{2\varepsilon} dW_t$$

$V : \mathbb{R}^d \rightarrow \mathbb{R}$ confining potential

$$\tau_y^x = \inf\{t > 0 : x_t \in \mathcal{B}_\varepsilon(y)\}$$

first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$,
when starting in x



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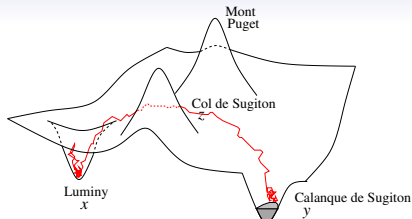
Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):

Eigenvalues of Hessian of V at minimum x : $0 < \nu_1 \leq \nu_2 \leq \dots \leq \nu_d$

Eigenvalues of Hessian of V at saddle z : $\lambda_1 < 0 < \lambda_2 \leq \dots \leq \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}} e^{[V(z)-V(x)]/\varepsilon} [1 + \mathcal{O}_\varepsilon(1)]$$



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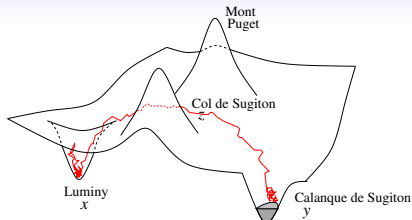
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Arrhenius' law: proved by Freidlin, Wentzell (1979) using large deviations

Eyring–Kramers law: Bovier, Eckhoff, Gayard, Klein (2004) using potential theory,

Helfer, Klein, Nier (2004) using Witten Laplacian, ...



Potential theory and Eyring–Kramers law

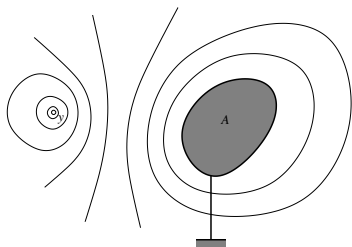
Consider first Brownian motion $W_t^x = x + W_t$

Fact 1: $w_A(x) = \mathbb{E}[\tau_A^x]$ satisfies

$$\begin{cases} -\frac{1}{2}\Delta w_A(x) = 1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$G_{A^c}(x, y)$ Green's function

$$\Rightarrow w_A(x) = \int_{A^c} G_{A^c}(x, y) dy$$



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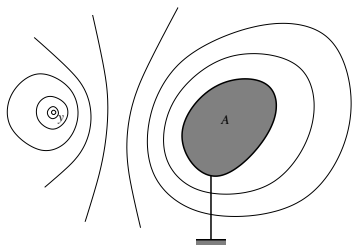
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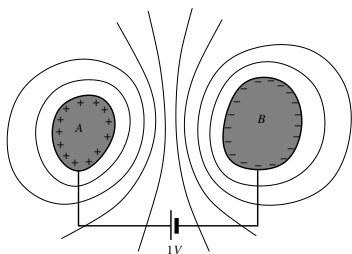


Fact 2: $h_{A,B}(x) = \mathbb{P}[\tau_A^x < \tau_B^x]$ satisfies

$$\begin{cases} -\frac{1}{2}\Delta h_{A,B}(x) = 0 & x \in (A \cup B)^c \\ h_{A,B}(x) = 1 & x \in A \\ h_{A,B}(x) = 0 & x \in B \end{cases}$$

$\rho_{A,B}$: "surface charge density" on ∂A

$$\Rightarrow h_{A,B}(x) = \int_{\partial A} G_{B^c}(x, y) \rho_{A,B}(dy)$$



Potential theory and Eyring–Kramers law

Capacity: $\text{cap}_A(B) = \int_{\partial A} \rho_{A,B}(dy) \Rightarrow \nu_{A,B}(dy) = \frac{\rho_{A,B}(dy)}{\text{cap}_A(B)}$ prob measure

Variational representation: Dirichlet form

$$\text{cap}_A(B) = \int_{(A \cup B)^c} \|\nabla h_{A,B}(x)\|^2 dx = \inf_{h \in \mathcal{H}_{A,B}} \int_{(A \cup B)^c} \|\nabla h(x)\|^2 dx$$

($\mathcal{H}_{A,B}$: set of sufficiently smooth functions satisfying b.c.)

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Key observation: let $C = \mathcal{B}_\varepsilon(x)$, then (using $G(y, z) = G(z, y)$)

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Harnack inequalities $\Rightarrow \mathbb{E}^x[\tau_A] \simeq \mathbb{E}^{\nu_{C,A}}[w_A] = \frac{1}{\text{cap}_C(A)} \int_{A^c} h_{C,A}(y) dy$

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General case: replace $\frac{1}{2}\Delta$ by $\varepsilon\Delta - \nabla V(x) \cdot \nabla$ and dx by $e^{-V(x)/\varepsilon} dx$

Eyring–Kramers law for SPDEs: heuristics

$$\dot{u}_t(x) = \Delta u_t(x) + f(u_t(x)) + \sqrt{2\varepsilon} \ddot{W}_{tx} \quad (\Delta \equiv \partial_{xx}, f(u) = u - u^3)$$

Initial condition: u_{in} near $u_- \equiv -1$ with eigenvalues ν_k

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0: \|u_t - u_+\|_{L^\infty}\} < \rho$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\text{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leq \beta\pi \\ u_1(x) \text{ } \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \end{cases} \quad \text{with ev } \lambda_k = \left(\frac{\beta k \pi}{L}\right)^2 - 1$$

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[Faris & Jona-Lasinio 82]: large-deviation principle

\Rightarrow Arrhenius law: $\mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$\Rightarrow \mathbb{E}^{u_{\text{in}}}[\tau_+] \simeq 2\pi \sqrt{\frac{1}{|\lambda_0| \nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}} e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}$

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- ▷ Rigorous proof?
- ▷ What happens when $L \rightarrow \beta\pi$ as then $\lambda_1 \rightarrow 0$?

Eyring–Kramers law for SPDEs: main result

Theorem 1: Neumann b.c. [B & Gentz, Elec J Proba 2013]

▷ If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0| \nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}} e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon} [1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2})]$$

▷ If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)

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- ▷ If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles)
- ▷ If $\pi - c \leq L \leq \pi$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{\lambda_1 + \sqrt{3\varepsilon/2L}}{|\lambda_0|\nu_0\nu_1} \prod_{k=2}^{\infty} \frac{\lambda_k}{\nu_k} \frac{e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}}{\Psi_+(\lambda_1/\sqrt{3\varepsilon/2L})}} [1 + R(\varepsilon)]$$

where Ψ_+ explicit, involves Bessel function $K_{1/4}$, $\lim_{\alpha \rightarrow \infty} \Psi_+(\alpha) = 1$

- ▷ If $\pi \leq L \leq \pi + c$, then same formula, with another function Ψ_- , involving Bessel functions $I_{\pm 1/4}$, $\lim_{\alpha \rightarrow \infty} \Psi_-(\alpha) = 2$

Sketch of proof: Spectral Galerkin approximation

$$u_t^{(d)}(x) = \frac{1}{\sqrt{L}} \sum_{k=-d}^d z_k(t) e^{i\pi kx/L} \quad \Rightarrow \quad dz_t = -\nabla V(z_t) dt + \sqrt{2\varepsilon} dW_t$$

Theorem [Blömker & Jentzen 13]

For all $\gamma \in (0, \frac{1}{2})$ there exists an a.s. finite r.v. $Z : \Omega \rightarrow \mathbb{R}_+$ s.t. $\forall \omega \in \Omega$

$$\sup_{0 \leq t \leq T} \|u_t(\omega) - u_t^{(d)}(\omega)\|_{L^\infty} < Z(\omega) d^{-\gamma} \quad \forall d \in \mathbb{N}$$

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Proposition (using potential theory)

$\exists \varepsilon_0 > 0 : \forall \varepsilon < \varepsilon_0 \exists d_0(\varepsilon) < \infty : \forall d \geq d_0 \exists \nu_d$ proba measure on $\partial B_r(u_-)$

$$\int_{\partial B_r(u_-)} \mathbb{E}^{\nu_0}[\tau_+^{(d)}] \nu_d(d\nu_0) = C(d, \varepsilon) e^{H(d)/\varepsilon} [1 + R(\varepsilon)]$$

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$$u_t^{(d)}(x) = \frac{1}{\sqrt{L}} \sum_{k=-d}^d z_k(t) e^{i\pi kx/L} \Rightarrow dz_t = -\nabla V(z_t) dt + \sqrt{2\varepsilon} dW_t$$

Theorem [Blömker & Jentzen 13]

For all $\gamma \in (0, \frac{1}{2})$ there exists an a.s. finite r.v. $Z : \Omega \rightarrow \mathbb{R}_+$ s.t. $\forall \omega \in \Omega$

$$\sup_{0 \leq t \leq T} \|u_t(\omega) - u_t^{(d)}(\omega)\|_{L^\infty} < Z(\omega) d^{-\gamma} \quad \forall d \in \mathbb{N}$$

Proposition (using potential theory)

$\exists \varepsilon_0 > 0 : \forall \varepsilon < \varepsilon_0 \exists d_0(\varepsilon) < \infty : \forall d \geq d_0 \exists \nu_d$ proba measure on $\partial \mathcal{B}_r(u_-)$

$$\int_{\partial \mathcal{B}_r(u_-)} \mathbb{E}^{\nu_0}[\tau_+^{(d)}] \nu_d(d\nu_0) = C(d, \varepsilon) e^{H(d)/\varepsilon} [1 + R(\varepsilon)]$$

Proposition (using large deviations and lots of other stuff)

$H_0 := V[u_{ts}] - V[u_-]$. $\forall \eta > 0 \exists \varepsilon_0, T_1, H_1 : \forall \varepsilon < \varepsilon_0 \exists d_0 < \infty$ s.t. $\forall d \geq d_0$

$$\sup_{\nu_0 \in \mathcal{B}_r(u_-)} \mathbb{E}^{\nu_0}[\tau_+^2] \leq T_1^2 e^{2(H_0 + \eta)/\varepsilon}, \quad \sup_{d \geq d_0} \sup_{\nu_0 \in \mathcal{B}_r(u_-)} \mathbb{E}^{\nu_0}[(\tau_+^{(d)})^2] \leq T_1^2 e^{2H_1/\varepsilon}$$

Main step of the proof

Set $T_{\text{Kr}} = C(\infty, \varepsilon) e^{H_0/\varepsilon}$

Let $B = \mathcal{B}_\rho(u_+)$ and define nested sets $B_- \subset B \subset B_+$ at L^∞ -distance δ

$$\Omega_{K,d} = \left\{ \sup_{t \in [0, KT_{\text{Kr}}]} \|v_t - v_t^{(d)}\|_{L^\infty} \leq \delta, \tau_{B_-}^{(d)} \leq KT_{\text{Kr}} \right\}$$

$$\mathbb{P}(\Omega_{K,d}^c) \leq \mathbb{P}\{Z > \delta d^\gamma\} + \frac{\mathbb{E}^{v_0^{(d)}}[\tau_{B_-}^{(d)}]}{KT_{\text{Kr}}} \stackrel{\text{Cauchy-Schwarz}}{\Rightarrow} \limsup_{d \rightarrow \infty} \mathbb{P}(\Omega_{K,d}^c) = \frac{M(\varepsilon)}{K}$$

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On $\Omega_{K,d}$ one has $\tau_{B_+}^{(d)} \leq \tau_B \leq \tau_{B_-}^{(d)}$

$$\Rightarrow \mathbb{E}^{v_0^{(d)}}[\tau_{B_+}^{(d)} \mathbf{1}_{\{\Omega_{K,d}\}}] \leq \mathbb{E}^{v_0}[\tau_B \mathbf{1}_{\{\Omega_{K,d}\}}] \leq \mathbb{E}^{v_0^{(d)}}[\tau_{B_-}^{(d)} \mathbf{1}_{\{\Omega_{K,d}\}}]$$

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Integrate against ν_d and use Cauchy-Schwarz to bound error terms:

$$\mathbb{E}^{v_0}[\tau_B \mathbf{1}_{\{\Omega_{K,d}^c\}}] \leq \sqrt{\mathbb{E}^{v_0}[\tau_B^2] \mathbb{P}(\Omega_{K,d}^c)}, \text{ take } d \rightarrow \infty \text{ and finally } K \text{ large}$$

Eyring–Kramers law for SPDEs: main result

Theorem 2: periodic b.c. [B & Gentz, Elec J Proba 2013]

▷ If $L < 2\pi - c$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon} [1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2})]$$

▷ If $2\pi - c \leq L \leq 2\pi$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \frac{\lambda_1 + \sqrt{3\varepsilon/L}}{\nu_1} \prod_{k=2}^{\infty} \frac{\lambda_k}{\nu_k} \frac{e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}}{\Theta_+(\lambda_1/\sqrt{3\varepsilon/L})} [1 + R(\varepsilon)]$$

where Θ_+ explicit, involves error function, $\lim_{\alpha \rightarrow \infty} \Theta_+(\alpha) = 1$

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where Θ_+ explicit, involves error function, $\lim_{\alpha \rightarrow \infty} \Theta_+(\alpha) = 1$

▷ Similar expression for $2\pi \leq L \leq 2\pi + c$ with $\lambda_1 = 0$

▷ If $L \geq 2\pi + c$, then

$$\mathbb{E} u_{\text{in}}[\tau_+] = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0}} \frac{\sqrt{2\pi\varepsilon\lambda_1}}{\nu_1} \prod_{k=2}^{\infty} \frac{\sqrt{\lambda_k\lambda_{-k}}}{\nu_k} \frac{e^{(V[u_{\text{ts}}] - V[u_-])/\varepsilon}}{L \|u'_{\text{ts}}\|_{L^2}} [1 + R(\varepsilon)]$$

$L \|u'_{\text{ts}}\|_{L^2} =$ “length of the saddle”

Outlook

- ▷ Potentials with **more than two wells**: essentially understood
- ▷ Potentials invariant under **symmetry group**: done for SDEs
- ▷ Allen–Cahn in **dimension $d \geq 2$** : only possible since Martin Hairer's theory of **regularity structures**

[Link]

- ▷ Open problem: **non-gradient systems**

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Thanks for your attention