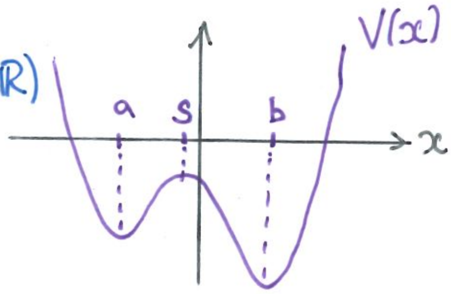
DK Seminar
TU Wien, 22/03/17

Metastability of stochastic Allen–Cahn equations on the torus

Collaborators: Barbara Gentz (Bielefeld), Giacomo Di Gesù (TU Wien), Hendrik Weber (Warwick)

1. Finite-dim case



Example 1: $dx_t = -V'(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t \quad (x \in \mathbb{R})$

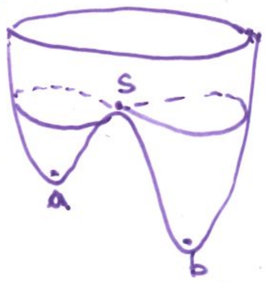
Invariant prob. density: $\frac{1}{Z} e^{-V(x)/\varepsilon}$

$\tau_A := \inf\{t > 0 : x_t \in A\}$ $\qquad \mathbb{E}^a[\tau_b] = ?$

Fact: $w_A(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases} \varepsilon w_A''(x) - V'(x) w_A'(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$

Solution for $A = [b, \infty)$: $\quad w_A(x) = \frac{1}{\varepsilon} \int_x^b \int_{-\infty}^z e^{[V(z)-V(y)]/\varepsilon}\,dy\,dz$

"Eyring-Kramers" $\qquad w_A(a) \overset{\text{Laplace}}{=} \frac{2\pi}{\sqrt{|V''(s)|\,V''(a)}}\, e^{[V(s)-V(a)]/\varepsilon}\,[1 + O(\varepsilon)]$



Example 2: $dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t \quad (x \in \mathbb{R}^n)$

$$\mathbb{E}^a[\tau_B] \cong 2\pi \sqrt{\frac{|\det \operatorname{Hess} V(s)|}{|\lambda_1(s)| \det \operatorname{Hess} V(a)}}\, e^{[V(s)-V(a)]/\varepsilon}$$

$\uparrow$ nbh of b

$\lambda_j(x)$: ev of $\operatorname{Hess} V(x)$

Potential-theoretic proof uses $\quad \mathbb{E}^{\mu_{A,B}}[\tau_B] = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} h_{A,B}(x) e^{-V(x)/\varepsilon}\,dx$

committor $\quad h_{A,B}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$

capacity $\quad \operatorname{cap}(A,B) = \varepsilon \int_{(A \cup B)^c} \|\nabla h_{A,B}(x)\|^2 e^{-V(x)/\varepsilon}\,dx$

equ. potential $\quad \mu_{A,B}$ concentrated on ∂A

2. Allen-Cahn on 1d torus

$$\partial_t u = u'' + u - u^3 + \sqrt{2\varepsilon}\,\xi \quad \leftarrow \text{space-time white noise}$$

$$\left(\mathbb{E}[\langle \xi, \varphi_1\rangle \langle \xi, \varphi_2\rangle] = \langle \varphi_1, \varphi_2\rangle\right)$$

$u = u(t,x)$, $t \geq 0$, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$

Potential: $$V[u] = \int_0^L \left[\frac{1}{2}u'(x)^2 - \frac{1}{2}u(x)^2 + \frac{1}{4}u(x)^4\right] dx$$

$$\Rightarrow \lim_{h \to 0} \frac{1}{h}\left(V[u+hv] - V[u]\right) = -\langle u'' + u - u^3, v\rangle$$

Stationnary solutions include $u_0(x) \equiv 0$ $\quad u_\pm(x) \equiv \pm 1$

Hessian: $$V[u] = \frac{1}{2}\langle -u'' - u, u\rangle + O(u^4)$$

$$\Rightarrow \underline{V''(0) = -\Delta - 1} \qquad \text{ev: } (-\Delta - 1)\,e^{ik2\pi x/L} = \underbrace{\left[\left(k\tfrac{2\pi}{L}\right)^2 - 1\right]}_{\lambda_k \in \mathbb{Z}} e^{ik2\pi x/L}$$

$$\underline{V''(\pm 1) = -\Delta + 2} \qquad \mu_k = \left(k\tfrac{2\pi}{L}\right)^2 + 2$$

$$\det\left([\underbrace{-\Delta_\perp}_{k\neq 0} - 1][-\Delta_\perp + 2]^{-1}\right) = \det\left([-\Delta_\perp + 2 - 3][-\Delta_\perp + 2]^{-1}\right)$$
$$= \det\left(\mathbb{1} - 3[-\Delta_\perp + 2]^{-1}\right) \qquad \underline{\text{Fredholm determinant}}$$

$$\log \det(\dots) = \operatorname{Tr}\log\left(1 - 3[-\Delta_\perp + 2]^{-1}\right) = -\sum_{n \geq 1} \frac{3^n}{n} \underbrace{\operatorname{Tr}\left[(-\Delta_\perp + 2)^{-n}\right]}_{\sim\,\left[\left(\frac{2\pi}{L}\right)^2 + 2\right]^{-n}}$$

3. Allen-Cahn on 2d torus

$$\partial_t u = \Delta u + u - u^3 + \sqrt{2\varepsilon}\,\xi \qquad \operatorname{Tr}\left[(-\Delta_\perp + 2)^{-1}\right] \sim \sum_{k \neq (0,0)} \frac{1}{\|k\|^2} = +\infty$$

$x \in \mathbb{T}^2$

[Da Prato & Debussche] $$\partial_t u = \Delta u + u - \left[u^3 - 3\varepsilon C_N u\right] + \sqrt{2\varepsilon}\,\underbrace{\xi_N}_{\text{mollified at scale } 1/N}$$

$$C_N = \sum_{|k| \leq N} \frac{1}{\lambda_k}$$

$$\Rightarrow \det\left([-\Delta_\perp - 1][-\Delta_\perp + 2]^{-1} e^{3C_N}\right) = \det\left(\mathbb{1} - 3[-\Delta_\perp + 2]^{-1}\right) e^{\operatorname{Tr}[-\Delta_\perp + 2]^{-1}}$$

<u>Carleman-Fredholm determinant</u>

requires operator to be only Hilbert-Schmidt