Operator scaling Gaussian random fields

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Outlines

1 Fractional Brownian motion

1.1 Definition

Kolmogorov (1940), Mandelbrot and Van Ness (1968).
For $H \in (0,1)$, the fractional Brownian motion $B_H = \left\{ B_H(x) \, ; x \in \mathbb{R}^d \right\}$ is a scalar values centered Gaussian random field with $B_H(0) = 0$ and

$$\mathrm{Cov}\left(B_H(x), B_H(x')\right) = C\left(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H}\right).$$

Properties :

- Stationary increments: $\forall x_0 \in \mathbb{R}^d$, $B_H(. + x_0) - B_H(x_0) \stackrel{fdd}{=} B_H(.)$
- Self-similar of order H: $\forall \lambda > 0$, $B_H(\lambda .) \stackrel{fdd}{=} \lambda^H B_H(.)$.
- Isotropic: $\forall R \in O_d(\mathbb{R})$, $B_H(R.) \stackrel{fdd}{=} B_H(.)$.

1.2 Harmonizable representation

Let W be a complex Brownian measure.
When $F \in L^2(\mathbb{R}^d)$ one can define

$$X_F = \mathrm{Re} \int_{\mathbb{R}^d} F(\xi) W(\mathrm{d}\xi) \sim \mathcal{N}\left(0, \int |F|^2\right).$$

Isometry property : if $F, G \in L^2(\mathbb{R}^d)$,

$$\mathrm{Cov}\,(X_F, X_G) = \mathrm{Re} \int_{\mathbb{R}^d} F(\xi)\overline{G(\xi)}\mathrm{d}\xi.$$

Harmonizable representation of fBm

$$B_H \overset{fdd}{=} \left\{ \mathrm{Re} \int_{\mathbb{R}^d} (e^{ix\cdot\xi} - 1)\|\xi\|^{-H-d/2} W(\mathrm{d}\xi); x \in \mathbb{R}^d \right\}.$$

Sketch of the proof.

Let us denote $\psi_H(\xi) = \|\xi\|^H$ and $v_H(x) = \int_{\mathbb{R}^d} \left|(e^{ix\cdot\xi} - 1)\psi_H(\xi)^{-1-d/2H}\right|^2 \mathrm{d}\xi$.
Then, using polar coordinates

$$v_H(x) = \int_{S^{d-1}} \int_0^{+\infty} \left|e^{ix\cdot r\theta} - 1\right|^2 r^{-2H-1} \mathrm{d}r\mathrm{d}\theta,$$

with

$$\left|e^{ix\cdot r\theta} - 1\right| \leq C(x)\min(r, 1).$$

$\hookrightarrow$ $v_H(x) < +\infty$ for $1/H > 1$.

Moreover,

$$\forall\lambda > 0, \qquad \psi_H\left(\lambda^{1/H}\xi\right) = \lambda\psi_H(\xi) \quad \curvearrowright \quad v_H(\lambda^{1/H}x) = \lambda^2 v_H(x),$$
$$\forall R \in O_d(\mathbb{R}), \qquad \psi_H(R\xi) = \psi_H(\xi) \quad \curvearrowright \quad v_H(Rx) = v_H(x).$$

$\hookrightarrow$ $v_H(x) = C\|x\|^{2H}$.

1.3 Hölder regularity

Definition Let $\gamma \in (0,1)$. A random field $\{X(x)\}_{x\in\mathbb{R}^d}$ admits γ as critical Hölder exponent on $B_d(0,1)$ if:

(a) $\forall s < \gamma$, a.s. X satisfies $H(s)$: $\exists A \geq 0$ random variable s.t. $\forall x, y \in B_d(0,1)$,

$$|X(x) - X(y)| \leq A\|x - y\|^s.$$

(b) $\forall s > \gamma$, a.s. X fails to satisfy $H(s)$.

Proposition:[Adler, 1981] Let $\{X(x)\}_{x\in\mathbb{R}^d}$ be a Gaussian random field. If

$$\mathbb{E}\left(X(x)-X(y)\right)^2 \asymp \|x-y\|^{2\gamma}$$

in the sense that $\forall\delta>0$, $\exists C_1, C_2>0$,

$$C_1\|x-y\|^{2\gamma+\delta} \leq \mathbb{E}\left(X(x)-X(y)\right)^2 \leq C_2\|x-y\|^{2\gamma-\delta}$$

➥ critical Hölder exponent on $B_d(0,1)=\gamma$.

S.I. case: $v_X(x)=\mathbb{E}\left(X(x)-X(0)\right)^2 \asymp \|x\|^{2\gamma}$.

Corollary: The fBm B_H admits H as critical Hölder exponent on $B_d(0,1)$ and, a.s.

$$\dim_{\mathcal{H}}\mathcal{G}(B_H)=\dim_{\mathcal{B}}\mathcal{G}(B_H)=d+1-H,$$

with $\mathcal{G}(B_H)=\{(x,B_H(x));x\in B_d(0,1)\}$.

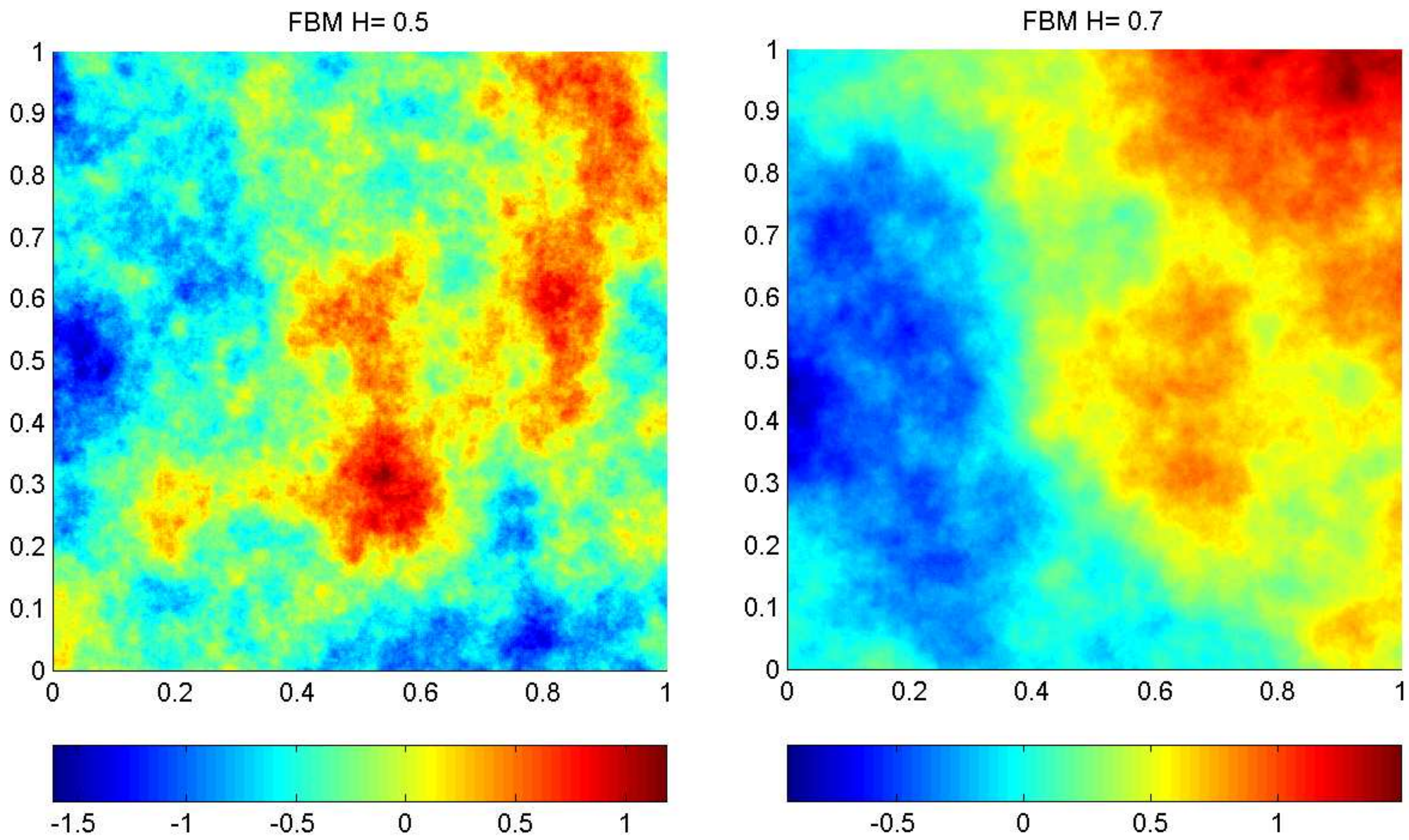

Figure 1: FBM for $d = 2$

2 Operator scaling Gaussian fields

Let $E \in \mathcal{M}_d(\mathbb{R})^{>1}$ be a real $d \times d$ matrix with real parts of the eigenvalues $\frac{1}{H_d} \geq \cdots \geq \frac{1}{H_1} > 1$. We want to define a Gaussian random field with stationary increments $X_E = \left\{X_E(x); x \in \mathbb{R}^d\right\}$ operator scaling w.r.t. E ie such that $\forall \lambda > 0$,

$$X_E(\lambda^E .) \overset{fdd}{=} \lambda X_E(.),$$

where $\lambda^E = \sum_{n=0}^{+\infty} \frac{\log(\lambda)^n}{n!} E^n$.

- For $H \in (0,1)$ let $E_H = \frac{1}{H} I_d$. Since $\lambda^{E_H} = \lambda^{1/H} I_d$, a self-similar field of order H is operator scaling w.r.t. E_H.
- If $E\theta_j = \frac{1}{H_j}\theta_j$, since $\lambda^E \theta_j = \lambda^{1/H_j}\theta_j$, the process $\{X_E(t\theta_j); t \in \mathbb{R}\}$ is self-similar of order H_j.

2.1 E-homogeneous functions

Definition: Let $\varphi : \mathbb{R}^d \to \mathbb{R}^+$. We say that φ is E-*homogeneous* if φ is continuous on $\mathbb{R}^d$ with $\varphi(x) \neq 0$ if $x \neq 0$ and

$$\forall \lambda > 0 \;\; \varphi(\lambda^E x) = \lambda \varphi(x).$$

Example: $\psi_H(x) = \|x\|^H$ is E_H- homogeneous.

Remark: $C_1 t^{1/H_d+\delta} \leq \|t^E\| \leq C_2 t^{1/H_1-\delta}$ for any $t \in (0,1)$, $\delta > 0$.

Polar coordinates with respect to E.

One can define a norm on $\mathbb{R}^d$ w.r.t. by $\|x\|_E = \int_0^1 \|t^E x\| \frac{\mathrm{d}t}{t}$.

➥ $S_E = \{x \in \mathbb{R}^d : \|x\|_E = 1\}$ sphere w.r.t. E.

Since $\|\lambda^E x\|_E = \int_0^\lambda \|t^E x\| \frac{\mathrm{d}t}{t}$, for any $x \neq 0$ the map $\lambda \mapsto \|\lambda^E x\|_E$ is a bijection from $(0,+\infty)$ onto $(0,+\infty)$. Hence $\exists!\ \lambda_0 > 0$ s.t. $\ell(x) = \lambda_0^E x \in S_E$. For $\tau(x) = \lambda_0^{-1}$ we get $x = \tau(x)^E \ell(x)$.

Example: $S_{E_H} = \frac{1}{H} S^{d-1}$, therefore

$$(\tau(x), \ell(x)) = \left(H^H \|x\|^H, \frac{1}{H} \frac{x}{\|x\|} \right) \text{ s.t. } x = \tau(x)^{E_H} \ell(x).$$

Change of variables. Let $f \in L^1(\mathbb{R}^d)$.

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) \mathrm{d}x &= \int_{S^{d-1}} \int_0^\infty f(r\theta) r^{d-1} \mathrm{d}r \mathrm{d}\theta \\ &= c_H \int_{S_{E_H}} \int_0^\infty f(r^{E_H} \theta) r^{d/H-1} \mathrm{d}r \mathrm{d}\theta. \end{aligned}$$

Proposition: Let $q = \operatorname{trace}(E)$, there exists a unique finite Radon measure σ on S_E such that

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_{S_E} \int_0^\infty f(r^E \theta)\, r^{q-1}\, \mathrm{d}r\, \sigma(\mathrm{d}\theta).$$

Examples of E-homogeneous functions.

- Diagonalizable case: let $\theta_1, \ldots, \theta_d$ a basis of $\mathbb{R}^d$, $0 < H_d \leq \ldots \leq H_1 < 1$ and $C_1, \ldots, C_d > 0$ and define for $\rho > 0$

$$\psi(x) = \left(\sum_{j=1}^{d} C_j |x \cdot \theta_j|^{\rho H_j} \right)^{1/\rho}.$$

Let E such that $E\theta_j = \frac{1}{H_j}\theta_j$ for $1 \leq j \leq d$ and remark that

$$\left(\lambda^{E^t} x\right) \cdot \theta_j = x \cdot \left(\lambda^E \theta_j\right) = \lambda^{1/H_j} \left(x \cdot \theta_j\right).$$

Therefore ψ is E^t-homogeneous.

➥ General case: Let $E \in \mathcal{M}_d(\mathbb{R})^{>1}$ and $M(d\theta)$ a finite non-negative measure on S_E such that

$$\overline{\text{Vect}\left\{r^E\theta : r > 0, \theta \in \text{supp}(M)\right\}} = \mathbb{R}^d.$$

Then

$$\psi(x) = \int_{S_E} \int_0^\infty \left(1 - \cos\left(x \cdot r^E\theta\right)\right) \frac{\mathrm{d}r}{r^2} M(\mathrm{d}\theta)$$

is a E^t-homogeneous function.

Remark: $\left|1 - \cos\left(x \cdot r^E\theta\right)\right| \leq C(x)\|r^E\|^2$. Since $\|r^E\| \leq C_1 r^{1/H_1}$ for $r \leq 1$, the function ψ is well defined as soon as $H_1 < 2$.

2.2 Harmonizable representation

Let $E \in \mathcal{M}_d(\mathbb{R})^{>1}$ with $\mathrm{Re}\,(\mathrm{sp}(E)) = \left\{\frac{1}{H_1}, \ldots, \frac{1}{H_d}\right\}$ for $0 < H_d \leq \ldots \leq H_1 < 1$ and $q = \mathrm{trace}(E)$.
Let W be a complex Brownian measure.
Theorem: Let $\psi : \mathbb{R}^d \to \mathbb{R}^+$ be a E^t-homogeneous function. Then the Gaussian random field

$$X(x) = \mathrm{Re} \int_{\mathbb{R}^d} \left(e^{ix\cdot\xi} - 1\right) \psi(\xi)^{-1-q/2} W(\mathrm{d}\xi)\ , x \in \mathbb{R}^d$$

is well-defined. Moreover, it is

- operator scaling w.r.t. E.
- with stationary increments.

Sketch of the proof.

Let us denote $v(x) = \int_{\mathbb{R}^d} \left|(e^{ix\cdot\xi} - 1)\psi(\xi)^{-1-q/2}\right|^2 \mathrm{d}\xi$.

Then, using polar coordinates w.r.t. E^t

$$v(x) = \int_{S_{E^t}} \int_0^{+\infty} \left|e^{ix\cdot r^{E^t}\theta} - 1\right|^2 \psi(r^{E^t}\theta)^{-2-q} r^{q-1} \mathrm{d}r\sigma(\mathrm{d}\theta),$$

with

$$\left|e^{ix\cdot r^{E^t}\theta} - 1\right| \leq C(x) \min\left(r^{1/H_1-\delta}, 1\right) \text{ and } \psi(r^{E^t}\theta) = r\psi(\theta).$$

$$\looparrowright \quad v(x) < +\infty \quad \text{for} \quad 1/H_1 > 1.$$

$$\forall\lambda > 0, \quad \psi\left(\lambda^{E^t}\xi\right) = \lambda\psi(\xi) \quad \curvearrowright \quad v(\lambda^E x) = \lambda^2 v(x).$$

Moreover, writing $x = \tau(x)^E \ell(x)$, since $x \cdot r^{E^t}\theta = \ell(x) \cdot (\tau(x)r)^{E^t}\theta$,

$$\looparrowright \quad v(x) = v(\ell(x))\tau(x)^2.$$

Remark: for $E = E_H$, we get $v(x) = v\left(\frac{x}{\|x\|}\right)\|x\|^{2H}$.

3 Hölder and directional regularity

3.1 Hölder regularity

We consider X whose variogramme is given by

$$v(x) = v(\ell(x))\tau(x)^2 \text{ with } 0 < C_1 \leq v(\ell(x)) \leq C_2.$$

Writing $\ell(x) = \tau(x)^{-E}x$ we get $\tau(x) \leq C\|x\|^{H_d-\delta}$ for $\|x\| \leq 1$.

↪ critical Hölder exponent $\geq H_d$.

Considering $x = \tau(x)^E \ell(x)$ we get $\tau(x) \geq C\|x\|^{H_1+\delta}$ for $\|x\| \leq 1$.

↪ critical Hölder exponent $\leq H_1$.

3.2 Directional regularity

Definition:[Bonami, Estrade, 2003] Let $\{X(x)\}_{x\in\mathbb{R}^d}$ be a Gaussian random field with stationary increments and $u \in S^{d-1}$. We say that X admits $\gamma(u) \in (0,1)$ as directional regularity in the direction u if

$$\mathbb{E}\left(X(x_0+tu) - X(x_0)\right)^2 \asymp |t|^{2\gamma(u)}.$$

Proposition:[BE, 2003] If $\exists \gamma : S^{d-1} \to (0,1)$ s.t. $\forall u \in S^{d-1}$, X admits $\gamma(u)$ as directional regularity in the direction u. Then γ takes at most d values. Moreover, if γ takes k values $\gamma_k < \ldots < \gamma_1$, there exist

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k := \mathbb{R}^d$$

$$\gamma(u) = \gamma_i \Leftrightarrow u \in (V_i \setminus V_{i-1}) \cap S^{d-1}.$$

Let $E \in \mathcal{M}_d(\mathbb{R})^{>1}$ with $\mathrm{Re}\,(\mathrm{sp}(E)) = \left\{\frac{1}{H_1}, \ldots, \frac{1}{H_p}\right\}$ for $0 < H_p < \ldots < H_1 < 1$. Let $W_1, \ldots, W_p$ the spectral decomposition of $\mathbb{R}^d$ w.r.t. E. For $i = 1, \ldots, p$, we denote $V_0 = \{0\}$ and

$$V_i = W_1 \oplus \ldots \oplus W_i.$$

For any $u \in V_i$, with $u = v_{i-1} + w_i \in V_{i-1} \oplus W_i$, when $|t| \leq 1$,

$$C_1 \|t w_i\|^{H_i+\delta} \leq \tau(tu) \leq C_2 |t|^{H_i-\delta}.$$

Theorem: Let X be a Gaussian random field with stationary increments and operator scaling w.r.t. E. Then,

- $\forall u \in V_i \setminus V_{i-1}$, X admits H_i directional regularity in the direction u.
- X admits H_p as critical Hölder exponent on $B_d(0,1)$ and, a.s.

$$\dim_{\mathcal{H}} \mathcal{G}(X) = \dim_{\mathcal{B}} \mathcal{G}(X) = d + 1 - H_p,$$

with $\mathcal{G}(X) = \{(x, X(x)); x \in B_d(0,1)\}$.

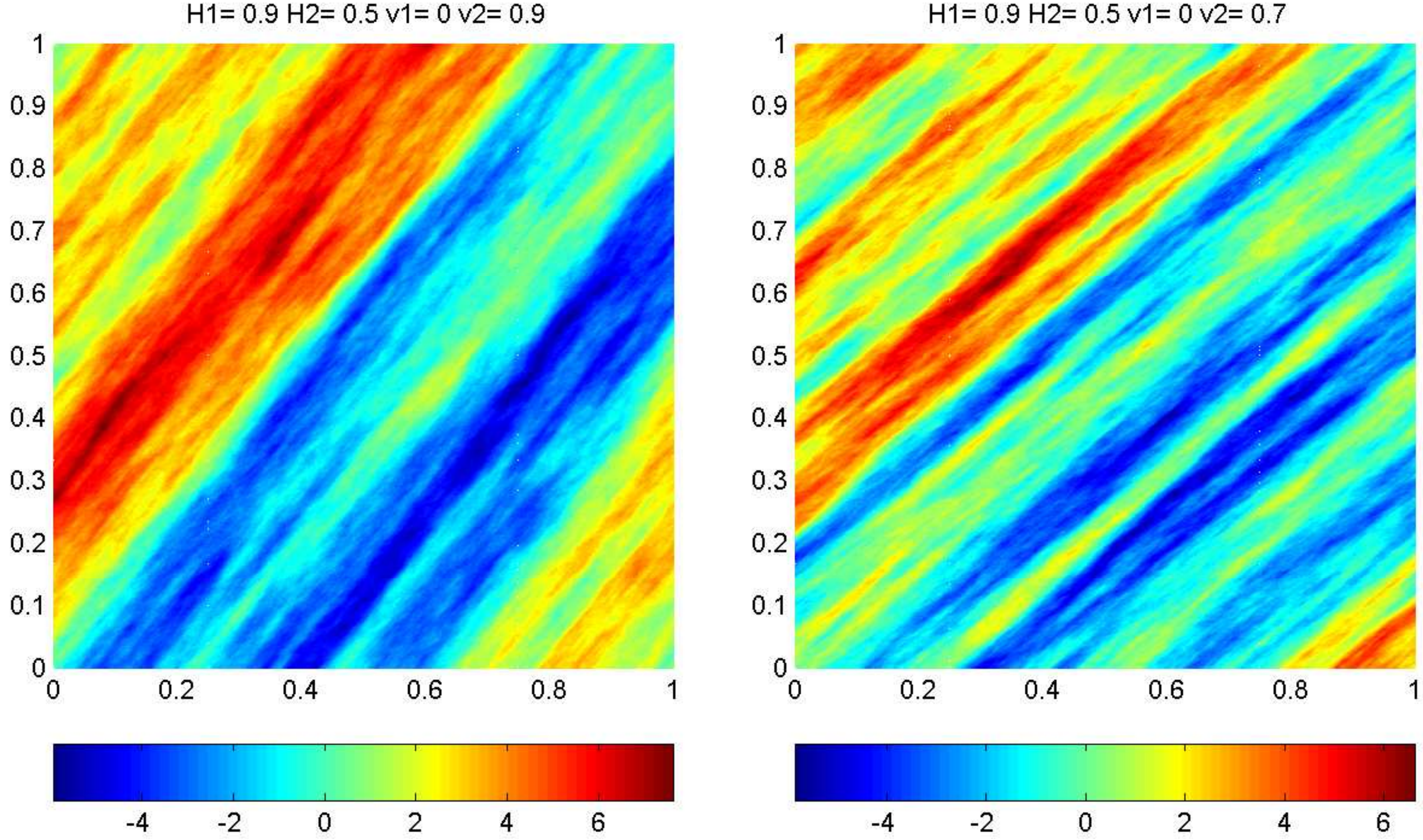

Figure 2: OSGRF for $d = 2$, $H_1 = 0.9$, $\theta_1 = 0$ et $H_2 = 0.5$

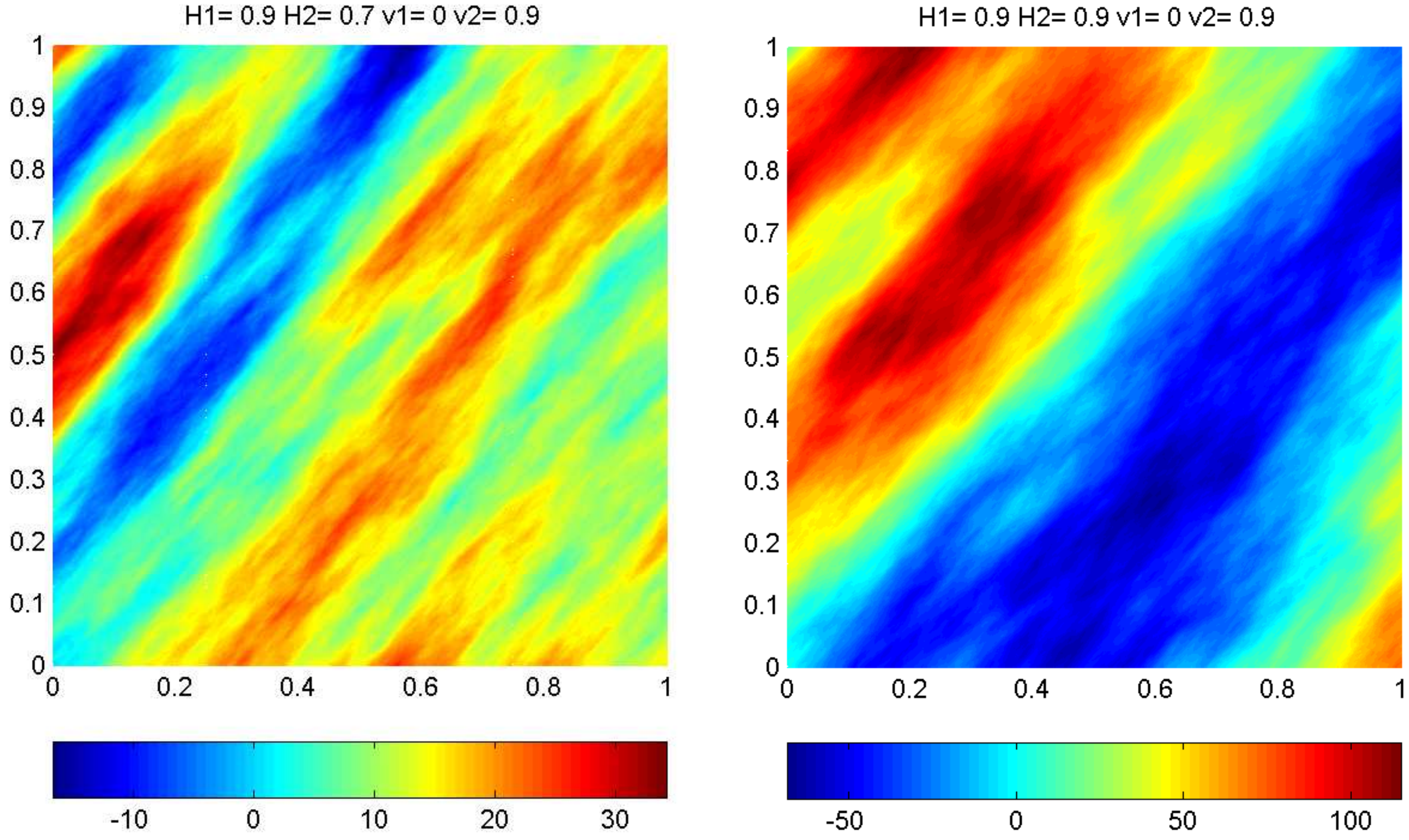

Figure 3: OSGRF for $d = 2$, $H_1 = 0.9$, $\theta_1 = 0$ et $\theta_2 \sim 51°$

4 Outlooks

- Simulations: speed of convergence and numerical obstructions.
- Estimation of the matrix E.
- Generalization to multifractional fields $E \curvearrowright E(x)$.
- Generalization to α-stable fields.