Injectivity of rotation invariant windowed Radon transforms

Hermine Biermé

MAPMO-UMR 6628, Département de Mathématiques, Université d'Orléans, 45067 Orléans cedex 2, France

Abstract

We consider rotation invariant windowed Radon transforms that integrate a function over hyperplanes by using a radial weight (called window). T. Quinto proved their injectivity for square integrable functions of compact support. This cannot be extended in general. Actually, when the Laplace transform of the window has a zero with positive real part δ, the windowed Radon transform is not injective on functions with a Gaussian decay at infinity, depending on δ. Nevertheless, we give conditions on the window that imply injectivity of the windowed Radon transform on functions with a more rapid decay than any Gaussian function.
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1. Introduction

With the classical Radon transform, one integrates a function over hyperplanes. The Radon transform has developed very rapidly in the early 1970s, with a lot of applications in medicine, optics, physics and other areas. It is well known that one can recover a function from its integrals along all hyperplanes, that is, the Radon transform is injective (see, for

example, [6,10]). It is no more the case when the Radon transform is replaced by a more realistic generalization, basically when one integrates with respect to different weights on the hyperplanes. Here we consider a weight that does not depend on the hyperplane.

Our starting point has been the work of Bonami and Estrade [3], in relation with the engineering department of the university of Orléans [8], on image processing related to bones radiographs. They model such radiographs by a Gaussian random field with stationary increments, characterized by a function F, called spectral density. They choose a convenient window ψ (smooth and rapidly decreasing for instance), and perform a windowed Radon transform of the radiographs. Then, for each direction $\theta \in S^1$, they obtain a Gaussian random process with stationary increments and spectral density $R_{|\hat{\psi}|^2} F(\theta, .)$. Thus, a natural question is the following: for which windows does one have injectivity? Since ψ and its Fourier transform are used as windows, the choice of a Gaussian window is natural. However we are interested in more general windows. Compared with the Radon transform, the main difficulty is the loss of translation invariance for the windowed Radon transform. One can force rotation invariance by choosing a radial window, which we do here. The question of injectivity (in law) is then given by the injectivity of the windowed Radon transform for spectral densities, which satisfy adapted integrability conditions at infinity. Let us recall that their asymptotic behaviour in power law is of particular interest since it gives the Hölder exponent for the corresponding field [3].

For such radial windows, Quinto [9] gave an injectivity result for square integrable functions with compact support. In the literature, injectivity for generalized Radon transforms is only studied for compactly supported functions (for instance, in the case of attenuated Radon transforms in the so-called Emission Tomography [2,12]...). There are two main reasons for this. On one hand, in general, such transforms appear in experiments and imply real objects, which have compact support. On the other hand, there is a mathematical obstruction for injectivity in a general setting. Actually, one can find windows for which the windowed Radon transform is not injective on square integrable functions with a Gaussian decay at infinity. Here, we proceed further with counter-examples. We state conditions on radial windows which guarantee the injectivity of the windowed Radon transform on square integrable functions that decrease faster than any Gaussian function. The rotation invariance allows us to restrict to a collection of operators defined on $L^2(\mathbb{R}^+, r^{d-1} e^{\delta_0 r^2} \, dr)$, with $\delta_0 \in \mathbb{R}$ depending on the integrability of the window. For each one, we find $\delta > \delta_0^+$ such that it is injective on $L^2(\mathbb{R}^+, r^{d-1} e^{\delta r^2} \, dr)$, where $\delta_0^+ := \max(\delta_0, 0)$.

The paper is organized as follows. In Section 2, we define the windowed Radon transform and recall Quinto's proof for injectivity results, which allows us to weaken his conditions on the windows. We emphasize in Section 3 the role of Gaussian functions. On one hand, they are examples of windows for which there is injectivity. On the other hand, they give counter-examples for injectivity, as test functions. In Section 4, we consider the special case of radial functions. This case is simpler since the windowed Radon transform can be reduced to an integral convolution operator. The general case is studied in Section 5. By Laplace transform we obtain an ordinary differential equation with holomorphic coefficients. Using the inverse Laplace transform, we can reduce to an application of the fixed point theorem. In a final remark, we mention that this injectivity question gives rise to an open problem on outer functions in the complex plane.

2. Definition and preliminary results

Let us first define the *windowed Radon transform* under consideration. We fix the dimension $d \geqslant 2$, as well as the *window* φ, which is assumed to be a smooth function on $\mathbb{R}$, such that, for $\delta \in \mathbb{R}$,

$$\int\limits_0^{+\infty} \varphi(r)^2 r^{(d-3)/2} e^{-\delta r} \, dr < \infty.$$

We call $\mathcal{W}_\delta$ the class of such windows φ. We define the windowed Radon transform (with the window φ), for $\theta \in S^{d-1}$ and $p \in \mathbb{R}$, by

$$R_\varphi F(\theta, p) = \int\limits_{x \in H(\theta, p)} F(x) \varphi\big(|x - p\theta|^2\big) \, dx_H, \tag{1}$$

when this make sense. Here, $H(\theta, p)$ is the hyperplane $\{x \in \mathbb{R}^d; \ \theta.x = p\}$, and dx_H is the Lebesgue measure on this hyperplane. When $F \in L^2_\delta(\mathbb{R}^d) := L^2(\mathbb{R}^d, e^{\delta |x|^2} \, dx)$, the second hand of (1) is well defined. This follows from the Cauchy–Schwarz inequality

$$\int\limits_{S^{d-1}} \int\limits_{\mathbb{R}} \big|R_\varphi F(\theta, p)\big|^2 e^{\delta p^2} \, dp \, d\theta \leqslant C_\varphi \int\limits_{\mathbb{R}^d} \big|F(x)\big|^2 e^{\delta |x|^2} \, dx,$$

with $d\theta$ the Lebesgue measure on S^{d-1} and $C_\varphi < +\infty$ when $\varphi \in \mathcal{W}_\delta$.

Thus, for $\varphi \in \mathcal{W}_\delta$, $R_\varphi : L^2_\delta(\mathbb{R}^d) \to L^2_\delta(S^{d-1} \times \mathbb{R}) := L^2(S^{d-1} \times \mathbb{R}, d\theta \otimes e^{\delta p^2} \, dp)$ is a bounded operator. Moreover, since $R_\varphi F(\theta, -p) = R_\varphi F(-\theta, p)$, we can restrict our study on $L^2_\delta(S^{d-1} \times \mathbb{R}^+)$. The choice of a radial window allows us to obtain the rotation invariance of the windowed Radon transform. Namely, for any rotation $k \in O(d)$ and $F \in L^2_\delta(\mathbb{R}^d)$, we have

$$R_\varphi F(k\theta, p) = \big(R_\varphi(F \circ k)\big)(\theta, p) \quad \text{for } (\theta, p) \in S^{d-1} \times \mathbb{R}^+.$$

Using this property, we can decompose $L^2_\delta(\mathbb{R}^d)$ into a Hilbertian sum of subspaces for which the windowed Radon transform simplifies. We denote by $L^2(S)$ the Hilbert space of square integrable functions on the sphere of $\mathbb{R}^d$. A spherical harmonic of degree l, for $l \in \mathbb{N}$, is the restriction to S of a homogeneous harmonic polynomial on $\mathbb{R}^d$ of degree l. As in [1, p. 80], we write $\mathcal{H}_l(S)$ the space of such functions. Then $L^2(S)$ is the Hilbertian sum of the spaces $\mathcal{H}_l(S)$, namely

$$L^2(S) = \bigoplus_{l=0}^\infty \mathcal{H}_l(S).$$

We proceed as in [13] to obtain decompositions of $L^2_\delta(\mathbb{R}^d)$ and $L^2_\delta(S^{d-1} \times \mathbb{R}^+)$ in Hilbertian sums. We define

$$\mathcal{H}_{l,\delta} = \text{Vect}\bigg(f(|x|) P\bigg(\frac{x}{|x|}\bigg), \ f \in L^2\big(\mathbb{R}^+, r^{d-1} e^{\delta r^2} \, dr\big), \ P \in \mathcal{H}_l(S) \bigg) \subset L^2_\delta(\mathbb{R}^d),$$

and, in a similar way,

$$\tilde{\mathcal{H}}_{l,\delta} = \text{Vect}\big(f(p)P(\theta), \ f \in L^2\big(\mathbb{R}^+, e^{\delta r^2}\, dr\big), \ P \in \mathcal{H}_l(S)\big) \subset L_\delta^2\big(S^{d-1} \times \mathbb{R}^+\big).$$

Using an orthonormal basis of $\mathcal{H}_l(S)$, we obtain the following result.

Proposition 2.1. *For $\delta \in \mathbb{R}$ we can write the Hilbertian decompositions*

$$L_\delta^2\big(\mathbb{R}^d\big) = \bigoplus_{l=0}^\infty \mathcal{H}_{l,\delta} \quad and \quad L_\delta^2\big(S^{d-1} \times \mathbb{R}^+\big) = \bigoplus_{l=0}^\infty \tilde{\mathcal{H}}_{l,\delta}.$$

The rotation invariance of the windowed Radon transform implies that R_φ maps $\mathcal{H}_{l,\delta}$ into $\tilde{\mathcal{H}}_{l,\delta}$. Let us define, for $f \in L^2(\mathbb{R}^+, r^{d-1}e^{\delta r^2}\, dr)$,

$$S_l f(p) = \int\limits_p^{+\infty} f(u)u^{d-2}\varphi\big(u^2 - p^2\big)C_l^{(d-2)/2}\left(\frac{p}{u}\right)\left(1 - \frac{p^2}{u^2}\right)^{(d-3)/2} du,$$

where $C_l^{(n-2)/2}$ is the Gegenbauer polynomial of degree l. Then, S_l maps $L^2(\mathbb{R}^+, r^{d-1}e^{\delta r^2}\, dr)$ into $L^2(\mathbb{R}^+, e^{\delta r^2}\, dr)$ and we can link S_l with R_φ by the following proposition [9].

Proposition 2.2. *Let $F(x) = f(|x|)P(x/|x|)$ be a function of $\mathcal{H}_{l,\delta}$. Then*

$$R_\varphi F(\theta, p) = \frac{m(S^{d-2})}{C_l^{(d-2)/2}(1)} S_l f(p)P(\theta) \in \tilde{\mathcal{H}}_{l,\delta}.$$

By Proposition 2.1, the windowed Radon transform R_φ is injective on $L_\delta^2(\mathbb{R}^d)$ if and only if, for all l, the operator S_l is injective on $L^2(\mathbb{R}^+, r^{d-1}e^{\delta r^2}\, dr)$. However, it is more convenient to consider the operators defined, for $f \in L^2(\mathbb{R}^+, r^{(d-2)/2}e^{\delta r}\, dr)$, by

$$T_l f(p) = \int\limits_p^{+\infty} f(u)\varphi(u - p)C_l^{(d-2)/2}\left(\sqrt{\frac{p}{u}}\right)(u - p)^{(d-3)/2}\, du.$$

Then, T_l maps $L^2(\mathbb{R}^+, r^{(d-2)/2}e^{\delta r}\, dr)$ into $L^2(\mathbb{R}^+, r^{-1/2}e^{\delta r}\, dr)$ and S_l is injective on $L^2(\mathbb{R}^+, r^{d-1}e^{\delta r^2}\, dr)$ if and only if T_l is injective on $L^2(\mathbb{R}^+, r^{(d-2)/2}e^{\delta r}\, dr)$.

Quinto [9] proved the injectivity of R_φ on the class of square integrable functions with compact support under the assumption that the window φ does not vanish. A careful reading of his proof leads to the following result.

Theorem 2.1. *Let $\delta \in \mathbb{R}$. Let $\varphi \in \mathcal{W}_\delta$ be a window that does not vanish at 0. Let $F \in L_\delta^2(\mathbb{R}^d)$ such that $R_\varphi F = 0$. If F has compact support, then $F \equiv 0$.*

Proof. Let $F \in L^2_\delta(\mathbb{R}^d)$ be compactly supported in the ball $B(0, M^{1/2}) = \{x \in \mathbb{R}^d; \ |x| < M^{1/2}\}$. For $l \in \mathbb{N}$, we choose $(Y_{lm})_{1 \leqslant N(l)}$ an orthonormal basis of $\mathcal{H}_l(S)$. Then, the orthogonal projection of F onto $\mathcal{H}_{l,\delta}$ is given by

$$\mathcal{P}_l F = \sum_1^{N(l)} f_{lm} Y_{lm} \quad \text{with } f_{lm}(r) = \int_{S^{d-1}} F(r\theta) Y_{lm}(\theta) \, d\theta.$$

So each coordinate f_{lm} has also its support in $[0, M^{1/2})$. We are reduced to prove the injectivity of T_l on functions $f \in L^2(\mathbb{R}^+, r^{(d-2)/2} e^{\delta r} \, dr)$ compactly supported in $[0, M)$. Let $\epsilon \in (0, M)$. By a change of variables, we write, for $t \in (\epsilon, M)$,

$$T_l f(t^{-1}) = \int_{1/M}^t W_l(s, t) s^{-(d+2)/4} f(s^{-1})(t - s)^{(d-3)/2} \, ds,$$

where

$$W_l(s, t) = s^{-d/4} t^{-(d-3)/2} \varphi\left(\frac{1}{s} - \frac{1}{t}\right) C_l^{(d-2)/2}\left(\sqrt{\frac{s}{t}}\right).$$

Then, we are lead to study the following integral equation

$$g(t) = \int_{1/M}^t f(s) W_l(s, t)(t - s)^{(d-3)/2} \, ds, \tag{2}$$

where $g, f \in L^2((1/M, 1/\epsilon))$ and W_l is a $\mathcal{C}^\infty$ function on $(1/M, \infty)^2$, which does not vanish on the diagonal. Existence and uniqueness results in L^2 are known for Volterra integral equations of the second kind [14, p. 10]. However, the kernel of the integral transform $W_l(s, t)(t - s)^{(d-3)/2}$ can vanish along the diagonal according to d. Thus, T. Quinto got rid off this difficulty by taking derivatives of (2). Let us recall that if $I = (1/M, 1/\epsilon)$, the Sobolev space $H^1(I)$ is defined by

$$H^1(I) = \left\{ u \in L^2(I); \ \exists v \in L^2(I) \text{ such that } \int_I u\psi' = -\int_I v\psi, \ \forall \psi \in \mathcal{C}_c^1(I) \right\},$$

while, for $m \geqslant 2$, $H^m(I)$ is defined by induction as $H^m(I) = \{u \in H^{m-1}(I); \ u' \in H^{m-1}(I)\}$. Let us write $n = (d - 3)/2$ for d odd, $n = d/2 - 1$ for d even. We are interested in the case when $g = 0$, so we assume that $g \in H^{n+1}((1/M, 1/\epsilon))$. We take n derivatives of (2),

$$g^{(n)}(t) = \int_{1/M}^t f(s) \frac{\partial^n}{\partial t^n} \left(W_l(s, t)(t - s)^{(d-3)/2} \right) ds. \tag{3}$$

If d is odd, taking one more derivative, we get

$$g^{(n+1)}(t) = \left(n! t^{-(3d-6)/4} C_l^{(d-2)/2}(1) \varphi(0) \right) f(t)$$

$$+ \int_{1/M}^{t} f(s) \frac{\partial^{n+1}}{\partial t^{n+1}} \left(W_l(s,t)(t-s)^n \right) ds,$$

which is a Volterra integral equation of the second kind, for which we have a unique solution since φ does not vanish at 0.

In the even case, we write

$$\frac{\partial^n}{\partial t^n} \left(W_l(s,t)(t-s)^{(d-3)/2} \right) = K_l(s,t)(s-t)^{-1/2}$$

such that

$$K_l(t,t) = \frac{(2n)!}{2^{2n}n!} t^{-(3d-6)/4} C_l^{(d-2)/2}(1)\varphi(0) \neq 0.$$

Similar arguments as in the previous case allow us to show existence and uniqueness of the solution of the generalized Abel integral equation (3) under the additional assumption that $g^{(n)}(1/M) = 0$ [9, Theorem B]. Hence, in all case, if $T_l f \equiv 0$, we get that $f \equiv 0$ on (ϵ, M) by uniqueness, for all $\epsilon \in (0, M)$. This concludes the proof. $\square$

We generalize this result to functions that do not have compact support, but decrease rapidly at infinity. Again, we give injectivity result for the collection of operators $(T_l)_{l \in \mathbb{N}}$. We will prove the following theorems.

Theorem 2.2. *Let $\delta_0 \in \mathbb{R}$, $\delta_0^+ = \max(\delta_0, 0)$ and $\varphi \in \mathcal{W}_{\delta_0}$ with $\varphi(0) \neq 0$. We assume that φ and all its derivatives have at most an exponential growth,*

$$\left| \varphi^{(k)}(r) \right| \leqslant C_k e^{\delta_0^+ r}, \tag{4}$$

for $r \geqslant 0$, where k is an integer and C_k a positive constant. Then, for l an integer, there exists $\delta > \delta_0^+$ (which depends on l) such that T_l is injective on $L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2} dr)$.

Thus, the windowed Radon transform is injective considered on the intersection.

Theorem 2.3. *Let $\delta_0 \in \mathbb{R}$. Let $\varphi \in \mathcal{W}_{\delta_0}$ be a window with $\varphi(0) \neq 0$. We assume that φ satisfies (4). Let $F \in L_\delta^2(\mathbb{R}^d)$ for all $\delta \geqslant \delta_0$. If $R_\varphi F = 0$, then $F \equiv 0$.*

The first operator T_0 is related to the action of the windowed Radon transform on radial functions. Before a careful study of this operator in Section 4, we consider the special case of Gaussian functions in the next part. The last part deals with the injectivity of T_l in the general case.

3. Gaussian functions

A natural generalization of the Radon transform is given by Gaussian windows. We consider windows of the form

$$\varphi_{\delta_0}(r) = e^{\delta_0 r/2}, \quad \text{with } \delta_0 \in \mathbb{R},$$

so that $\varphi_{\delta_0}(r^2)$ is a Gaussian function when $\delta_0 < 0$. Obviously $\varphi_{\delta_0} \in \mathcal{W}_\delta$, when $\delta > \delta_0$. Let $\delta > \delta_0$. Then, we can define the windowed Radon transform with the window φ_{δ_0} for functions in $L^2_\delta(\mathbb{R}^d)$. Moreover, when $F \in L^2_\delta(\mathbb{R}^d)$,

$$R_{\varphi_{\delta_0}}(F)(\theta, p) = e^{-\delta_0 p^2/2} R\left(Fe^{\delta_0|x|^2/2}\right)(\theta, p),$$

where R is the classical Radon transform. Then, using injectivity of the Radon transform on $L^1(\mathbb{R}^d)$, we can state the following theorem.

Theorem 3.1. *Let $\delta_0 \in \mathbb{R}$. Then the windowed Radon transform $R_{\varphi_{\delta_0}}$ is injective on $L^2_\delta(\mathbb{R}^d)$, when $\delta > \delta_0$.*

Let us remark that, with further smoothness assumptions, we can also extend to $R_{\varphi_{\delta_0}}$ the classical inversion formulas of the Radon transform.

On the other hand, Gaussian functions give counter-examples for injectivity as test functions. Let $\delta_0 \in \mathbb{R}$ and $\varphi \in \mathcal{W}_{\delta_0}$. From above, the windowed Radon transform is injective on $\mathcal{H}_{\delta_0,0}$ if and only if S_0 is injective on $L^2(\mathbb{R}^+, r^{d-1} e^{\delta_0 r^2} dr)$. Since the Gegenbauer polynomial $C_0^{(d-2)/2}$ is a constant c, we have a simpler expression of S_0. Let $z \in \mathbb{C}$, after a change of variables, we obtain the image of the function e^{-zr^2}. For $\Re(z) > \delta_0$,

$$S_0\left(e^{-zr^2}\right)(p) = \frac{c}{2} e^{-zp^2} \Phi(z), \quad \text{where } \Phi(z) = 2 \int\limits_0^{+\infty} e^{-zr^2} \varphi(r^2) r^{d-2} \, dr$$

is the Laplace transform of $\varphi(r) r^{(d-3)/2}$. Since $\varphi \in \mathcal{W}_{\delta_0}$, the abscissa of convergence of Φ is lower than δ_0. It is obvious that if Φ has a zero z_0 on the half plane Π_{δ_0}, S_0 is not injective on $L^2(\mathbb{R}^+, r^{d-1} e^{\delta r^2} dr)$ for $\delta \in (\delta_0, \Re(z_0))$.

Let us remark that the Laplace transform of $\varphi_{\delta_0}(r) r^{(d-3)/2}$ is, for $\Re(z) > \delta_0/2$, $\Phi_{\delta_0}(z) = \Gamma((d-1)/2)(z - \delta_0/2)^{-(d-1)/2}$. Thus Φ_{δ_0} does not vanish on $\overline{\Pi_\delta}$, where Π_δ denotes the half plane $\{z \in \mathbb{C}; \Re(z) > \delta\}$. The next section gives a kind of converse.

4. Injectivity for radial functions

Let $\delta_0 \in \mathbb{R}$ and $\varphi \in \mathcal{W}_{\delta_0}$ with $\varphi(0) \neq 0$. We will prove Theorem 2.2 for $l = 0$. Under the growth conditions (4) on φ, we will find $\delta \geqslant \delta_0^+$ such that, when $F \in L^2_\delta(\mathbb{R}^d)$ is radial and $R_\varphi F \equiv 0$, then $F \equiv 0$. Let us remark that the assumption $\varphi(0) \neq 0$ is a natural one when compared with Theorem 2.1. From above, we need a control on the zeros of the holomorphic function Φ. This is given in the next proposition.

Proposition 4.1. *Let $\delta_0 \in \mathbb{R}$ and $\varphi \in \mathcal{W}_{\delta_0}$ with $\varphi(0) \neq 0$. Under the assumption (4), there exists $\delta_1 > \delta_0^+$ such that Φ does not vanish in Π_{δ_1}.*

Actually, the assumptions on the growth of φ and its derivatives allow us to give growth results on Φ and its derivatives. We give here a stronger result that we will need later.

Lemma 4.1. *Let $\varphi \in \mathcal{W}_{\delta_0}$ that satisfies* (4). *Then Φ may be written as*

$$\Phi(z) = \varphi(0)\Gamma\left(\frac{d-3}{2}+1\right)z^{-(d-3)/2-1} + \Psi(z),$$

where Ψ is a holomorphic function on $\Pi_{\delta_0^+}$ such that, when $\delta > \delta_0^+$,

$$\left|\Psi^{(k)}(z)\right| \leqslant C_k|1+z|^{-(d+1)/2-k}, \tag{5}$$

for $z \in \Pi_\delta$. Here k is an integer and C_k is a positive constant (*depending on δ*).

Proof. We write $\psi = \varphi - \varphi(0)$ and Ψ for the Laplace transform of the function $\psi(t)t^{(d-3)/2}\mathbf{1}_{t>0}$. Therefore, the abscissa of convergence of Ψ is lower than δ_0^+. Then, when $z \in \mathbb{C}$ is such that $\Re(z) > \delta_0^+$,

$$\Phi(z) = \varphi(0)\Gamma\left(\frac{d-3}{2}+1\right)z^{-(d-3)/2-1} + \Psi(z),$$

since $\Gamma((d-3)/2+1)z^{-(d-3)/2-1}$ is the Laplace transform of $t^{(d-3)/2}\mathbf{1}_{t>0}$.

For $\delta > \delta_0^+$, we will prove that Ψ satisfies (5) for all $z \in \overline{\Pi_\delta}$. In fact, if $\Re(z) > \delta_0^+$, the function $\psi(t)t^{(d-3)/2}e^{-zt}$ is integrable over $\mathbb{R}^+$. Thus, Ψ is holomorphic on $\Pi_{\delta_0^+}$. Moreover, for all $k \in \mathbb{N}$, the function $(1+z)^{(d+1)/2+k}\Psi^{(k)}(z)$ has a polynomial growth, and we apply the Phragmen Lindelöf method (see, for instance, [11]). To obtain a uniform upper bound over the domain Π_δ, it is sufficient to obtain a uniform upper bound over the line $\{\delta + is; \ s \in \mathbb{R}\}$, which follows from the next lemma.

Lemma 4.2. *Let $k \in \mathbb{N}$ and $\lambda \in (k-1; k]$. Let ψ be a function in $\mathcal{C}^{k+2}((0,+\infty))$. We assume there exists $\delta_0^+ \geqslant 0$ and $C > 0$ such that, for all $j = 0, \ldots, k+2$, for all $t > 0$,*

$$\left|\psi^{(j)}(t)\right| \leqslant Ct^{\lambda-j}e^{\delta_0^+ t}.$$

Then, for all $\delta > \delta_0^+$,

$$\int_0^{+\infty} e^{-(\delta+is)t}\psi(t)\,dt = \mathcal{O}_{|s|\to+\infty}\left(|s|^{-\lambda-1}\right).$$

Proof. The scheme for proving such estimates is well known. We sketch the proof for completeness. We may assume that $s > 1$ and prove this lemma by induction on $k \in \mathbb{N}$. We write $\int_0^{+\infty} e^{-(\delta+is)t}\psi(t)\,dt = \int_0^{1/s} + \int_{1/s}^{+\infty}$. By assumptions on ψ, we obtain the upper bound for the first term. For the second one, we use an integration by parts. Assumptions on ψ are sufficient to conclude for one of the two terms. When $k = 0$, another integration by part and the assumptions on ψ' and ψ'' give right upper bound for the integral one. Afterwards, we use the induction on ψ', which satisfies the growth conditions for $k-1$, to conclude. $\quad\square$

The assumptions on φ show that $t^{k+(d-3)/2}\psi(t)$ satisfies the conditions of Lemma 4.2, with $\lambda = k + (d-1)/2$. Thus we can find a positive constant C_k such that, for all $s \in \mathbb{R}$,

$$\left| \left(1 + (\delta + is)\right)^{(d+1)/2+k} \int_0^{+\infty} e^{-(\delta+is)t} t^{k+(d-3)/2} \psi(t)\, dt \right| \leqslant C_k.$$

Finally, by the Phragmen Lindelöf method, we obtain the required upper bounds for Ψ and its derivatives. $\quad\square$

The proof of Proposition 4.1 follows. Since $\varphi(0) \neq 0$, one can find constants $C > 0$ and $\delta_1 > \delta_0^+$ such that Φ does not vanish in $\overline{\Pi_{\delta_1}}$ since, for $z \in \overline{\Pi_{\delta_1}}$,

$$\left| \Phi(z) \right| \geqslant C|z|^{-(d-3)/2-1}. \tag{6}$$

Now we prove the injectivity of T_0. From Proposition 4.1, there exists $\delta_1 > \delta_0^+$ such that Φ does not vanish in the half plane Π_{δ_1}. We will find $\delta > \delta_1$ such that T_0 is injective on $L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$, or equivalently, such that its adjoint T_0^* has a dense range in $L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$. For $a > 0$, we write $L_a^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$ (respectively $L_a^2(\mathbb{R}^+, e^{\delta r} r^{-1/2}\, dr)$) for the space of functions in $L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$ (respectively $L^2(\mathbb{R}^+, e^{\delta r} r^{-1/2}\, dr)$) with support in $(a, +\infty)$. Since T_0 is injective on compactly supported functions by Theorem 2.1, we only have to prove the following proposition.

Proposition 4.2. *Under the assumptions of Theorem 2.2, there exists $\delta > \delta_0^+$ such that, when $a > 0$, $T_0^*(L_a^2(\mathbb{R}^+, e^{\delta r} r^{-1/2}\, dr))$ is dense in $L_a^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$.*

Let us denote by $\mathcal{C}_c^\infty((a, +\infty))$ the space of smooth functions with compact support in $(a, +\infty)$. Since $\mathcal{C}_c^\infty((a, +\infty))$ is dense in $L_a^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2}\, dr)$, it is enough to prove that, when $h \in \mathcal{C}_c^\infty((a, +\infty))$, there exists $g \in L_a^2(\mathbb{R}^+, e^{\delta r} r^{-1/2}\, dr)$ such that

$$
\begin{aligned}
h(u) &= T_0^*(g)(u) \\
&= cu^{-(d-2)/2} e^{-\delta u} \int_0^u g(p) p^{-1/2} e^{\delta p} \varphi(u-p)(u-p)^{(d-3)/2}\, dp.
\end{aligned} \tag{7}
$$

Then (7) is equivalent to the next convolution integral equation.

Proposition 4.3. *Let $\varphi \in \mathcal{W}_{\delta_0}$ be a window which satisfies (4), with $\varphi(0) \neq 0$. Then, there exists $\delta > \delta_0^+$ such that, when $a > 0$ and $h \in \mathcal{C}_c^\infty((a, +\infty))$, the equation*

$$h(u) = \int_0^u g(p) \varphi(u-p)(u-p)^{(d-3)/2}\, dp \tag{8}$$

has a unique solution $g \in L_a^2(\mathbb{R}^+, e^{-\delta r} r^{1/2}\, dr)$.

Proof. We assume that $g \in L_a^2(\mathbb{R}^+, e^{-\delta r} r^{1/2}\, dr)$ satisfies (8). Taking the Laplace transform of (8) we have, for $z \in \mathbb{C}$ such that $\Re(z) > \delta$, $H(z) = G(z)\Phi(z)$, where H and G are

the Laplace transforms of h and g. Since the Laplace transform is injective, Eq. (8) has at most one solution, which has support in $(a, +\infty)$. Since φ satisfies the assumption (4), there exists $\delta_1 > \delta_0^+$ such that Φ does not vanish in Π_{δ_1}. Thus, if $\delta > \delta_1$, for all $z \in \Pi_\delta$,

$$G(z) = H(z)\Phi(z)^{-1}.$$

We will take the inverse Laplace transform of the above equation. From [5, p. 36], it is sufficient to have holomorphic functions that decay faster than $|z|^{-\lambda}$, with λ strictly greater than 1. Since h is smooth, with compact support in $\mathbb{R}^+$, for all $n \in \mathbb{N}$, $H(z) = \mathcal{L}(h)(z) = z^{-n}\mathcal{L}(h^{(n)})(z)$, with $\mathcal{L}(h^{(n)})$ holomorphic and bounded in Π_{δ_1}. Thus, from (6), one can find a positive constant C such that, for all $z \in \Pi_{\delta_1}$,

$$\left| H(z)\Phi(z)^{-1} \right| \leqslant C \left\| \mathcal{L}(h^{(d)}) \right\|_\infty |z|^{-(d+1)/2}.$$

Therefore, we can define, for $b > \delta_1$, the function

$$g(t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \frac{H(b+iu)}{\Phi(b+iu)} e^{(b+iu)t} \, du.$$

Finally, when $\delta > \delta_1$, then $g \in L_a^2(\mathbb{R}^+, e^{-\delta r} r^{1/2} \, dr)$ and has $H\Phi^{-1}$ for Laplace transform in Π_δ. The Laplace transform injectivity allows to conclude for the proof. □

Therefore, when $\delta > \delta_1$, Eq. (7) has a unique solution and T_0^* has a dense range in $L_a^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2} \, dr)$. Since T_0 is injective on compactly supported functions, Theorem 2.2 is proved for $l = 0$.

5. General case

We give here similar injectivity results for the operators $(T_l)_{l \in \mathbb{N}}$ and prove Theorem 2.2. Let $\delta_0 \in \mathbb{R}$ and $\varphi \in \mathcal{W}_{\delta_0}$ with $\varphi(0) \neq 0$. We assume that φ satisfies (4). For $l \in \mathbb{N}$ we will find $\delta > \delta_0^+$ (which depends on l) such that T_l is injective on $L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2} \, dr)$.

We follow the scheme of the proof for the radial case. By Theorem 2.1, it is still sufficient to find $\delta > \delta_0^+$ such that, if $f \in L^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2} \, dr)$ satisfies $T_l f \equiv 0$, then f has compact support. Let $a > 0$. As previously, we will find $\delta > \delta_0^+$ such that, $T_l^*(L_a^2(\mathbb{R}^+, e^{\delta r} r^{-1/2} \, dr))$ is dense in $L_a^2(\mathbb{R}^+, e^{\delta r} r^{(d-2)/2} \, dr)$. Here, T_l^*, the dual operator of T_l, is given by

$$T_l^* g(u)$$
$$= u^{-(d-2)/2} e^{-\delta u} \int\limits_0^u g(p) p^{-1/2} e^{\delta p} \varphi(u-p) C_l^{(d-2)/2}\left(\sqrt{\frac{p}{u}}\right)(u-p)^{(d-3)/2} \, dp.$$

Thus, for $h \in \mathcal{C}_c^\infty((a, +\infty))$, it is sufficient to find $\delta > \delta_0^+$ such that $h(u) = T_l^* g(u)$, with $g \in L_a^2(\mathbb{R}^+, e^{\delta r} r^{-1/2} \, dr)$. For $l \geqslant 2$ this is no more a convolution equation. Nevertheless we use the particular structure of the Gegenbauer polynomial. If l is odd (respectively even), $C_l^{(d-2)/2}$ is odd (respectively even). We sketch the proof in the even case (the odd

case is similar). Let us assume that $l = 2n$, $n \in \mathbb{N}^*$, and write $C_l^{(d-2)/2}(X) = \sum_{k=0}^n \alpha_k X^{2k}$. When $h \in \mathcal{C}_c^\infty((a, +\infty))$, the function $u^{n+(d-2)/2} e^{\delta u} h$ belongs also to $\mathcal{C}_c^\infty((a, +\infty))$. As in the radial case, we are reduced to prove the following result.

Proposition 5.1. *Let* $h \in \mathcal{C}_c^\infty((a, +\infty))$*. Let* $\varphi \in \mathcal{W}_{\delta_0}$ *be a window that satisfies (4) with* $\varphi(0) \neq 0$*. Then, for* $(\alpha_k)_{0 \leqslant k \leqslant n} \in \mathbb{R}^{n+1}$ *with* $\sum_{k=0}^n \alpha_k \neq 0$*, there exists* $\delta > \delta_0^+$ *such that the equation*

$$h(u) = \sum_{k=0}^n \alpha_k u^{n-k} \int_0^u p^k g(p) \varphi(u - p)(u - p)^{(d-3)/2} \, dp \tag{9}$$

has a unique solution $g \in L_a^2(\mathbb{R}^+, e^{-\delta r} r^{1/2} \, dr)$*.*

Proof. Let us take the Laplace transform of both members of Eq. (9). With the former notations, since $\mathcal{L}(p^k g) = (-1)^k G^{(k)}$, we obtain, for all $\Re(z) > \delta$,

$$H(z) = \sum_{j=0}^n A_j G^{(j)}(z) \Phi^{(n-j)}(z),$$

with $A_j = (-1)^n \sum_{k=0}^j \alpha_k C_{n-k}^{j-k}$. From Proposition 4.1 we can choose $\delta > \delta_0^+$ such that, for $\Re(z) > \delta$, $\Phi(z) \neq 0$. Since $A_n = (-1)^n \sum_{k=0}^n \alpha_k \neq 0$,

$$G^{(n)}(z) = \frac{H(z)}{A_n \Phi(z)} - \sum_{j=0}^{n-1} A_j G^{(j)}(z) \frac{\Phi^{(n-j)}(z)}{A_n \Phi(z)}. \tag{10}$$

Thus, we are lead to solve a differential equation of order n whose coefficients are holomorphic functions. To come back to the initial problem we need a growth control of the solution. Such equations can be solved by taking the inverse Laplace transform (see, for instance, [4]). Let us prove that the coefficients satisfy growth conditions that allow to take inverse Laplace transforms. We choose $\delta_1 > \delta_0^+$ sufficiently large. Then, from Lemma 4.1, for $k \in \{1, \dots, n\}$, one can find $c_k \neq 0$ and Ψ_k a holomorphic function on Π_{δ_1} such that

$$\frac{\Phi^{(k)}(z)}{\Phi(z)} = c_k z^{-k} + \Psi_k(z), \quad \text{with } |\Psi_k(z)| \leqslant C_k |z|^{-k-1}.$$

Since $z^{-k} = \mathcal{L}(\frac{t^{k-1}}{(k-1)!})(z)$, we can define, for any $b > \delta_1$, the function

$$\varphi_k(t) = -\frac{A_{n-k}}{A_n} \left(\frac{c_k t^{k-1}}{(k-1)!} + \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_k(b + iu) e^{(b+iu)t} \, du \right).$$

Then, when $\delta > \delta_1$, $\varphi_k \in L^2(\mathbb{R}^+, e^{-\delta r} \, dr)$, and φ_k admits $-\frac{A_{n-k} \Phi^{(k)}}{A_n \Phi}$ for Laplace transform in Π_δ. Similarly, with the same arguments as in the radial case, there exists $g_0 \in L_a^2(\mathbb{R}^+, e^{-\delta r} \, dr)$ with Laplace transform $H(z)/A_n \Phi(z)$ in Π_δ. We take the inverse Laplace transform of Eq. (10) to obtain

$$(-1)^n t^n g(t) = g_0(t) + \sum_{j=0}^{n-1} (-1)^j \int_0^t s^j g(s) \varphi_{n-j}(t - s) \, ds, \tag{11}$$

where g_0 and g have support in $(a, +\infty)$. For $\delta > 0$ and $n \in \mathbb{N}^*$, we write $L^2_{n,\delta}((a, +\infty))$ for functions in $L^2(\mathbb{R}^+, (1+t)^{2n-2}e^{-\delta t})$ with support in $(a, +\infty)$. For $g \in L^2_{n,\delta}((a, +\infty))$, let us define on $\mathbb{R}^+$,

$$
\mathcal{N}(g)(t) = \begin{cases} \sum_{j=0}^{n-1}(-1)^{j+n}t^{-n}\int_0^t s^j g(s)\varphi_{n-j}(t-s)\,ds & \text{if } t \geqslant a, \\ 0 & \text{else.} \end{cases}
$$

Lemma 5.1. *There exists $\delta_2 > \delta_1$ such that, for all $\delta \geqslant \delta_2$, the operator*

$$
\mathcal{N} : L^2_{n,\delta}\big((a, +\infty)\big) \to L^2_{n,\delta}\big((a, +\infty)\big)
$$

has norm strictly smaller than 1.

Proof. Let $\delta > \delta_1$ and $\delta' \in (\delta_1, \delta)$. By the Cauchy–Schwarz inequality,

$$
\int_a^{+\infty} t^{-2n}\left(\int_0^t s^j g(s)\varphi_{n-j}(t-s)\,ds\right)^2 (1+t)^{2n-2}e^{-\delta t}\,dt
$$

$$
\leqslant a^{-2n}C(\varphi_{n-j}, \delta)\int_a^t g(s)^2(1+s)^{2n-2}e^{-\delta s}\,ds,
$$

where

$$
C(\varphi_{n-j}, \delta) = \left(\int_0^{+\infty} \big|\varphi_{n-j}(u)\big|e^{-(\delta/2)u}\,du\right)^2 \leqslant \frac{1}{\delta - \delta'}\int_0^{+\infty} \varphi_{n-j}(u)^2 e^{-\delta' u}\,du.
$$

Thus, since $\varphi_{n-j} \in L^2(\mathbb{R}^+, e^{-\delta' r}\,dr)$, $\mathcal{N}(L^2_{n,\delta}((a, +\infty))) \subset L^2_{n,\delta}((a, +\infty))$ and one can find $C_{n,a} > 0$ such that

$$
\|\mathcal{N}\| \leqslant \frac{C_{n,a}}{\delta - \delta'}.
$$

Then it is sufficient to choose $\delta_2 > \delta' + C_{n,a}$ such that $\mathcal{N}$ has norm strictly smaller than 1. $\quad\square$

Therefore, when $\delta > \delta_2$, Eq. (11) has a unique solution g in $L^2_{n,\delta_2}((a, +\infty)) \subset L^2_a(\mathbb{R}^+, e^{-\delta r}r^{1/2}\,dr)$. Moreover, by taking Laplace transform of (11), the Laplace transform of g satisfies Eq. (10) and g is the unique solution of Eq. (9). $\quad\square$

Finally, $T_l^*(L^2_a(\mathbb{R}^+, e^{\delta r}r^{-1/2}\,dr))$ is dense in $L^2_a(\mathbb{R}^+, e^{\delta r}r^{(d-2)/2}\,dr)$ and Theorem 2.2 is proved.

Final remark. Let us mention that this study leads to a natural problem of complex analysis. We have given sufficient conditions on the window φ such that R_φ is injective on $\bigcap_{\delta > \delta_0} L^2_\delta(\mathbb{R}^d)$. More precisely, under these assumptions, we have found, for each l, an abscissa $\delta(l) > \delta_0$ such that T_l is injective on $L^2(\mathbb{R}^+, e^{\delta(l)r}r^{(d-2)/2}\,dr)$. We would like to

find necessary and sufficient conditions on the window φ for injectivity of R_φ on $L^2_\delta(\mathbb{R}^d)$ for a fixed $\delta > \delta_0$. The radial case emphasizes the necessary condition that Φ, the Laplace transform of φ, must have no zero on Π_δ (a condition which cannot be written easily on φ itself). Thus, one may consider this problem on the Laplace transform domain, using complex analysis methods. When considering the Laplace transform, it is natural to work first with Hardy spaces H^2 and H^∞ of the half-plane $\{z \in \mathbb{C};\ \Re(z) > 0\}$. We can state the following characterization which relies on the theory of outer functions and invariant subspaces (see, for instance, [7]).

Proposition 5.2. *Let $d = 2$ and $\varphi \in \mathcal{W}_{\delta_0}$. Then, for $\delta > \delta_0$, T_0 is injective on $L^2(\mathbb{R}^+, e^{\delta r}\, dr)$ if and only if $\Phi(z) \neq 0$ for all $z \in \{z \in \mathbb{C};\ \Re(z) \geqslant \delta/2\}$, where*

$$\Phi(z) = \mathcal{L}\big(\varphi t^{-1/2}\big)(z) = \int\limits_0^{+\infty} e^{-zt} \varphi(t) t^{-1/2}\, dt.$$

Proof. We have already seen that T_0 is injective on $L^2(\mathbb{R}^+, e^{\delta r}\, dr)$ if and only if T_0^* has a dense range in $L^2(\mathbb{R}^+, e^{\delta r}\, dr)$. Since $L^2(\mathbb{R}^+) \cap L^2(\mathbb{R}^+, r^{1/2}\, dr)$ is dense in $L^2(\mathbb{R}^+, r^{1/2}\, dr)$ and $L^2(\mathbb{R}^+, dr)$, T_0^* has a dense range in $L^2(\mathbb{R}^+, e^{\delta r}\, dr)$ if and only if

$$\left\{ \int\limits_0^u g(p)\varphi(u - p)(u - p)^{(d-3)/2} e^{-(\delta(u-p))/2}\, dp;\ g \in L^2\big(\mathbb{R}^+\big) \right\}$$

is dense in $L^2\big(\mathbb{R}^+\big)$.   (12)

By Laplace transform we have a correspondence between $L^2(\mathbb{R}^+)$ and H^2, the Hardy space of the half-plane $\{z \in \mathbb{C};\ \Re(z) > 0\}$. Actually, the Paley–Wiener theorem [7, p. 131] states that $g \in L^2(\mathbb{R}^+)$ if and only if its Laplace transform belongs to H^2. Since $\varphi \in \mathcal{W}_{\delta_0}$, with $\delta_0 < \delta$, $\tau_{\delta/2}\Phi(z) := \Phi(z + \delta/2) \in H^\infty$. Then (12) holds if and only if

$$\big\{ \tau_{\delta/2}\Phi G;\ G \in H^2 \big\} \quad \text{is dense in } H^2.$$

By the Lax theorem [7, p. 107] this holds if and only if $\tau_{\delta/2}\Phi$ is an outer function of H^∞. But, since $\tau_{\delta/2}\Phi$ is continuous on the imaginary axis, it is an outer function of H^∞ if and only if $\tau_{\delta/2}\Phi(z) \neq 0$ for all $z \in \{z \in \mathbb{C};\ \Re(z) \geqslant 0\}$. $\square$

We do not know whether the injectivity holds for R_φ under the assumptions of Proposition 5.2. An easy modification of the proof above has allowed us to prove the injectivity of T_1, T_2 and T_3 on $L^2(\mathbb{R}^+, e^{\delta r}\, dr)$.

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